FLEXIBILITY OF TORIC AFFINE VARIETIES

MIKHAIL ZAIDENBERG

1. FLEXIBLE VARIETIES

1.1. DEFINITIONS.

- **We work over a base field $\Bbbk = \bar{\Bbbk}$, char$(\Bbbk) = 0$ (e.g., $\Bbbk = \mathbb{C}$).**
- **X is an affine algebraic variety of dimension ≥ 2.**
- **$x \in X_{\text{reg}}$ is *FLEXIBLE* if $T_x X$ is spanned by tangent vectors to the orbits $U.x$ where $U \subseteq \text{Aut}(X)$, $U \cong \mathbb{G}_a$, is a one-parameter unipotent subgroup.**
- **X is *FLEXIBLE* if any smooth point $x \in X_{\text{reg}}$ is.**
- **The *SPECIAL AUTOMORPHISM GROUP* is**

$$\text{SAut}(X) := \langle U | U \subset \text{Aut}(X) \rangle \,.$$

1.2. FIRST EXAMPLES.

- **$\text{SAut}(\mathbb{A}^n) \supset \text{Transl}(\mathbb{A}^n)$, hence $X = \mathbb{A}^n$ with $n \geq 2$ is flexible.**
- **$\text{SAut}(\mathbb{A}^1) = \text{Transl}(\mathbb{A}^1) \subsetneq \text{Aff}(\mathbb{A}^1)$ is an algebraic group acting transitively, but not 2-transitively on $\mathbb{A}^1$.**
- **In contrast, $\text{SAut}(\mathbb{A}^n)$ for $n \geq 2$ is a non-algebraic group. Indeed, it is ∞-*TRANSITIVE* on $\mathbb{A}^n$, i.e. m-transitive $\forall m \geq 1$.**
- **For instance, $\text{SAut}(\mathbb{A}^2)$ contains the *shears* (elementary transformations)**

$$(x, y) \mapsto (x, y + P(x))\,, \quad P \in \Bbbk[x]\,.$$

- **$\text{SAut}(\mathbb{A}^2) = \text{JAut}(\mathbb{A}^2)$ where**

$$\text{JAut}(\mathbb{A}^2) := \{f \in \text{Aut}(\mathbb{A}^2) \,|\, \text{Jac}(f) = 1\}.$$

- **$\forall n \geq 3$ one has $\text{SAut}(\mathbb{A}^n) \subset \text{JAut}(\mathbb{A}^n)$.**
  However, it is unknown whether the equality holds.
- **If X_{reg} carries an algebraic volume form ω, that is, if $K_{X_{\text{reg}}}$ is trivial, then $\text{SAut}(X)$ preserves ω, that is, $\text{SAut}(X) \subset \text{JAut}(X)$.**

1.3. FIRST RESULTS.

THEOREM (AFKKZ '13)

The following are equivalent:

- ***X is flexible;***
- ***$SAut(X)$ is transitive in X_{reg};***
- ***$SAut(X)$ is ∞-transitive in X_{reg}.***

REMARK

- **An algebraic group G cannot act ∞-transitively.**
- **(A. Borel - F. Knop) G cannot act 3-transitively on an affine X.**

THEOREM (Gromov-Winkelmann for $X = \mathbb{A}^n$; FKZ '16)
If $Y \subset X$ is a closed subset of $\operatorname{codim}_X Y \geq 2$ then $\operatorname{Stab}_{\mathrm{SAut}(X)}(Y)$ acts ∞-transitively in $X_{\text{reg}} \setminus Y$.

1.4. FLEXIBILITY OF SUSPENSIONS.

DEFINITION

A *SUSPENSION* X over Y is a hypersurface in $Y \times \mathbb{A}^2$ given by

$$uv - f(y) = 0, \quad f \in \mathcal{O}_Y(Y), \quad f \neq \text{cst}.$$

THEOREM (KZ '99, AKZ '12)

If Y is flexible then $X = \operatorname{Susp}_f(Y)$ is.

THEOREM (AFKKZ '13) ***X flexible $\Rightarrow (TX)^{\otimes m} \otimes (T^*X)^{\otimes n}$ is $\forall m, n$.***

1.5. FLEXIBILITY OF ORBITS.

THEOREM

- ***Any $\mathrm{SAut}(X)$-orbit O with $\dim O \geq 2$ is a flexible quasi-affine variety.***
- ***An open orbit O (if any) consists of all flexible points in X_{reg}, and $\mathrm{SAut}(X)$ is ∞-transitive in O.***

HINT: O is locally closed in X (Ramanujam).

ROSENLICHT THEOREM ON INVARIANTS
$\exists f_1, \ldots, f_m \in \mathcal{O}_X(X)^{\mathrm{SAut}(X)}$ ***separating general $\mathrm{SAut}(X)$-orbits.***

1.6. FLEXIBILITY OF HOMOGENEOUS VARIETIES.

THEOREM (AFKKZ '13)
Let G be an algebraic group without characters, i.e. $G^\vee = \{1\}$. If G/H is affine with $\dim G/H \geq 2$ then G/H is flexible.

REMARK
$G^\vee = \{1\}$ if G is unipotent, or semisimple, or an extention of one by another.

THEOREM (AKZ '12)
Let G/P be a flag variety, and let $G/P \hookrightarrow \mathbb{P}^n$ be an equivariant embedding. Then $X = \mathrm{AffCone}(G/P)$ is flexible.

1.7. FLEXIBILITY OF QUASIHOMOGENEOUS VARIETIES.

THEOREM (AKZ '12)
Any non-degenerate toric affine variety is flexible.

THEOREM (AFKKZ '13)
Let G be semisimple, and let X be a smooth G-variety. If G acts on X with an open orbit then X is flexible.

HINT: $\exists \tilde{G} \supseteq G$ s.t. $\tilde{G}^\vee = \{1\}$ and $\tilde{G} : X$ is transitive (Luna's Étale Slice Theorem).

1.8. FLEXIBLE NON-HOMOGENEOUS VARIETIES: EXAMPLE.

Consider a smooth del Pezzo surface Y_d of degree $d \in \{1, \ldots, 9\}$ and a pluri-anticanonical embedding $Y_d \hookrightarrow \mathbb{P}^n$. Then Y_d is a toric affine 3-fold for $6 \leq d \leq 9$, while $\mathrm{Aut}(Y_d)$ is finite for $d \leq 5$.

THEOREM (KPZ '14, Perepechko '13, Cheltsov-Park-Won '14-'15)
Let $X_d = \mathrm{AffCone}\,(Y_d) \subseteq \mathbb{A}^{n+1}$.

- ***X_d is flexible $\Leftrightarrow d \geq 4$.***
- ***For $d \leq 3$, X_d does not admit any effective $\mathbb{G}_a$-action.***

1.9. LNDs.

DEFINITION

Let $A = \mathcal{O}_X(X)$ be an affine algebra. A *LOCALLY NILPOTENT DERIVATION* (LND, for short) on A is a derivation $\partial \in \mathrm{Der}\, A$ such that

$$\forall a \in A \ \exists n \in \mathbb{N} : \partial^n(a) = 0.$$

The associated one parameter unipotent subgroup with infinitesimal generator ∂ is

$$U = \exp(\Bbbk\partial) \subseteq \mathrm{SAut}(A).$$

The element $h_t \in U$, $t \in \mathbb{G}_a$ acts on A via

$$h_t(a) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \partial^{(k)}(a), \quad a \in A\,.$$

1.10. REPLICAS.

DEFINITION

A *REPLICA* of ∂ is a derivation $f\partial \in \mathrm{LND}(A)$ where $f \in A^U = \ker\partial$. The associated one parameter unipotent subgroup $U_f = \exp(\Bbbk f\partial)$ is also called a *REPLICA* of U.

EXAMPLE (NAGATA AUTOMORPHISM) **Set**

$$X = \mathbb{A}^3 = \mathrm{Spec}\,\Bbbk[X, Y, Z]$$

$$\partial = X\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial Z}$$

$$f = Y^2 - 2XZ \in \ker\partial$$

THEOREM (“Nagata Conjecture”, Shestakov and Umirbaev ′04)

The replica* $\exp(f\partial) \in \mathrm{SAut}(\mathbb{A}^3)$ *is* wild*, that is, it is not a product of affine and triangular transformations.

References

[AFKKZ13] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, and M. Zaidenberg. Flexible varieties and automorphism groups. *Duke Math. J.*, 162(4):767–823, 2013.

[AKZ12] I. Arzhantsev, K. Kuyumzhiyan, and M. Zaidenberg. Flag varieties, toric varieties, and suspensions: Three instances of infinite transitivity. *Sb. Math.*, 203(7):3–30, 2012.

[FKZ16] H. Flenner, S. Kaliman, and M. Zaidenberg. The Gromov–Winkelmann theorem for flexible varieties. *J. Eur. Math. Soc.* 18(11):2483–2510, 2016.

[KZ99] S. Kaliman and M. Zaidenberg. Affine modifications and affine varieties with a very transitive automorphism group. *Transform. Groups* 4:53–95, 1999.

[KPZ14] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg. Unipotent Group Actions On Del Pezzo Cones. *Algebraic Geometry (Foundation Compositio Math.)* 1:46–56, 2014.