Characterization of affine toric varieties by their automorphism groups

Alvaro Liendo

Joint work with Andriy Regeta and Christian Urech

Warsaw, September 28, 2018
Is a geometric object uniquely determined by its symmetry group?
Regular polygons

\[
\text{Sym}(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1, rs = sr^{-1} \rangle
\]

\[
\text{Sym}(S) = D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle
\]
Regular polygons

\[ \text{Sym}(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1, \; rs = sr^{-1} \rangle \]

\[ \text{Sym}(S) = D_8 = \langle r, s \mid r^4 = s^2 = 1, \; rs = sr^{-1} \rangle \]
Regular polygons

Sym(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1, \hspace{0.5em} rs = sr^{-1} \rangle

Sym(S) = D_8 = \langle r, s \mid r^4 = s^2 = 1, \hspace{0.5em} rs = sr^{-1} \rangle
Regular polygons

\[ \text{Sym}(P_n) = D_{2n} = \langle r, s \mid r^n = s^2 = 1, \; rs = sr^{-1} \rangle \]

\[ |\text{Sym}(P_n)| = 2n \]
Regular polygons are uniquely determined by their symmetry group.
Regular polytopes
Regular polytopes
Regular polytopes

\[ \text{Sym}(C) \cong \text{Sym}(O) \]
Regular polytopes are **not** uniquely determined by their symmetry group.
Is a geometric object uniquely determined by its symmetry group?

In algebraic geometry there are at least two possibilities for the symmetry group:

Regular automorphism group \( \sim \rightarrow \text{Aut}(X) \)
Birational automorphism group \( \sim \rightarrow \text{Bir}(X) \)

\( \text{Aut}(X) \subseteq \text{Bir}(X) \)
Theorem (Hacon, McKernan, Xu)

Let $n$ be a positive integer. Then there is a constant $C = C(n)$ such that for any projective $n$-dimensional variety $X$ of general type, the order of $\text{Bir}(X)$ is bounded by $C \cdot \text{Vol}(X, K_X)$.

Theorem (Hurwitz)

The order of $\text{Bir}(X) = \text{Aut}(X)$ for a smooth algebraic curve of genus $g \geq 2$ is bounded by $84 \cdot (g - 1)$.

Algebraic varieties are not uniquely determined by their symmetry group.
Theorem (Cantat)

$\text{Bir}(\mathbb{P}^n) \simeq \text{Bir}(\mathbb{P}^m)$ if and only if $n = m$. 
Theorem (Cantat)
\[ \text{Bir}(\mathbb{P}^n) \cong \text{Bir}(\mathbb{P}^m) \text{ if and only if } n = m. \]

Theorem (Cantat)

Let \( X \) be an \( n \)-dimensional variety. If \( \text{Bir}(X) \) is isomorphic to \( \text{Bir}(\mathbb{P}^n) \), then \( X \) is rational.

Rational varieties are uniquely determined (up to birational equivalence) among \( n \)-dimensional varieties by their birational automorphism group.
A toric variety is a normal algebraic variety endowed with a faithful action of an algebraic torus $T$ having an open orbit.

Demazure (and later Cox) gave a description of $\text{Aut}(X)$ for a complete toric variety.

For most complete toric varieties, we have $\text{Aut}(X) = T$.

Toric varieties are not uniquely determined by their automorphism group.
Theorem (Regeta, Urech, L.)

Let $S$ and $S'$ be normal affine surfaces with $S$ toric. If $\text{Aut}(S) \cong \text{Aut}(S')$, then $S \cong S'$. 

Affine toric surfaces are uniquely determined among normal affine surfaces by their regular automorphism group.
Proposition (Regeta, Urech, L.)

Let $S$ be an affine surface. If $\text{Aut}(S) \simeq \text{Aut}(\mathbb{A}^2)$ then $S \simeq \mathbb{A}^2$.

Proposition (Díaz, L.)

Let $S$ be a toric surface different from $\mathbb{A}^2$, $\mathbb{A}^1 \times \mathbb{A}^1$ and $\mathbb{A}^1_* \times \mathbb{A}^1_*$. Then, there exists a non-normal toric surface $S'$ such that $\text{Aut}(S) \simeq \text{Aut}(S')$. 
Idea of proof

Theorem (Regeta, Urech, L.)

Let $S$ and $S'$ be normal affine surfaces with $S$ toric. If $\text{Aut}(S) \cong \text{Aut}(S')$ then $S \cong S'$. 
Let $A$ be a variety and $f: A \times X \to A \times X$ be an $A$-birational map, i.e.,

- $f$ is the identity in the first factor, and
- induces an isomorphism between open subsets $U$ and $V$ of $A \times X$ such that the projections from $U$ and from $V$ to $A$ are both surjective.

This yields a map $A \to \text{Bir}(X)$ that we call a morphism.

The Zariski topology on $\text{Bir}(X)$ is the finest topology making all such morphisms continuous.
Definition

An algebraic subgroup of Bir($X$) is the image of a morphism $G \to \text{Bir}(X)$ that is also an homomorphism.

An element $g \in \text{Bir}(X)$ is called algebraic if it is contained in an algebraic subgroup.
Divisibility in Bir(S)

Definition
Let $G$ be a group.
- An element $f$ in is called divisible by $n$ if there exists an element $g \in G$ such that $g^n = f$.
- An element is called divisible if it is divisible by all $n \in \mathbb{Z}_{>0}$.

Lemma

Let $S$ be a surface and $f \in \text{Bir}(S)$.
Then the following two conditions are equivalent:
- There exists a $k > 0$ such that $f^k$ is divisible; and
- $f$ is algebraic.
Algebraic elements in $\text{Aut}(S)$

**Definition**

An algebraic subgroup of $\text{Aut}(X)$ is the image of a regular action $G \to \text{Aut}(X)$ of an algebraic group. An element $g \in \text{Aut}(X)$ is called algebraic if it is contained in an algebraic subgroup.

**Lemma**

Let $S$ be a normal affine surface and let $g \in \text{Aut}(S)$ be an automorphism. Then $g$ is an algebraic element in $\text{Bir}(S)$ if and only if $g$ is an algebraic element in $\text{Aut}(S)$. 
Proposition

Let $S$ and $S'$ be normal affine surfaces, $\varphi : \text{Aut}(S) \to \text{Aut}(S')$ a group homomorphism, and $g \in \text{Aut}(S)$ an algebraic element.

Then $\varphi(g)$ is an algebraic element in $\text{Aut}(S')$. 

Torus goes to a 2-dimensional torus

Lemma

Let $S$ and $S'$ be normal affine surfaces with $S$ toric, and $\varphi: \text{Aut}(S) \to \text{Aut}(S')$ a group isomorphism.

Then $\varphi(T)$ is a 2-dimensional torus in $\text{Aut}(S')$. 
Definition

Let $T \subset \text{Aut}(X)$ be a maximal torus in $\text{Aut}(X)$. An algebraic subgroup $U \subset \text{Aut}(X)$ isomorphic to $\mathbb{G}_a$ is called a root subgroup with respect to $T$ if the normalizer of $U$ in $\text{Aut}(X)$ contains $T$.

This is equivalent to saying that $T$ and $U$ span an algebraic group isomorphic to $\mathbb{G}_a \rtimes \chi$ with $\chi : T \to \mathbb{G}_m$ character.

A root subgroup is also uniquely determined by a homogeneous derivation of $\partial$ of $\mathcal{O}_X$ (with some integrability conditions).

Demazure’s description of $\text{Aut}^0(X)$ is based on a description of root subgroups of non-necessarily complete toric variety.
Lemma

Let $S$ and $S'$ be normal affine surfaces with $S$ toric, \( \varphi : \text{Aut}(S) \rightarrow \text{Aut}(S') \) a group isomorphism, and \( U \subset \text{Aut}(S) \) a root subgroup

Then \( \varphi(U) \) is a root subgroup in \( \text{Aut}(S') \) with respect to \( \varphi(T) \).
End of the proof

We know now that $S'$ is a toric surface and we have a bijection on the root subgroups of $S$ and $S'$ with respect to $T$ and $\varphi(T)$.

Hence, to conclude the proof, it is enough to show that we can recover a toric surface $S$ from the abstract group structure of its root subgroups and their relationship with the torus.

Recall that any affine toric surface $S$ without torus factor is isomorphic to $V_{d,e}$, the quotient of $\mathbb{A}^2$ under the $\mathbb{Z}/d\mathbb{Z}$-action

$$g: (x, y) \mapsto (\xi^e x, \xi y)$$

where $\xi$ is a $d$-th primitive root of unity $0 \leq e < d$, $(e, d) = 1$.

$V_{d,e}$ is isomorphic to $V_{d',e'}$ if and only if $d = d'$ and $e = e'$ or $d = d'$ and $e \cdot e' = 1 \mod d$. 
End of the proof

The center of $\mathbb{G}_a \rtimes \chi T$ is $\{0\} \times \ker \chi$, so we can recover $\ker \chi$.

There are two families $\mathcal{K}, \mathcal{L}$ of commuting root subgroups in $\text{Aut}(S)$. We define the following subsets of $\mathbb{Z}_{>0}$:

\[ K_U = \left\{ |\ker \chi \cap \ker \chi'|, \forall U' \in \mathcal{L} \right\} \quad \forall U \in \mathcal{K} \]

\[ L_U = \left\{ |\ker \chi \cap \ker \chi'|, \forall U' \in \mathcal{K} \right\} \quad \forall U \in \mathcal{L} \]

After some finite part, they form arithmetic progressions. The two shortest common differences in this arithmetic progressions are:

\[ d \text{ and } d + e \quad \text{ or } \quad d \text{ and } d + e' \text{ with } e \cdot e' = 1 \mod d \]

Hence, these sets uniquely determine $S$. \qed
What about higher dimensional toric varieties?
An ind-variety is a set $V$ together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V$ such that the following conditions are satisfied:

- $V = \bigcup_{k \geq 0} V_k$;
- each $V_k$ has is an algebraic variety;
- for every $k \in \mathbb{Z}_{\geq 0}$, the embedding $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

Morphisms of ind-varieties:

$V = \bigcup_k V_k$ and $W = \bigcup_m W_m$

A map $\psi: V \to W$ such that for any $k$ there is an $m \in \mathbb{Z}_{\geq 0}$ such that $\psi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties.
Definition
An ind-group is a group object in the category of ind-varieties, i.e., an ind-variety endowed with a group structure such that multiplication and inversion are morphisms.

Theorem (Kraft)
Let $X$ be an affine variety. Then $\text{Aut}(X)$ has a natural structure of an ind-group such that for any algebraic group $G$, a regular $G$-action on $X$ induces an ind-group homomorphism $G \to \text{Aut}(X)$.
Theorem (Regeta, Urech, L.)

Let $X$ be an affine toric variety different from the algebraic torus and let $Y$ be a normal affine variety.

If $\text{Aut}(X)$ and $\text{Aut}(Y)$ are isomorphic as ind-groups, then $X$ and $Y$ are isomorphic.

Theorem (Regeta, Urech, L.)

Let $T$ be an algebraic torus and let $C$ be a smooth affine curve. If $C$ has trivial automorphism group and no invertible global functions, then $\text{Aut}(T)$ and $\text{Aut}(C \times T)$ are isomorphic as ind-groups.
Idea of proof

Lemma (Kraft)

Let $Y$ be a normal affine variety and let $H \subset \text{Aut}(Y)$ be a torus. If there exists a root subgroup $U \subset \text{Aut}(Y)$ with respect to $H$ such that $\mathcal{O}(X)^U$ is multiplicity-free, then $\dim H \leq \dim Y \leq \dim H + 1$.

Let $\varphi : \text{Aut}(X) \to \text{Aut}(Y)$ be an isomorphism of ind-groups.

For every algebraic subgroup $G \subset \text{Aut}(X)$, the isomorphism $\varphi$ restricts to an isomorphism of algebraic groups.

We find a subtorus $H \subset \varphi(T)$ of codimension 1 and a root subgroup $U$ satisfying the lemma to conclude that $\dim Y \leq \dim X$.

Both $X$ and $Y$ have a faithful action of $T$ with $\dim T = \dim X$. So $\dim Y = n$ and $Y$ is toric.
Idea of proof

The weight of root subgroups (roots) are also preserved by $\varphi$.

The sets of roots of $X$ and $Y$ are identified by the isomorphism $\varphi$ restricted to $T$.

Finally, a combinatorial computation in terms of the cones defining the toric varieties proves that a toric variety is determined by its set of roots.
¡Gracias!