Compensating operator of diffusion process with variable diffusion in semi-Markov space

Rosa W., Chabanyuk Y., Khimka U. T.

Zakopane, XLV KZM 2016

September 9, 2016

Presentation plan

Due to the wide use of stochastic diffusion problem arose establish conditions of stability and control of such systems. The paper [6] sufficient conditions of stability of stochastic systems via Lyapunov function properties and obtained estimates of large deviations of linear diffusion systems.

Due to the wide use of stochastic diffusion problem arose establish conditions of stability and control of such systems. The paper [6] sufficient conditions of stability of stochastic systems via Lyapunov function properties and obtained estimates of large deviations of linear diffusion systems. On the other hand, it is important asymptotic behavior of diffusion processes that considered in [20] and [21].

Due to the wide use of stochastic diffusion problem arose establish conditions of stability and control of such systems. The paper [6] sufficient conditions of stability of stochastic systems via Lyapunov function properties and obtained estimates of large deviations of linear diffusion systems. On the other hand, it is important asymptotic behavior of diffusion processes that considered in [20] and [21]. Construction of semi-Markov processes and investigation of asymptotic properties of random processes with semi-Markov switching are devoted [1, 2, 3, 4].

In this paper, we consider dynamical system with semi-Markov switchings using small series parameter. $x(t), t \geq 0$ is a semi-Markov process in the standard phase space of states $(X, \mathcal{E})$, generated by renewal Markov process $x_n, \tau_n, n \geq 0$ defined by a semi-Markov kernel:

$$Q(t, x, B) = P(x, B) G_x(t),$$

where the stochastic kernel

$$P(x, B) := P\{x_{n+1} \in B | x_n = x\}, B \in \mathcal{E}$$

defines an embedded Markov chain $x_n = x(\tau_n)$ at renewal moments:

$$\tau_n = \sum_{k=1}^{n} \theta_k, n \geq 0, \tau_0 = 0,$$

with intervals $\theta_{k+1} = \tau_{k+1} - \tau_k$ between renewal moments. θ_n are defined by the distribution functions

$$G_x(t) = P\{\theta_{n+1} \leq t | x_n = x\} =: P\{\theta_x \leq t\}.$$

A semi-Markov process is defined by the relation:

$$x(t) = x_{\nu(t)}, t \geq 0,$$

where the counting process $\nu(t)$ is defined by the formula:

$$\nu(t) := \max\{n : \tau_n \leq t\}, t \geq 0.$$

We consider a semi-Markov process $x(t), t \geq 0$ that is regular and uniformly ergodic with stationary distribution $\pi(B), B \in \mathcal{E}$:

$$\pi(dx) = \rho(dx)m(x)/m.$$

Here $\rho(B), B \in \mathcal{E}$, is a stationary distribution of Markov chain attached.

Problem

Diffusion process $u^\varepsilon(t) \in R^d$ in averaging scheme with a small parameter $\varepsilon > 0$ defined by stochastic differential equation

$$du^\varepsilon(t) = C\left(u^\varepsilon(t); x\left(\frac{t}{\varepsilon}\right)\right)dt + \sigma(u^\varepsilon(t))dw(t) \tag{1}$$

where: $u^\varepsilon(t), t \geq 0$ - random evolution in a diffusion process (1) [11][9, 14, 15];
$x(t), t \geq 0$ - semi-Markov process [11][8, 12, 13];
$w(t)$ - Wiener process [3,4, 5].

Semigroup $\mathbf{C}_{t+s}^{t}(x), t \geq 0, s \geq 0, x \in X$ accompanying systems

$$du_x(t) = C(u_x(t); x)dt + \sigma(u_x(t))dw(t), u_x(0) = u, \tag{2}$$

defined by the relation

$$\mathbf{C}_{t+s}^{t}(x)\varphi(u) = \varphi(u_x(t+s)), u_x(t) = u \tag{3}$$

where

$$u_x(t+s) := u_x(t+s, u), u_x(t) := u_x(t, u) \tag{4}$$

(4) - semigroup property.

Generating operator $\mathbf{C}(x)$ semigroup $\mathbf{C}_{t+s}^{t}(x)$ is defined by form

$$\mathbf{C}(x)\varphi(u) = C(u,x)\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u), \tag{5}$$

where $\varphi(u) \in C^2(R^d)$.

Theorem

Let regression function $C(u,x)$ *and variation* $\sigma(u)$ *satisfy the following conditions:*
C1: $C(u,\cdot) \in C^2(R^d)$
C2: $\sigma(u) \in C^2(R^d)$.
C3: the distribution functions $G_x(t), t \geq 0, x \in X$ *satisfy the Cramer condition uniformly in* $x \in X$,

$$\sup_{x \in X} \int_0^\infty e^{ht} \overline{G}_x(t) dt \leq H < +\infty, h > 0$$

Then the solution $u^\varepsilon(t), t \geq 0$ *of the equation (1) converges weakly to the limit diffusion process* $\zeta(t), t \geq 0$ *as* $\varepsilon \to 0$, *which is defined by the generator*

$$\boldsymbol{L}\varphi(u) = C(u)\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u),$$

MRP

We introduce advanced Markov renewal process (MRP) [11], which given sequence:

$$u_n^\varepsilon = u^\varepsilon(\tau_n^\varepsilon), x_n^\varepsilon = x^\varepsilon(\tau_n^\varepsilon), \tau_n^\varepsilon = \varepsilon\tau_n, \tag{6}$$

where $\tau_n = \sum_{k=1}^n \theta_k$, $n \geq 0$, $\tau_0 = 0$, there are times renewal semi-Markov process $x(t), t \geq 0$, [6] determined by the distribution function of the time spent in the state x.

Compensating operator

Definition 1. [11][16] *Compensating operator advanced MRP (6) is defined by the form*

$$\mathbf{L}_t^{\varepsilon}(x)\varphi(u,x,t) = \varepsilon^{-1}[E\{\varphi(u_{n+1}^{\varepsilon}, x_{n+1}^{\varepsilon}, \tau_{n+1}^{\varepsilon}) | u_n^{\varepsilon} = u, x_n^{\varepsilon} = x, \tau_n^{\varepsilon} = t\} -$$

$$- \varphi(u,x,t)]/g(x). \tag{7}$$

Lemma (1)

Compensating operator (7) on test-functions $\varphi(u)$ is defined by formula:

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(u,x) =$$

$$\varepsilon^{-1}q(x)\left[\int_0^{\infty} G_x(ds)\,\boldsymbol{C}_{t+s}^{t}(x)\int_X P(x,dy)\varphi(u,y) - \varphi(u,x)\right] \qquad (8)$$

where

$$q(x) = \frac{1}{g(x)}, g(x) = E\theta_x = \int_0^{\infty}(1 - G_x(t))dt.$$

Proof: Given point u_1 we have:

$$\mathbb{E}\varphi(u_1^\varepsilon, x_1^\varepsilon) = \mathbb{E}\mathbf{C}_{\theta_{x_0}}^t(x)\varphi(u, x_1^\varepsilon) = \int\limits_0^\infty G_x(ds)\mathbf{C}_{t+s}^t(x)\int\limits_X P(x, dy)\varphi(u, y)$$

Here we have (9).

Lemma (2)

Compensating operator $\boldsymbol{L}_t^{\varepsilon}(x)$ *is defined by form*

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(u,x) = \varepsilon^{-1}\boldsymbol{Q}\varphi(u,x) + \varepsilon^{-1}\left[\boldsymbol{G}_t^{\varepsilon}(x) - I\right]\boldsymbol{Q}_0\varphi(u,x), \qquad (9)$$

where $\boldsymbol{G}_t^{\varepsilon}(x) = \int\limits_0^{\infty} G_x(ds) C_{t+\varepsilon s}^{t}(x).$

Proof. From (8) have

$$\begin{aligned}\mathbf{L}_t^{\varepsilon}(x)\varphi(u,x) =& \varepsilon^{-1}q(x)\left[\int_0^{\infty} G_x(ds)\mathbf{C}_{t+\varepsilon s}^{t}(x)\int_X P(x,dy)\varphi(u,y)-\varphi(u,x)\right] \\ =& \varepsilon^{-1}q(x)\int_X P(x,dy)\left[\varphi(u,y)-\varphi(u,x)\right] \\ &+\varepsilon^{-1}q(x)\int_0^{\infty} G_x(ds)\left[\mathbf{C}_{t+\varepsilon s}^{t}-I\right]\int_X P(x,dy)\varphi(u,y).\end{aligned}$$

Then we obtain (9) □

Lemma (3)

Compensating operator $\boldsymbol{L}_t^{\varepsilon}(x)$ *has the asymptotic representation*

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(u,x) = \varepsilon^{-1}\boldsymbol{Q}\varphi(u,x) + \theta_1^{\varepsilon}(x)\boldsymbol{P}\varphi(u,x) \tag{10}$$

$$\boldsymbol{L}_t^{\varepsilon}(x)\varphi(u,x) = \varepsilon^{-1}\boldsymbol{Q}\varphi(u,x) + \boldsymbol{C}(x)\varphi(u,x) + \varepsilon\theta_2^{\varepsilon}(x)\varphi(u,x) \tag{11}$$

where

$$\theta_1^{\varepsilon}(x) = q(x)\,\boldsymbol{C}(x)\int_0^{\infty}\overline{G}_x(s)\,\boldsymbol{C}_{t+\varepsilon s}^{t}(x)ds$$

$$\theta_2^{\varepsilon}\varphi(u,x) = q(x)(\boldsymbol{C}(x))^2\,\boldsymbol{G}_{t,2}^{\varepsilon}(x)\varphi(u,x),$$

and

$$\boldsymbol{G}_{t,2}^{\varepsilon}(x)\varphi(u,x) = \int_0^{\infty}\overline{G}_x^{(2)}(s)\,\boldsymbol{C}_{t+\varepsilon s}^{t}(x)ds$$

.

Proof. We have semigroup equation $\mathbf{C}_{t+\varepsilon s}^{t}(x), t \geq 0, x \in X$,

$$d\mathbf{C}_{t+\varepsilon s}^{t}(x) = \varepsilon \mathbf{C}(x)\mathbf{C}_{t+\varepsilon s}^{t} ds.$$

Integrating by parts we have:

$$\mathbf{G}_t^{\varepsilon}(x) - I = \int_0^{\infty} G_x(ds)\left[\mathbf{C}_{t+s}^{t}(x) - I\right] = \left| \begin{matrix} u = \mathbf{C}_{t+\varepsilon s}^{t} & dv = G_x(ds \\ du = \varepsilon \mathbf{C}(x)\mathbf{C}_{t+\varepsilon s}^{t} ds & v = -\overline{G}_x(\end{matrix} \right.$$

$$= -\overline{G}_x(s)\left[\mathbf{C}_{t+\varepsilon s}^{t}(s) - I\right]\Big|_0^{\infty} + \varepsilon \int_0^{\infty} \overline{G}_x(s)\mathbf{C}(x)\mathbf{C}_{t+s}^{t}(x) ds$$

Given Kramer condition we have:

$$\mathbf{G}_t^{\varepsilon}(x) - I = \varepsilon \int\limits_0^{\infty} \overline{G}_x(s)\mathbf{C}(x)\mathbf{C}_{t+\varepsilon s}^t(x)ds = \varepsilon \mathbf{C}(x) \int\limits_0^{\infty} \overline{G}_x(s)\mathbf{C}_{t+\varepsilon s}^t(x)ds$$

Hence we have (10).

For

$$\mathbf{G}^{\varepsilon}_{t,1}(x) = \int\limits_0^{\infty} \overline{G}(x)\mathbf{C}^t_{t+\varepsilon s}(x)dx$$

integrating by parts we have:

$$\mathbf{G}^{\varepsilon}_{t,1}(x) = \int\limits_0^{\infty} \overline{G}(x)\mathbf{C}^t_{t+\varepsilon s}(x)dx = \left| \begin{matrix} u = \mathbf{C}^t_{t+\varepsilon s} & dv = \overline{G}_x(ds) \\ du = \varepsilon \mathbf{C}(x)\mathbf{C}^t_{t+\varepsilon s}ds & v = -\overline{G}^{(2)}_x(s) \end{matrix} \right|$$

$$= -\,\mathbf{C}^t_{t+\varepsilon s}(x)\cdot \overline{G}^{(2)}_x(s)|_0^{\infty} + \varepsilon \int\limits_0^{\infty} \mathbf{C}(x)\mathbf{C}^t_{t+\varepsilon s}(x)\cdot \overline{G}^{(2)}_x(s)ds$$

$$= m(x)I + \mathbf{C}(x)\varepsilon \int\limits_0^{\infty} \overline{G}^{(2)}_x \mathbf{C}^t_{t+\varepsilon s}ds.$$

Thus we have

$$\mathbf{G}_{t,1}^{\varepsilon}(x) = m(x)I + \mathbf{C}(x)\varepsilon\mathbf{G}_{t,2}^{\varepsilon}(x)$$

where:

$$\mathbf{G}_{t,2}^{\varepsilon}(x) = \int_0^\infty \overline{G}_x^{(2)}(s)\mathbf{C}_{t+\varepsilon s}^{t}(s)ds$$

$$\overline{G}_x^{(2)}(s) := \int_s^\infty \overline{G}_x^{(1)}(t)dt.$$

Hence:

$$\mathbf{C}(x)\mathbf{G}_{t,1}^{\varepsilon}(x) = \mathbf{C}(x)\left[m(x)I + \varepsilon\mathbf{C}(x)\mathbf{G}_{t,2}^{\varepsilon}(x)\right]$$

$$= m(x)\mathbf{C}(x) + \varepsilon(\mathbf{C}(x))^2\mathbf{G}_{t,2}^{\varepsilon}(x).$$

$$\begin{aligned}\mathbf{L}_t^{\varepsilon}(x)\varphi(u,x) =&\varepsilon^{-1}\mathbf{Q}\varphi(u,x) + q(x)\left[m(x)\mathbf{C}(x) + \varepsilon(\mathbf{C}(x))^2\mathbf{G}_{t,2}^{\varepsilon}(x)\right]\varphi(u,x)\\ =&\varepsilon^{-1}\mathbf{Q}\varphi(u,x) + \mathbf{C}(x)\varphi(u,x) + \varepsilon q(x)(\mathbf{C}(x))^2\mathbf{G}_{t,2}^{\varepsilon}(x)\varphi(u,x)\\ =&\varepsilon^{-1}\mathbf{Q}\varphi(u,x) + \mathbf{C}(x)\varphi(u,x) + \varepsilon\theta_2^{\varepsilon}(x)\varphi(u,x),\end{aligned}$$

where

$$\theta_2^{\varepsilon}(x)\varphi(u,x) = q(x)(\mathbf{C}(x))^2\mathbf{G}_{t,2}^{\varepsilon}(x)\varphi(u,x).$$

□

Lemma (4)

Compensating operator $\boldsymbol{L}_t^\varepsilon(x)$ *has the asymptotic representation in the function* $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x)$

$$\boldsymbol{L}_t^\varepsilon(x)\varphi^\varepsilon(u,x) = \boldsymbol{Q}\varphi_1(u,x) + \varepsilon^{-1}\boldsymbol{Q}\varphi(u) + \boldsymbol{C}(x)\varphi(u) + \varepsilon\theta^\varepsilon(x)\varphi(u)$$

where $\theta^\varepsilon(x)\varphi(u) = \theta_1^\varepsilon(x)\boldsymbol{P}\varphi_1(u,x) + \theta_2^\varepsilon(x)\varphi(u)$

Proof. We have

$$\mathbf{L}_t^\varepsilon(x)\left[\varphi(u)+\varepsilon\varphi_1(u,x)\right]=$$

$$\varepsilon\left[\varepsilon^{-1}\mathbf{Q}+\theta_1^\varepsilon\mathbf{P}\right]\varphi_1(u,x)+\left[\varepsilon^{-1}\mathbf{Q}+\mathbf{C}(x)+\varepsilon\theta_2^\varepsilon(x)\right]\varphi(u)=$$

$$\mathbf{Q}\varphi_1(u,x)+\varepsilon\theta_1^\varepsilon(x)\mathbf{P}\varphi_1(u,x)+\varepsilon^{-1}\mathbf{Q}\varphi(u)+\mathbf{C}(x)\varphi(u)+\varepsilon\theta_2^\varepsilon(x)\varphi(u).$$

□

Lemma (5)

Given singular perturbation problem[6, 14,18], limit generator $\boldsymbol{L}$ *is defined by formula:*

$$\boldsymbol{L}\varphi(u) = C(u)\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u)$$

Proof. From what $\varphi(u) \in N_Q$ we have

$$\mathbf{Q}\varphi(u) = 0.$$

Using formula from lemma 4 we have:

$$\mathbf{Q}\varphi_1(u,x) + \mathbf{C}(x)\varphi(u) = \mathbf{L}\varphi(u),$$

where $\mathbf{L} = \Pi\mathbf{C}(x)\Pi$

$$\mathbf{Q}\varphi_1(u,x) = (\mathbf{C}(x) - \mathbf{L})\varphi(u) = \widetilde{\mathbf{L}}(x)\varphi(u)$$

where

$$\widetilde{\mathbf{L}}(x) = \mathbf{C}(x) - \mathbf{L}.$$

Hence

$$\varphi_1(u,x) = \mathbf{R}_0\widetilde{\mathbf{L}}(x)\varphi(u). \qquad (12)$$

We lemma 5 statement [6].

Theorem

[11] Pattern limit theorem: If the following conditions holds: (C1): The family of embedded Markov renewal process $\xi_t^\varepsilon, x_t^\varepsilon, t \geq 0, \varepsilon > 0$, is relatively compact
(C2): There exists a family of test functions $\varphi^\varepsilon(u, x)$ in $C_0^\infty(\mathbb{R}^d \times E)$, such that

$$\lim_{\varepsilon \to 0} \varphi^\varepsilon(u, x) = \varphi(u),$$

uniformly on u, x.
(C3): The following convergence holds

$$\lim_{\varepsilon \to 0} \boldsymbol{L}^\varepsilon \varphi^\varepsilon(u, x) = \boldsymbol{L}\varphi(u),$$

uniformly on u, x. The family of functions $\boldsymbol{L}^\varepsilon \varphi^\varepsilon, \varepsilon > 0$ is uniformly bounded and $\boldsymbol{L}^\varepsilon \varphi^\varepsilon$ and $\boldsymbol{L}\varphi$ belong to $C(\mathbb{R}^d \times E)$.
(C4): The convergence of the initial values holds, that is,

$$\xi_0^\varepsilon \xrightarrow{P} \xi_0, \varepsilon \to 0$$

Pattern limit Theorem

Then the weak convergence:

$$\xi_t^\varepsilon \Rightarrow \xi_t, \varepsilon \to 0$$

takes place. The limit process $\xi_t, t \geq 0$ with generator $\mathbf{L}$ and is characterized by the martingale:

$$\mu_t = \varphi(\xi_t) - \int_0^t \mathbf{L}\varphi(\xi_s)ds, t \geq 0.$$

□

Corollary

The diffusion process $\zeta(t), t \geq 0$ is the solution of the stochastic differential equation:

$$d\zeta(t) = C(\zeta(t))dt + \sigma(\zeta(t))dw(t)$$

Conclusions. This result can be used in Poisson Aproximation scheme [17, 18] for the diffusion process with semi-Markov switching.

Theorem

Let regression function $C(u,x)$ *and variation* $\sigma(u,x)$ *satisfy the following conditions:*
C1: $C(u,\cdot) \in C^2(R^d)$
C2: $\sigma(u,\cdot) \in C^2(R^d)$.
C3: the distribution functions $G_x(t), t \geq 0, x \in X$ *satisfy the Cramer condition uniformly in* $x \in X$,

$$\sup_{x \in X} \int_0^\infty e^{ht} \overline{G}_x(t) dt \leq H < +\infty, h > 0$$

Then the solution $u^\varepsilon(t), t \geq 0$ *of the equation*

$$du^\varepsilon(t) = C\left(u^\varepsilon(t); x\left(\frac{t}{\varepsilon}\right)\right) dt + \sigma\left(u^\varepsilon(t); x\left(\frac{t}{\varepsilon}\right)\right) dw(t)$$

converges weakly to the limit diffusion process $\zeta(t), t \geq 0$ as $\varepsilon \to 0$, which is defined by the generator

$$\mathbf{L}\varphi(u) = C(u)\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u),$$

where

$$\sigma^2(u) := \int_X \pi(dx)\sigma^2(u,x), \sigma^2(u,x) = \sigma^*(u,x)\sigma(u,x).$$

- Anisimov, V.V.: Limit Theorems for Switching Processes and their Applications. Cybernetics 14(6), 917-929 (1978).
- Anisimov, V.V.: Limit Theorems for Switching Processes. Theory Probab. and Math. Statist. 37, 1-5 (1988)
- Anisimov, V.V.: Switching Processes: Averaging Principle, Diffusion Approxima- tion and Applications. Acta Applicandae Mathematicae 40, 95-141 (1995)
- Anisimov, V.V.: Averaging Methods for Transient Regimes in Overloading Retrial Queuing Systems. Mathematical and Computing Modelling 30(3/4), 65-78 (1999).
- Anisimov, V.V.: Switching Processes in Queueing Models. Wiley, Sons, ISTE, London (2008).
- Blankenship G.L., Papanicolaou G.C. Stability and Control of stochastic systems with wide band noise disturbances.// SIAM. Appl. Math.– 1978.– 34.– P.437-476.

- Chabanyuk, Ya. M. Stability of a dynamical system with semi-Markov switchings under conditions of diffusion approximation. Ukrainian Mathematical Journal (2007-09-01) 59: 1441-1452, September 01, 2007.
- Chabanyuk, Ya. M.Continuous stochastic approximation with semi-Markov switchings in the diffusion approximation scheme. Cybernetics and Systems Analysis (2007-07-01) 43: 605-612, July 01, 2007.
- Chabanyuk, Ya. M.Convergence of a jump procedure in a semi-Markov environment in diffusion-approximation scheme. Cybernetics and Systems Analysis (2007-11-01) 43: 866-875, November 01, 2007.
- Korolyuk V. S. Stability of stochastic systems in the diffusion-approximation scheme. Ukrainian Mathematical Journal (1998-01-01) 50: 40-54 , January 01, 1998.

Korolyuk V.S, Limnios N. Stochastic Systems in Merging Phase Space. - World Scientific, Singapore. - 2005. - 330 p.

Korolyuk, V. S. Problem of large deviations for Markov random evolutions with independent increments in the scheme of asymptotically small. Ukrainian Mathematical Journal (2010-11-01) 62: 739-747 , November 01, 2010.

Korolyuk, V.S., Swishchuk, A.V.: Random Evolutions. Kluwer Acad. Publ., Dordrecht (1994).

Korolyuk, V.S., Korolyuk, V.V.: Stochastic Models of Systems. Kluwer, Dordrecht (1999)

Korolyuk, V.S., Korolyuk, V.V., Limnios, N.: Queueing Systems with Semi-Markov Flow in Average and Diffusion Approximation Schemes. Methodol. Comput. Appl. Probab. 11, 201-209 (2009).

- Korolyuk, V. S.; Chabanyuk, Ya. M. Stability of a Dynamical System with Semi-Markov Switchings under Conditions of Stability of the Averaged System. Ukrainian Mathematical Journal (2002-02-01) 54: 239-252, February 01, 2002.
- Korolyuk V. S., Limnios N., Samoilenko I.V. Poisson Aproximation of Recurent Process with Semi-Markov Switching. Stochastic Analisys and Applications. 29: 769-778. 2011.
- Korolyuk V. S., Limnios N., Samoilenko I.V. Poisson Aproximation of Recurent Process with Locally Independent Increments and Semi-Markov Switching – Toward Application in Reliability. Advances in Degradation Modeling. January: 105-116. 2010.
- Kushner H.J. Optimality conditions for the average cost per unit time problem with a diffusion model. Siam J. Control and Optimization. Vol. 16. No.2. March 1978. PP. 330-346.
- Skorokhod A.V. (1989). Asymptotic Methods in the Theory of Stochastic Differen- tial Equations, AMS, vol. 78, Providence.

- Stroock D.W., Varadhan S.R.S. (1979). Multidimensional Diffusion Processes, Springer-Verlag, Berlin.
- Sviridenko M.N. (1986). Martingale approach to limit theorems for semi-Markov processes, Theor. Probab. Appl., pp 540-545.

Thank you!