Lecture 1

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Gauss' hypergeometric function

Euler and Gauss defined

$$ {}_2F_1\left(\begin{matrix}\alpha \ \beta \\ \gamma\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}\ z^n $$

where $(x)_n = x(x+1)\cdots(x+n-1)$ (Pochhammer symbol).
Examples

1. ${}_2F_1\left(\begin{matrix}1 \ 1 \\ 2\end{matrix}\middle| z\right) = -\frac{1}{z}\log(1-z)$
2. ${}_2F_1\left(\begin{matrix}1/2 \ 1 \\ 1\end{matrix}\middle| z\right) = (1-z)^{-1/2}$
3. ${}_2F_1\left(\begin{matrix}1/2 \ 1/2 \\ 1\end{matrix}\middle| z\right) = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-zt^2)}}$

We will use the notation $F(\alpha, \beta, \gamma | z)$.

Differential equation

$$z(z-1)F'' + ((\alpha+\beta+1)z-\gamma)F' + \alpha\beta F = 0$$

This is a Fuchsian differential equation of order 2 with singularities at $0, 1, \infty$.

Local solutions at $z = 0$:

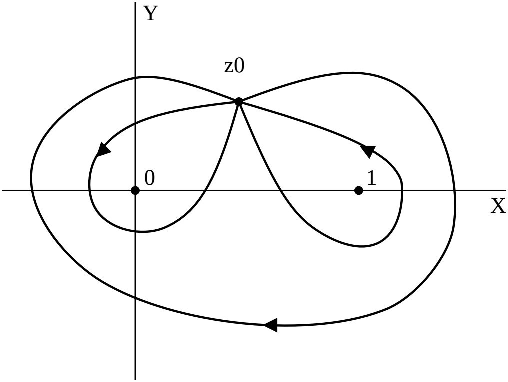
- $F(\alpha, \beta, \gamma | z)$
- $z^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma | z)$

At $z = \infty$

- $(1/z)^{\alpha} F(\alpha, \alpha+1-\gamma, \alpha+1-\beta | 1/z)$
- $(1/z)^{\beta} F(\beta, \beta+1-\gamma, \beta+1-\alpha | 1/z)$

Monodromy

Let V be solution space of hypergeometic equation. Analytic continuation gives us the monodromy representation $\rho : \pi_1(\mathbb{C} \setminus \{0, 1\}) \to GL(V)$



Monodromy matrices: $M_i := \rho(g_i), \ i = 0, 1, \infty$ with relation $M_\infty M_1 M_0 = 1$.

Monodromy properties

Denote $e(x) = \exp(2\pi ix)$. Eigenvalues

- $M_0: \; 1, e(-\gamma)$
- $M_1: \; 1, e(\gamma - \alpha - \beta)$
- $M_\infty: \; e(\alpha), e(\beta)$

Proposition

Let $A, B \in GL(2, \mathbb{C})$ with eigenvalues a_1, a_2 resp b_1, b_2 and such that $A^{-1}B$ has eigenvalue 1. Let $G = \langle A, B \rangle$. Then

$$G \text{ irreducible} \iff \{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset.$$

In that case A, B are uniquely determined up to common conjugation.

Application: $A = M_0^{-1}, B = M_\infty$.
So, monodromy irreducible $\iff$ $\{\alpha, \beta\} \pmod{\mathbb{Z}}$ and $\{0, \gamma\} \pmod{\mathbb{Z}}$ disjoint.

Explicit matrices

Characteristic polynomial

- of M_0^{-1} is $x^2 - (1 + e(\gamma))x + e(\gamma)$
- of M_∞ is $x^2 - (e(\alpha) + e(\beta))x + e(\alpha + \beta)$.

Up to common conjugation:

$$M_0^{-1} = \begin{pmatrix} 0 & -e(\gamma) \\ 1 & 1 + e(\gamma) \end{pmatrix} \qquad M_\infty = \begin{pmatrix} 0 & -e(\alpha+\beta) \\ 1 & e(\alpha) + e(\beta) \end{pmatrix}.$$

Theorem

Suppose $\alpha, \beta, \gamma \in (0, 1]$. Then there exists a Hermitian form F on $\mathbb{C}^2$ such that $F(g\mathbf{x}, g\mathbf{y}) = F(\mathbf{x}, \mathbf{y})$ for all $g \in \langle M_0, M_\infty \rangle$. This form is definite if and only if γ lies between α and β.

Schwarz's list

In 1873 H.A. Schwarz gave a list of all parameter triples α, β, γ such that ${}_2F_1\left(\begin{matrix}\alpha\ \beta\\ \gamma\end{matrix}\middle| z\right)$ is algebraic in z. All triples are in $\mathbb{Q}$.

An example, ${}_2F_1\left(\begin{matrix}19/60\ 49/60\\ 4/5\end{matrix}\middle| z\right)$ is algebraic of degree 720. Its Galois group is a central extension of the alternating group A_5 by a cyclic group of order 60.

Such functions were used in F.Klein's "*Vorlesungen über das Ikosaeder*".

Clausen-Thomae functions

Let $\alpha_1, \dots, \alpha_d$ and $\beta_1, \dots, \beta_{d-1}$ be any parameters and $\beta_d = 1$. Define

$$ {}_dF_{d-1}\left(\begin{matrix}\alpha_1, \dots, \alpha_d \\ \beta_1, \dots, \beta_{d-1}\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_d)_k}{(\beta_1)_k \cdots (\beta_{d-1})_k k!} z^k $$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the *Pochhammer symbol*.
Hypergeometric equation

$$ z(D+\alpha_1)\cdots(D+\alpha_d)F = (D+\beta_1-1)\cdots(D+\beta_d-1)F, \quad D = z\frac{d}{dz} $$

This is a Fuchsian differential equation of order d with singularities at $0, 1, \infty$.

Monodromy

Theorem

Monodromy irreducible $\iff$ $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_d\}$ disjoint modulo $\mathbb{Z}$.

Levelt's theorem (1960)

Write $\prod_{i=1}^{d}(x - e(\beta_i)) = x^d + B_1 x^{d-1} + \cdots + B_d$ and $\prod_{i=1}^{d}(x - e(\alpha_i)) = x^d + A_1 x^{d-1} + \cdots + A_d$. Then up to common conjugation M_∞ and M_0^{-1} equal

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & -A_d \\ 1 & 0 & \ldots & 0 & -A_{d-1} \\ 0 & 1 & \ldots & 0 & -A_{d-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -A_1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & \ldots & 0 & -B_d \\ 1 & 0 & \ldots & 0 & -B_{d-1} \\ 0 & 1 & \ldots & 0 & -B_{d-2} \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & -B_1 \end{pmatrix}$$

Invariant Hermitean form

Suppose that $\alpha_i, \beta_j \in \mathbb{R}$ for all i, j.

Theorem

Then there exists a unique (up to scalars) monodromy invariant Hermitean form F. That is, $F(gx, gy) = F(x, y)$ for all monodromy matrices g.

Theorem

The Hermitian form F is definite if and only if the sets $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_d\}$ interlace modulo $\mathbb{Z}$.

Interlacing

Interlacing sets in $[0, 1)$ when $d = 4$,

Two sets $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_d\}$ are said to interlace modulo $\mathbb{Z}$ if the sets $\{\alpha_i - \lfloor \alpha_i \rfloor\}_{i=1,\ldots,d}$ and $\{\beta_i - \lfloor \beta_i \rfloor\}_{i=1,\ldots,d}$ interlace in $[0, 1)$.

Finite monodromy

Suppose $\{\alpha_1, \ldots, \alpha_d\}$ and $\{\beta_1, \ldots, \beta_d\}$ are sets of rational numbers disjoint modulo $\mathbb{Z}$. Let N be a common denominator.

Suppose the monodromy group is finite. Then there is an invariant definite Hermitian form. Hence the parameter sets interlace mod $\mathbb{Z}$.

Monodromy matrices have elements in $\mathbb{Z}[e(1/N)]$. Apply Galois element $\zeta_N \to \zeta_N^p, \gcd(p, N) = 1$ to monodromy matrices. Get monodromy with parameter sets $\{p\alpha_i\}$ and $\{p\beta_i\}$. Hence they interlace modulo $\mathbb{Z}$.

Algebraic hypergeometric functions

Converse also holds.

Theorem (Beukers-Heckman, 1986)

A hypergeometric group is finite if and only if the sets $\{p\alpha_1, \ldots, p\alpha_d\}$ and $\{p\beta_1, \ldots, p\beta_d\}$ interlace mod $\mathbb{Z}$ for every integer p with $\gcd(p, N) = 1$.

Example:

$$F(x) = {}_8F_7\left(\begin{matrix} 1/30\ 7/30\ 11/30\ 13/30\ 17/30\ 19/30\ 23/30\ 29/30 \\ 1/5\ 1/3\ 2/5\ 1/2\ 3/5\ 2/3\ 4/5 \end{matrix}\,\middle|\, x\right)$$

which equals

$$\sum_{n\geq 0} \frac{(30n)!n!}{(15n)!(10n)!(6n)!}\left(\frac{z}{2^{14}3^9 5^5}\right)^n.$$

Appell's functions

Consider

$$F_1(\alpha,\beta,\beta',\gamma,x,y)=\sum_{m,n\geq 0}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}}\,x^m y^n$$

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',x,y)=\sum_{m,n\geq 0}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n}\,x^m y^n$$

$$F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y)=\sum_{m,n\geq 0}\frac{(\alpha)_m(\alpha)_n(\beta)_m(\beta')_n}{m!n!(\gamma)_{m+n}}\,x^m y^n$$

$$F_4(\alpha,\beta,\gamma,\gamma',x,y)=\sum_{m,n\geq 0}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!(\gamma)_m(\gamma')_n}\,x^m y^n$$

These are the Appell hypergeometric functions in two variables, introduced in 1880.

Appell differential equation

The Appell functions satisfy a system of partial linear differential equations of order 2. For example, $F_4(\alpha, \beta, \gamma, \gamma', x, y)$ satisfies

$$x(1-x)F_{xx} - y^2 F_{yy} - 2xyF_{xy} + \gamma F_x - (\alpha+\beta+1)(xF_x + yF_y) = \alpha\beta F$$

$$y(1-y)F_{yy} - x^2 F_{xx} - 2xyF_{xy} + \gamma' F_y - (\alpha+\beta+1)(xF_x + yF_y) = \alpha\beta F$$

Studied by Picard and Goursat.

Lauricella functions

Further generalisation by Lauricella (1893),

$$F_A(a,\mathbf{b},\mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad |x_1| + \cdots + |x_n| < 1$$

$$F_B(\mathbf{a},\mathbf{b},c|\mathbf{x}) = \sum_{\mathbf{m}\geq 0} \frac{(\mathbf{a})_{\mathbf{m}}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad \forall i : |x_i| < 1$$

$$F_C(a,b,\mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(b)_{|\mathbf{m}|}}{(\mathbf{c})_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad |\sqrt{x_1}| + \cdots + |\sqrt{x_n}| < 1$$

$$F_D(a,\mathbf{b},c|\mathbf{x}) = \sum_{\mathbf{m}\geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}} \quad \forall i : |x_i| < 1$$

When $n = 2$ these functions coincide with Appell's F_2, F_3, F_4, F_1 respectively. When $n = 1$, they all coincide with Gauss' ${}_2F_1$.

The A-polytope

Start with a finite subset $A \subset \mathbb{Z}^r \subset \mathbb{R}^r$. We assume

- The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$
- There is a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Define a vector of parameters

$$\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$$

Remember:

The set A and the vector α will completely characterise a so-called A-hypergeometric system of differential equations.

Lattice of relations

Write $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$. The lattice of relations $L \subset \mathbb{Z}^N$ is formed by all $\mathbf{l} = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that

$$l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}.$$

Let h be the form such that $h(\mathbf{a}_i) = 1$ for $i = 1, \ldots, r$.
Apply h to any relation $l_1\mathbf{a}_1 + \cdots + l_N\mathbf{a}_N = \mathbf{0}$.
Then we get $\sum_{i=1}^{N} l_i = 0$ for all $\mathbf{l} \in L$.

Formal A-hypergeometric series

Choose $\gamma_1, \ldots, \gamma_N$ such that

$$\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N.$$

Note that $\gamma = (\gamma_1, \ldots, \gamma_N)$ is determined modulo $L \otimes \mathbb{R}$.
Let $v_1, \ldots, v_N$ be variables and consider

$$\Phi = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdots \Gamma(l_N + \gamma_N + 1)}.$$

Homogeneity equations

For any $j = 1, \ldots, N$ write $\mathbf{a}_j = (a_{1j}, \ldots, a_{rj})^t$.
Note that $a_{i1} l_1 + \cdots + a_{iN} l_N = 0$ for every $\mathbf{l} \in L$ and every i.
For $i = 1, \ldots, r$ define the differential operator

$$Z_i = a_{i1} v_1 \frac{\partial}{\partial v_1} + \cdots + a_{iN} v_N \frac{\partial}{\partial v_N}$$

Note that

$$\begin{aligned} Z_i(v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}) &= (a_{i1}(l_1+\gamma_1) + \cdots + a_{iN}(l_N+\gamma_N))\mathbf{v}^{\mathbf{l}+\gamma} \\ &= \alpha_i \mathbf{v}^{\mathbf{l}+\gamma} \end{aligned}$$

Hence $(Z_i - \alpha_i)\Phi = 0$.
These equations reflect the homogeneity property

$$\Psi(\mathbf{t}^{\mathbf{a}_1} v_1, \cdots, \mathbf{t}^{\mathbf{a}_N} v_N) = \mathbf{t}^{\alpha} \Psi(v_1, \ldots, v_N)$$

for any solution Ψ and any $\mathbf{t} \in (\mathbb{C}^*)^r$. Here $\mathbf{t}^{\mathbf{a}}$ denotes $t_1^{a_1} \cdots t_r^{a_r}$.

Box equations

Let $(\lambda_1, \ldots, \lambda_N) \in L$. Define the operator

$$\Box_\lambda = \prod_{\lambda_i > 0} \left(\frac{\partial}{\partial v_i}\right)^{\lambda_i} - \prod_{\lambda_i < 0} \left(\frac{\partial}{\partial v_i}\right)^{-\lambda_i}$$

Let λ^+ be the vector with components $\max(0, \lambda_i)$ and λ^- with components $\min(0, -\lambda_i)$. Then $\lambda = \lambda^+ - \lambda^-$.
Notice that

$$\Box_\lambda \frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})} = \frac{\mathbf{v}^{\mathbf{l}+\gamma-\lambda^+}}{\Gamma(\mathbf{l} + \gamma - \lambda^+ + \mathbf{1})} - \frac{\mathbf{v}^{\mathbf{l}+\gamma+\lambda^-}}{\Gamma(\mathbf{l} + \gamma + \lambda^- + \mathbf{1})}.$$

Since $\lambda^+ - \lambda^- = \lambda \in L$ summation over L gives equal sums that cancel.

A-hypergeometric system of equations

The system of differential equations

$$\Box_\lambda \Phi = 0, \qquad \lambda \in L$$

and

$$(Z_i - \alpha_i)\Phi = 0, \qquad i = 1, 2, \ldots, r$$

was first explicitly described by Gel'fand, Kapranov and Zelevinsky around 1988. They called these equations *A-hypergeometric equations* and their analytic solutions *A-hypergeometric functions*.
We denote the system by $H_A(\alpha)$.
In his book on *Generalised hypergeometric equations*, which appeared in 1990, B.Dwork independently arrives at the same equations, but in the language of differential modules.

Example 1, Gauss ${}_2F_1$

Gauss $F(\alpha, \beta, \gamma | z)$ is proportional to

$$\sum_{n \geq 0} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+\gamma)\Gamma(n+1)} \ z^n.$$

Application of Γ-identities gives

$$\sum_{n \geq 0} \frac{z^n}{\Gamma(-n+1-\alpha)\Gamma(-n+1-\beta)\Gamma(n+\gamma)\Gamma(n+1)}.$$

The lattice L is spanned by $(-1, -1, 1, 1)$. A set A is given by

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

A-hypergeometric equations for ${}_2F_1$

Recall that $L = \langle(-1,-1,1,1)\rangle$ and $F(\alpha,\beta,\gamma|z)$ is proportional to

$$\sum_{n\geq 0} \frac{z^n}{\Gamma(-n+1-\alpha)\Gamma(-n+1-\beta)\Gamma(n+\gamma)\Gamma(n+1)}.$$

Formal A-hypergeometric solution:

$$\sum_{n\geq 0} \frac{v_1^{-n-\alpha} v_2^{-n-\beta} v_3^{n+\gamma-1} v_4^n}{\Gamma(-n+1-\alpha)\Gamma(-n+1-\beta)\Gamma(n+\gamma)\Gamma(n+1)}.$$

The A-hypergeometric equations read

$$(\partial_1\partial_2 - \partial_3\partial_4)\Phi = 0$$

$$\begin{aligned}
(v_1\partial_1 + v_4\partial_4 + \alpha)\Phi &= 0 \\
(v_2\partial_2 + v_4\partial_4 + \beta)\Phi &= 0 \\
(-v_3\partial_3 + v_4\partial_4 + \gamma - 1)\Phi &= 0
\end{aligned}$$

Classical equations for ${}_2F_1$

Reduction of the A-hypergeometric system gives, after setting $v_1 = v_2 = 1, v_3 = 1, v_4 = z$,

$$z(z-1)F'' + ((\alpha+\beta+1)z - \gamma)F' + \alpha\beta F = 0$$

Example 2, Appell F_1

Appell $F_1(\alpha, \beta, \beta', \gamma | x, y)$ is proportional to

$$\sum_{m,n \geq 0} \frac{\Gamma(m+n+\alpha)\Gamma(m+\beta)\Gamma(n+\beta')}{\Gamma(m+n+\gamma)\Gamma(m+1)\Gamma(n+1)} x^m y^n.$$

Application of Γ-identities gives

$\sum_{m,n \geq 0} \frac{x^m y^n}{\Gamma(-m-n+1-\alpha)\Gamma(-m+1-\beta)\Gamma(-n+1-\beta')\Gamma(m+n+\gamma)\Gamma(m+1)\Gamma(n+1)}$

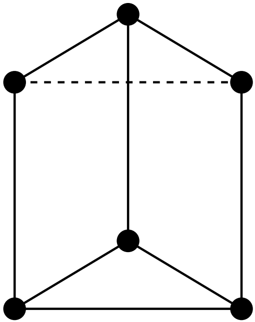
The lattice L is spanned by

$$(-1, -1, 0, 1, 1, 0) \quad \text{and} \quad (-1, 0, -1, 1, 0, 1).$$

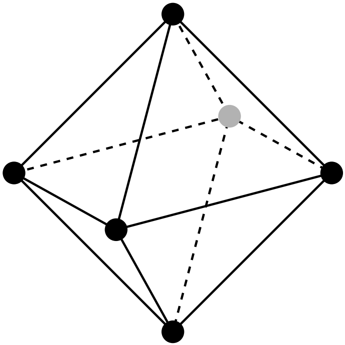
A corresponding set A,

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_1 - \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_1 - \mathbf{e}_4 \in \mathbb{R}^4$$

F_1 and F_4 polytope



F1



F4

A-hypergeometric equations for F_1

Recall $L = \langle(-1,-1,0,1,1,0),(-1,0,-1,1,0,1)\rangle$ and F_1 proportional to

$$\sum_{m,n\geq 0} \frac{x^m y^n}{\Gamma(-m-n+1-\alpha)\Gamma(-m+1-\beta)\Gamma(-n+1-\beta')\Gamma(m+n+\gamma)\Gamma(m+1)\Gamma(n+1)}$$

Formal A-hypergeometric solution:

$$\sum_{m,n\in\mathbb{Z}} \frac{v_1^{-m-n-\alpha} v_2^{-m-\beta} v_3^{-n-\beta'} v_4^{m+n+\gamma} v_5^m v_6^n}{\Gamma(-m-n+1-\alpha)\Gamma(-m+1-\beta)\Gamma(-n+1-\beta')\Gamma(m+n+\gamma)\Gamma(m+1)\Gamma(n+1)}$$

Denote $\partial_i = \frac{\partial}{\partial v_i}$. The A-hypergeometric equations read

$$\partial_1\partial_2\Phi - \partial_4\partial_5\Phi = 0, \quad \partial_1\partial_3\Phi - \partial_4\partial_6\Phi = 0, \quad \partial_2\partial_6\Phi - \partial_3\partial_5\Phi = 0$$

$$\begin{aligned}
(v_1\partial_1 + v_5\partial_5 + v_6\partial_6 + \alpha)\Phi &= 0 \\
(v_2\partial_2 + v_5\partial_5 + \beta)\Phi &= 0 \\
(v_3\partial_3 + v_6\partial_6 + \beta')\Phi &= 0 \\
(v_4\partial_4 - v_5\partial_5 - v_6\partial_6 - \gamma + 1)\Phi &= 0
\end{aligned}$$

Classical equations for F_1

The A-hypergeometric system for F_1 can be reduced to the following system, where we have set $v_1 = v_2 = v_3 = v_4 = 1, v_5 = x, v_6 = y$,

$$\begin{aligned} x(1-x)F_{xx} + y(1-x)F_{xy} + [\gamma - (\alpha+\beta+1)x]F_x - \beta y F_y & \\ -\alpha\beta F &= 0 \\ y(1-y)F_{yy} + x(1-y)F_{xy} + [\gamma - (\alpha+\beta'+1)y]F_y - \beta' x F_x & \\ -\alpha\beta' F &= 0 \\ (x-y)F_{xy} - \beta' F_x + \beta F_y &= 0 \end{aligned}$$

In classical literature the last equation is usually presented as a (non-trivial) consequence of the first two.

The rank of an A-hypergeometric system

The *toric ideal* associated to A is the ideal in $\mathbb{C}[\partial_1, \ldots, \partial_N]$ generated by all $\Box_\lambda$ with $\lambda \in L$. Notation: I_A.

The *A-polytope* is the convex hull of the set A. Notation: Q_A. We assign to Q_A a volume normalised such that the volume of a standard simplex is 1. Notation: $\mathrm{Vol}(Q_A)$.

Theorem (GKZ 1989)

The A-hypergeometric system $H_A(\alpha)$ has finite rank. Suppose that $\mathbb{C}[\partial_i]/I_A$ satisfies the Cohen-Macaulay condition. Then the rank equals $\mathrm{Vol}(Q_A)$.

By a theorem of Hochster the Cohen-Macaulay condition is satisfied if A is *saturated*, that is,

$$\mathbb{Z}_{\geq 0}A = \mathbb{Z}^r \cap \mathbb{R}_{\geq 0}A.$$

Rank jumps

A.Adolphson (1994) pointed out that without the Cohen-Macaulay condition the GKZ-theorem on the ranks need not be true.

Example, consider $A \subset \mathbb{R}^2$ given by the columns of

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}$$

Then the rank of $H_A(\alpha, \beta)$ equals 5 if $\alpha = 1, \beta = 2$ and 4 otherwise.

It is known that the rank of any A-hypergeometric system is finite and $\geq \mathrm{Vol}(Q_A)$. L.Matusevich and U.Walther (2005) showed that this difference can be arbitrarily large.

Irreducibility

Theorem (GKZ 1990)

Suppose $\alpha + \mathbb{Z}^r$ has trivial intersection with the faces of $C(A)$ (non-resonance). Then $H_A(\alpha)$ is irreducible.

Theorem (F.B, Walther 2011)

Suppose that A is saturated and Q_A is not a pyramid. If there exists a point of $\alpha + \mathbb{Z}^r$ contained in a face of $C(A)$, then the A-hypergeometric system is reducible.

Reducibility of Gauss' equation

A-matrix

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

and parameters $(-\alpha, -\beta, \gamma - 1)$.
Faces are given by

- $\mathbf{a}_1, \mathbf{a}_3$, equation $x_2 = 0$
- $\mathbf{a}_1, \mathbf{a}_4$, equation $x_2 + x_3 = 0$
- $\mathbf{a}_2, \mathbf{a}_3$, equation $x_1 = 0$
- $\mathbf{a}_2, \mathbf{a}_4$, equation $x_1 + x_3 = 0$

Non-resonance condition: None of $\beta, \beta - \gamma, \alpha, \alpha - \gamma$ is an integer.

Reducibility of F_1

The set A associated to F_1 is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}$$

Parameter vector is given by $(-\alpha, -\beta, -\beta', \gamma - 1)^t$. Q_A has 5 faces,

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_1 + x_4 = 0,\ x_2 + x_3 + x_4 = 0.$$

Non-resonance: none of the following numbers is an integer,

$$\alpha,\ \beta,\ \beta',\ \alpha - \gamma,\ \beta + \beta' - \gamma.$$