FLEXIBILITY OF TORIC AFFINE VARIETIES II

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1. FLEXIBILITY CRITERIA

We fix an affine variety $X = \mathrm{Spec}\,(A)$ of $\dim X = n \geq 2$ over a field $\Bbbk = \bar{\Bbbk}$ with $\mathrm{char}\,\Bbbk = 0$.

1.1. NON-ALGEBRAICITY OF THE AUTOMORPHISM GROUP.

REMARK

If there exists $\partial \in \mathrm{LND}(A) \setminus \{0\}$ then

$$\exp\left((\ker \partial)\partial\right) \subset \mathrm{SAut}(X)$$

is an infinite-dimensional unipotent Abelian subgroup. Indeed,

$$\mathrm{tr.deg}\,[A : \ker \partial] = 1.$$

CONJECTURE

If $\mathrm{LND}(A) = \{0\}$ ***then*** $\mathrm{Aut}^0\,(X)$ ***is an algebraic torus*** $\mathbb{G}_m^k$ ***of dimension*** $k \leq \dim\,X$***.***

True if $\dim X = 2$ (Perepechko-Z., unpublished).

1.2. FINITENESS CONJECTURE.

DEFINITION

X is called GENERICALLY FLEXIBLE if $\mathrm{SAut}\,(X)$ acts on X with an open orbit $\mathcal{O}_X$ and is infinitely transitive on $\mathcal{O}_X$.

CONJECTURE

Any generically flexible affine variety X admits a finite collection of $\mathbb{G}_a$-subgroups $H_1, \ldots, H_N$ of $\mathrm{Aut}(X)$ such that the group $G = \langle H_1, \ldots, H_N \rangle$ acts on X with an open orbit $\mathcal{O}_G$ and is infinitely transitive on $\mathcal{O}_G$.

REMARK

The conjecture is true if one replaces 'finite' by 'countable' (AKZ '18).

1.3. MAIN RESULTS.

THEOREM 1

For any toric affine variety X of dimension at least 2 with no toric factor and smooth in codimention 2 one can find a finite collection of $\mathbb{G}_a$-subgroups $H_1, \ldots, H_k$ such that the group $G = \langle H_1, \ldots, H_k \rangle$ acts infinitely transitively in the smooth locus $\operatorname{reg}(X)$.

THEOREM 2

For any $n \geq 2$ one can find $\mathbb{G}_a$-subgroups $H_1, H_2, H_3 \subset \operatorname{Aut}(\mathbb{A}^n)$ s.t. $G = \langle H_1, H_2, H_3 \rangle$ acts infinitely transitively on $\mathbb{A}^n$.

1.4. GENERIC FLEXIBILITY: A CRITERION.

The next is a refined version of a result from AFKKZ ′13.

THEOREM 0

Let a set $\partial_1, \ldots, \partial_k \in \operatorname{LND}(X)$ contains n linearly independent derivations $\partial_1, \ldots, \partial_n$. Let also $A_i \subset \ker \partial_i$, $i = 1, \ldots, k$, be a finitely generated subalgebra such that $[\operatorname{Frac}(A_i) : \operatorname{Frac}(\ker \partial_i)] < +\infty$. Assume one of the following holds:

- (α) ***$\mathcal{O}_X(X)$ is generated by $A_1, \ldots, A_k$;***
- (β) ***$[\operatorname{Frac}(\ker \partial_1) : \operatorname{Frac}(A_1)] = 1$;***
- (γ) ***$[\operatorname{Frac}(\ker \partial_i) : \operatorname{Frac}(A_i)] > 1$ $\forall i$ and there is an extra element $b_1 \in \ker \partial_1$ such that $\operatorname{Frac}(\ker \partial_1)$ is generated by b_1 and $\operatorname{Frac}(A_1)$.***

Let G be the subgroup of $\operatorname{SAut}(X)$ generated by $H_0 = \exp(\Bbbk b_1 \partial_1)$ and $H_i(a_i) = \exp(\Bbbk a_i \partial_i)$, $a_i \in A_i$, $i = 1, \ldots, k$. Then G acts on X with an open orbit $\mathcal{O}_G$ and the action of G on $\mathcal{O}_G$ is infinitely transitive.

1.5. ORBITS OF THE CLOSURE OF A SUBGROUP.

LEMMA

- (a) ***The closure $\overline{G}$ of a subgroup $G \subset \operatorname{Aut}(X)$ is a closed ind-subgroup of $\operatorname{Aut}(X)$.***
- (b) ***If $\rho \colon \mathbb{A}^1 \to \operatorname{Aut}(X)$ is a morphism such that $\rho(t) \in G$ for $t \neq 0$ then $\rho(0) \in \overline{G}$.***
- (c) ***Any G-invariant closed subset $Y \subset X$ is $\overline{G}$-invariant.***
- (d) ***If G acts on X with an open orbit $\mathcal{O}_G$ then $\mathcal{O}_G$ coincides with the open orbit $\mathcal{O}_{\overline{G}}$ of $\overline{G}$.***
- (e) ***If a normal subgroup $G \subset \operatorname{Aut}(X)$ acts on X with an open orbit $\mathcal{O}_G$ then $\mathcal{O}_G = \mathcal{O}_{\operatorname{Aut}(X)}$.***

DEFINITION

Let $G \subset \operatorname{Aut}(X)$ be subgroup. It is called ALGEBRAICALLY

GENERATED if it is generated by a family of connected algebraic subgroups of $\mathrm{Aut}(X)$. The orbits of G are locally closed subsets of X (AFKKZ '13).

PROPOSITION

Let $G \subset \mathrm{Aut}(X)$ be an algebraically generated subgroup. Then the following hold.

(a) ***The orbits of G and of $\overline{G}$ in X are the same. In particular, if $\overline{G}$ acts on X with an open orbit $\mathcal{O}_{\overline{G}}$ then G does and $\mathcal{O}_G = \mathcal{O}_{\overline{G}}$.***
(b) ***If $\overline{G}$ acts m-transitively on $\mathcal{O}_{\overline{G}}$ then also G does.***
(c) ***If $\overline{G}$ acts infinitely transitively on $\mathcal{O}_{\overline{G}}$ then also G does.***

2. TORIC AFFINE VARIETIES

Fix the following objects:

- **M – a lattice of rank $n \geq 2$;**
- **$M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ – a vector space over $\mathbb{Q}$ of dimension n;**
- **$\sigma^\vee \subset M_{\mathbb{Q}}$ – a rational convex cone with a nonempty interior (called the WEIGHT CONE);**
- **a base of M;**
- **$\forall m = (m_1, \ldots, m_n) \in M$ the Laurent monomial $\chi^m = x_1^{m_1} \ldots x_n^{m_n}$;**
- **the graded affine algebra**

$$A = \bigoplus_{m \in M \cap \sigma^\vee} \Bbbk \chi^m;$$

- **the toric affine variety $X = \mathrm{Spec}\, A$, $\dim X = n$, where**
- **the action of the n-torus $\mathbb{T} = \mathbb{G}_m^n$ on X is defined by the grading.**

REMARKS

- **$\mathbb{T} = \mathrm{Hom}\,(M, \mathbb{G}_m)$ is the torus of characters of M.**
- **By duality, M is the character lattice of $\mathbb{T}$.**
- **In fact, any toric affine variety arises in this way.**

2.1. **DUAL CONE. Consider also the following associated objects:**

- **the dual lattice $N = \mathrm{Hom}\,(M, \mathbb{Z})$;**
- **the dual cone**

$$\sigma \subset N_{\mathbb{Q}}, \quad \sigma = \{x \in N_{\mathbb{Q}} \,|\, \langle x, y\rangle \geq 0 \ \ \forall y \in \sigma^\vee\};$$

- **the set $\Xi = \{\rho_1, \ldots, \rho_k\}$ of RAY GENERATORS of σ, that is, the primitive lattice vectors on the extremal rays of σ.**

LEMMA ***TFAE:***

- $\sigma^\vee$ ***is a pointed cone, that is,*** $\sigma^\vee$ ***contains no line;***
- σ ***is of full dimension, that is,*** Ξ ***contains a basis of*** $N_{\mathbb{Q}}$***;***
- X ***has no toric factor, that is,*** X ***cannot be decomposed into a product*** $\mathbb{G}_m \times Y$ ***where*** Y ***is another toric variety.***

CONVENTION

Assume in the sequel that the conidions of the lemma are fulfilled.

2.2. DEMAZURE ROOTS AND DEMAZURE FACETS.

DEFINITIONS

A DEMAZURE ROOT belonging to a primitive ray generator $\rho_i \in \Xi$ **is a vector** $e \in M$ **such that**

(i) $\langle \rho_i, e\rangle = -1$;
(ii) $\langle \rho_j, e\rangle \geqslant 0 \ \forall j \neq i$.

The DEMAZURE FACET $\mathcal{S}_i$ **of** $\sigma^\vee$ **is the rational convex polyhedron defined by inequalities (ii) in the affine hyperplane**

$$\mathcal{H}_i = \{\langle \rho_i, e\rangle = -1\}.$$

Thus, the Demazure roots which belong to the ray generator $\rho_i \in \Xi$ **are the points in** $\mathcal{S}_i \cap M$**.**

The ROOT SUBGROUP associated with a Demazure root $e \in \mathcal{S}_i$ **is**

$$H_e = \exp(\Bbbk \partial_{\rho_i,e}) \subset \mathrm{SAut}(X),$$

see the formula for $\partial_{\rho_i,e}$ **below.**

2.3. HOMOGENEOUS DERIVATIONS.

DEFINITION

A derivation $\partial \in \mathrm{Der}\,(A)$ **is called *homogeneous* if** ∂ **respects the grading, that is, sends any graded piece to another one.**

Given $\rho \in N$, $e \in M$, **let**

$$\partial_{\rho,e}(\chi^m) := \langle \rho, m\rangle \chi^{m+e} \quad \forall m \in M\,.$$

Then $\partial_{\rho,e}$ **extends to a homogeneous derivation of** A**;**
the lattice vector $e \in M$ **is called the DEGREE of** $\partial_{\rho,e}$**.**

2.4. HOMOGENEOUS LNDs.

LEMMA (Liendo '10)

- ***If*** $\partial \in \mathrm{Der}\,(A)$ ***is homogeneous then*** $\partial = \lambda \partial_{\rho,e}$ ***for some*** $\lambda \in \Bbbk$***,*** $\rho \in N$***, and*** $e \in \Sigma^\vee \cap M$ ***where***

$$\Sigma^\vee = \sigma^\vee \cup \bigcup_{i=1}^{k} \mathcal{S}_i\,.$$

  If $e \in \mathcal{S}_i \cap M$ ***then*** $\rho = \rho_i$***;***
- $\partial_{\rho,e} \in \mathrm{LND}\,(A) \Leftrightarrow e \in \mathcal{S}_i$ ***and*** $\rho = \rho_i$ ***for some*** $i \in \{1, \ldots, k\}$;
- $\ker(\partial_{\rho,e}) = \mathrm{span}\,(\chi^m \,|\, m \in \tau_\rho)$ ***where***

$$\tau_\rho = \{m \in \sigma^\vee \cap M \,|\, \langle \rho, m\rangle = 0\};$$

- $\tau_{\rho_i} =: \tau_i$ ***is the facet of*** $\sigma^\vee$ ***parallel to*** $\mathcal{S}_i$***.***

2.5. GRADING ON $\mathrm{Der}\,(A)$.

LEMMA (Liendo '10)

- ***Any derivation*** $\partial \in \mathrm{Der}\,(A)$ ***admits a decomposition***

$$\partial = \sum_{e \in \Sigma^\vee \cap M} \partial_e$$

  where ∂_e ***is a homogeneous derivation of degree*** e***.***
- ***The set*** $\{e \in \Sigma^\vee \cap M | \partial_e \neq 0\}$ ***is finite. Its convex hull*** $N(\partial)$ ***is called the NEWTON POLYTOPE of*** ∂***.***
- ***Let*** $\partial \in \mathrm{LND}\,(A)$***. Then for any face*** τ ***of*** $N(\partial)$ ***one has***

$$\partial_\tau := \sum_{e \in \tau \cap M} \partial_e \in \mathrm{LND}\,(A)\,.$$

  In particular, for any vertex e ***of*** $N(\partial)$ ***one has*** $\partial_e \in \mathrm{LND}\,(A)$***.***

2.6. REPLICAS OF HOMOGENEOUS LNDs.

LEMMA

- ***The semigroup*** $\mathcal{S}_i \cap M$ ***is a finitely generated*** $(\tau_i \cap M)$***-module.***
- ***For any*** $e' \in \tau_i \cap M$ ***one has***

$$\chi^{e'} \partial_{\rho_i,e} = \partial_{\rho_i,e+e'} \in \mathrm{LND}\,(A)\,.$$

2.7. COMMUTATORS OF HOMOGENEOUS LNDs.

LEMMA (Romaskevich '14)

- ***Let*** $\partial = \partial_{\rho,e}$ ***and*** $\partial' = \partial_{\rho',e'}$***. Then*** $[\partial, \partial'] = \partial_{\hat{\rho},\hat{e}}$ ***where***
$$\hat{\rho} = \langle \rho, e' \rangle \rho' - \langle \rho', e \rangle \rho \in N \quad \textit{and} \quad \hat{e} = e + e' \in M\,.$$
- ***If*** $\hat{\rho} \neq 0$ ***then*** $\deg([\partial, \partial']) = e + e' \in \Sigma^\vee \cap M$***.***
- ∂ ***and*** ∂' ***commute, that is,*** $\hat{\rho} = 0$ ***if and only if one of the following holds:***
  - ρ ***and*** ρ' ***are collinear and*** $\langle \rho, e \rangle = \langle \rho, e' \rangle$ ***(this holds, in particular, if*** $e, e' \in \mathcal{S}_i$ ***for some*** $i \in \{1, \ldots, k\}$***);***
  - ρ ***and*** ρ' ***are non-collinear and*** $\langle \rho', e \rangle = \langle \rho, e' \rangle = 0$***.***

LEMMA $\mathrm{Der}\,(A) = \bigoplus_{e \in \Sigma^\vee \cap M} \mathcal{L}_e$ ***is a graded Lie algebra, where*** $\mathcal{L}_e$ ***is the span of all the homogeneous derivations of*** A ***of degree*** e***.***

2.8. ITERATED COMMUTATORS.

LEMMA (Manetti '12)

- ***Given*** $U = \partial_1$ ***and*** $V = \partial_2 \in \mathrm{Der}\,(A)$ ***consider***
$$\mathrm{ad}_U^m(V) = [U, [U, \ldots [U, V] \ldots]]$$
where U ***is repeated*** m ***times. Then*** $\mathrm{ad}_U^m \in \mathrm{End}\,(\mathrm{Der}\,(A))$ ***and***
$$\mathrm{ad}_U^m(V) = \sum_{i=0}^{m} \binom{m}{i} \partial_1^{m-i} \partial_2 (-\partial_1)^i.$$
- ***Let*** $U \in \mathrm{LND}(A)$***. Then*** $\mathrm{ad}_U \in \mathrm{End}\,(\mathrm{Der}\,(A))$ ***is locally nilpotent, that is, for any*** $V \in \mathrm{Der}\,(A)$***,***
$$\mathrm{ad}_U^m(V) = 0 \;\; \forall m \gg 1;$$
- ***(a version of the Baker-Campbell-Hausdorff formula)***
$$\mathrm{Ad}_{\exp(U)}(V) = \exp(\mathrm{ad}_U)(V) = \sum_{m=0}^{N(U)} \frac{1}{m!} \mathrm{ad}_U^m(V) \in \mathrm{LND}(A)\,.$$

2.9. NEWTON POLYTOPE OF A CONJUGATE LND.

LEMMA ***Let*** $U = \partial_{\rho_1,e_1}$ ***and*** $V = \partial_{\rho_2,e_2} \in \mathrm{LND}\,(A)$ ***where*** $e_i \in \mathcal{S}_i \cap M$***,*** $i = 1, 2$***. Let also***
$$c_2 = \langle \rho_2, e_1 \rangle, \quad d_1 = \langle \rho_1, e_2 \rangle, \;\; \textit{and} \;\; \delta = d_1 + 1.$$

- ***Assume that*** $c_2 \geq 1$***. Then***
$$\mathrm{ad}_U^m(V) = \partial_{r_m, e_2 + m e_1} \quad \forall m = 0, \ldots, d_1.$$

- ***If*** $d_1 \geq 0$ ***then***
$$\mathrm{ad}_U^m(V) = 0 \quad \forall m \geq \delta + 1$$
and
$$\mathrm{ad}_U^{\delta}(V) = -c_2\delta!\partial_{\rho_1, e_2+\delta e_1} \in \mathrm{LND}(A)$$
where
$$e_2 + \delta e_1 \in \mathcal{S}_1 \cap M.$$
- ***If*** $\partial = \mathrm{Ad}_{\exp(U)}(V)$ ***then*** $N(\partial) = [e_2, e_2 + \delta e_1]$.

2.10. ROOT SUBGROUPS IN THE CLOSURE.

LEMMA ***Consider a subgroup*** $G \subset \mathrm{Aut}(X)$ ***normalized by the torus*** $\mathbb{T}$***. Let*** $\partial \in \mathrm{LND}\,(A)$ ***be s.t.*** $H = \exp(\Bbbk\partial) \subset G$***. Then any vertex*** e ***of the Newton polytope*** $N(\partial)$ ***belongs to some Demazure facet*** $\mathcal{S}_i$***, and the root subgroup*** H_e ***is contained in*** $\overline{G}$***.***

LEMMA

Consider two roots $e_i \in \mathcal{S}_i \cap M$***,*** $i = 1, 2$***. Let*** $\delta = \langle \rho_1, e_2 \rangle + 1$***. Suppose that*** $\delta e_1 + e_2 \in \mathcal{S}_1$***, that is,*** $\langle \rho_2, e_1 \rangle \geq 1$***. Then*** $H_{\delta e_1 + e_2} \subset \overline{\langle H_{e_1},\, H_{e_2} \rangle}$.

Remind our

THEOREM 2

For any $n \geq 2$ ***one can find*** $\mathbb{G}_a$***-subgroups*** $U_1, U_2, U_3 \subset \mathrm{SAut}(\mathbb{A}^n)$ ***such that***
$$G = \langle U_1, U_2, U_3 \rangle \subset \mathrm{SAut}(\mathbb{A}^n)$$
acts infinitely transitively on $\mathbb{A}^n$***.***

LEMMA (Chistopolskaya '18)

For any nilpotent $x \in \mathrm{sl}(n, \Bbbk)$ ***there exists a nilpotent*** $y \in \mathrm{sl}(n, \Bbbk)$ ***such that*** $\mathrm{sl}(n, \Bbbk) = \mathrm{lie}\,\langle x, y \rangle$***.***

HINT OF THE PROOF:

Assume $n \geq 3$**; the case** $n = 2$ **is left as an exercise. Consider the root vectors**
$$e_1 = (-1, 0, \ldots, 0) \in \mathcal{S}_1, \quad e_2 = (0, -1, 0, \ldots, 0) \in \mathcal{S}_2,$$
and
$$u = (-1, 2, 0, \ldots, 0) \in \mathcal{S}_1.$$
By the lemma preceding the theorem one has
$$H_{u+e_2} = H_{e_1 - e_2} = \exp(\Bbbk x) \subset \overline{\langle H_u,\, H_{e_2} \rangle} \cap \mathrm{SL}(n, \Bbbk)$$

where $x \in \mathrm{sl}(n, \Bbbk)$ is the nilpotent generator of $H_{e_1-e_2} = U_x = \exp(\Bbbk x)$. Let $y \in \mathrm{sl}(n, \Bbbk)$ be a nilpotent matrix such that $\mathrm{sl}(n, \Bbbk) = \mathrm{lie}\,\langle x, y\rangle$. It follows that

$$\mathrm{SL}(n, \Bbbk) = \langle U_x, U_y\rangle \text{ where } U_y = \exp(\Bbbk y).$$

By virtue of the inclusion above one has

$$\mathrm{SAff}_n = \langle U_x,\, U_y,\, H_{e_2}\rangle \subset \overline{\langle U_y,\, H_{e_2},\, H_u\rangle}$$

where $H_u \not\subset \mathrm{Aff}_n$.

Let $G = \langle U_y,\, H_{e_2},\, H_u\rangle$. One shows that the subgroup $\langle \mathrm{SAff}_n,\, H_u\rangle \subset \overline{G}$ acts infinitely transitively on $\mathbb{A}^n$. Hence $\overline{G}$ does. Then the same holds for G. □

REMARK

Andrist '18 has found, for any $n \geq 2$, three explicit locally nilpotent derivations (vector fields) x, y, z on $\mathbb{A}^n$ such that the group $\langle U_x, U_y, U_z\rangle$ acts infinitely transitively on $\mathbb{A}^n$.

2.11. SMOOTHNESS IN CODIMENSION 2.

DEFINITION

We say that X is SMOOTH IN CODIMENSION 2 if the singular locus of X has codimension ≥ 3 in X.

LEMMA

A toric affine variety X is smooth in codimension 2 if and only if, for any two-dimensional face τ of the cone $\sigma_X \subset N_{\mathbb{Q}}$, the ray generators $(\rho_i,\, \rho_j)$ of τ can be included in a base of the lattice N.

2.12. INFINITE TRANSITIVITY ON TORIC VARIETIES.

Recall our

THEOREM 1

Let X be a toric affine variety of dimension $n \geq 2$ with no toric factor and smooth in codimension 2. Then one can find root subgroups $H_1, \ldots, H_N$ such that the group $G = \langle H_1, \ldots, H_N\rangle$ acts infinitely transitively in the regular locus $\mathrm{reg}(X)$.

HINT OF THE PROOF:

If $n = 2$ then X is smooth, hence $X \cong \mathbb{A}^2$. Suppose $n \geq 3$.

We use the Cox ring construction. It replaces our initial toric variety X by the spectrum $\mathrm{Cox}(X)$ of its Cox ring, which is just the polynomial ring in k variables. The linear forms $\rho_1, \ldots, \rho_k \in \Xi$ define

the TOTAL COORDINATES on $\mathbb{A}^k = \mathrm{Cox}(X)$. The procedure now is very similar to the one in the case of the affine space.

One can find a finite collection of root subgroups $H_1, \ldots, H_r$ such that the group generated by $H_1, \ldots, H_r$ acts transitively in $\mathrm{reg}\,(X)$ (AFKKZ ′13). To get infinite transitivity we need to enlarge this collection.

Assume that $[\rho_1, \rho_2]$ and $[\rho_1, \rho_3]$ are incident faces of σ. Using the assumption of smoothness in codimension 2 one constructs

- **a cone $\omega \subset \tau_1$ of dimension $n-1$ with ray generators $v_1, \ldots, v_{n-1}$;**
- **the submonoid $\mathcal{M}_1 = \mathbb{Z}_{\geq 0}v_1 + \ldots + \mathbb{Z}_{\geq 0}v_{n-1}$ of ω of rank $n-1$;**
- **a subgroup**
$$G_1 = \langle H_{e_1}, H_{u_1}, H_{u_2}, \ldots, H_{u_{n-1}}, H_{e_3} \rangle \subset \mathrm{SAut}(X)$$
where $u_i = v_i - e_3 \in \mathcal{S}_2 \cap M$ are such that
- **$H_w \subset \overline{G}_1$ for any root $w \in e_1 + \mathcal{M}_1 \subset \mathcal{S}_1 \cap M$.**

Letting $\partial_1 = \partial_{\rho_1, e_1} \in \mathrm{LND}(A)$ consider the subalgebra
$$A_1 = \Bbbk[\chi^v \,|\, v \in \mathcal{M}_1] = \Bbbk[\chi^{v_1}, \ldots, \chi^{v_{n-1}}] \subset \ker(\partial_1).$$
For any $f \in A_1$ the replica $\exp(\Bbbk f \partial_1)$ of H_{e_1} is a subgroup of $\overline{G}_1$. Since $\mathrm{rank}\,(\mathcal{M}_1) = n-1$ one has
$$[\mathrm{Frac}(\ker(\partial_1)) : \mathrm{Frac}(A_1)] < +\infty\,.$$
Hence there exists $b_1 \in \ker\partial_1$ such that $\mathrm{Frac}\,(\ker\partial_1)$ is generated by b_1 and $\mathrm{Frac}\,(A_1)$. One can write $b_1 = \sum_{j=1}^s c_j \chi^{m_j}$ where $m_j \in \tau_1 \cap M$. Then $H_0 = \exp(\Bbbk b_1 \partial_1)$ is contained in the product of the root subgroups $H_{r+j} := \exp(\Bbbk \chi^{m_j} \partial_1)$, $j = 1, \ldots, s$.

Choose linearly independent ray generators $\rho_1, \ldots, \rho_n \in \Xi$. Repeating the same construction one obtains for any $i = 1, 2, \ldots, n$ a triple (G_i, ∂_i, A_i) with properties similar to the ones of (G_1, ∂_1, A_1).

Let now
$$G = \langle H_1, \ldots, H_{r+s}, G_1, \ldots, G_n \rangle \subset \mathrm{SAut}(X).$$
The group $\overline{G}$ satisfies (γ) from Theorem 0 (a criterion of infinite transitivity). Due to this criterion, $\overline{G}$ acts infinitely transitively on its open orbit $\mathscr{O}_{\overline{G}} = \mathscr{O}_G = \mathrm{reg}(X)$. Then the same is true for G. $\square$

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