Exercises around Waring Rank
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Two definitions. Let $S = \Bbbk[x_1, \dots, x_n]$, $T = \Bbbk[\partial_1, \dots, \partial_n]$, with T acting on S by differentiation (where ∂_i acts as $\frac{\partial}{\partial x_i}$), denoted $\lrcorner$, as in $\partial_i \lrcorner x_i^3 = 3x_i^2$.

For $F \in S$ homogeneous of degree d, a *power sum decomposition of* F is an expression $F = \lambda_1 \ell_1^d + \dots + \lambda_r \ell_r^d$, $\lambda_i \in \Bbbk$, $\ell_i \in S_1$ linear forms. The *Waring rank*, denoted $\operatorname{rank}(F)$, is the least number of terms r in a power sum decomposition of F.

The *apolar ideal* or *annihilating ideal* of F is the ideal $F^\perp = \{\Theta \in T : \Theta \lrcorner F = 0\}$.

We assume always that $\Bbbk$ has characteristic 0 or $> d$. We also assume that $\Bbbk$ has any needed roots, roots of unity, etc.

Discussion Exercises

These are exercises suggested for discussion during the workshop.

1. (a) In a decomposition $F = \sum_{i=1}^r \lambda_i \ell_i^d$, if r is minimal, then the λ_i are uniquely determined, and can be found by linear algebra.
(b) The set of ranks achievable by forms of a given degree and number of variables is an interval; that is, if there is a form of rank r, then there are forms of rank $1, 2, \dots, r$.

2. Find examples of forms $F_t \to F_0 \neq 0$ so that for $t \neq 0$, $\operatorname{rank}(F_t)$ is constant, but (a) $\operatorname{rank} F_0 < \operatorname{rank} F_t$, and (b) $\operatorname{rank} F_0 > \operatorname{rank} F_t$.

3. Let $F \in S_d$ be a homogeneous form of degree d. Let $P = (c_1, \dots, c_n)$ be a point in affine space and let $D_P = c_1\partial_1 + \dots + c_n\partial_n \in T_1$, $\ell_P = c_1x_1 + \dots + c_nx_n \in S_1$.
(a) $D_P^d \lrcorner F = d!\, F(P)$.
(b) Let $D \in T_k$. For all $d \geq k$, $D(P) = 0$ as a polynomial if and only if $D \lrcorner \ell_P^d = 0$.
(c) $D_P^k \in F^\perp$ if and only if F vanishes to order (at least) $d + 1 - k$ at the point P, that is, every partial derivative of F of order $\leq d - k$ vanishes at the point P.

4. Sylvester's algorithm: finding power sum decompositions for binary forms, that is, polynomials of 2 variables.

Let $F(x, y)$ be a homogeneous form of degree d in 2 variables. We write α for $\partial/\partial x$ and β for $\partial/\partial y$.
(a) Suppose $t_1, \dots, t_{d+1} \in \Bbbk$ are any pairwise distinct numbers. Show that $(x+t_1y)^d, \dots, (x+t_{d+1}y)^d$ are linearly independent and in fact span the vector space of homogeneous forms of degree d in 2 variables.
(b) Suppose $F = \lambda_1(a_1x+b_1y)^d + \dots + \lambda_r(a_rx+b_ry)^d$. Let $h(\alpha, \beta) = (b_1\alpha - a_1\beta)(b_2\alpha - a_2\beta)\cdots(b_r\alpha - a_r\beta)$. Show that $h \in F^\perp$.
(c) Conversely, if $h(\alpha, \beta) = (b_1\alpha - a_1\beta)(b_2\alpha - a_2\beta)\cdots(b_r\alpha - a_r\beta) \in F^\perp$ with distinct roots (i.e., no repeated or proportional factors), $h \neq 0$, then there exist λ_i such that $F = \lambda_1(a_1x + b_1y)^d + \dots + \lambda_r(a_rx + b_ry)^d$.
(d) Deduce Sylvester's algorithm: the rank of F is the least r such that $F^\perp$ contains a form of degree r with projectively distinct roots.
(e) Let $F = x^3y$. Find $F^\perp$. Show that there is no $h \in F^\perp$ of degree 3 or less with distinct roots, therefore F cannot be written as a linear combination of 3 powers of linear forms. Find an $h \in F^\perp$ of degree 4 with distinct roots and write F as a linear combination of 4 powers of linear forms.

5. Some exercises about the apolar ideal.
(a) $F^\perp$ is an $\mathfrak{m}$-primary ideal, and if F is homogeneous then so is $F^\perp$. $T/F^\perp \cong \operatorname{Derivs}(F)$ as a T-module (graded, if F is homogeneous).
(b) $(x_1^{a_1} \cdots x_n^{a_n})^\perp = (\partial_1^{a_1+1}, \dots, \partial_n^{a_n+1})$.
(c) For $F \in S$ and $\Theta \in T$, $(\Theta \lrcorner F)^\perp = F^\perp : \Theta$. (Recall for any ring R, ideal $I \subseteq R$, and $a \in R$, $I : a = \{r \in R : ar \in I\}$.)

(d) The polynomials $x^d + hy^d \to x^d$ as $h \to 0$. What is the limit of the ideals $(x^d + hy^d)^\perp$ and how does it compare to $(x^d)^\perp$? What about $xyz + hw^3 \to xyz$, and $\lim(xyz + hw^3)^\perp$ versus $(xyz)^\perp$?

(e) Find the apolar ideal, and its Hilbert function, for the following forms: $x_1^d + \cdots + x_n^d$, an elementary symmetric polynomial, the $d \times d$ generic determinant, the $d \times d$ generic permanent, and the Hilbert function for the following form of degree 4 in 13 variables: $x^3t_1 + x^2yt_2 + x^2zt_3 + xy^2t_4 + xyzt_5 + xz^2t_6 + y^3t_7 + y^2zt_8 + yz^2t_9 + z^3t_{10}$. (This last one is *not unimodal*.)

6. Let $x_1, \ldots, x_n, y$ be independent variables and $F = F(x_1, \ldots, x_n)$ a form of degree $d > 1$. Then $\operatorname{rank}(F + y^d) = \operatorname{rank}(F) + 1$. [3]

7. For an $n \times n$ symmetric matrix M, let Q be the quadratic form given by

$$Q(x_1, \ldots, x_n) = \sum_{1 \le i,j \le n} x_i m_{ij} x_j = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Show $\operatorname{rank} Q = \operatorname{rank} M$.

8. The following is a conjecture of Shitov: Let $S = \Bbbk[x_1, \ldots, x_n]$ and let u be a new variable. Let $F \in S_d$ and $G \in S_{d-1}$, with $G \neq 0$. Then

$$\operatorname{rank}(F + uG) \ge d + \min_{v \in S_1} \operatorname{rank}(F + vG),$$

with equality if G is a power of a linear form. [12]

This conjecture is open, but the following two special cases are exercises.

(a) The case $F = 0$.

(b) The case $d = 2$.

Additional Exercises

These are additional exercises. Some of them are more difficult; some of them are easy but perhaps tedious; some of them require a bit of computation.

9. Let S be the set of solutions to the equation $(ax + by)^2 - (cx + dy)^2 = xy$ (in variables a, b, c, d, over $\Bbbk$). Find equations and a parametrization for the affine variety $S \subset \mathbb{A}^4$. (There are two irreducible components.) What is the closure of S in $\mathbb{P}^4$?

10. Write out explicitly the system of equations in a, b given by the equation $(ax + by)^4 = x^3y$ and show that the system has no solutions. Do the same for the equation $(a_1x + b_1y)^4 + (a_2x + b_2y)^4 = x^3y$ (in variables a_1, b_1, a_2, b_2).

The "brute force" approach to determining Waring rank of a form such as x^3y is to find the first time that there is a solution to the equations $(ax + by)^4 = x^3y$, $(a_1x + b_1y)^4 + (a_2x + b_2y)^4 = x^3y$, $(a_1x + b_1y)^4 + (a_2x + b_2y)^4 + (a_3x + b_3y)^4 = x^3y$, etc. Is this a feasible approach in general? Is it even feasible for the specific form x^3y?

11. For a binary form F of degree ≤ 3, $\operatorname{rank}(F)$ is determined by the multiplicities of the roots of F (whether F has 1, 2, or 3 distinct roots), independently of the location of the roots. In particular in degree 3, a cubic form with 3 distinct roots has rank 2, a form with 2 distinct roots (one double root and one single root) has rank 3, and a form with 1 root (a triple root) has rank 1. But in degree ≥ 4 this is no longer the case: $\operatorname{rank} xy(x^2 + y^2) = 2$ while $\operatorname{rank} xy(y + x)(y - 2x) = 3$.

However if F has a root of multiplicity m, and F is not a dth power of a linear form, then $\operatorname{rank} F > m$. [13]

12. Here are power sum decompositions of a few special polynomials. The exercise is to verify them.

(a)

$$y_1 \cdots y_d = \frac{1}{2^{d-1} d!} \sum_{\substack{\epsilon \in \{\pm 1\}^d \\ \epsilon_1 = 1}} \left(\prod_{i=1}^{d} \epsilon_i \right) \left(\sum_{i=1}^{d} \epsilon_i y_i \right)^d$$

(b) More generally, let $M = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial of degree d with $1 \leq a_1 \leq \cdots \leq a_n$. For $2 \leq i \leq n$ let ζ_i be a primitive $(a_i + 1)$th root of unity (assume that $\Bbbk$ contains enough roots of unity). Then

$$M = \frac{1}{C} \sum_{\substack{0 \leq k_i \leq a_i \\ 2 \leq i \leq n}} (x_1 + \zeta_2^{k_2} x_2 + \cdots + \zeta_n^{k_n} x_n)^d (\zeta_2^{k_2} \cdots \zeta_n^{k_n})$$

where $C = \binom{d}{a_1, a_2, \ldots, a_n} (a_1 + 1) \cdots (a_n + 1)$, where $\binom{d}{a_1, \ldots, a_n}$ is the multinomial coefficient $d!/(a_1! \cdots a_n!)$. [2]

(c) More generally in a different direction, let $e_{n,d}$ be the elementary symmetric polynomial of degree d in n variables. Then for $d = 2k+1$ odd, $n > d$, we have

$$2^{d-1} d! e_{n,d} = \sum_{I \subseteq [n], |I| \leq k} (-1)^{|I|} \binom{n-k-|I|-1}{k-|I|} (\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n)^d$$

where $[n] = \{1, 2, \ldots, n\}$ and $\delta(I, i) = -1$ if $i \in I$, 1 if $i \notin I$. And for $d = 2k$ even, $n > d$, we have

$$2^d (n-d)! e_{n,d} = \sum_{I \subseteq [n], |I| \leq k} (-1)^{|I|} \binom{n-k-|I|-1}{k-|I|} (n - 2|I|) (\delta(I,1)x_1 + \delta(I,2)x_2 + \cdots + \delta(I,n)x_n)^d.$$

See [7].

13. Here are two identities for the determinant and permanent.

Let X_d be the $d \times d$ matrix with entries $x_{i,j}$. Let $\det_d = \det X_d$ and $\mathrm{per}_d = \mathrm{per}\, X_d$.

(a) Derksen found the identity

$$\begin{aligned} \det_3 = \frac{1}{2}\Big(&(x_{13} + x_{12})(x_{21} - x_{22})(x_{31} + x_{32}) \\ &+ (x_{11} + x_{12})(x_{22} - x_{23})(x_{32} + x_{33}) \\ &+ 2x_{12}(x_{23} - x_{21})(x_{33} + x_{31}) \\ &+ (x_{13} - x_{12})(x_{22} + x_{21})(x_{32} - x_{31}) \\ &+ (x_{11} - x_{12})(x_{23} + x_{22})(x_{33} - x_{32})\Big). \end{aligned}$$

(More precisely, Derksen found a tensor identity in terms of the columns of X_3. See [5]. The above is a polynomial version of Derksen's identity.)

(b) Glynn's identity for the permanent is

$$\mathrm{per}_d = \frac{1}{2^{d-1}} \sum_{\substack{\epsilon \in \{\pm 1\}^d \\ \epsilon_1 = 1}} \prod_{i=1}^{d} \sum_{j=1}^{d} \epsilon_i \epsilon_j x_{i,j},$$

see [6]. For example,

$$\begin{aligned} \mathrm{per}_3 = \frac{1}{4}\Big\{ &(x_{1,1} + x_{1,2} + x_{1,3})(x_{2,1} + x_{2,2} + x_{2,3})(x_{3,1} + x_{3,2} + x_{3,3}) \\ &- (x_{1,1} + x_{1,2} - x_{1,3})(x_{2,1} + x_{2,2} - x_{2,3})(x_{3,1} + x_{3,2} - x_{3,3}) \\ &- (x_{1,1} - x_{1,2} + x_{1,3})(x_{2,1} - x_{2,2} + x_{2,3})(x_{3,1} - x_{3,2} + x_{3,3}) \\ &+ (x_{1,1} - x_{1,2} - x_{1,3})(x_{2,1} - x_{2,2} - x_{2,3})(x_{3,1} - x_{3,2} - x_{3,3})\Big\}. \end{aligned}$$

14. Conjugate rank. (From Greg Blekherman.)

Let F be a homogeneous real form. Let us compare the following three distinct values:

- The complex Waring rank of F, the least length of an expression $F = \sum \lambda_i \ell_i^d$ with $\lambda_i \in \mathbb{C}$, and ℓ_i having complex coefficients.
- The real Waring rank of F, the least length of an expression $F = \sum \lambda_i \ell_i^d$ with $\lambda_i \in \mathbb{R}$, and ℓ_i having real coefficients.
- The "*conjugate rank*" of F, the least length of an expression $F = \sum \lambda_i \ell_i^d$ with $\lambda_i \in \mathbb{R}$, ℓ_i having complex coefficients, and the ℓ_i occuring in conjugate pairs (or real). Equivalently, the set $\{\ell_1, \ldots, \ell_r\}$ should be fixed under complex conjugation.

With the obvious choice of notation, $\operatorname{rank}_{\mathbb{R}}(F) \geq \operatorname{rank}_{\text{conjugate}}(F) \geq \operatorname{rank}_{\mathbb{C}}(F)$.

(a) Show by example that $\operatorname{rank}_{\mathbb{R}}(F) > \operatorname{rank}_{\mathbb{C}}(F)$ can occur.
(b) What happens when F is a binary form?

15. Let $Q_n = x_1^2 + \cdots + x_n^2$.
(a) Every form of degree $\leq m$ is a derivative of Q_n^m. [10, Theorem 3.10]
(b) Show

$$Q_n^2 = \frac{1}{6} \sum_{i<j} (x_i \pm x_j)^4 + \frac{4-n}{3} \sum_{i=1}^n x_i^4,$$

thus $\operatorname{rank} Q_n^2 \leq n^2 \ll$ the general rank of a form of degree 4 in n variables, for $n \gg 0$.
(c) Similarly,

$$60 Q_n^3 = \sum_{i<j<k} (x_i \pm x_j \pm x_k)^6 + 2(5-n) \sum_{i<j} (x_i \pm x_j)^6 + 2(n^2 - 9n + 38) \sum x_i^6,$$

so $\operatorname{rank} Q_n^3 \leq 4\binom{n}{3} + 2\binom{n}{2} + n \ll$ the general rank, for $n \gg 0$.

16. What is the Waring rank of $F = x^2y + y^2z$, the plane cubic consisting of a smooth conic plus a tangent line? More generally, what is the Waring rank of any reducible cubic (in any number of variables)? [4]

Problems

17. B. Segre in his book "The Non-Singular Cubic Surfaces" [11], Section 96–97, determined the Waring rank of some quaternary cubics (forms of degree 3 in 4 variables); he showed that all of them have rank less than or equal to 7, and that rank 7 does occur. It was left to the reader to complete the determination of Waring ranks of all cubic surfaces.

18. What are the Waring ranks of the forms $Q^m = (x_1^2 + \cdots + x_n^2)^m$? These are known in only a handful of cases, see Reznick [9]. What about $Q^m L^k$ for a linear form L which may be tangent to Q or not?

19. Fixing the degree $d = 3$, what is the maximum rank of cubics in n variables as a function of n? For $n = 1, 2, 3, 4$, the maximum rank is $1, 3, 5, 7$, see [8]. (The generic ranks for $n = 2, 3, 4$ are $2, 3, 5$.) For $n = 5$ the generic rank is 8, and $\operatorname{rank} x_1(x_1x_2 + x_3^2 + x_4^2 + x_5^2) = 9$, but the maximum rank is unknown.

20. The 4×4 determinant be written as a sum of three determinants of 2×2 matrices of quadratic forms. Can this be done with fewer than three terms?

21. Is there any irreducible nondegenerate projective variety $X \subset \mathbb{P}^N$ such that the maximum rank with respect to X is equal to 2 times the general rank, over the complex numbers $\mathbb{C}$, or any algebraically closed field? (An example over $\mathbb{R}$ is given in [1].)

22. The Sylvester bound followed immediately from the trivial observation that if $F \in \operatorname{span}\{\ell_1^d, \ldots, \ell_r^d\}$, then $\operatorname{Derivs}(F)_{d-1} \subseteq \operatorname{span}\{\ell_1^{d-1}, \ldots, \ell_r^{d-1}\}$. Does the converse hold?

23. A conjecture of Shitov: Let $F \in S_d$ and $G \in S_{d-1}$, with $G \neq 0$, and let u be a new variable. Then

$$\operatorname{rank}(F + uG) \geq d + \min_{v \in S_1} \operatorname{rank}(F + vG),$$

with equality if G is a power of a linear form. [12]

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