Functional analysis — Exam (problem-solving part) June 25, 2019

- Duration: **120 minutes**.
- Solutions can be written either in English or Polish.
- You do not have to write different solutions on separate sheets.
- Each sheet should be **signed**.

Problem 1. Consider an operator $T \in \mathscr{L}(C[0,1])$ defined on the Banach space of continuous real-valued functions on $[0,1]$ by the formula

$$Tf(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Show that T is compact and determine its spectrum $\sigma(T)$. [10 points]

Problem 2. In the Hilbert space $L_2[0,1]$ consider the subspace

$$V = \left\{ f \in L_2[0,1] \colon \int_0^1 f(t)\,\mathrm{d}t = 0 \text{ and } \int_0^1 t f(t)\,\mathrm{d}t = 0 \right\}.$$

Calculate the distance $\mathrm{dist}(g, V)$, where $g(t) = t^5$. [10 points]

Problem 3. Let $1 \leqslant p < q < \infty$. Of course, ℓ_p is a subset of ℓ_q. Prove that the set difference $\ell_q \setminus \ell_p$ is of second category in the Banach space $(\ell_q, \|\cdot\|_q)$. [15 points]

Problem 4. Prove that for every $n \in \mathbb{N}$, $n \geqslant 2$ the convolution

$$\mathbb{1}_{[-1,1]} * \mathbb{1}_{[-2,2]} * \ldots * \mathbb{1}_{[-n,n]}$$

is the Fourier transform of a certain function $g_n \in L_1(\mathbb{R})$. Determine g_3 explicitely and calculate its L_2-norm $\|g_3\|_2$. [15 points]