## **Functional analysis — Midterm test** April 11, 2019

SOLUTIONS

**1.** (a) Using the dualities: $c_0^* \cong \ell_1$, $\ell_1^* \cong \ell_\infty$, $L_1(\mathbb{R})^* \cong L_\infty(\mathbb{R})$ we show that $c_0$, $\ell_1$ and $L_1(\mathbb{R})$ are not smooth. For example, for $x = \mathrm{e}_1 + \mathrm{e}_2 \in S_{c_0}$ we have two functionals $f_1 = \mathrm{e}_1$ and $f = \mathrm{e}_2$ in $S_{\ell_1}$ with $f_1(x) = f_2(x) = 1$. For $y = \mathrm{e}_1 \in S_{\ell_1}$ consider e.g. $g_1 = \mathrm{e}_1$ and $g_2 = (1, 1, 1, \ldots)$ in $S_{\ell_\infty}$ which satisfy $g_1(y) = g_2(y) = 1$. In the unit sphere of $L_1(\mathbb{R})$ take $z = \mathbb{1}_{[0,1]}$ and functionals $h_1 = \mathbb{1}_{[0,1]}$ and $h_2 = \mathbb{1}_{\mathbb{R}}$ from the unit sphere of $L_\infty(\mathbb{R})$. Again, $h_1(z) = h_2(z) = 1$.

To see that $L_2[0,1]$ is smooth, recall that the duality $(L_2[0,1])^* \cong L_2[0,1]$ identifies any $g \in L_2[0,1]$ with the functional $\varphi_g(f) = \int_{[0,1]} f(x)g(x)\,\mathrm{d}x$. If $\|f\|_2 = \|g\|_2 = 1$, then the Cauchy–Schwarz inequality yields $\varphi_g(f) \leqslant 1$ and in order to have equality $f$ and $g$ must be proportional a.e. on $[0,1]$. However, since $\|f\|_2 = \|g\|_2 = 1$, we have $f(x) = g(x)$ a.e. Therefore, for a fixed $f \in S_{L_2[0,1]}$ there exists only one $g \in S_{L_2[0,1]}$ (namely $g = f$) satisfying $\varphi_g(f) = 1$.

(b) First, notice that for all $x, z \in X \setminus \{0\}$ we have $f_z(z) = \|z\|^2 f_{z/\|z\|}(z/\|z\|) = \|z\|^2$ and $f_z(x) = \|z\| f_{z/\|z\|}(x) \leqslant \|z\|\|x\|$. Hence,

$$\begin{aligned}\frac{f_x(y)}{\|x\|} &= \frac{f_x(\lambda y)}{\lambda} = \frac{f_x(x+\lambda y) - f_x(x)}{\lambda} \leqslant \frac{\|f_x\|\|x+\lambda y\| - 1}{\lambda} \\ &= \frac{\|x+\lambda y\| - \|x\|}{\lambda} \\ &\qquad = \frac{\|x+\lambda y\|^2 - \|x\|\|x+\lambda y\|}{\lambda\|x+\lambda y\|} \leqslant \frac{\|x+\lambda y\|^2 - f_{x+\lambda y}(x)}{\lambda\|x+\lambda y\|} \\ &\qquad = \frac{f_{x+\lambda y}(x+\lambda y) - f_{x+\lambda y}(x)}{\lambda\|x+\lambda y\|} = \frac{f_{x+\lambda y}(y)}{\|x+\lambda y\|}.\end{aligned}$$

**2.** The subspace $Y$ is given by the equation $x = y$ and hence we have

$$\|f\| = \max\{|2x - z| \colon 2x^2 + z^2 \leqslant 1\}.$$

By the Cauchy–Schwarz inequality, $|2x - z| = |\sqrt{2}\cdot\sqrt{2}x + (-1)\cdot z| \leqslant \sqrt{3}\sqrt{2x^2+z^2}$ which implies $\|f\| \leqslant \sqrt{3}$. In fact, we have equality for any vector $(x,y,z)$ parallel to $(1,1,-1)$, thus $\|f\| = \sqrt{3}$. Any extension $F$ must be of the form $F(x,y,z) = Ax + By + Cz$ and satisfies $F(x,y,z) = 2x - z$ provided that $x = y$. Hence, $B = 2 - A$ and $C = -1$, thus $F(x,y,z) = Ax + (2-A)y - z$. By the duality $(\ell_2^3)^* \cong \ell_2^3$, we have $\|F\| = \|(A, 2-A, -1)\|_2$, so $\|F\| \leqslant \sqrt{3}$ if and only if $A^2 + (2-A)^2 + 1 \leqslant 3$, which happens only if $A = 1$. Consequently, the Hahn–Banach extension is unique and is given by the formula $F(x,y,z) = x + y - z$.

**3.** As we know, the Riesz lemma implies that there is a sequence $(x_n)_{n=1}^\infty \subset B_X$ such that $\|x_n - x_m\| \geqslant \frac{1}{2}$ for all $m \neq n$. Hence, $B_X$ contains infinitely many mutually disjoint balls of positive radius. By a suitable translation, we infer that any ball in $X$ has the same property. Assuming that there is a measure $\mu$ satisfying the required assumptions, we obtain that the measure of every open ball is infinite; a contradiction.

**4.** (a) Let $n = 2m$. For any $f \in C[0,1]$ we have

$$\begin{aligned}\Lambda_n f &= \sum_{0\leqslant j<m} \int_0^{1/n} f\Big(t + \frac{2j}{n}\Big)\,\mathrm{d}t - \sum_{0\leqslant j<m} \int_0^{1/n} f\Big(t + \frac{2j+1}{n}\Big)\,\mathrm{d}t \\ &= \sum_{0\leqslant k<n} \int_{k/n}^{(k+1)/n} (-1)^k f(t)\,\mathrm{d}t = \int_0^1 f\,\mathrm{d}g,\end{aligned}$$

where the last integral is the Riemann–Stieltjes integral with respect to an absolutely continuous function $g\colon [0,1]\to\mathbb{R}$ such that $g'(t)=(-1)^k$ for $t\in(\frac{k}{n},\frac{k+1}{n})$. We can define $g$ to be piecewise linear and such that $g(\frac{k}{n})=0$ for even $0\leqslant k\leqslant n$ and $g(\frac{k+1}{n})=\frac{1}{n}$ for odd $1\leqslant k<n$. Then $g\in\mathrm{NBV}([0,1])$ represents $\Lambda_n$ by means of the Riesz Representation Theorem. Hence, $\Lambda_n\in(C[0,1])^*$ and $\|\Lambda_n\|=V_0^1(g)=n\cdot\frac{1}{n}=1$.

(b) Consider any even indices $n<N$ and let $\varrho_{n,N}=g_n-g_N$, where $g_n\in\mathrm{NBV}([0,1])$ represents the functional $\Lambda_n$ as in the first part. For every $0\leqslant j<\frac{n}{2}$ we have

$$\varrho_{n,N}\Big(\frac{2j}{n}\Big)\leqslant g_n\Big(\frac{2j}{n}\Big)-0=0\quad\text{ and }\quad\varrho_{n,N}\Big(\frac{2j+1}{n}\Big)\geqslant g_n\Big(\frac{2j+1}{n}\Big)-\frac{1}{N}=\frac{1}{n}-\frac{1}{N}.$$

Hence,

$$\|\Lambda_n-\Lambda_N\|=V_0^1(\varrho_{n,N})\geqslant n\Big(\frac{1}{n}-\frac{1}{N}\Big)=1-\frac{n}{N}\xrightarrow[N\to\infty]{}1$$

and it shows that no subsequence of $(\Lambda_{2m})_{m=1}^\infty$ satisfies the Cauchy condition.

**5.** (a) For every $x=(x_n)_{n=1}^\infty\in\ell_2$ we have $\|Tx\|^2\leqslant\sum_{n=1}^\infty n^{-2}|x_n|^2\leqslant\|x\|^2$, hence $\|T\|\leqslant 1$. Taking $x=\mathrm{e}_1$ we see that $T\mathrm{e}_1=\mathrm{e}_2$ has norm one, so $\|T\|=1$. Obviously, every $x\in B_{\ell_2}$ satisfies $|x_n|\leqslant 1$ for each $n\in\mathbb{N}$, thus $T(B_X)\subseteq\{(y_n)_{n=1}^\infty\colon |y_n|\leqslant\frac{1}{n}\ \text{ for } n\in\mathbb{N}\}$ which, as we know, is a compact set. It follows that $T(B_X)$ is totally bounded, whence $T$ is compact.

By the Riesz–Schauder theorem, $0\in\sigma(T)$ and every nonzero $\lambda\in\sigma(T)$ must be an eigenvalue of $T$, i.e. $Tx=\lambda x$ for some $x\in\ell_2$, $x\neq 0$. But this means that $x_1=0$ and $\frac{1}{k}x_k=\lambda x_{k+1}$ for each $k\in\mathbb{N}$ which implies that $x=0$. Therefore, $T$ has no eigenvalues, thus $\sigma(T)=\{0\}$.

(b) Of course, $U$ is compact for the same reason as $T$ is compact. Assume that $\lambda\neq 0$ is an eigenvalue of $U$, that is, there exists a nonzero $x\in\ell_2$ such that $Tx=\lambda x$. Then, $\frac{1}{k}x_k=\lambda x_k$, i.e. $(\lambda-\frac{1}{k})x_k=0$ for every $k\in\mathbb{N}$. If $\lambda\neq\frac{1}{k}$ for all $k\in\mathbb{N}$, then the last condition implies $x=0$, so such a $\lambda$ is not an eigenvalue. However, if $\lambda=\frac{1}{k}$ for some $k\in\mathbb{N}$, then the condition $Tx=\lambda x$ is equivalent to $x$ being proportional to the $k^{\text{th}}$ canonical vector $\mathrm{e}_k$. We have thus proved that $\sigma_{\mathrm{p}}(U)=\{\frac{1}{k}\colon k\in\mathbb{N}\}$ (it is easily seen that $0\notin\sigma_{\mathrm{p}}(U)$), where each eigenvalue has multiplicity one: $\ker(U-\frac{1}{k}I)=\mathrm{lin}(\mathrm{e}_k)$. Finally, $\sigma(U)=\{0\}\cup\{\frac{1}{k}\colon k\in\mathbb{N}\}$.

Since $\mathrm{i}\notin\sigma(U)$, the operator $U-\mathrm{i}I$ is invertible. Notice that $(U-\mathrm{i}I)(x)=((\frac{1}{k}-\mathrm{i})x_k)_{k=1}^\infty$ and an easy estimate (see part (a)) shows that

$$\|U-\mathrm{i}I\|=\sup_{k\in\mathbb{N}}\Big|\frac{1}{k}-\mathrm{i}\Big|=\sqrt{2}.$$

In view of the 'invertibility result' (Corollary 4.2), if $V\in\mathscr{L}(\ell_2)$ has norm smaller that $\|U-\mathrm{i}I\|^{-1}$, then $U+V-\mathrm{i}I$ is invertible.