Functional analysis — Exercises*

Part 6: The Banach–Steinhaus, open mapping and closed graph theorems; Completeness of the trigonometric system in $L_2[0,2\pi]$; Fourier series; Fourier transform

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Recall that a subset A of a topological space X is called *nowhere dense* if its closure has empty interior, i.e. $\operatorname{int}\overline{A}=\varnothing$. A set $B\subset X$ is called a set *of first category* (or *meager*), provided that $B=\bigcup_{n=1}^{\infty}A_n$, where each A_n is nowhere dense. If B is not of first category, we call it a set *of second category*. Therefore, any topological space X can be either of first category in itself, which means that we can express it as a countable union of nowhere dense sets, or of second category in itself. The Baire category theorem says that every complete metric space, in particular, every Banach space, is of second category in itself.

It is easy to show that the collection of first category subsets of X forms a σ-ideal, that is, it is closed under taking subsets and countable unions. Note also the following useful characterization which can be proved easily by de Morgan's laws: A set $A\subset X$ is of first category if and only if its complement is the intersection of countable many sets with dense interiors.

● **Problem 6.1.** Let $1\leq p<q<\infty$. Show that for each $r>0$ the set

$$\{f\in L_q[0,1]\colon \|f\|_q\leq r\},$$

as a subset of the space $(L_p[0,1],\|\cdot\|_p)$, is closed and has empty interior. Conclude that $L_q[0,1]$ is a first category subset of $(L_p[0,1],\|\cdot\|_p)$.

● **Problem 6.2.** Using the Baire category theorem show that $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$. More generally, if X is a complete metric space without isolated points, then there are no countable dense G_δ subsets of X.

● **Problem 6.3.** Show that an infinite-dimensional Banach space cannot have a countable Hamel basis.

● **Problem 6.4.** Let X be a normed space, Y be a Banach space and $\Lambda_n\in\mathscr{L}(X,Y)$ for $n\in\mathbb{N}$. Suppose that there exists a dense subset $D\subseteq X$ such that the sequence $(\Lambda_n x)_{n=1}^{\infty}$ is norm convergent in Y for each $x\in D$. Prove that $(\Lambda_n)_{n=1}^{\infty}$ is pointwise convergent on the whole of X.

● **Problem 6.5.** Let $(X,\|\cdot\|)$ be an infinite-dimensional normed space.

(a) Construct an unbounded injective linear operator from $(X,\|\cdot\|)$ onto itself.

(b) Define a new norm on X by $\|x\|_T=\|Tx\|$ and show that $(X,\|\cdot\|)$ and $(X,\|\cdot\|_T)$ are isometrically isomorphic, even though the topologies induced by $\|\cdot\|$ and $\|\cdot\|_T$ are different.

● **Problem 6.6.** Prove that on every infinite-dimensional (real or complex) vector space one can define an incomplete norm.

● **Problem 6.7.** Let X and Y be Banach spaces and $T\colon X\to Y$ be a linear operator. Show that T is bounded if and only if $y^*\circ T\in X^*$ for every $y^*\in Y^*$.

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*Evaluation: ●=2pt, ●=3pt, ●=4pt

● **Problem 6.8.** Let Y and Z be closed linear subspaces of a Banach space X such that $Y \cap Z = \{0\}$ and $Y + Z = X$. Show that

$$\inf\Big\{\|y - z\|\colon y \in S_Y,\, z \in S_Z\Big\} > 0.$$

● **Problem 6.9.** Let $\sum_{n=1}^{\infty} x_n$ be a series in a Banach space X whose every subseries is weakly convergent. Prove that

$$\sup_{x^* \in B_{X^*}} \sum_{n=1}^{\infty} |\langle x_n, x^* \rangle| < \infty.$$

We denote by $\mathbb{T}$ the unit circle. Any function $F\colon \mathbb{T} \to \mathbb{C}$ is identified with a 2π-periodic function on the real line $\mathbb{R}$ given by $f(t) = F(e^{\mathrm{i}t})$. In particular, any $f \in C(\mathbb{T})$ can be regarded naturally as a continuous 2π-periodic function on $\mathbb{R}$. Similarly, we identify $L_p(\mathbb{T})$ with $L_p[-\pi,\pi]$ or $L_p[0, 2\pi]$, but keep in mind that our convention is to consider *normalized* Lebesgue measures. For example, $L_p[-\pi,\pi]$ stands here for the space $L_p(\mu)$, where the measure μ on $[-\pi,\pi]$ is given by $\mathrm{d}\mu = (2\pi)^{-1}\,\mathrm{d}x$.

For any $f \in L_1(\mathbb{T})$ we define its *Fourier coefficients* with respect to the trigonometric orthonormal system $(e^{\mathrm{i}nx}\colon n \in \mathbb{Z})$ (which happens to be complete in the Hilbert space $L_2[0, 2\pi]$ due to Fejér's theorem):

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-\mathrm{i}nx}\,\mathrm{d}x \quad (n \in \mathbb{Z}).$$

Recall that the partial Fourier sum $s_N(f; x) = \sum_{n=-N}^{N} \widehat{f}(n) e^{\mathrm{i}nx}$ can be expressed as the integral

$$s_N(f; x) = \frac{1}{2\pi}\int_0^{2\pi} f(t) D_N(x - t)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(x - t) D_N(t)\,\mathrm{d}t,$$

where $(D_N(t))_{N=1}^{\infty}$ if the *Dirichlet kernel*, that is, a sequence of real-valued functions on $\mathbb{R}$ defined as

$$D_N(t) = \sum_{n=-N}^{N} e^{\mathrm{i}nt} = \frac{\sin\big(N + \frac{1}{2}\big)t}{\sin\frac{t}{2}}.$$

For any $f \in L_1(\mathbb{R})$ we define its *Fourier transform* by the formula

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t) e^{-\mathrm{i}\xi t}\,\mathrm{d}t$$

and then $f \mapsto \widehat{f}$ maps $L_1(\mathbb{R})$ into the Banach space $C_0(\mathbb{R})$ of continuous functions vanishing at infinity. Recall also that if $f, g \in L_1(\mathbb{R})$, then the formula

$$f * g(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x - y) g(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(y) g(x - y)\,\mathrm{d}y$$

defines the *convolution* of f and g which is again a function from $L_1(\mathbb{R})$ satisfying $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.

● **Problem 6.10.** Show that a function $f \in L_1(\mathbb{T})$ is real-valued if and only if

$$\widehat{f}(n) = \overline{\widehat{f}(-n)} \quad \text{for every } n \in \mathbb{Z}.$$

● **Problem 6.11.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function of period 1. Show that for every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t)\,\mathrm{d}t.$$

Problem 6.12. Give an example of a Banach space which contains two dense (obviously not closed) linear subspaces Y and Z such that $Y \cap Z = \{0\}$.

Problem 6.13. Let $f \in C(\mathbb{T})$.
(a) Suppose that $f(0) = 0$ and f is differentiable at 0. Show that

$$g(x) := \frac{f(x)}{e^{-\mathrm{i}x} - 1} \in C(\mathbb{T})$$

and

$$\sum_{n=-N}^{N} \widehat{f}(n) = \widehat{g}(N+1) - \widehat{g}(-N) \quad \text{for every } N \in \mathbb{N}.$$

Conclude that $s_N(f;0) \to f(0)$.
(b) Show that if f is differentiable at a point x, then

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n) e^{inx} = f(x).$$

Problem 6.14. Show that the L_1-norms (under the normalized Lebesgue measure on $[-\pi, \pi]$) of the Dirichlet kernel $(D_n(t))_{n=1}^{\infty}$ satisfy

$$\|D_n\|_1 < 3 + \frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k}.$$

Recall that we have shown (Lemma 8.7) that the L_1-norms of the Dirichlet kernel satisfy the lower estimate

$$\frac{4}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} < \|D_n\|_1$$

which, of course, implies that $\|D_n\|_1 \to \infty$. This was a crucial property for answering negatively the two important questions: (a) Is the Fourier series of any function $f \in C(\mathbb{T})$ convergent pointwise at each point? (b) Is the map $L_1(\mathbb{T}) \ni f \mapsto (\widehat{f}(n))_{n\in\mathbb{Z}}$ onto $c_0(\mathbb{Z})$?

Problem 6.15. By considering the Fourier series of a suitable continuous periodic function, prove the following formula:

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi nx}{n^2} = \pi^2\Big(x^2 - x + \frac{1}{6}\Big) \quad \text{for every } x \in [0,1].$$

You are allowed to use the criterion of pointwise convergence of $(s_n(f;x))_{n=1}^{\infty}$ given in Problem 6.13, namely, that the Fourier series converges to the value of a given function f at every point of differentiability of f.

Problem 6.16. Calculate the Fourier transform $\widehat{f}$ for the following functions:

(a) $f(x) = e^{-a|x|}$ $(a > 0)$,

(b) $f(x) = \mathbb{1}_{[-L,L]}(x)$ $(L > 0)$,

(c) $f(x) = \begin{cases} L - |x| & \text{for } |x| \le L \\ 0 & \text{for } |x| > L. \end{cases}$

● **Problem 6.17.** For any $f \in L_1(\mathbb{R})$ consider a function $g(x) = -\mathrm{i}xf(x)$. Show that if $g \in L_1(\mathbb{R})$, then the Fourier transform $\widehat{f}$ is differentiable and satisfies

$$\widehat{f}\,' = \widehat{g}.$$

● **Problem 6.18.** Let $f(x) = e^{-1/x^2}$ for $x \in \mathbb{R}$. Show that both f and $\widehat{f}$ satisfy the same differential equation $f' + xf = 0$ and conclude that $\widehat{f} = f$.

● **Problem 6.19.** Let X and Y be Banach spaces and $T \in \mathscr{L}(X, Y)$ be a surjective operator. Show that there exists $\varepsilon > 0$ such that for every $S \in \mathscr{L}(X, Y)$ satisfying $\|T - S\| < \varepsilon$ the range $S(X)$ is dense in Y.

● **Problem 6.20.** Let $n_1 < n_2 < \ldots$ be a strictly increasing sequence of natural numbers and define E to be the set of all $x \in [0, 2\pi]$ for which the sequence $(\sin n_k x)_{k=1}^{\infty}$ is convergent. Using the Riemann–Lebesgue lemma show that E is of Lebesgue measure zero.

● **Problem 6.21.** Mimicking the original proof of the Banach–Steinhaus theorem prove the following version of the uniform boundedness principle on compact convex sets: Let X and Y be normed spaces, $K \subset X$ be a compact convex set and $\{T_\alpha \colon \alpha \in A\} \subset \mathscr{L}(X, Y)$ be a family of operators such that

$$\sup_{\alpha \in A} \|T_\alpha x\| < \infty \quad \text{for every } x \in K.$$

Then there exists a bounded set $B \subset Y$ such that $T_\alpha(K) \subseteq B$ for every $\alpha \in A$.

According to Theorem 8.6, for every $x_0 \in \mathbb{R}$ there exists a dense G_δ subset E of the Banach space $C(\mathbb{T})$ such that for every $f \in E$ the Fourier series of f is divergent at x_0. More precisely, for every $f \in E$ we have $\sup_n |s_n(f; x_0)| = \infty$. Using the Baire category theory one can strengthen this result in the way that for a large set of continuous functions the Fourier series diverges in uncountably many points. This is the content of the next problem.

● **Problem 6.22.** Show that there exists a dense G_δ set $E \subset C(\mathbb{T})$ such that for every $f \in E$ the set

$$\bigl\{x \in \mathbb{R} \colon \sup_n |s_n(f; x)| = \infty\bigr\}$$

is a dense G_δ subset of $\mathbb{R}$.

● **Problem 6.23.** Define D to be the set of all functions in $C[0, 1]$ for which there is at least one point $x \in [0, 1)$ such that the right derivative $f'_+(x)$ exists and is finite. Show that D is a set of first category in $C[0, 1]$. Hence, by the Baire category theorem, the set of nowhere differentiable continuous functions on $[0, 1]$ is of second category.

● **Problem 6.24.** For any $f \in L_2(\mathbb{T})$ and $n \in \mathbb{N}$ define

$$\Lambda_n f = \sum_{k=-n}^{n} \widehat{f}(k).$$

Show that the set of those $f \in L_2(\mathbb{T})$ for which the sequence $(\Lambda_n f)_{n=1}^{\infty}$ is convergent is a dense subset of first category in $L_2(\mathbb{T})$. Therefore, the set of functions in $L_2(\mathbb{T})$ for which the series of Fourier coefficients is divergent is of second category.

● **Problem 6.25.** Let X be a closed subspace of $C[0,1]$ such that every function in X is continuously differentiable on $[0,1]$. Show that $\dim X < \infty$.

● **Problem 6.26.** Let $1 \le p < \infty$ and let X be a closed subspace of $L_p[0,1]$ which is contained in $L_\infty[0,1]$. Show that $\dim X < \infty$ (*cf.* Problem 5.28).

● **Problem 6.27.** Let X be a Banach space and $(x_n)_{n=1}^\infty \subset X$. Suppose that for some $1 \le p < \infty$ we have $T(x^*) := (\langle x_n, x^* \rangle)_{n=1}^\infty \in \ell_q$ for every $x^* \in X^*$, where $\frac{1}{p} + \frac{1}{q} = 1$. Prove the following assertions:

(i) $T \in \mathscr{L}(X^*, \ell_q)$;

(ii) the series $\sum_{n=1}^\infty c_n x_n$ converges unconditionally (i.e. $\sum_{n=1}^\infty c_{\sigma(n)} x_{\sigma(n)}$ converges in norm for every permutation σ of $\mathbb{N}$) for each $(c_n)_{n=1}^\infty$;

(iii) $U \in \mathscr{L}(\ell_p, X)$, where $U((c_n)_{n=1}^\infty) = \sum_{n=1}^\infty c_n x_n$;

(iv) $T = U^*$.

● **Problem 6.28.** Let X be an infinite-dimensional Banach space and $\{x_i : i \in I\}$ be a Hamel basis of X, so that every $x \in X$ can be written uniquely as a finite linear combination $x = \sum_{i \in I} \alpha_i(x) x_i$ (that is, $\alpha_i(x) \neq 0$ for finitely many i's). Each map $x \mapsto \alpha_i(x)$ is a linear funtional X and $\{\alpha_i : i \in I\}$ is the family of *coordinate functionals* associated with the Hamel basis $\{x_i : i \in I\}$.

(a) Show by example that it is possible that $\alpha_i \in X^*$ (i.e. α_i is continuous) for some particular index $i \in I$.

(b) Define $I' = \{i \in I : \alpha_i \in X^*\}$; show that $\sup_{i \in I'} \|\alpha_i\| < \infty$.

(c) Show that the set I' is finite.

● **Problem 6.29.** Give an example of an infinite-dimensional normed space which possesses a Hamel basis for which all the coordinate functionals are continuous. (In view of the previous problem, such a space cannot be complete.)

● **Problem 6.30.** For each $n \in \mathbb{N}$ calculate the convolution $\mathbb{1}_{[-n,n]} * \mathbb{1}_{[-1,1]}$. Prove that it is the Fourier transform of a certain function $f_n \in L_1(\mathbb{R})$ which can be expressed as

$$f_n(x) = c_n \cdot \frac{\sin x \cdot \sin nx}{x^2}$$

with some constant c_n. Show that $\|f_n\|_1 \to \infty$.

It follows that the Fourier transform $f \mapsto \widehat{f}$ maps $L_1(\mathbb{R})$ onto a proper subspace of $C_0(\mathbb{R})$ (i.e. it is not surjective likewise the map $L_1(\mathbb{T}) \ni f \to (\widehat{f}(n))_{n \in \mathbb{Z}} \in c_0(\mathbb{Z})$).

● **Problem 6.31.** Let $f \in C(\mathbb{T})$ and $(s_n(f;x))_{n=1}^\infty$ be the sequence of partial sums of the Fourier series of f at a point $x \in \mathbb{R}$. Prove that $s_n(f;x)/\log n$ converges uniformly to zero, that is,

$$\lim_{n \to \infty} \frac{\|s_n(f)\|_\infty}{\log n} = 0.$$

On the other hand, show that for every sequence $(\lambda_n)_{n=1}^\infty \subset \mathbb{R}$ with $\lambda_n/\log n \to 0$ there exists $f \in C(\mathbb{T})$ such that the sequence $(s_n(f;0)/\lambda_n)_{n=1}^\infty$ is unbounded.

● **Problem 6.32.** We say that a sequence $(\gamma_n)_{n\in\mathbb{Z}}$ of complex numbers has a *multiplier property*, provided that for every function $f \in C(\mathbb{T})$ there exists $\Lambda f \in C(\mathbb{T})$ satisfying

$$\widehat{\Lambda f}(n) = \gamma_n \widehat{f}(n) \quad \text{for each } n \in \mathbb{Z}.$$

Prove that $(\gamma_n)_{n=1}^{\infty}$ has the multiplier property if and only if there exists a complex Borel measure μ on $\mathbb{T}$ such that

$$\gamma_n = \int\limits_{\mathbb{T}} z^n \, \mathrm{d}\mu(z).$$

(The last integral can be also written as $\int_{[0,2\pi)} e^{\mathrm{i}n\theta} \, \mathrm{d}\mu(\theta)$ provided we identify every complex number $z = e^{\mathrm{i}\varphi} \in \mathbb{T}$ with the angle $\varphi \in [0, 2\pi)$ and redefine μ correspondingly.)

● **Problem 6.33.** Let $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Prove the following inequality which can be interpreted as a quantitative version of Heisenberg's uncertainity principle:

$$\|f\|^2_{L^2(\mathbb{R})} \leq 4\pi \cdot \inf_{y\in\mathbb{R}} \left(\int_{\mathbb{R}} |x-y|^2 |f(x)|^2 \, \mathrm{d}x \right)^{1/2} \cdot \inf_{z\in\mathbb{R}} \left(\int_{\mathbb{R}} |\xi - z|^2 \left| \widehat{f}(\xi) \right|^2 \mathrm{d}\xi \right)^{1/2}.$$