Lecture 2

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The A-polytope

Start with a finite subset $A \subset \mathbb{Z}^r \subset \mathbb{R}^r$. We assume

- The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$
- There is a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Define a vector of parameters

$$\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$$

Remember:

The set A and the vector α will completely characterise a so-called A-hypergeometric system of differential equations.

Lattice of relations

Write $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$. The lattice of relations $L \subset \mathbb{Z}^N$ is formed by all $\mathbf{l} = (l_1, \ldots, l_N) \in \mathbb{Z}^N$ such that

$$l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}.$$

Let h be the form such that $h(\mathbf{a}_i) = 1$ for $i = 1, \ldots, r$.
Apply h to any relation $l_1\mathbf{a}_1 + \cdots + l_N\mathbf{a}_N = \mathbf{0}$.
Then we get $\sum_{i=1}^{N} l_i = 0$ for all $\mathbf{l} \in L$.

Formal A-hypergeometric series

Choose $\gamma_1, \ldots, \gamma_N$ such that

$$\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N.$$

Note that $\gamma = (\gamma_1, \ldots, \gamma_N)$ is determined modulo $L \otimes \mathbb{R}$.
Let $v_1, \ldots, v_N$ be variables and consider

$$\Phi = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1+\gamma_1} \ldots v_N^{l_N+\gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdots \Gamma(l_N + \gamma_N + 1)}.$$

Power series solutions, Gauss' equation

Consider the set $A \subset \mathbb{Z}^3$ given by

$$\mathbf{a}_1 = \mathbf{e}_1, \quad \mathbf{a}_2 = \mathbf{e}_2, \quad \mathbf{a}_3 = \mathbf{e}_3, \quad \mathbf{a}_4 = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$$

and the parameter triple $(-a, -b, c-1)$.
Lattice of relations L is generated by $(-1,-1,1,1)$. Choose $\gamma = (-a, -b, c-1, 0) + \tau(-1,-1,1,1)$ for some τ. We choose τ such that one of the components of γ vanishes. Let us take $\tau = 0$.
Formal solution:

$$\Phi = \sum_{n \in \mathbb{Z}} \frac{v_1^{-n-a} v_2^{-n-b} v_3^{n+c-1} v_4^n}{\Gamma(-n-a+1)\Gamma(-n-b+1)\Gamma(n+c)\Gamma(n+1)}$$

Notice that $n \geq 0$. Standard identities for Γ yield

$$\Phi \sim v_1^{-a} v_2^{-b} v_3^{c-1} \sum_{n \geq 0} \frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+c)\Gamma(n+1)} \left(\frac{v_3 v_4}{v_1 v_2}\right)^n$$

This is ${}_2F_1\left(\begin{matrix} a \; b \\ c \end{matrix} \middle| z\right)$, when we put $v_1 = v_2 = v_3 = 1, v_4 = z$.

Power series, second solution at $z = 0$

Same example, but now $\tau = 1 - c$ so that $\gamma = (c - a - 1, c - b - 1, 0, 1 - c)$. We get

$$\Phi = \sum_{n \in \mathbb{Z}} \frac{v_1^{-n+c-a-1} v_2^{-n+c-b-1} v_3^n v_4^{n+1-c}}{\Gamma(-n+c-a)\Gamma(-n+c-b)\Gamma(n+1)\Gamma(n+2-c)}$$

Notice that $n \geq 0$. Standard identities for Γ yield

$$\Phi \sim v_1^{c-a-1} v_2^{c-b-1} v_4^{1-c} \sum_{n \geq 0} \frac{\Gamma(n+a+1-c)\Gamma(n+b+1-c)}{\Gamma(n+1)\Gamma(n+2-c)} \left(\frac{v_3 v_4}{v_1 v_2}\right)^n$$

This is $z^{1-c}\,{}_2F_1\left(\begin{matrix} a+1-c \;\; b+1-c \\ 2-c \end{matrix} \,\middle|\, z \right)$, when we put $v_1 = v_2 = v_3 = 1, v_4 = z$.

Power series, solution at $z = \infty$

Same example, but now $\tau = a$ so that
$\gamma = (0, a-b, c-a-1, -a)$. We get

$$\Phi = \sum_{n\in\mathbb{Z}} \frac{v_1^{-n} v_2^{-n+a-b} v_3^{n+c-a-1} v_4^{n-a}}{\Gamma(-n+1)\Gamma(-n+a-b+1)\Gamma(n+c-a)\Gamma(n-a+1)}$$

Notice that $n \le 0$. Replace $n \to -n$. Standard identities for Γ yield

$$\Phi \sim v_2^{a-b} v_3^{c-a-1} v_4^{-a} \sum_{n\geq 0} \frac{\Gamma(n+1+a-c)\Gamma(n+a)}{\Gamma(n+1)\Gamma(n+a-b+1)} \left(\frac{v_1 v_2}{v_3 v_4}\right)^n$$

This is $z^{-a}\,{}_2F_1\left(\begin{smallmatrix} 1+a-c & a \\ a-b+1 & \end{smallmatrix} \middle| \frac{1}{z}\right)$, when we put
$v_1 = v_2 = v_3 = 1, v_4 = z$.

Appell F_1 again

Appell $F_1(\alpha, \beta, \beta', \gamma | x, y)$ is proportional to
$\sum_{m,n\in\mathbb{Z}} \frac{x^m y^n}{\Gamma(-m-n+1-\alpha)\Gamma(-m+1-\beta)\Gamma(-n+1-\beta')\Gamma(m+n+\gamma)\Gamma(m+1)\Gamma(n+1)}$

The lattice L is spanned by rows of

$$\begin{pmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Denote i-th column by $\mathbf{b}_i$. We call this the B-matrix.
Then solution becomes
$\sum_{\mathbf{s}\in\mathbb{Z}^2} \frac{x^{\mathbf{b}_5\cdot\mathbf{s}} y^{\mathbf{b}_6\cdot\mathbf{s}}}{\Gamma(\mathbf{b}_1\cdot\mathbf{s}+1-\alpha)\Gamma(\mathbf{b}_2\cdot\mathbf{s}+1-\beta)\Gamma(\mathbf{b}_3\cdot\mathbf{s}+1-\beta')\Gamma(\mathbf{b}_4\cdot\mathbf{s}+\gamma)\Gamma(\mathbf{b}_5\cdot\mathbf{s}+1)\Gamma(\mathbf{b}_6\cdot\mathbf{s}+1)}$

In general: $d := N - r$ (rank of L) and γ_i are fixed. We write formal solution as

$$\Phi_\sigma = \sum_{\mathbf{s}\in\mathbb{Z}^d} \prod_{i=1}^{N} \frac{v_i^{\mathbf{b}_i\cdot(\mathbf{s}+\sigma)+\gamma_i}}{\Gamma(\mathbf{b}_i\cdot(\mathbf{s}+\sigma)+\gamma_i+1)},$$

where $\sigma \in \mathbb{R}^d$ is arbitary.
Choose a subset $\mathscr{I} \subset \{1, 2, \ldots, N\}$ with $|\mathscr{I}| = d$ such that $\mathbf{b}_i$ with $i \in \mathscr{I}$ are linearly independent.
Then choose σ such that $\mathbf{b}_i \cdot \sigma + \gamma_i = 0$ for $i \in \mathscr{I}$. Then Φ_σ becomes Laurent series with support in $\mathbf{b}_i.\mathbf{s} \geq 0$ for $i \in \mathscr{I}$.

An example, F_1

Recall that the rows of L are given by

$$\begin{pmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Then Φ_σ equals sum over $\mathbf{s} \in \mathbb{Z}^2$ of

$$\frac{x^{\mathbf{b}_5 . (\mathbf{s}+\sigma)} y^{\mathbf{b}_6 . (\mathbf{s}+\sigma)}}{\Gamma(\mathbf{b}_1 \cdot (\mathbf{s}+\sigma) + 1 - \alpha)\Gamma(\mathbf{b}_2 \cdot (\mathbf{s}+\sigma) + 1 - \beta)\Gamma(\mathbf{b}_3 \cdot (\mathbf{s}+\sigma) + 1 - \beta'}$$
$$\times \frac{1}{\Gamma(\mathbf{b}_4 \cdot (\mathbf{s}+\sigma) + \gamma)\Gamma(\mathbf{b}_5 \cdot (\mathbf{s}+\sigma) + 1)\Gamma(\mathbf{b}_6 \cdot (\mathbf{s}+\sigma) + 1)}.$$

Choose σ such that $\mathbf{b}_1 \cdot \sigma - \alpha = 0$ and $\mathbf{b}_2 \cdot \sigma - \beta = 0$. Explicitly, $-\sigma_1 - \sigma_2 = \alpha$ and $-\sigma_1 = \beta$. So $\sigma_1 = -\beta$ and $\sigma_2 = \beta - \alpha$.

F_1 continued

We get

$$\Phi_{1,2} = \sum_{s_1,s_2\in\mathbb{Z}} \frac{x^{s_1-\beta} y^{s_2+\beta-\alpha}}{\Gamma(-s_1-s_2+1)\Gamma(-s_1+1)\Gamma(-s_2+1-\beta+\alpha-\beta')} \times \frac{1}{\Gamma(s_1+s_2+\gamma-\alpha)\Gamma(s_1+1-\beta)\Gamma(s_2+1+\beta-\alpha)}.$$

Laurent series with support $-s_1-s_2 \geq 0, -s_1 \geq 0$.Setting $m = -s_1 - s_2, n = -s_1$ gives

$$\Phi_{1,2} = \sum_{m,n\geq 0} \frac{(y/x)^{n+\beta} y^{-(m+\alpha)}}{\Gamma(m+1)\Gamma(n+1)\Gamma(m-n+1-\beta+\alpha-\beta')} \times \frac{1}{\Gamma(-m+\gamma-\alpha)\Gamma(-n+1-\beta)\Gamma(n-m+1+\beta-\alpha)}.$$

Triangulations

For any subset J of $\{1, 2, \ldots, N\}$ we denote by Σ_J the convex hull of $\{a_j\}_{j \in J}$.

Definition

A triangulation of $Q(A)$ is a subset

$$T \subset \{J \subset \{1, 2, \ldots, N\} \mid |J| = r \text{ and } \mathrm{rank}(\Sigma_J) = r\}$$

such that

- $Q(A) = \cup_{J \in T} \Sigma_J$
- for all $J, J' \in T$: $\Sigma_J \cap \Sigma_{J'} = \Sigma_{J \cap J'}$.

Basis of solutions

Theorem (GKZ)

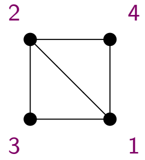
Let T be a (regular) triangulation of $Q(A)$. Then the Laurent series Φ_{J^c} with $J \subset T$ form a basis of solutions having a common domain of convergence.

Gauss' hypergeometric function with A-matrix

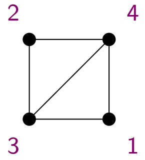
$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{matrix}$$

Triangulations of $Q(A)$,

T_1 :



T_2 :



Algebraic Appell functions

Schwarz's list has been extended to Appell's functions in the following cases

- F_1 and higher generalisations (Lauricella F_D) by T.Sasaki (1977) and P.Cohen, J.Wolfart (1992)
- F_2 (and F_3) by Mitsuo Kato (2000)
- F_4 by Mitsuo Kato (1997)

Examples of algebraic Appell functions;

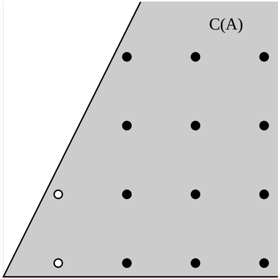
- $F_2(1/2, 5/6, 1/6, 2/3, 1/3, x, y)$ with Galois group of order 192.
- $F_2(-1/10, 3/10, 1/10, 3/5, 1/5, x, y)$ with Galois group of order 14400.

Apexpoints

Let $C(A)$ be the positive real cone spanned by the elements of A and $\alpha \in \mathbb{R}^r$. Consider the set

$$K(\alpha, A) := (\alpha + \mathbb{Z}^r) \cap C(A)$$

A point $\mathbf{p} \in K(\alpha, A)$ is called *apexpoint* if there is no $\mathbf{q} \in K(\alpha, A)$, distinct from $\mathbf{p}$, such that $\mathbf{p} - \mathbf{q} \in C(A)$.



Maximal apexpoints

Lemma

The number of apexpoints in $K(\alpha, A)$ is at most equal to the volume of the convex hull of A.

We say that the number of apexpoints of $K(\alpha, A)$ is *maximal* if it equals this upper bound.

Algebraicity

Consider the A-hypergeometric system $H_A(\alpha)$ with $\alpha \in \mathbb{Q}^r$.
Suppose the normality condition is satisfied and that the GKZ-system is irreducible.
Let N be the smallest positive integer such that $N\alpha \in \mathbb{Z}^r$.

Theorem (FB, 2006)

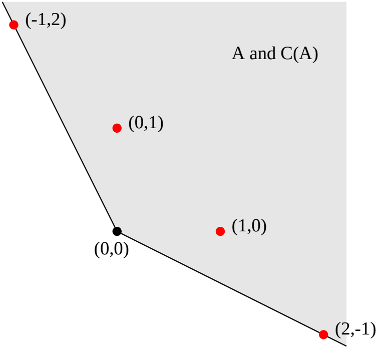
The GKZ-system has a solution space consisting of algebraic functions $\iff$ the number of apex points in $(k\alpha + \mathbb{Z}^r) \cap C(A)$ is maximal for all integers k with $\gcd(k, N) = 1$.

Remark: Using this criterion it is possible to extend Schwarz's list to all algebraic Lauricella functions and two variable Horn functions (E.Bod, 2009).

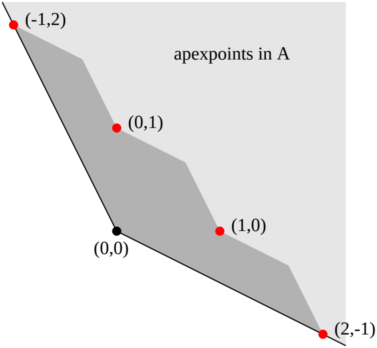
The Horn series G_3

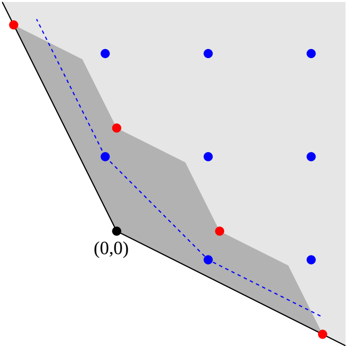
Consider

$$G_3(a,b,x,y) = \sum_{m,n\geq 0} \frac{(a)_{2m-n}(b)_{2n-m}}{m!n!}\, x^m y^n$$



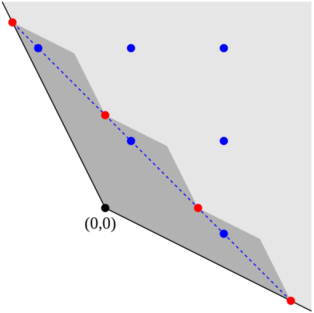
Apexpoints for G_3



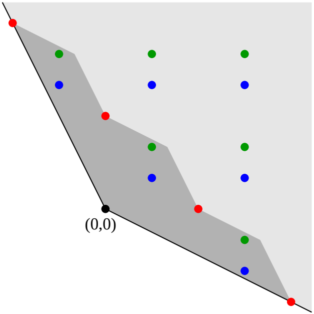


Algebraic G_3

Let $\alpha \in \mathbb{Q}$ and choose $a = \alpha, b = 1 - \alpha$.



Let $a = 1/2, b = 1/3$ and $a = 1/2, b = 2/3$.



G_3-list

It is proven by J.Schipper that the only $a, b \in \mathbb{Q}$ for which the system for $G_3(a, b, x, y)$ is irreducible with finite monodromy is given by the following cases

1. $a + b \in \mathbb{Z}$ and $a, b \notin \mathbb{Z}$
2. $a \equiv 1/2 (\mathrm{mod}\ \mathbb{Z}), \quad b \equiv 1/3, 2/3 (\mathrm{mod}\ \mathbb{Z})$ or vice versa.

The first case explicitly,

$$G_3(a, 1-a, x, y) = f(x, y)^a \sqrt{\frac{g(x, y)}{\Delta}}$$

where

$$\Delta = 1 + 4x + 4y + 18xy - 27x^2y^2$$

and

$$xf^3 - y = f - f^2, \qquad g(g - 1 - 3x)^2 = x^2\Delta$$

Steps of the proof

- The combinatorial condition is equivalent to the statement that for almost all primes p the GKZ-system modulo p has a maximal set of independent (over $\mathbb{F}_p[v_1^p, \ldots, v_N^p]$) polynomial solutions in $\mathbb{F}_p[v_1, \ldots, v_N]$.
- This is equivalent to the statement that the D-module associated to the GKZ-system has vanishing p-curvature for all almost all primes p.
- A conjecture of Grothendieck asserts that vanishing p-curvature for almost all p is equivalent to finite monodromy, hence a solution space consisting of algebraic functions.
- Grothendieck's conjecture has been proven by N.Katz (1972) in the case of systems of equation which are factors of Gauss-Manin systems, i.e systems that are associated to families of algebraic varieties.
- Any GKZ-system with rational parameters 'comes from algebraic geometry'.

Monodromy computation

Solutions for the Gauss hypergeometric equation.
Solution base in $|z| < 1$:

- $f_0 = {}_2F_1\left(\left.{\alpha,\beta \atop \gamma}\right| z\right)$
- $f_1 = z^{1-\gamma}\,{}_2F_1\left(\left.{1+\alpha-\gamma,1+\beta-\gamma \atop 2-\gamma}\right| z\right)$

Solution base in $|z| > 1$ (locally around $z = \infty$):

- $g_0 = z^{-\alpha}\,{}_2F_1\left(\left.{\alpha,1+\alpha-\gamma \atop 1+\alpha-\beta}\right| \frac{1}{z}\right)$
- $g_1 = z^{-\beta}\,{}_2F_1\left(\left.{\beta,1+\beta-\gamma \atop 1+\beta-\alpha}\right| \frac{1}{z}\right)$

Mellin-Barnes integrals

Define

$$M(z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(-\alpha+s)\Gamma(-\beta+s)\Gamma(1-\gamma-s)\Gamma(-s)z^s ds$$

where $i=\sqrt{-1}$. Converges whenever $-2\pi < \mathrm{Arg}(z) < 2\pi$.

Theorem

When $\alpha, \beta > 0, \gamma < 1$ this is solution of hypergeometric equation. Moreover, different argument choices for z yield two independent solutions.

Transition matrices

Let $z \in \mathbb{C} \setminus \mathbb{R}$. Let $M_1(z)$ be the Mellin-Barnes integral with $\mathrm{Arg}(z) \in (-2\pi, 0)$ and $M_2(z)$ with $\mathrm{Arg}(z) \in (0, 2\pi)$.
After analytic continuation along a loop counter clockwise around 0 we get $M_1(z) \to M_2(z)$.
Let $f_1(z), z^{1-\gamma} f_2(z)$ be basis of hypergeometric solutions around $z = 0$ with f_1, f_2 analytic. There exist $\mu_1, \mu_2 \in \mathbb{C}$ such that

$$M_1(z) = \mu_1 f_1(z) + \mu_2 z^{1-\gamma} f_2(z).$$

After analytic continuation around $z = 0$,

$$M_2(z) = \mu_1 f_1(z) + \mu_2 e^{2\pi i(1-\gamma)} z^{1-\gamma} f_2(z).$$

Note $\mu_1, \mu_2 \neq 0$ and we can renormalize f_1, f_2 to get

$$\begin{aligned} M_1(z) &= f_1(z) + z^{1-\gamma} f_2(z) \\ M_2(z) &= f_1(z) + c z^{1-\gamma} f_2(z), \ c = e^{2\pi i(1-\gamma)} \end{aligned}$$

Monodromy matrix at 0

Previously,

$$\begin{aligned} M_1(z) &= f_1(z) + z^{1-\gamma} f_2(z) \\ M_2(z) &= f_1(z) + c z^{1-\gamma} f_2(z),\ c = e^{2\pi i(1-\gamma)} \end{aligned}$$

So we have a transition matrix X_0 between the bases M_1, M_2 and $f_1, z^{1-\gamma} f_2$, namely

$$X_0 = \begin{pmatrix} 1 & 1 \\ 1 & c \end{pmatrix}.$$

After closed loop around 0: $f_1 \to f_1$ and $z^{1-\gamma} f_2 \to c z^{1-\gamma} f_2$. Hence,

$$\begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} \to X_0 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} X_0^{-1} \begin{pmatrix} M_1(z) \\ M_2(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & c+1 \end{pmatrix} \begin{pmatrix} M1(z) \\ M_2(z) \end{pmatrix}.$$

Monodromy matrix at ∞

Letting $z^{-\alpha}g_1(1/z), z^{-\beta}g_2(1/z)$ be suitably normalized basis around $z = \infty$,

$$\begin{aligned} M_1(z) &= z^{-\alpha}g_1(1/z) + z^{-\beta}g_2(1/z) \\ M_2(z) &= az^{-\alpha}g_1(1/z) + bz^{-\beta}g_2(1/z),\ a = e^{-2\pi i\alpha}, b = e^{-2\pi i\beta} \end{aligned}$$

Transition matrix

$$X_\infty = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}.$$

After a closed loop around ∞: $z^{-\alpha}g_1 \to az^{-\alpha}g_1$ and $z^{-\beta}g_2 \to bz^{-\beta}g_2$. Hence,

$$\begin{pmatrix} M1(z) \\ M_2(z) \end{pmatrix} \to X_\infty \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} X_\infty^{-1} \begin{pmatrix} M1(z) \\ M_2(z) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -ab & a+b \end{pmatrix} \begin{pmatrix} M1(z) \\ M_2(z) \end{pmatrix}.$$

Riemann's monodromy

With respect to the Mellin-Barnes basis of solutions M_1, M_2 the monodromy group G of the Gauss hypergeometric equation is generated by

$$\begin{pmatrix} 0 & 1 \\ -c & c+1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -ab & a+b \end{pmatrix}$$

where $a = e^{-2\pi i\alpha}, b = e^{-2\pi i\beta}, c = e^{-2\pi i\gamma}$.
When $\alpha, \beta, \gamma \in \mathbb{R}$ there is G-invariant Hermitian form

$$H = \begin{pmatrix} c - ab & a + b - (c+1) \\ (a+b)c + ab(c+1) & c - ab \end{pmatrix}$$

(i.e. $\overline{g}^T H g = H$ for all $g \in G$).

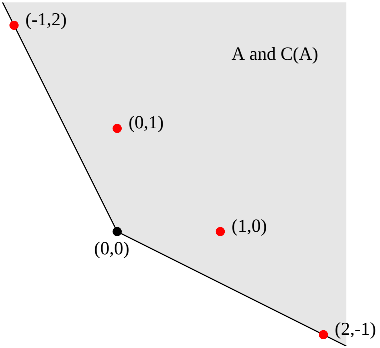
Horn G_3

Consider the Horn G_3-function

$$G_3(a,b,x,y) = \sum_{m,n\geq 0} \frac{(a)_{2n-m}(b)_{2m-n}}{m!n!} x^m y^n$$

We have

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{pmatrix}$$



The B-matrix of G_3

Recall

$$A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \end{pmatrix}.$$

Lattice of relations is generated by the rows of

$$B = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

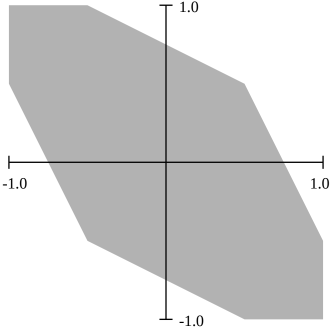
We call this matrix the *B-matrix*.

The B-zonotope

Start with a set $A \subset \mathbb{Z}^r$ and construct a B-matrix. The number of rows is $N - r$ which we denote by d (number of essential variables). Denote its columns by $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N \in \mathbb{R}^d$. Define the *B-zonotope* by

$$Z_B = \left\{ \frac{1}{4} \sum_{j=1}^{N} \lambda_j \mathbf{b}_j \quad ; \quad -1 < \lambda_j < 1 \right\}$$

Picture for G_3,



Mellin-Barnes in general

Given an A-hypergeometric system A, α. Choose $\gamma_1, \ldots, \gamma_N$ such that

$$\gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N = \alpha$$

and define

$$M(\mathbf{v}) = \frac{1}{(2\pi i)^d} \int_{i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s}} d\mathbf{s}$$

where $\mathbf{s} = (s_1, \ldots, s_d)$ and $d\mathbf{s} = ds_1 \wedge \cdots \wedge ds_d$.

Convergence

Recall

$$M(\mathbf{v}) = \frac{1}{(2\pi i)^d} \int_{i\mathbb{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s}} d\mathbf{s}$$

Theorem

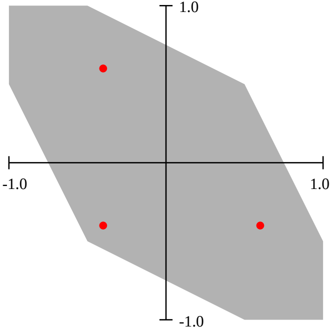
For $j = 1, \ldots, N$ let θ_j be an argument choice for v_j. Then the integral for $M(\mathbf{v})$ converges if

$$\frac{\theta_1}{2\pi}\mathbf{b}_1 + \frac{\theta_2}{2\pi}\mathbf{b}_2 + \cdots + \frac{\theta_N}{2\pi}\mathbf{b}_N \in Z_B.$$

Moreover, if $\gamma_j < 0$ for all j, then $M(\mathbf{v})$ is a solution of the hypergeometric system A, α.

Mellin-Barnes for G_3

The Horn system has solution space of dimension 3. Consider the B-zonotope



Notice we have a basis of Mellin-Barnes solutions.

Questions, invariant form

Hypotheses underlying the monodromy calculation.

1. We need a Mellin-Barnes basis of solutions.
2. Is the global monodromy group generated by the local contributions?

Theorem

Let $M \subset GL_D(\mathbb{C})$ be the monodromy group of an irreducible A-hypergeometric system. Then there exists a non-trivial Hermitean matrix H such that $\overline{g}^t Hg = H$ for all $g \in M$.

Signature

Signature in the case G_3. Take following triangulation of A, i.e.

$$\left\{\begin{pmatrix}-1\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\},\quad \left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\right\}$$

Then write the parameter vector $(-a,-b)$ as linear combination of each of these pairs

$$a\begin{pmatrix}-1\\2\end{pmatrix}+(-b-2a)\begin{pmatrix}0\\1\end{pmatrix},\ -b\begin{pmatrix}0\\1\end{pmatrix}-a\begin{pmatrix}1\\0\end{pmatrix},\ (-a-2b)\begin{pmatrix}1\\0\end{pmatrix}+b\begin{pmatrix}2\\-1\end{pmatrix}.$$

Then the signs of
$\sin\pi a\cdot\sin\pi(-b-2a),\ \sin\pi(-a)\cdot\sin\pi(-b),\ \sin\pi(-a-2b)\cdot\sin\pi b$
determine the signature.