Functional analysis — Exam (problem-solving part) September 9, 2019

Second term

- Duration: **120 minutes**.
- Solutions can be written either in English or Polish.
- You do not have to write different solutions on separate sheets.
- Each sheet should be **signed**.

Problem 1. Let $C^{(1)}[0,1]$ be the space of real-valued continuously differentiable functions on $[0,1]$ equipped with the standard norm

$$\|f\|_1 = \|f\|_\infty + \|f'\|_\infty,$$

where $\|\cdot\|_\infty$ is the usual the supremum norm. Let $T\colon (C^{(1)}[0,1], \|\cdot\|_1) \to (C[0,1], \|\cdot\|_\infty)$ be an operator defined by

$$Tf(x) = f'(x) + x^2 f(x).$$

Prove that T is a Fredholm operator and find its Fredholm index. [10 points]

Hint. The corresponding differential equation can be solved explicitly by considering first its homogeneous form and then treating the constant of integration as a variable.

Problem 2. In the Hilbert space $L_2[0,1]$ consider the subspace

$$V = \left\{ f \in L_2[0,1]\colon \int_0^1 f(t)\,\mathrm{d}t = 0 \text{ and } \int_0^1 t f(t)\,\mathrm{d}t = 0 \right\}.$$

Let $P\colon L_2[0,1] \to V$ be the orthogonal projection onto V. Calculate $P(g)$ for the function $g(t) = t^2$. [10 points]

Problem 3. Suppose X is a closed subspace of the real Banach space $C[0,1]$ such that every function in X is continuously differentiable. Prove that X is finite-dimensional. [15 points]

Hint. Apply the closed graph theorem to a suitable operator.

Problem 4. It is known that every function $f \in C(\mathbb{T})$ (which can be identified with a continuous 2π-periodic function on $\mathbb{R}$), differentiable at a given point $x \in \mathbb{R}$ can be expressed as the sum of its Fourier series at that point x. In other words, if f is differentiable at x, then

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \widehat{f}(n) e^{\mathrm{i}nx} = f(x).$$

Using this fact prove the formula

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{n^2} = \pi^2 \Big(x^2 - x + \frac{1}{6} \Big)$$

for every $x \in [0,1]$. [15 points]