Skew-symmetric tensor decomposition

[Arrondo, –, Macias Marques, Mourrain]

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Symmetric-rank

$$\mathbb{C}[x_0, \dots, x_n]_d \ni F = \sum_{i=1}^{r} \lambda_i L_i^d$$

$$S^d \mathbb{C}^{n+1} \ni F = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d}$$

this corresponds to find the minimum number r of points

$$P_1 = [L_1^d] = [v_1^{\otimes d}], \dots, P_r = [L_r^d] = [v_r^{\otimes d}]$$

on the d-th Veronese embedding of $\mathbb{P}^n$ such that

$$[F] \in \langle P_1, \dots, P_r \rangle$$

Skew-symmetric rank:

$$\bigwedge^{d} \mathbb{C}^{n+1} \ni T = \sum_{i=1}^{r} \lambda_i v_1^{(i)} \wedge \cdots \wedge v_d^{(i)}$$

this corresponds to find the minimum number r of points

$$P_i = [v_1^{(i)} \wedge \cdots \wedge v_d^{(i)}],\ i = 1, \dots, r$$

on the Grassmannian:
$Gr(\mathbb{P}^{d-1}, \mathbb{P}^n) := \{[v_1 \wedge \cdots \wedge v_d],\ v_i \in \mathbb{C}^n\} \subset \mathbb{P}(\bigwedge^d \mathbb{C}^{n+1})$ such that

$$[T] \in \langle P_1, \dots, P_r \rangle$$

Symmetric case

One of the first classical tools used in the symmetric case to compute the rank and the decomposition of an $F \in S^d$ is

APOLARITY

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and let $R = \mathbb{C}[y_0, \ldots, y_n]$ be its dual ring acting on S by differentiation:

$$y_j(x_i) = \frac{d}{dx_j}(x_i) = \delta_{ij}. \tag{1}$$

Symmetric case

The annihilator of a homogeneous polynomial $F \in S_d$

$$F^{\perp} = \{G \in R \mid G(F) = 0\}$$

is an ideal of R.

A subscheme $X \subset \mathbb{P}(S_1)$ is apolar to $F \in S_d$ if its homogeneous ideal $I_X \subset R_d$ is contained in the annihilator of F.

Symmetric case

Useful tools to get the apolar ideal of a polynomial $F \in S^d\mathbb{C}^{n+1}$ are

catalecticant matrices

$$C_F^{i,d-i} \in \mathrm{Hom}(R_i, S_{d-i})$$

$$C_F^{i,d-i}(y_0^{i_0} \cdots y_n^{i_n}) = \frac{\partial^i}{\partial x_0^{i_0} \cdots x_n^{i_n}}(F)$$

with $\sum_{j=0}^{n} i_j = i$, for $i = 0, \dots, d$.

Symmetric case

Lemma (Apolarity Lemma)

A homogeneous polynomial $F \in S^d\mathbb{C}^{n+1}$ can be written as

$$F = \sum_{i=1}^{r} \lambda_i L_i^d,$$

with $L_1, \ldots, L_r$ linear forms, if and only if the ideal of the scheme $X = \{[L_1^*], \ldots, [L_r^*]\} \subset \mathbb{P}(R)$ is contained in $F^\perp$:

$$I_X \subset F^\perp.$$

Skew-symmetric case

Does this have an extension in the skew-symmetric setting?

[Catalisano, Geramita, Gimigiliano, 02] used a skew-symmetric version of apolarity defined by the perfect paring

$$\bigwedge^{d} \mathbb{C}^{n+1} \times \bigwedge^{n+1-d} \mathbb{C}^{n+1} \to \bigwedge^{n+1} \mathbb{C}^{n+1} \simeq \mathbb{C}$$

induced by the $\wedge$-multiplication:

If $Y \subset \bigwedge^d \mathbb{C}^{n+1}$ is any subspace then
$Y^{\perp} = \{w \in \bigwedge^{n+1-d} \mathbb{C}^{n+1} \mid w \wedge v = 0, \text{ for all } v \in Y\}$.

1. Defined for points in $\bigwedge^d \mathbb{C}^{n+1}$ (not subspaces).
2. Definition of Derivation " $\cdot$ " $\neq \wedge$.
3. Defined in any degree: $T \in \bigwedge^d \mathbb{C}^{n+1}$

$$T^{\perp} := \left\{ h \in \bigwedge^{i \leq d} \mathbb{C}^{n+1*} \mid h \cdot t = 0 \right\}.$$

We already see that there will be a big difference from the symmetric case where $F^{\perp}$ is a classical ideal so it has elements in any degree while here we don't have anything in degree bigger than $n+1$.

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Definition

Let $\mathbf{h}_{\{1,\ldots,i\}} = h^{(1)} \wedge \cdots \wedge h^{(i)} \in \bigwedge^i V^*$ and $\mathbf{v}_{\{1,\ldots,i\}} = v^{(1)} \wedge \cdots \wedge v^{(i)} \in \bigwedge^i V$. For those elements the *skew-apolarity* action is defined as the determinant among $\bigwedge^i V^*$ and $\bigwedge^i V$:

$$\mathbf{h}_{\{1,\ldots,i\}}(\mathbf{v}_{\{1,\ldots,i\}}) = \begin{vmatrix} h^{(1)}(v^{(1)}) & \cdots & h^{(1)}(v^{(i)}) \\ \vdots & & \vdots \\ h^{(i)}(v^{(1)}) & \cdots & h^{(i)}(v^{(i)}) \end{vmatrix}. \tag{2}$$

Definition

For $s \leq d$:

$$\mathbf{h}_{\{1,\ldots,s\}} \cdot \mathbf{v}_{\{1,\ldots,d\}} := \sum_{\substack{R \subset \{1,\ldots,d\} \\ |R|=s}} \operatorname{sign}(R) \cdot \mathbf{h}_{\{1,\ldots,s\}}(\mathbf{v}_R)\mathbf{v}_{\underline{R}},$$

We define the apolarity action extending this by linearity. Now we can define the *skew-catalecticant matrices* $\mathcal{C}_T^{s,d-s} \in \mathrm{Hom}(\bigwedge^s V^*, \bigwedge^{d-s} V)$ associated to any element $T \in \bigwedge^d V$ as $\mathcal{C}_T^{s,d-s}(\mathbf{h}) = \mathbf{h} \cdot T$.

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Lemma

For $\mathbf{v} = v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$ *and* $\mathbf{h} = h_1 \wedge \cdots \wedge h_s \in \bigwedge^s V^*$ *such that* $\vec{\mathbf{h}}^{\perp}$ *and* $\vec{\mathbf{v}}$ *intersect properly,*

$$\mathcal{C}_{\mathbf{v}}^{s,d-s}(\mathbf{h}) = \mathbf{h} \cdot \mathbf{v} = u_1 \wedge \cdots \wedge u_{d-s}$$

where $\langle u_1, \ldots, u_{d-s} \rangle = \vec{\mathbf{h}}^{\perp} \cap \vec{\mathbf{v}}$.

Lemma

For $\mathbf{v} = v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$,

$$\ker \mathcal{C}_{\mathbf{v}}^{s,d-s} = (\vec{\mathbf{v}}^{\perp})_s, \ \operatorname{img} \mathcal{C}_{\mathbf{v}}^{s,d-s} = \bigwedge^{d-s} \vec{\mathbf{v}}.$$

The skew-symmetric analog of an ideal of "indecomposable" points:

Definition

Let $\mathbf{v}_i := v_i^{(1)} \wedge \cdots \wedge v_i^{(d)} \in \bigwedge^d V$ for $i = 1, \ldots, r$ be r points. We define

$$I^{\wedge}(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \bigcap_{i=1}^{r} (\vec{\mathbf{v}}_i^{\perp}). \tag{3}$$

Remark

If $T = \sum_i \lambda_i \mathbf{v}_i$ with $\mathbf{v}_i \in Gr(d, V)$, then by the previous Lemma

$$I^{\wedge}(\mathbf{v}_1, \ldots, \mathbf{v}_r)_s = \bigcap_{i=1}^{r} (\vec{\mathbf{v}}_i^{\perp})_s \subset \ker \mathcal{C}_T^{s,d-s}$$

for $s \leq d$. In particular, for $s = d$,

$$\bigcap_{i=1}^{r} (\vec{\mathbf{v}}_i^{\perp})_d = \bigcap_{i=1}^{r} \ker \mathcal{C}_{\mathbf{v}_i}^{d,0} = \left\{ \mathbf{h} \in \bigwedge^d V^* \mid \mathbf{h}(\mathbf{v}_i) = 0, i = 1, \ldots, r \right\}.$$

Lemma (Skew-apolarity lemma)

The following are equivalent:

1. *An element* $T \in \bigwedge^d V$, *can be written as*

$$T = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i$$

where $\mathbf{v}_i = v_i^{(1)} \wedge \cdots \wedge v_i^{(d)} \in Gr(d, V) \subset \bigwedge^d V$;

2. $\cap_i (\vec{\mathbf{v}}_i^{\perp}) \subset T^{\perp}$;
3. $\cap_i (\vec{\mathbf{v}}_i^{\perp})_d \subset (T^{\perp})_d$.

Essential variables

Like in the symmetric case, one can define *essential variables* for an element $T \in \bigwedge^d V$ to be a basis of the smallest vector subspace $W \subseteq V$ such that $T \in \bigwedge^d W$. (I.e. check if T is *concise*.)

We can check this by computing the kernel of the first catalecticant $\mathcal{C}_T^{1,d-1}$.

The behavior of the skew-symm. ideal of points is different from the symm. case. **For example**: the ideal I of r independent points in $\mathbb{P}^N$:

- In the symmetric case the $H(R/I, d) = r$ for any $d \geq r - 1$
- In the skew-symmetric case... Let's do the easiest case:

  r points in $Gr(d, \mathbb{C}^{n+1}) \subset \bigwedge^d \mathbb{C}^{n+1}$ with $rd \leq n+1$,

  $$\mathbf{v}_1 = e_1 \wedge \cdots \wedge e_d, \ \mathbf{v}_2 = e_{d+1} \wedge \cdots \wedge e_{2d}, \ \ldots, \ \mathbf{v}_r = e_{(r-1)d+1} \wedge \cdots \wedge e_{rd}.$$

  $I^{\wedge}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is generated by $e^*_{rd+1}, \ldots, e^*_{n+1}$, in degree 1 only if $rd < n+1$,

  and degree-two elements $e^*_i \wedge e^*_j$ such that $1 \leq i \leq sd < j \leq rd$, for some $s < r$.

  It is clear that these elements are in the ideal, and it is straightforward to see that they are enough to generate it.

Another example:

- If $[L_1^d] \neq [L_2^d] \in \nu_d(\mathbb{P}(V))$ and $d > 1$ then $rk([L_1^d] + [L_2^d]) = 2$.
- If $[\mathbf{v}_1] \neq [\mathbf{v}_2] \in \mathbb{G}(d, V)$, then $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ has rank one

$$\Leftrightarrow$$

the line passing through $\mathbf{v}_1$ and $\mathbf{v}_2$ is contained in $\mathbb{G}(d, V)$

$$\Leftrightarrow$$

the intersection of the subspaces $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ has dimension at least $d - 1$.

We are interested in the skew-symmetric rank of $T \bigwedge^d(V)$. The first interesting case is $T \in \bigwedge^3(V)$.

Example (Triky)

$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \in \bigwedge^3 \mathbb{C}^6$.

$$\mathbf{v}_1 = f_0 \wedge f_1 \wedge f_2, \qquad \mathbf{v}_2 = f_0 \wedge f_3 \wedge f_4, \qquad \text{and } \mathbf{v}_3 = f_1 \wedge f_3 \wedge f_5,$$

$\ker \mathcal{C}_{\mathbf{v}}^{1,2} = I^{\wedge}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)_1 = 0$, and

$$\ker \mathcal{C}_{\mathbf{v}}^{2,1} = \langle f_0^* \wedge f_2^* + f_3^* \wedge f_5^*,\ f_0^* \wedge f_4^* - f_1^* \wedge f_5^*,\ f_0^* \wedge f_5^*,\ f_1^* \wedge f_2^* - f_3^* \wedge f_4^*, \\ f_1^* \wedge f_4^*,\ f_2^* \wedge f_3^*,\ f_2^* \wedge f_4^*,\ f_2^* \wedge f_5^*,\ f_4^* \wedge f_5^* \rangle.$$

We claim that $\mathbf{v}$ has rank 3. If NOT $\mathbf{v} = \mathbf{v}_4 + \mathbf{v}_5$, where $\mathbf{v}_4 = g_0 \wedge g_1 \wedge g_2$ and $\mathbf{v}_5 = g_3 \wedge g_4 \wedge g_5$. Since $\ker \mathcal{C}_{\mathbf{v}}^{1,2} = 0$, we must have $\mathbb{C}^6 = \langle g_0, \dots, g_5 \rangle$. So $g_0, \dots, g_5$ are independent, and therefore

$$I^{\wedge}(\mathbf{v}_4, \mathbf{v}_5)_2 = (\mathbf{v}_4{}^{\perp})_1 \wedge (\mathbf{v}_5{}^{\perp})_1 = \langle g_0^*, g_1^*, g_2^* \rangle \wedge \langle g_3^*, g_4^*, g_5^* \rangle.$$

Since $I^{\wedge}(\mathbf{v}_4, \mathbf{v}_5)_2 \subseteq \ker \mathcal{C}_{\mathbf{v}}^{2,1}$ and both spaces have dimension 9, equality must hold.

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Example

Then, $I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3 = \ker \mathcal{C}_{\mathbf{v}}^{2,1} * \{\text{all linear forms}\}$

$$I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3 = \left\langle \begin{array}{l} f_0^* \wedge f_1^* \wedge f_4^*, f_0^* \wedge f_1^* \wedge f_5^*, f_0^* \wedge f_2^* \wedge f_3^*, f_0^* \wedge f_2^* \wedge f_4^*, \\ f_0^* \wedge f_2^* \wedge f_5^*, f_0^* \wedge f_3^* \wedge f_5^*, f_0^* \wedge f_4^* \wedge f_5^*, f_2^* \wedge f_3^* \wedge f_4^*, \\ f_2^* \wedge f_3^* \wedge f_5^*, f_2^* \wedge f_4^* \wedge f_6^*, f_3^* \wedge f_4^* \wedge f_5^*, \\ f_0^* \wedge f_1^* \wedge f_2^* - f_1^* \wedge f_3^* \wedge f_5^*, f_0^* \wedge f_3^* \wedge f_4^* - f_1^* \wedge f_3^* \wedge f_5^* \end{array} \right\rangle \quad (4)$$

and

$$I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3^{\perp} = \langle f_0 \wedge f_1 \wedge f_2 + f_0 \wedge f_3 \wedge f_4 + f_1 \wedge f_3 \wedge f_5, f_0 \wedge f_1 \wedge f_3 \rangle.$$

But, if $\mathbf{v} = \mathbf{v}_4 + \mathbf{v}_5$ and $\vec{\mathbf{v}}_4 \cap \vec{\mathbf{v}}_5 = \{0\}$ then $I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3^{\perp} = \langle \mathbf{v}_4, \mathbf{v}_5 \rangle$. This implies that $\mathbb{P}(I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3^{\perp}) \cap Gr(3,V) = \{\mathbf{v}_4,\mathbf{v}_5\}$.
An explicit computation shows that
$\lambda(f_0 \wedge f_1 \wedge f_2 + f_0 \wedge f_3 \wedge f_4 + f_1 \wedge f_3 \wedge f_5) + \mu f_0 \wedge f_1 \wedge f_3 \in Gr(3,V)$ with $\lambda, \mu \in K$ implies that $\lambda = 0$. This contradicts the property that $\mathbb{P}(I^{\wedge}(\mathbf{v}_4,\mathbf{v}_5)_3^{\perp}) \cap Gr(3,V) = \{\mathbf{v}_4,\mathbf{v}_5\}$. Therefore, $\mathbf{v}$ must be of rank 3.

Cases: $\bigwedge^3 \mathbb{C}^3$ and $\bigwedge^3 \mathbb{C}^4$

There is only one possibility:

$$[\mathbf{v}] = [v_0 \wedge v_1 \wedge v_2]. \tag{II}$$

If $\mathbf{v} \in \bigwedge^3 \mathbb{C}^4$ then $I^\wedge(\mathbf{v}) = I^\wedge(\mathbf{v})_1$ in particular $I^\wedge(\mathbf{v}) = (v_3^*)$ where $\langle v_3 \rangle = \langle v_0, v_1, v_2 \rangle^\perp \subset V$.

Therefore if one wants to find its decomposition as in (II), one has simply to compute a basis $\{v_0, v_1, v_2\}$ of $I^\wedge(\mathbf{v})_1^\perp$, and such a basis will be good for the presentation of $\mathbf{v}$ as a tensor of skew-symmetric rank 1 as in (II).

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Case $\bigwedge^3 \mathbb{C}^5$

Any tri-vector $\mathbf{v} \in \bigwedge^3 \mathbb{C}^5$ is divisible by some vector say v_0:
$\mathbf{v} = v_0 \wedge \mathbf{v}'$ where $\mathbf{v}' \in \bigwedge^2 \mathbb{C}^5$, hence there are only 2 possibilities:

- $rk_\wedge(\mathbf{v}) = 1$ and $\mathbf{v} = v_0 \wedge v_1 \wedge v_2$,
- $rk_\wedge(\mathbf{v}) = 2$ and $\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4$.

$$\dim \ker \mathcal{C}^{1,2} = \begin{cases} \neq 0 & \text{if } rk_\wedge(\mathbf{v}) = 1, \\ = 0 & \text{if } rk_\wedge(\mathbf{v}) = 2 \end{cases}$$

Case $\bigwedge^3 \mathbb{C}^5$

Look for a decomposition.

If we find generators of $I^{\wedge}(\mathbf{v})$ in degree 1, say $\{v_4^*, v_5^*\}$, then $\mathbf{v}$ is of the form $\mathbf{v} = v_0 \wedge v_1 \wedge v_2$ such that $\langle v_0, v_1, v_2 \rangle = \langle v_4, v_5 \rangle^{\perp}$.

If we do not find any generators in degree 1 and we want to recover the decomposition of $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 = v_0 \wedge v_1 \wedge v_2$ and $\mathbf{v}_2 = v_0 \wedge v_3 \wedge v_4$, we have to look at $I^{\wedge}(\mathbf{v}_1, \mathbf{v}_2)_2$:
$I^{\wedge}(\mathbf{v}_1, \mathbf{v}_2)$ is generated in degree 2 by

$$(v_1^* \wedge v_3^*, v_1^* \wedge v_4^*, v_2^* \wedge v_3^*, v_2^* \wedge v_4^*, v_2^* v_4^* - v_1^* v_2^*). \tag{5}$$

Hence $\mathbf{v}_1 = v_0 \wedge v_1 \wedge v_2$ and $\mathbf{v}_2 = v_0 \wedge v_3 \wedge v_4$ where $v_i = (v_i^*)^*$ for $i = 1, \ldots, 4$ and $\langle v_0 \rangle = \langle v_1, v_2, v_3, v_4 \rangle^{\perp}$.

Case $\bigwedge^3 \mathbb{C}^5$

Look for a decomposition.

If we find generators of $I^{\wedge}(\mathbf{v})$ in degree 1, say $\{v_4^*, v_5^*\}$, then $\mathbf{v}$ is of the form $\mathbf{v} = v_0 \wedge v_1 \wedge v_2$ such that $\langle v_0, v_1, v_2 \rangle = \langle v_4, v_5 \rangle^{\perp}$.

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Hence $\mathbf{v}_1 = v_0 \wedge v_1 \wedge v_2$ and $\mathbf{v}_2 = v_0 \wedge v_3 \wedge v_4$ where $v_i = (v_i^*)^*$ for $i = 1, \ldots, 4$ and $\langle v_0 \rangle = \langle v_1, v_2, v_3, v_4 \rangle^{\perp}$.

Case $\bigwedge^3 \mathbb{C}^5$

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If we find generators of $I^{\wedge}(\mathbf{v})$ in degree 1, say $\{v_4^*, v_5^*\}$, then $\mathbf{v}$ is of the form $\mathbf{v} = v_0 \wedge v_1 \wedge v_2$ such that $\langle v_0, v_1, v_2 \rangle = \langle v_4, v_5 \rangle^{\perp}$.

If we do not find any generators in degree 1 and we want to recover the decomposition of $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 = v_0 \wedge v_1 \wedge v_2$ and $\mathbf{v}_2 = v_0 \wedge v_3 \wedge v_4$, we have to look at $I^{\wedge}(\mathbf{v}_1, \mathbf{v}_2)_2$:
$I^{\wedge}(\mathbf{v}_1, \mathbf{v}_2)$ is generated in degree 2 by

$$(v_1^* \wedge v_3^*, v_1^* \wedge v_4^*, v_2^* \wedge v_3^*, v_2^* \wedge v_4^*, v_2^* v_4^* - v_1^* v_2^*). \qquad (5)$$

Hence $\mathbf{v}_1 = v_0 \wedge v_1 \wedge v_2$ and $\mathbf{v}_2 = v_0 \wedge v_3 \wedge v_4$ where $v_i = (v_i^*)^*$ for $i = 1, \ldots, 4$ and $\langle v_0 \rangle = \langle v_1, v_2, v_3, v_4 \rangle^{\perp}$.

Case $\bigwedge^3 \mathbb{C}^6$

- $\mathbf{v} = v_0 \wedge v_1 \wedge v_2$, ($\checkmark$)
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 3 \Rightarrow$ rank 1 and $I(\mathbf{v})_1^{\perp}$ suffices for the decomposition:
  $\langle v_0, v_1, v_2 \rangle = (\ker \mathcal{C}_{\mathbf{v}}^{1,2})^{\perp}$
- $\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4$, ($\checkmark$)
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 1 \Rightarrow$ rank 2. Now $I(\mathbf{v})_1^{\perp}$ suffices only to say that $\langle v_0, \ldots, v_4 \rangle = (\ker \mathcal{C}_{\mathbf{v}}^{1,2})^{\perp} = \langle v_5^* \rangle$ but doesn't say who is v_0, for that we need $I(\mathbf{v})_2$: $\ker \mathcal{C}_{\mathbf{v}}^{2,1} = \langle v_1^* \wedge v_3^*, v_1^* \wedge v_4^*, v_2^* \wedge v_3^*, v_2^* \wedge v_4^*, v_2^* v_4^* - v_1^* v_2^* \rangle$
- $\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_3 \wedge v_4 \wedge v_5$,
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 0$, this is not sufficient to say that the rank is 2,
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{2,1} = 9$, also this is not sufficient to say that rank 2,
  $\ker \mathcal{C}_{\mathbf{v}}^{2,1} = \langle v_i^* \wedge v_j^* \rangle_{i \in \{0,1,2\}, j \in \{3,4,5\}}$
- $\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4 + v_1 \wedge v_3 \wedge v_5$, (triky)
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 0$, this is not sufficient to say that the rank is 3,
  $\dim \ker \mathcal{C}_{\mathbf{v}}^{2,1} = 9$, also this is not sufficient to say that the rank is 3,

$$\ker \mathcal{C}_{\mathbf{v}}^{2,1} = \langle v_0^* \wedge v_2^* + v_3^* \wedge v_5^*,\ v_0^* \wedge v_4^* - v_1^* \wedge v_5^*,\ v_0^* \wedge v_5^*,\ v_1^* \wedge v_2^* - v_3^* \wedge v_4^*,$$
$$v_1^* \wedge v_4^*,\ v_2^* \wedge v_3^*,\ v_2^* \wedge v_4^*,\ v_2^* \wedge v_5^*,\ v_4^* \wedge v_5^* \rangle.$$

Using this idea of computing the rank and the decomposition of the

normal forms

is going to work only for 2 more cases: $\bigwedge^3 \mathbb{C}^7$ and $\bigwedge^3 \mathbb{C}^8$ since

for $\mathbb{C}^{n+1>8}$ the number of normal forms in $\bigwedge^3 \mathbb{C}^{n+1}$ is infinite.

Case $\bigwedge^3 \mathbb{C}^7$

In this case the classification of normal forms of tri-vectors is due to [Schouten '31]. More than the previous classes there are other 5 classes:

$$[a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p], \tag{VI}$$

$$[q \wedge r \wedge s + a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p], \tag{VII}$$

$$[a \wedge b \wedge c + q \wedge r \wedge s + a \wedge q \wedge p], \tag{VIII}$$

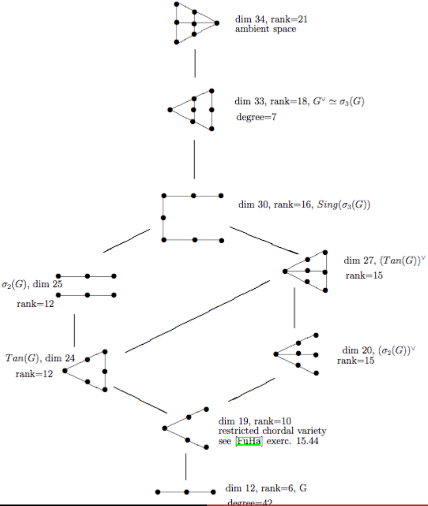
$$[a \wedge b \wedge c + q \wedge r \wedge s + a \wedge q \wedge p + b \wedge r \wedge p], \tag{IX}$$

$$[a \wedge b \wedge c + q \wedge r \wedge s + a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p]. \tag{X}$$

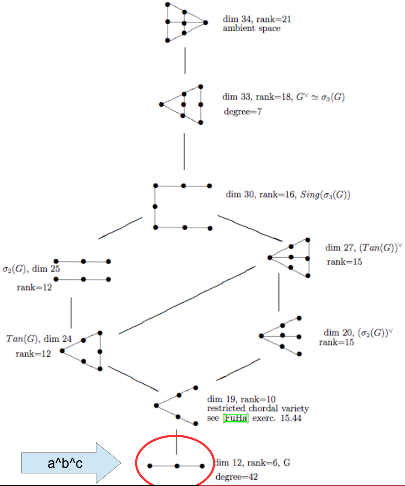
The containment diagram of the orbits of those normal forms is described in [Abo, Ottaviani, Peterson '09].

Case $\bigwedge^3 \mathbb{C}^7$

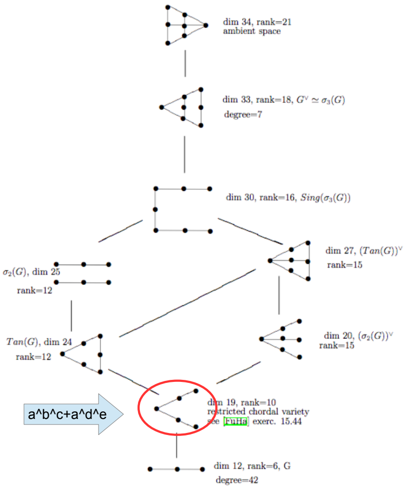
[Abo Ottaviani Peterson '09]



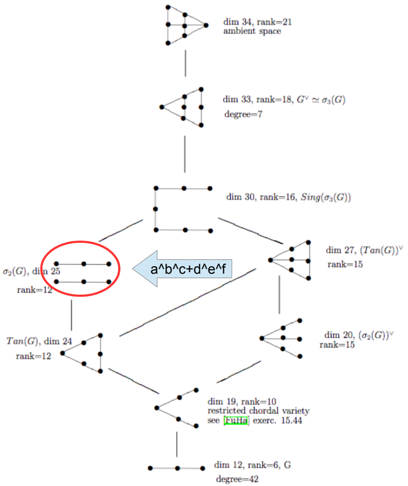
Case $\bigwedge^3 \mathbb{C}^7$



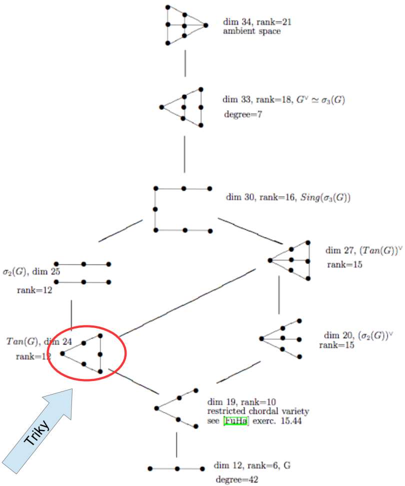
Case $\bigwedge^3 \mathbb{C}^7$



Case $\bigwedge^3 \mathbb{C}^7$



Case $\bigwedge^3 \mathbb{C}^7$

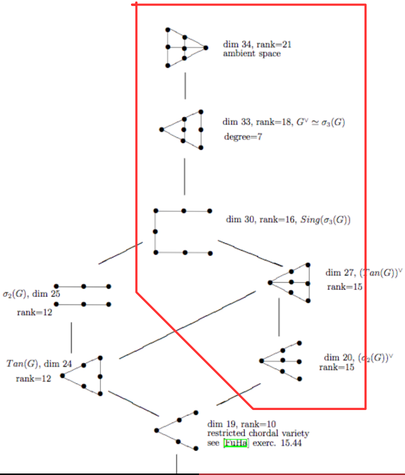


Case $\bigwedge^3 \mathbb{C}^7$

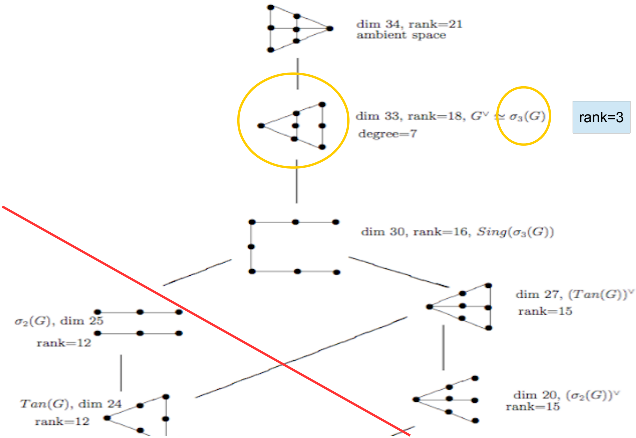
All the previous cases do not involve all the variables, so, after having re-written the tensor in the minimum number of variables ($\ker \mathcal{C}_{\mathbf{v}}^{1,2}$ will give the *essential variables*) we can use the previous technique to tackle them.

Case $\bigwedge^3 \mathbb{C}^7$

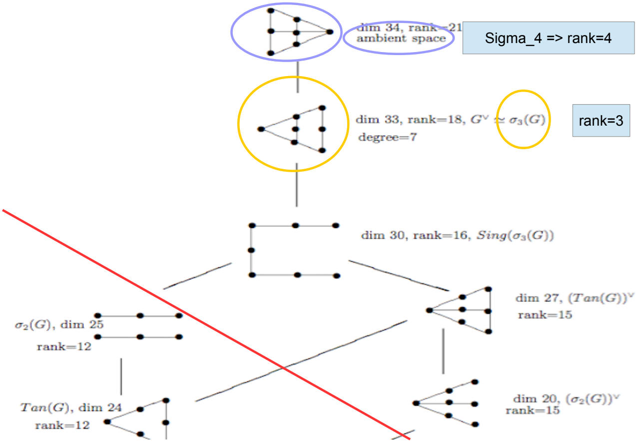
We are left with:



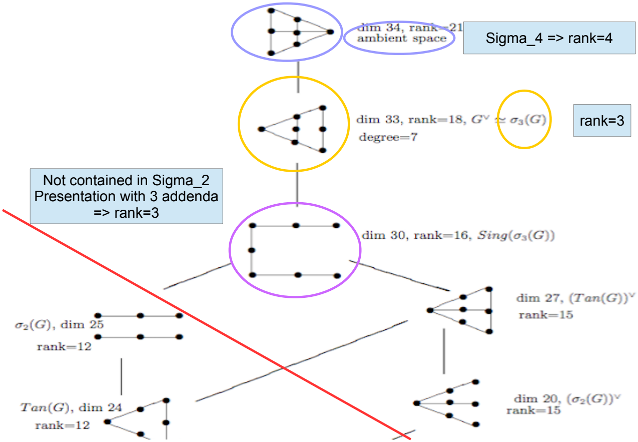
Case $\bigwedge^3 \mathbb{C}^7$



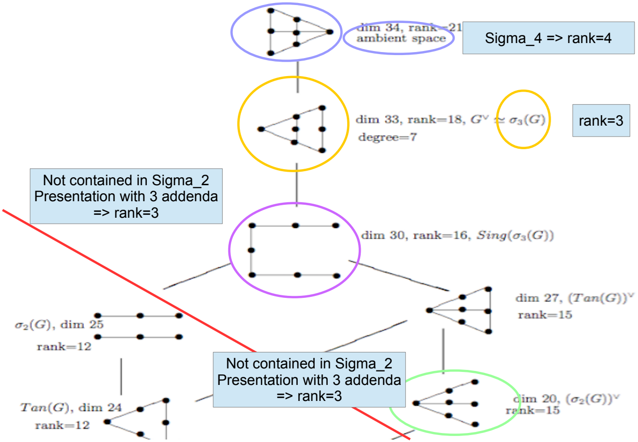
Case $\bigwedge^3 \mathbb{C}^7$



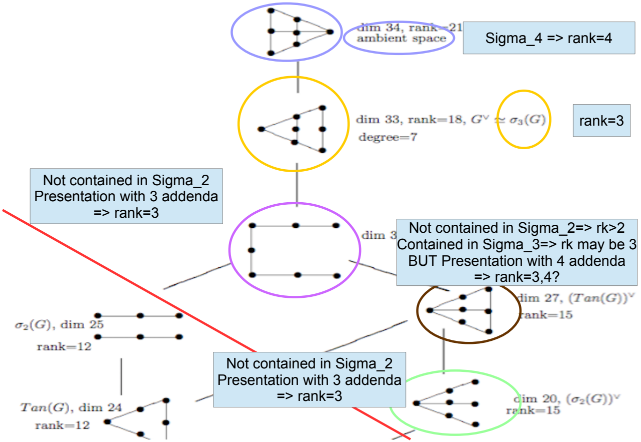
Case $\bigwedge^3 \mathbb{C}^7$



Case $\bigwedge^3 \mathbb{C}^7$



Case $\bigwedge^3 \mathbb{C}^7$



Case $\bigwedge^3 \mathbb{C}^7$

We want to understand the rank of

$$q \wedge r \wedge s + a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p$$

First Remark: There are 5 normal forms involving all the variables,
There are only 4 of them which MAY have rank 3.

Case $\bigwedge^3 \mathbb{C}^7$

Let

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \in \bigwedge^3 \mathbb{C}^7$$

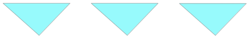
with $\mathbf{v}_i \in \mathbb{G}(3, \mathbb{C}^7)$ be a minimal presentation of skew-symmetric rank 3 of $\mathbf{v}$.

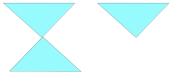
Let $[\vec{\mathbf{v}}_i]$ the planes in $\mathbb{P}^6$ corresponding to $\mathbf{v}_i$, $i = 1, 2, 3$ and assume that $\langle [\vec{\mathbf{v}}_1], [\vec{\mathbf{v}}_2], [\vec{\mathbf{v}}_3] \rangle = \mathbb{P}^6$ (hence we are using all the variables).

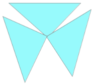
We consider another invariant that is preserved by the action of $SL(7)$, namely the intersection of the $[\vec{\mathbf{v}}_i]$'s.

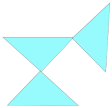
$[\vec{\mathbf{v}}_i] \cap [\vec{\mathbf{v}}_j]$ is at most a point since otherwise $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ won't be a minimal presentation of skew-symmetric rank 3 for $\mathbf{v}$; in fact $\mathbf{v}_i + \mathbf{v}_j$ would have skew-symmetric rank 1.

Case $\bigwedge^3 \mathbb{C}^7$





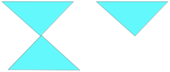


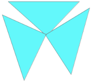


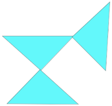
Case $\bigwedge^3 \mathbb{C}^7$

e0 ∧e1 ∧e2 +e3 ∧e4 ∧e5 +e6 ∧(e0 +···+e6)∧l;

sigma_3 



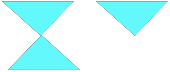




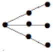
Case $\bigwedge^3 \mathbb{C}^7$

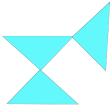
$e0 \wedge e1 \wedge e2 + e3 \wedge e4 \wedge e5 + e6 \wedge (e0 + \cdots + e6) \wedge l;$

sigma_3 



$a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p$

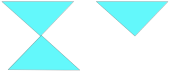




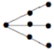
Case $\bigwedge^3 \mathbb{C}^7$

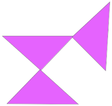
$e0 \wedge e1 \wedge e2 + e3 \wedge e4 \wedge e5 + e6 \wedge (e0 + \cdots + e6) \wedge l;$

sigma_3

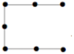


$a \wedge q \wedge p + b \wedge r \wedge p + c \wedge s \wedge p$





$a \wedge b \wedge c + q \wedge r \wedge s + a \wedge q \wedge$



Case $\bigwedge^3 \mathbb{C}^7$

e0 ∧e1 ∧e2 +e3 ∧e4 ∧e5 +e6 ∧(e0 +···+e6)∧l;

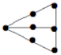
sigma_3 

e0^e1^e2+
e0^e3^e4 +
e5^e6^(e0+..+e4)

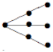
This has to be an orbit
Among those whit all vars

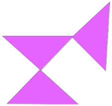
It has to be of rank 3

There is only one orbit left that
May have rank 3

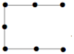


a ∧ q ∧ p + b ∧ r ∧ p + c ∧ s ∧ p





a ∧ b ∧ c + q ∧ r ∧ s + a ∧ q ∧



Case $\bigwedge^3 \mathbb{C}^7$

Compute the rank and the decomposition of $\mathbf{v} \in \bigwedge^3 \mathbb{C}^7$:

1. If $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} > 0$, then $\mathbf{v} \in \bigwedge^3 \mathbb{C}^{n<7}$ and we reduce to previous cases. ✓
2. If $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 0$, then check if $\mathbf{v} \in \sigma_3(Gr(3, \mathbb{C}^7))$ by checking if $\mathbf{v}$ satisfy its degree 7 equation (cf. [AOP '09]).
    1. If NO, then its $rk_{\wedge}(\mathbf{v}) = 4$ and to get a decomposition
        - choose an element $P \in Gr(3, \mathbb{C}^7)$,
        - choose one of the points $Q_i \in \langle P, \mathbf{v} \rangle \cap \sigma_3(Gr(3, \mathbb{C}^7))$,
        - then $\mathbf{v} = P + Q_i$ for an $i \in \{1, \ldots, 7\}$.
        - If something goes wrong, repeat. ✓
    2. If YES, then $rk_{\wedge}(\mathbf{v}) = 3$ and to get a decomposition we have to understand to which orbit $\mathbf{v}$ belongs.

Case $\bigwedge^3 \mathbb{C}^7$

Compute the rank and the decomposition of $\mathbf{v} \in \bigwedge^3 \mathbb{C}^7$:

1. If $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} > 0$, then $\mathbf{v} \in \bigwedge^3 \mathbb{C}^{n<7}$ and we reduce to previous cases. ✓
2. If $\dim \ker \mathcal{C}_{\mathbf{v}}^{1,2} = 0$, then check if $\mathbf{v} \in \sigma_3(Gr(3, \mathbb{C}^7))$ by checking if $\mathbf{v}$ satisfy its degree 7 equation (cf. [AOP '09]).
   1. If NO, then its $rk_\wedge(\mathbf{v}) = 4$ and to get a decomposition
      - choose an element $P \in Gr(3, \mathbb{C}^7)$,
      - choose one of the points $Q_i \in \langle P, \mathbf{v} \rangle \cap \sigma_3(Gr(3, \mathbb{C}^7))$,
      - then $\mathbf{v} = P + Q_i$ for an $i \in \{1, \ldots, 7\}$.
      - If something goes wrong, repeat. ✓
   2. If YES, then $rk_\wedge(\mathbf{v}) = 3$ and to get a decomposition we have to understand to which orbit $\mathbf{v}$ belongs.

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      - If something goes wrong, repeat. $\checkmark$
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Case $\bigwedge^3 \mathbb{C}^7$

Compute Kernel and Image of the multiplication map by $\mathbf{v} \in \bigwedge^3 \mathbb{C}^7$:

$$\mathbb{C}^7 \xrightarrow{\wedge \mathbf{v}} \bigwedge^4 \mathbb{C}^7 \qquad (6)$$

- If $\ker(\wedge \mathbf{v}) = \langle v_0 \rangle \neq \{0\}$ then $\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4 + v_0 \wedge v_5 \wedge v_6$

and $v_1, \ldots, v_6$ are given by $\ker \mathcal{C}_{\mathbf{v}}^{2,1}$. ✓

In the 3 remaining cases $\ker(\wedge \mathbf{v}) = \{0\}$ and we use the intersection of the Image of $\wedge \mathbf{v}$ with the Grassmannian.

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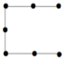
In the 3 remaining cases $\ker(\wedge \mathbf{v}) = \{0\}$ and we use the intersection of the Image of $\wedge \mathbf{v}$ with the Grassmannian.

Case $\bigwedge^3 \mathbb{C}^7$

- If $\ker(\wedge \mathbf{v}) = \{0\}$ and $Im(\wedge \mathbf{v})$ meets the Grassmannian in 2 points then

$$\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4 + v_3 \wedge v_5 \wedge v_6$$

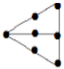


where v_0 and v_3 are pre-images of the 2 points in the Grassmannian,
and v_1, v_2, v_4, v_5, v_6 are given by $\ker \mathcal{C}_{\mathbf{v}}^{2,1}$.

Case $\bigwedge^3 \mathbb{C}^7$

- If $\ker(\wedge \mathbf{v}) = \{0\}$ and $Im(\wedge \mathbf{v})$ meets the Grassmannian in 1 point then

$$\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_0 \wedge v_3 \wedge v_4 + v_5 \wedge v_6 \wedge (v_0 + \cdots + v_6)$$

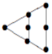


where v_0 is the pre-images of the point in the Grassmannian, and $v_1, \ldots, v_6$ are given by $\ker \mathcal{C}_{\mathbf{v}}^{2,1}$.

Case $\bigwedge^3 \mathbb{C}^7$

- If $\ker(\wedge \mathbf{v}) = \{0\}$ and $Im(\wedge \mathbf{v})$ doesn't meet the Grassmannian then

$$\mathbf{v} = v_0 \wedge v_1 \wedge v_2 + v_3 \wedge v_4 \wedge v_5 + v_6 \wedge (v_0 + \cdots + v_6) \wedge l$$



where $v_0, \ldots, v_6, l$ can be computed via the pre-images of the Grassmannian.

Case $\bigwedge^3 \mathbb{C}^8$

In order to get a complete classification of the ranks we used the [Gurevich '64] classification of normal forms I: $w = 0$, $\dim I = 0$,
II: $[qrs]$, $\dim II = 16$, $Gr(\mathbb{P}^2, \mathbb{P}^7)$,
III: $[aqp][brp]$, $\dim III = 25$,
IV: $[aqr][brp][cpq]$, $\dim IV = 31$,
V: $[abc][pqr]$, $\dim V = 32$, $\sigma_2(Gr(\mathbb{P}^2, \mathbb{P}^7))$,
VI: $[aqp][brp][csp]$, $\dim VI = 28$,
VII: $[abc][prq][aps]$, $\dim VII = 35$,
VIII: $[abc][qrs][aqp]$, $\dim VIII = 38$,
IX: $[abc][qrs][aqp][brp]$, $\dim IX = 41$,
X: $[abc][qrs][aqp][brp][csp]$, $\dim X = 42$
XI: $[aqp][brp][csp][crt]$, $\dim XI = 40$,
XII: $[qrs][aqp][brp][csp][crt]$, $\dim XII = 43$,
XIII: $[abc][qrs][aqp][crt]$, $\dim XIII = 44$,
XIV: $[abc][qrs][aqp][brp][crt]$, $\dim XIV = 46$,
XV: $[abc][qrs][aqp][brp][csp][crt]$, $\dim XV = 48$,
XVI: $[aqp][bst][crt]$, $\dim XVI = 41$,
XVII: $[aqp][brp][bst][crt]$, $\dim XVII = 47$,
XVIII: $[qrs][aqp][brp][bst][crt]$, $\dim XVIII = 50$,
XIX: $[aqp][brp][csp][bst]$, $\dim XIX = 48$,
XX: $[qrs][aqp][brp][csp][bst][crt]$, $\dim XX = 52$,
XXI: $[abc][qrs][aqp][bst]$, $\dim XXI = 53$,
XXII: $[abc][qrs][aqp][brp][bst][crt]$, $\dim XXII = 55$,

XXIII: $[abc][qrs][aqp][brp][csp][bst][crt]$, $\dim XXIII = 56$, $\mathbb{P}^{55}$.

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X: $[abc][qrs][aqp][brp][csp]$, $\dim X = 42$
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XV: $[abc][qrs][aqp][brp][csp][crt]$, $\dim XV = 48$,
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XXII: $[abc][qrs][aqp][brp][bst][crt]$, $\dim XXII = 55$,

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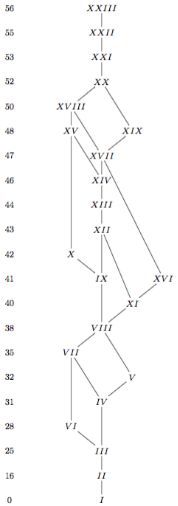
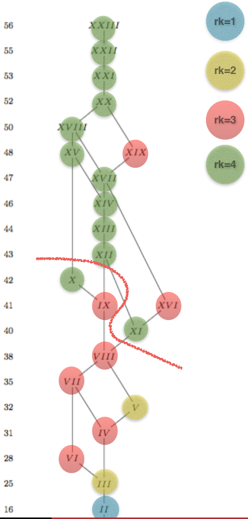
Case $\bigwedge^3 \mathbb{C}^8$

and the containment diagram of their orbit closures [– Vanzo, '18]



Case $\bigwedge^3 \mathbb{C}^8$



THANKS