Functional analysis — Midterm test

April 11, 2019

- Each problem is worth **10 points**.
- Duration: **180 minutes**.
- Solutions can be written either in English or Polish.
- You do not have to write different solutions on separate sheets.
- Each sheet should be **signed**.

Problem 1. (a) A normed space X is called *smooth* provided that for every $x \in S_X$ there exists a unique functional $f_x \in S_{X^*}$ such that $f_x(x) = 1$. Decide whether the following normed spaces over $\mathbb{R}$ are smooth: c_0, ℓ_1, $L_1(\mathbb{R})$, $L_2[0,1]$.

(b) If X is smooth, there exists a mapping $x \mapsto f_x$ from $X \setminus \{0\}$ to $X^* \setminus \{0\}$ such that $\|f_x\| = f_x(x) = 1$ for every $x \in S_X$ and $f_{\lambda x} = \lambda f_x$ for all $x \in X \setminus \{0\}$ and $\lambda > 0$. Prove that for all $x, y \in S_X$ and $\lambda > 0$ with $x + \lambda y \neq 0$ we have

$$\frac{f_x(y)}{\|x\|} \leqslant \frac{\|x+\lambda y\| - \|x\|}{\lambda} \leqslant \frac{f_{x+\lambda y}(y)}{\|x+\lambda y\|}.$$

Problem 2. Let Y be the 2-dimensional subspace of $\mathbb{R}^3$ that contains the vectors $(1,1,0)$ and $(0,0,1)$. Define a functional $f\colon Y \to \mathbb{R}$ by $f(x,y,z) = 2x - z$. We equip $\mathbb{R}^3$ (and also Y) with the Euclidean norm $\|\cdot\|_2$. Calculate $\|f\|$ and find a Hahn–Banach extension $F\colon \mathbb{R}^3 \to \mathbb{R}$ of f, i.e. a linear functional satisfying $F|_Y = f$ and $\|F\| = \|f\|$. Is this extension unique?

Problem 3. Let X be an infinite-dimensional Banach space. Prove that there does not exist a translation invariant positive Borel measure μ on X such that $\mu(V) > 0$ for every nonempty open set $V \subseteq X$ and $\mu(U) < \infty$ for at least one open set $U \subseteq X$. (A measure μ is *translation invariant* if $\mu(E + x) = \mu(E)$ for every Borel set $E \subseteq X$ and any $x \in X$.)

Problem 4. For every even $n \in \mathbb{N}$, consider a linear functional Λ_n defined on the real Banach space $C[0,1]$ by the formula

$$\Lambda_n f = \int_0^{1/n} \Big[f\big(t\big) - f\Big(t + \frac{1}{n}\Big) + f\Big(t + \frac{2}{n}\Big) - \ldots + f\Big(t + \frac{n-2}{n}\Big) - f\Big(t + \frac{n-1}{n}\Big) \Big]\, \mathrm{d}t.$$

(a) Show that $\Lambda_n \in (C[0,1])^*$ and $\|\Lambda_n\| = 1$ for each $n = 2, 4, \ldots$

(b) Prove that the sequence $(\Lambda_{2m})_{m=1}^{\infty}$ does not contain subsequence which is norm convergent in $(C[0,1])^*$.

Problem 5. (a) Consider an operator $T\colon \ell_2 \to \ell_2$ defined on the complex Banach space ℓ_2 by

$$T(x_1, x_2, \ldots) = \Big(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\Big).$$

Show that $\|T\| = 1$, T is compact and $\sigma(T) = \{0\}$.

(b) Define $U \in \mathscr{L}(\ell_2)$ by

$$U(x_1, x_2, \ldots) = \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\Big).$$

Determine the spectrum $\sigma(U)$ and show that for every $V \in \mathscr{L}(\ell_2)$ with $\|V\| < \sqrt{\frac{1}{2}}$ the operator $U + V - \mathrm{i}I$ is invertible.