Characterization of affine toric varieties by their automorphism groups

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Joint work with Andriy Regeta and Christian Urech

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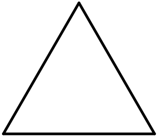
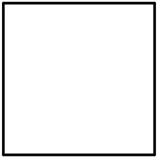
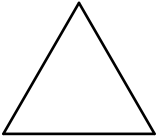
Is a geometric object uniquely determined
by its symmetry group?

Regular polygons

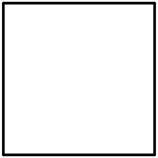




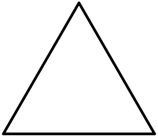
Regular polygons



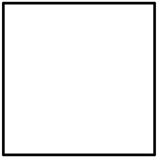
$$\mathrm{Sym}(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1,\ rs = sr^{-1} \rangle$$



Regular polygons

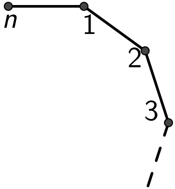


$$\mathrm{Sym}(T) = D_6 = \langle r, s \mid r^3 = s^2 = 1,\ rs = sr^{-1} \rangle$$



$$\mathrm{Sym}(S) = D_8 = \langle r, s \mid r^4 = s^2 = 1,\ rs = sr^{-1} \rangle$$

Regular polygons

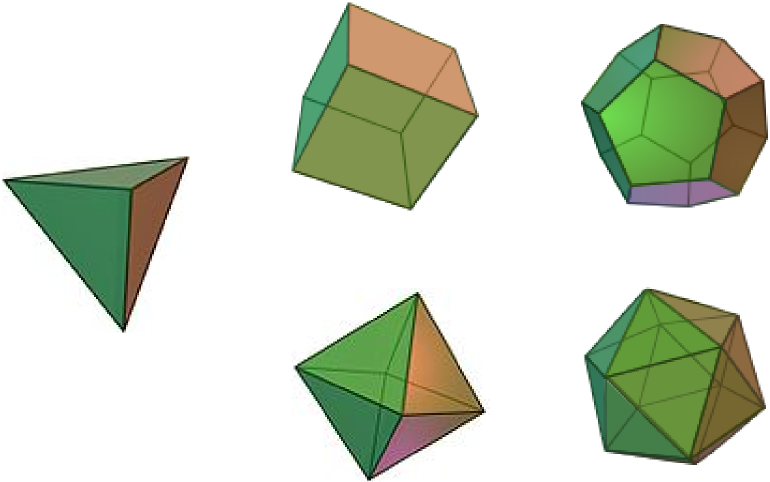


$$\mathrm{Sym}(P_n) = D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle$$

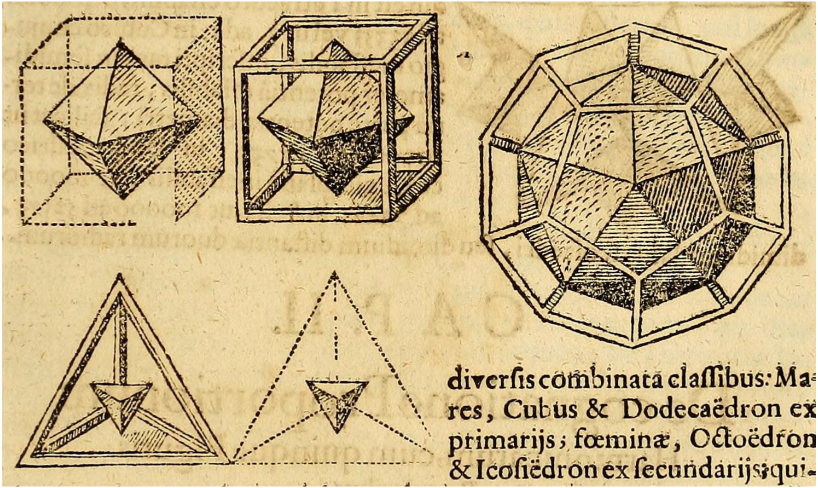
$$|\mathrm{Sym}(P_n)| = 2n$$

Regular polygons are uniquely determined
by their symmetry group

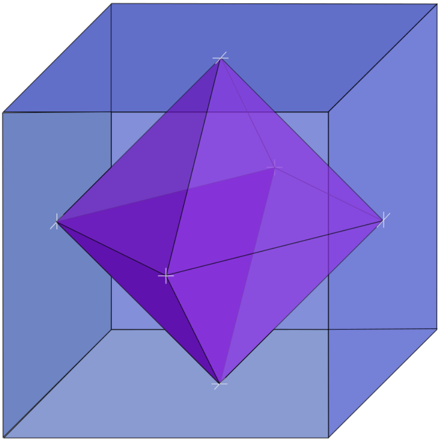
Regular polytopes



Regular polytopes



Regular polytopes



$$\mathrm{Sym}(C) \simeq \mathrm{Sym}(O)$$

Regular polytopes are not uniquely determined by their symmetry group

Is a geometric object uniquely determined
by its symmetry group?

In algebraic geometry there are at least two possibilities for the symmetry group:

Regular automorphism group $\rightsquigarrow \mathrm{Aut}(X)$

Birational automorphism group $\rightsquigarrow \mathrm{Bir}(X)$

$$\mathrm{Aut}(X) \subseteq \mathrm{Bir}(X)$$

Theorem (Hacon, McKernan, Xu)

Let n be a positive integer.
Then there is a constant $C = C(n)$ such that for any projective n-dimensional variety X of general type, the order of $\mathrm{Bir}(X)$ is bounded by $C \cdot \mathrm{Vol}(X, K_X)$.

Theorem (Hurwitz)

The order of $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ for a smooth algebraic curve of genus $g \geq 2$ is bounded by $84 \cdot (g-1)$.

Algebraic varieties are not uniquely determined
by their symmetry group

Theorem (Cantat)

$\mathrm{Bir}(\mathbb{P}^n) \simeq \mathrm{Bir}(\mathbb{P}^m)$ *if and only if* $n = m$.

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Theorem (Cantat)

Let X *be an* n*-dimensional variety.*
If $\mathrm{Bir}(X)$ *is isomorphic to* $\mathrm{Bir}(\mathbb{P}^n)$, *then* X *is rational.*

Rational varieties are uniquely determined (up to birational equivalence) among n-dimensional varieties by their birational automorphism group

A toric variety is a normal algebraic variety endowed with with a faithful action of an algebraic torus T having an open orbit.

Demazure (and later Cox) gave a description of $\mathrm{Aut}(X)$ for a complete toric variety.

For most complete toric varieties, we have $\mathrm{Aut}(X) = T$.

Toric varieties are not uniquely determined
by their automorphism group

Theorem (Regeta, Urech, L.)

Let S and S' be normal affine surfaces with S toric.
If $\mathrm{Aut}(S) \simeq \mathrm{Aut}(S')$ *then* $S \simeq S'$.

Affine toric surfaces are uniquely determined among normal affine surfaces by their regular automorphism group

Proposition (Regeta, Urech, L.)

Let S be an affine surface. If $\mathrm{Aut}(S) \simeq \mathrm{Aut}(\mathbb{A}^2)$ *then* $S \simeq \mathbb{A}^2$.

Proposition (Díaz, L.)

Let S be a toric surface different from $\mathbb{A}^2$, $\mathbb{A}^1 \times \mathbb{A}^1_*$ and $\mathbb{A}^1_* \times \mathbb{A}^1_*$. Then, there exits a non-normal toric surface S' such that $\mathrm{Aut}(S) \simeq \mathrm{Aut}(S')$.

Idea of proof

Theorem (Regeta, Urech, L.)

Let S and S' be normal affine surfaces with S toric.
If $\mathrm{Aut}(S) \simeq \mathrm{Aut}(S')$ then $S \simeq S'$.

Topology on $\mathrm{Bir}(X)$

Let A be a variety and $f\colon A \times X \dashrightarrow A \times X$ be an A-birational map, i.e.,

- f is the identity in the first factor, and
- induces an isomorphism between open subsets U and V of $A \times X$ such that the projections from U and from V to A are both surjective.

This yields a map $A \to \mathrm{Bir}(X)$ that we call a morphism.

The Zariski topology on $\mathrm{Bir}(X)$ is the finest topology making all such morphisms continuous.

Algebraic elements in Bir(X)

Definition

An algebraic subgroup of Bir(X) is the image of a morphism $G \to \mathrm{Bir}(X)$ that is also an homomorphism.

An element $g \in \mathrm{Bir}(X)$ is called algebraic if it is contained in an algebraic subgroup.

Divisibility in $\mathrm{Bir}(S)$

Definition

Let G be a group.

- An element f in is called divisible by n if there exists an element $g \in G$ such that $g^n = f$.
- An element is called divisible if it is divisible by all $n \in \mathbb{Z}_{>0}$.

Lemma

Let S be a surface and $f \in \mathrm{Bir}(S)$.
Then the following two conditions are equivalent:

- *There exists a $k > 0$ such that f^k is divisible; and*
- *f is algebraic.*

Algebraic elements in $\mathrm{Aut}(S)$

Definition

An algebraic subgroup of $\mathrm{Aut}(X)$ is the image of a regular action $G \to \mathrm{Aut}(X)$ of an algebraic group.

An element $g \in \mathrm{Aut}(X)$ is called algebraic if it is contained in an algebraic subgroup.

Lemma

Let S be a normal affine surface and let $g \in \mathrm{Aut}(S)$ be an automorphism. Then

g is an algebraic element in $\mathrm{Bir}(S)$ if and only if g is an algebraic element in $\mathrm{Aut}(S)$.

Algebraic elements are preserved

Proposition

Let S and S' be normal affine surfaces,
$\varphi\colon \operatorname{Aut}(S) \to \operatorname{Aut}(S')$ a group homomorphism, and
$g \in \operatorname{Aut}(S)$ an algebraic element.

Then $\varphi(g)$ is an algebraic element in $\operatorname{Aut}(S')$.

Torus goes to a 2-dimensional torus

Lemma

Let S and S' be normal affine surfaces with S toric, and $\varphi\colon \operatorname{Aut}(S) \to \operatorname{Aut}(S')$ a group isomorphism.

Then $\varphi(T)$ is a 2-dimensional torus in $\operatorname{Aut}(S')$.

Root subgroups

Definition

Let $T \subset \mathrm{Aut}(X)$ be a maximal torus in $\mathrm{Aut}(X)$. An algebraic subgroup $U \subset \mathrm{Aut}(X)$ isomorphic to $\mathbb{G}_{\mathrm{a}}$ is called a root subgroup with respect to T if the normalizer of U in $\mathrm{Aut}(X)$ contains T.

This is equivalent to saying that T and U span an algebraic group isomorphic to $\mathbb{G}_{\mathrm{a}} \rtimes_{\chi} T$ with $\chi : T \to \mathbb{G}_{\mathrm{m}}$ character.

A root subgroup is also uniquely determined by a homogeneous derivation of ∂ of $\mathcal{O}_X$ (with some integrability conditions).

Demazure's description of $\mathrm{Aut}^0(X)$ is based on a description of root subgroups of non-necessarily complete toric variety.

Root subgroups go to root subgroups

Lemma

Let S *and* S' *be normal affine surfaces with* S *toric,*
$\varphi\colon \mathrm{Aut}(S) \to \mathrm{Aut}(S')$ *a group isomorphism, and*
$U \subset \mathrm{Aut}(S)$ *a root subgroup*

Then $\varphi(U)$ *is a root subgroup in* $\mathrm{Aut}(S')$ *with respect to* $\varphi(T)$*.*

End of the proof

We know now that S' is a toric surface and we have a bijection on the root subgroups of S and S' with respect to T and $\varphi(T)$.

Hence, to conclude the proof, it is enough to show that we can recover a toric surface S from the abstract group structure of its root subgroups and their relationship with the torus.

Recall that any affine toric surface S without torus factor is isomorphic to $V_{d,e}$, the quotient of $\mathbb{A}^2$ under the $\mathbb{Z}/d\mathbb{Z}$-action

$$g\colon (x, y) \mapsto (\xi^e x, \xi y)$$

where ξ is a d-th primitive root of unity $0 \leq e < d$, $(e, d) = 1$.

$V_{d,e}$ is isomorphic to $V_{d',e'}$ if and only if
$d = d'$ and $e = e'$ or $d = d'$ and $e \cdot e' = 1 \mod d$.

End of the proof

The center of $\mathbb{G}_{\mathrm{a}} \rtimes_\chi T$ is $\{0\} \times \ker\chi$, so we can recover $\ker\chi$.

There are two families $\mathcal{K}$, $\mathcal{L}$ of commuting root subgroups in $\mathrm{Aut}(S)$. We define the following subsets of $\mathbb{Z}_{>0}$:

$$K_U = \left\{ \left| \ker\chi \cap \ker\chi' \right|, \forall U' \in \mathcal{L} \right\} \quad \forall U \in \mathcal{K}$$

$$L_U = \left\{ \left| \ker\chi \cap \ker\chi' \right|, \forall U' \in \mathcal{K} \right\} \quad \forall U \in \mathcal{L}$$

After some finite part, they form arithmetic progressions.

The two shortest common differences in this arithmetic progressions are:

$$d \text{ and } d+e \qquad \text{or} \qquad d \text{ and } d+e' \text{ with } e \cdot e' = 1 \mod d$$

Hence, these sets uniquely determine S. $\square$

What about higher dimensional toric varieties?

An ind-variety is a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V$ such that the following conditions are satisfied:

- $V = \bigcup_{k \geq 0} V_k$;
- each V_k has is an algebraic variety;
- for every $k \in \mathbb{Z}_{\geq 0}$, the embedding $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

Morphisms of ind-varieties:

$V = \bigcup_k V_k$ and $W = \bigcup_m W_m$

A map $\psi \colon V \to W$ such that for any k there is an $m \in \mathbb{Z}_{\geq 0}$ such that $\psi(V_k) \subset W_m$ and such that the induced map $V_k \to W_m$ is a morphism of algebraic varieties.

Definition

An ind-group is a group object in the category of ind-varieties, i.e., an ind-variety endowed with a group structure such that multiplication and inversion are morphisms.

Theorem (Kraft)

Let X be an affine variety. Then $\mathrm{Aut}(X)$ has a natural structure of an ind-group such that for any algebraic group G, a regular G-action on X induces an ind-group homomorphism $G \to \mathrm{Aut}(X)$.

Theorem (Regeta, Urech, L.)

Let X be an affine toric variety different from the algebraic torus and let Y be a normal affine variety.

If $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(Y)$ *are isomorphic as ind-groups, then X and Y are isomorphic.*

Theorem (Regeta, Urech, L.)

Let T be an algebraic torus and let C be a smooth affine curve. If C has trivial automorphism group and no invertible global functions, then $\mathrm{Aut}(T)$ *and* $\mathrm{Aut}(C \times T)$ *are isomorphic as ind-groups.*

Idea of proof

Lemma (Kraft)

Let Y be a normal affine variety and let $H \subset \mathrm{Aut}(Y)$ be a torus. If there exists a root subgroup $U \subset \mathrm{Aut}(Y)$ with respect to H such that $\mathcal{O}(X)^U$ is multiplicity-free, then $\dim H \leq \dim Y \leq \dim H + 1$.

Let $\varphi : \mathrm{Aut}(X) \to \mathrm{Aut}(Y)$ be an isomorphism of ind-groups.

For every algebraic subgroup $G \subset \mathrm{Aut}(X)$, the isomorphism φ restricts to an isomorphism of algebraic groups.

We find a subtorus $H \subset \varphi(T)$ of codimension 1 and a root subgroup U satisfying the lemma to conclude that $\dim Y \leq \dim X$.

Both X and Y have a faithful action of T with $\dim T = \dim X$. So $\dim Y = n$ and Y is toric.

Idea of proof

The weight of root subgroups (roots) are also preserved by φ.

The sets of roots of X and Y are identified by the isomorphism φ restricted to T.

Finally, a combinatorial computation in terms of the cones defining the toric varieties proves that a toric variety is determined by its set of roots.

¡Gracias!