EXERCISES FOR THE "MORI DREAM SPACES AND BLOW-UPS" LECTURES

1. Prove that the blow-up of $\mathbb{P}^2$ at 9 sufficiently general points contains infinitely many (-1)-divisors.

2. Let X_n be the blow-up of $\mathbb{P}^2$ at $n \leq 8$ general points. Prove that X_n has finitely many (-1) curves. Prove that if $n \leq 7$, then the Cox ring $\mathrm{Cox}(X_n)$ is generated as a $\mathbb{C}$-algebra by sections corresponding to the (-1)-curves on it. (For $n = 8$, you may also prove that the Cox ring is generated by the above set together with two linearly independent sections of $-K_X$.)

3. Let X be the blow-up of $\mathbb{P}^3$ at 4 or 5 points such that any 4 points span $\mathbb{P}^3$. Prove that the Cox ring of X is generated by sections (unique up to a constant) corresponding to the exceptional divisors corresponding to the points and the proper transforms of the planes passing through three of the points.

4. Prove that the Cox ring of a smooth projective toric variety is isomorphic to a polynomial algebra $\mathbb{C}[w_1, \dots, w_r]$, where r is the number of 1-dimensional rays in the fan of X.

5. Let X be the blow-up of $\mathbb{P}^3$ at two distinct points p and q. Find (a) the cone of nef divisors, (b) the cone generated by effective divisors, (c) the cone generated by movable divisors (all cones in $\mathrm{Pic}(X) \otimes \mathbb{R} = \mathbb{R}^3$). Describe the movable cone as a union of the nef cone of X and the nef cone of Y, the variety obtained by flipping the line spanned by p and q.

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