EXERCISES FOR THE COURSE
"AUTOMORPHISMS OF PROJECTIVE VARIETIES"

1) Let X be a complete variety, and $\eta : \tilde{X} \to X$ the normalization. Show that the action of $\mathrm{Aut}(X)$ on X lifts to a unique action on $\tilde{X}$.

2) Let X be the finite scheme $\operatorname{Spec} k[t]/(t^n)$, where $n \geq 2$. Assuming that $\mathrm{char}(k) = 0$, show that $\mathrm{Aut}(X)$ is the semi-direct product of a normal unipotent group of dimension $n-2$ with $\mathbb{G}_m$.

3) Describe the automorphism group schemes of the smooth projective curves of genus ≤ 1.

4) Let E be an elliptic curve, and L a line bundle of degree 0 over E.

(i) Show that $\tau_x^*(L) \simeq L$ for all $x \in E(k)$, where $\tau_x : E \to E$, $y \mapsto x+y$ denotes the translation by x.

(ii) Show that each translation of E lifts to an automorphism of the variety L which commutes with the action of $\mathbb{G}_m$.

(iii) Let $X := \mathbb{P}(L \oplus \mathcal{O}_E)$ be the projective completion of L, so that X is a ruled surface over E. Construct an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}^0(X) \longrightarrow E \longrightarrow 1.$$

(iv) Show that $\mathrm{Aut}^0(X)$ is commutative, and $\pi_0 \mathrm{Aut}(X)$ is finite.

5) Let E, F be elliptic curves and assume that $E(k)$ has a point x_0 of order 2 (equivalently, E is ordinary if $\mathrm{char}(k) = 2$). Consider the involution σ of $E \times F$ defined by $\sigma(x,y) = (x + x_0, -y) = (\tau_{x_0}(x), -y)$.

(i) Show that the quotient $X := (E \times F)/\sigma$ is a smooth projective surface, and the quotient map $\pi : E \times F \to X$ is a finite étale morphism.

(ii) The projection $E \times F \to E$ induces a morphism $f : X \to E'$, where E' denotes the quotient of E by the translation τ_{x_0}. Show that f is the Albanese morphism of X. In particular, X is not an abelian surface.

(iii) Assume that $\mathrm{char}(k) = 2$. Show that $\pi^*(T_X)$ is trivial as a vector bundle on $E \times F$ equipped with an action of σ, where T_X denotes the tangent bundle of X. Deduce that T_X is trivial as well, and Aut_X is non-reduced.