Polish Academy of Sciences, Institute of Mathematics

Doctoral dissertation

Applications of algebraic methods in geometric tomography

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Jun 30, 2023

2020 *Mathematics Subject Classification.* Primary 52A20, 53A05, 53A07; Secondary 12D05, 12D10, 12H05, 12H20, 13A18, 13N10, 15A03, 52A38, 53A15

Key words and phrases. Geometric tomography, convex body, hyperplanar section, body of affine revolution, quadratic surface, ruled surface, generalized Wronskian, differential algebra, Gröbner basis, symbolic computation, isotropic volume function, Fourier transform, valued fields

I would like to express my special gratitude to my supervisor Prof. Michał Wojciechowski for his immense support, guidance, and opportunities he provided me during my doctoral journey. You have always been so enthusiastic about mathematics and full of innovative ideas. Your doors have always been open to me, and You have been extremely generous with Your time and advice. Working under Your supervision has been very enjoyable and made me grow both academically and personally.

Contents

Summary of professional accomplishments

Geometric tomography is a branch of mathematics dealing with the retrieval of information about a high-dimensional geometric object from its low-dimensional characteristics [**11**]. Most often these are either its sections by hyperplanes or its projections (shadows) on hyperplanes. The subject naturally overlaps with convex geometry and employs many of its tools. In my research, I follow the idea of associating with every low-dimensional characteristic an algebraic object (e.g. the group of its affine symmetries). Assuming that the hypersurface is sufficiently smooth, this allows us to find certain constraints satisfied by its Taylor polynomial and thus rephrase a geometric assumption in the language of (non)commutative algebra. Such a homological approach usually requires combining many different fields of mathematics, ranging from general topology and abstract algebra to partial differential equations to differential and algebraic geometry.

My dissertation consists of three independent chapters. Each of them illustrates an application of the previously described general paradigm to some particular problem in geometric tomography. I will outline them in the following sections.

1. On star-convex bodies with rotationally invariant sections

A compact domain $K \subset \mathbb{R}^n$, $n \geq 1$, is called a body of affine revolution if its symmetry group contains a subgroup affinely conjugated to $\mathrm{O}(n-1, \mathbb{R})$. In the first chapter of the dissertation, we will consider bounded domains $K \subset \mathbb{R}^n$, $n \geq 4$, whose sections by codimension 1 hyperplanes are bodies of affine revolution. Our main result is the following characterization theorem:

Theorem I.1.2. *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be an origin-symmetric star-convex body. Assume that the boundary ∂K is a submanifold of class C^3. If every hyperplane section of K passing through the origin is a body of affine revolution, then K itself is a body of affine revolution.*

At the root of this problem lies the celebrated isometric conjecture of S. Banach:

Question 1.1 (cf. [**5**, Remarques au chapitre XII, propriété (5)]). *Let B^n be a Banach space of finite dimension n and let k be a natural number satisfying the inequalities $1 < k < n$. If all the k-dimensional subspaces of B^n are isometrically isomorphic to each other, is B^n a Hilbert space?*

After more than five decades since the seminal work of H. Auerbach, S. Mazur, and S. Ulam [**4**], who settled Question 1.1 for $n = 3$, and M. Gromov [**12**], who generalized their approach and settled Question 1.1 for odd dimensions n, the idea of reducing the structure groups of certain fiber bundles was taken up again by L. Montejano et al. [**7, 8, 15, 16**]. Finally, in [**7**] the authors settled Question 1.1 for even dimensions n of the form $4k + 2 \geq 6$, $n \neq 134$. The key element of their proof was to show that all the hyperplanar sections of the unit ball of B^n must be bodies of affine revolution, which prompted them to ask the following, somewhat more general question:

Question I.1.1 (cf. [**7**, Remark 2.9]). *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be a convex body containing the origin O in its interior. If every hyperplane section of K passing through O is a body of affine revolution, is K necessarily a body of affine revolution?*

Theorem I.1.2 gives a positive answer to Question I.1.1 under certain restrictions on K.

The idea of the proof is rather elementary and follows the paradigm described in the introduction. For any point p on ∂K at which the second fundamental form of ∂K is positive definite, we normalize the extrinsic coordinate system so that ∂K is locally parametrized as a graph of some function of the form

$$f(\boldsymbol{x}) = \frac{1}{2}\langle \boldsymbol{x}, \boldsymbol{x}\rangle + O(\|\boldsymbol{x}\|)^3.$$

Now, for any hyperplane $H^{n-2} \in \mathrm{Gr}(n-2, T_p\partial K)$, the restriction $f|_{H^{n-2}}$ is a parametrization of a certain section of K, which by assumption is invariant under affine action of $\mathrm{O}(n-1,\mathbb{R})$. We show that there exists a hyperplane $H^{n-3} \in \mathrm{Gr}(n-3, H^{n-2})$, being precisely the hyperplane of revolution, such that the 3$^{\text{rd}}$ order Taylor polynomial of $f|_{H^{n-3}}$ vanishes. It follows that there exists a hyperplane $H_0^{n-2} \in \mathrm{Gr}(n-2, T_p\partial K)$ such that the 3$^{\text{rd}}$ order Taylor polynomial of $f|_{H_0^{n-2}}$ vanishes. Interestingly, the proof of the latter for $n = 4$ uses a topological argument and is essentially different from the proof for $n \geq 5$ based on algebraic geometry.

This algebraic fact is a key to comprehending the geometry of K. For there exists an open subset $V \subseteq \mathrm{Gr}(n-2, T_p\partial K)$ such that for every $H^{n-2} \in V$, $K \cap \langle H^{n-2}, O\rangle$ is invariant under the action of $\mathrm{O}(n-3,\mathbb{R})$ with the hyperplane of revolution $H^{n-2} \cap H_0^{n-2}$. Further, we show that either $V = \mathrm{Gr}(n-2, T_p\partial K)$ or the 3$^{\text{rd}}$ order Taylor polynomial of f vanishes, again considering the cases $n = 4$ and $n \geq 5$ separately. Finally, if the first case holds on an open subset of ∂K, we directly construct a group isomorphic to $\mathrm{O}(n-1,\mathbb{R})$ that acts on K. Otherwise, we obtain an intrinsic differential equation on ∂K, which by the classical result of Berwald [**18**, Theorem II.4.5] implies that K is an ellipsoid.

Presumably, the superfluous symmetry assumption can be disposed of, but this will significantly complicate any proof along our lines and most likely it will also lose its nice geometric flavor to the intensive computation of general affine differential invariants. Nevertheless, this direction seems promising. The smoothness assumption is an inherent element of our argument and therefore can not be easily relaxed.

One element of the proof is the following lemma, interesting on its own, which may be considered a counterpart of [**7**, Lemma 2.3]:

Lemma I.4.5. *A body of affine* 2*-revolution $K \subset \mathbb{R}^m$, $m \geq 4$, admitting three different codimension* 2 *hyperplanes of affine revolution, admits a codimension* 1 *hyperplane of affine revolution (i.e. is a body of affine* 1*-revolution).*

We state then a more general question:

Question I.4.7. *Does a compact domain $K \subset \mathbb{R}^m$, $m \geq 4$, admitting $k+1$ different codimension k hyperplanes of affine revolution, admit a codimension $k-1$ hyperplane of affine revolution, $0 < k < m$?*

To our best knowledge, the answer is not known.

The chapter was already published as an article [**21**].

2. Differential characterization of quadratic surfaces

Let $f \in W^{3,1}_{\text{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. In the second chapter of the dissertation, we will show that its graph is contained in a quadratic surface if and only if f is a weak solution to a certain system of 3rd order partial differential equations. Our main result is the following theorem:

THEOREM II.1.2. *Let $f \in W^{3,1}_{\text{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. Suppose that the Hessian determinant of f is not non-positive. Then f is a weak solution to the system of partial differential equations*

$$\begin{aligned} &f^{(3,0)}{f^{(0,2)}}^2 - 3f^{(1,2)}f^{(2,0)}f^{(0,2)} + 2f^{(0,3)}f^{(1,1)}f^{(2,0)} = 0, \\ &f^{(0,3)}{f^{(2,0)}}^2 - 3f^{(2,1)}f^{(0,2)}f^{(2,0)} + 2f^{(3,0)}f^{(1,1)}f^{(0,2)} = 0 \end{aligned} \tag{II.1.3}$$

if and only if its graph is contained in a quadratic surface.

The assumption on the Hessian determinant is not purely technical, as the following holds:

THEOREM II.1.4. *Let $f \in W^{3,1}_{\text{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. Suppose that the Hessian determinant of f is non-positive. Then f is a weak solution to the system of partial differential equations* (II.1.3) *if and only if Ω contains a countable sum of disjoint open connected subsets Ω_i such that:*

(1) *on each Ω_i the graph of f is contained in either:*
   (a) *a doubly-ruled surface[1], or*
   (b) *a developable surface[2], or*
   (c) *a Catalan surface[3] with directrix plane XZ, or*
   (d) *a Catalan surface with directrix plane YZ,*
(2) *the union $\bigcup \Omega_i$ is dense in Ω.*

Such a characterization of quadratic surfaces of positive Gaussian curvature as the only solutions to certain partial differential equations without boundary conditions turns out to be helpful when applying the paradigm described in the introduction to convex geometry. Theorem II.1.2 acts then as a bridge between intrinsic and extrinsic differential equations on ∂K. On the one hand, to prove that K is a quadric, it is enough to check that a local parametrization of ∂K satisfies two particular partial differential equations formulated in the extrinsic coordinate system, which usually does not require the use of any advanced tools of affine differential geometry. On the other hand, it is enough to verify this condition pointwise in any local coordinate system, as Theorem II.1.2 implies the following corollary:

COROLLARY II.5.9. *Let $S \subset \mathbb{R}^3$ be a convex surface of class C^3 such that for every $\boldsymbol{x} \in S$ there is a quadratic surface having 3rd order contact with S at $\boldsymbol{x}$. Then S is itself a quadratic surface.*

Problems of this type were considered already by W. Blaschke, who proved that conics are the only planar curves with constant equiaffine curvature. Since in this simple case the equiaffine curvature can be easily expressed in the extrinsic coordinate system, we get the following equivalent formulation:

THEOREM 2.1 ([**6**, p. 18]). *Let $f \in C^5(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}$. Then f is a solution to the ordinary differential equation*

$$9f''(x)^2 f^{(5)}(x) - 45f''(x)f^{(3)}(x)f^{(4)}(x) + 40f^{(3)}(x)^3 = 0$$

[1] A ruled surface that contains two families of rulings.
[2] A ruled surface having Gaussian curvature $K = 0$ everywhere.
[3] A ruled surface all of whose rulings are parallel to a fixed plane, called the directrix plane of the surface.

if and only if its graph is contained in a conic.

Afterward, it was proved by H. Maschke (for analytic surfaces), G. A. Pick (for surfaces), and L. Berwald (for hypersurfaces) [**18**, Theorem 4.5] that hyperquadrics can be characterized by vanishing of the certain intrinsically defined cubic form [**18**, ch. II, s. 4]. However, this time it can hardly be expressed in the extrinsic coordinate system. It is also unclear what minimal smoothness we need to assume. Theorem II.1.2 answers both of these issues, by providing explicit equations formulated in the extrinsic coordinate system, which are minimal in terms of the order of differentiation and the assumed smoothness of the solution. It gives also an alternative proof of Corollary II.5.9, which is a classical result.

This chapter is quite digressive and throughoutly explores many interesting aspects both of the problem itself and of techniques used to solve it. At the beginning, we observe that a graph of a function f is contained in a quadratic surface if and only if the set of functions

$$\left\{x^2, \quad xy, \quad xf(x,y), \quad y^2, \quad yf(x,y), \quad f(x,y)^2, \quad x, \quad y, \quad f(x,y), \quad 1\right\} \tag{II.3.2}$$

is linearly dependent. That is how the concept of generalized Wronskian for functions of several variables enters play. For it is clear that each generalized Wronskian of (II.3.2) may be viewed as a polynomial partial differential equation satisfied by a local parametrization of any quadratic surface. By a direct construction, we show that the system (II.1.3) is minimal in the sense that the left-hand sides form a reduced Gröbner basis of the differential ideal of all polynomial partial differential equations satisfied by local parametrizations of quadratic surfaces. This fact is useful on its own, as it reduces the problem of checking if the solution space of such an equation contains parametrizations of all quadratic surfaces to the well-studied problem of ideal membership (see e.g. [**9**, ch. 2, s. 8]). After showing that (II.1.3) enjoys the smoothing property, which follows from the tricky fact that the specially devised functions

$$\begin{aligned} u(x,y) &:= \frac{f^{(2,0)}(x,y) - f^{(0,2)}(x,y)}{\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{3/4}}, \\ v(x,y) &:= \frac{2f^{(1,1)}(x,y)}{\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{3/4}}, \end{aligned} \tag{II.4.2}$$

turn out to satisfy the Cauchy-Riemann equations, we may finally apply the delicate result of K. Wolsson [**20**, Theorem 2] to deduce the linear dependence of a set of functions from the vanishing of their generalized Wronskians.

To perform lengthy computations, we employ a widely used technical computing system Wolfram Mathematica [**13**]. Nevertheless, the proof remains human-surveyable.

3. On separably integrable symmetric convex bodies

An infinitely smooth symmetric convex body $K \subset \mathbb{R}^d$ is called k-separably integrable if its k-dimensional isotropic volume function $V_{K,H}(t) = \mathcal{H}^d(\{\boldsymbol{x} \in K : \operatorname{dist}(\boldsymbol{x}, H^\perp) \leq t\})$ can be written as a finite sum of products in which the dependence on $H \in \operatorname{Gr}(k, \mathbb{R}^d)$ and $t \in \mathbb{R}$ is separated. In the third chapter of the dissertation, we will obtain the complete classification of such bodies. Our main result is the following theorem:

THEOREM III.2.6. *Let $K \subseteq \mathbb{R}^d$ be an origin-symmetric convex body with infinitely smooth boundary ∂K. If K is locally k-separably integrable, then $d - k$ is even and K is an ellipsoid or $d - k$ is odd and K is a Euclidean ball.*

The history of the problem goes back to I. Newton, who argued in *Principia* that the areas of segments of planar convex bodies with an infinitely smooth boundary cut off by straight lines are not expressible in terms of algebraic equations [**17**, s. VI, Lemma XXVIII]. On the 300$^{\text{th}}$ anniversary of the first publication of *Principia*, V. I. Arnold asked several related questions in his famous seminar at Moscow State University [**3**, 1987-14, 1988-13, 1990-27]. Only after almost three decades, V. A. Vassiliev [**19**] showed that if $K \subset \mathbb{R}^d$ is a bounded domain in an even-dimensional space then the volume $V^{\pm}_{K,H}(t)$ cut off by a hyperplane parallel to $H \in \mathrm{Gr}(d-1, \mathbb{R}^d)$ at distance $\pm t \in \mathbb{R}$ from the origin is not an algebraic function of H and t, thus extending Newton's result to arbitrary even-dimensional space. Soon after M. L. Agranovsky introduced the concept of polynomial integrability and showed that in an even-dimensional space, the volume $V^{\pm}_{K,H}(t)$ is not a polynomial in t [**1**, Theorem 2]. On the one hand, he constrained the dependence of $V^{\pm}_{K,H}(t)$ on t, but on the other relaxed the dependence on H. The idea was further developed by A. Koldobsky, A. S. Merkurjev, and V. Yaskin, who showed that if K is polynomially integrable then the space is odd-dimensional and K is an ellipsoid [**14**, Proposition 3.1, Theorem 3.7]. Finally, M. Agranovsky, A. Koldobsky, D. Ryabogin, and V. Yaskin proved a similar result assuming that $V^{\pm}_{K,H}(t)$ is of the more general form $P(\boldsymbol{\xi}, t)\sqrt{Q(\boldsymbol{\xi}, t)}$, where P, Q are polynomials in t and $\deg Q = 2$ [**2**]. If we restrict ourselves to symmetric convex bodies, Theorem III.2.6 contains the results of [**14**] as a special case. However, the improvement in [**2**] goes in a completely different direction. Nevertheless, Theorem III.2.6 seems to indicate the crux of polynomial integrability. Namely, it is not so much the rigidity of polynomials that makes [**14**, Theorem 3.7] hold as the fact that the linear space of polynomials of fixed degree is finite-dimensional. This phenomenon prompts us to ask the following question, which contains all the aforementioned results, including ours:

Question III.2.7. *Let K be a bounded domain in $\mathbb{R}^d$ with an infinitely smooth boundary ∂K. If the k-dimensional isotropic volume function $V_{K,H}(t)$ can be locally expressed in the form*

$$V_{K,H}(t) = \Phi(a_1(H), a_2(H), \ldots, a_m(H), b_1(t), b_2(t), \ldots, b_n(t))$$

on some open neighborhood of $\mathrm{Gr}(k, \mathbb{R}^d) \times \{0\}$, where $\Phi : \mathbb{R}^{m+n} \to \mathbb{R}$ is algebraic and $a_i : \mathrm{Gr}(k, \mathbb{R}^d) \to \mathbb{R}$, $b_i : [0, \infty) \to \mathbb{R}$ are smooth, is K necessarily an ellipsoid?

To our best knowledge no counterexample is known so far.

Our proof begins along the lines of [**14**, Theorem 3.7]. We rewrite the assumption in terms of the Fourier transform of the iterated Laplace operator applied to the powers of the Minkowski functional $\|\cdot\|_K$ to obtain a family of equations of the form

$$\sum_{i=0}^{n} c_{s,i} \Delta^i \|\boldsymbol{x}\|_K^{-d+2s+2i+k} = P_s(\boldsymbol{x}), \tag{3.1}$$

where $n \in \mathbb{N}$ is a fixed constant, $c_{s,i}$ are some scalars and $P_s(\boldsymbol{x})$ is a homogeneous polynomial of degree $-d + 2s + k$. Using properties of the Fourier transform and (simple but tedious) linear algebra, we infer that the sequence $\{P_s(\boldsymbol{x})\|\boldsymbol{x}\|_K^{d-2s-k}\}_{s \geq \lceil d/2 \rceil}$ spans a finite-dimensional subspace of $C(\mathbb{R}^d, \mathbb{R})$. Further, we prove that only finitely many elements of $\{P_s(\boldsymbol{x})\}_{s \geq \lceil d/2 \rceil}$ are zero, unless K is a Euclidean ball. This way we pass from differential to algebraic equations and reduce the problem to solving an infinite system of polynomial equations in $\zeta := \|\boldsymbol{x}\|_K^{-2}$ over the rational function field $\mathbb{R}(\boldsymbol{x})$. For this purpose, we employ techniques from field theory. Denote the minimal polynomial of ζ by

$$\mu_\zeta(\lambda) = \mu_{\zeta,0} + \mu_{\zeta,1}\lambda + \ldots + \mu_{\zeta,m-1}\lambda^{m-1} + \lambda^m,$$

where $\mu_{\zeta,i} \in \mathbb{R}(\boldsymbol{x})$ and $m = [\mathbb{R}(\boldsymbol{x},\zeta) : \mathbb{R}(\boldsymbol{x})]$ is the degree of a field extension. We observe that multiplication by ζ defines an $\mathbb{R}(\boldsymbol{x})$-linear operator $T_\zeta : \mathbb{R}(\boldsymbol{x},\zeta) \to \mathbb{R}(\boldsymbol{x},\zeta)$. Using the well-known Cayley-Hamilton theorem we deduce that the sequence $\{P_s(\boldsymbol{x})[T_\zeta]^{2s}\}_{s\geq\lceil d/2\rceil}$ spans a finite-dimensional subspace of $\mathbb{R}(\boldsymbol{x})^{m\times m}$, where $[T_\zeta] \in \mathbb{R}(\boldsymbol{x})^{m\times m}$ denotes the matrix of T_ζ in certain basis.

As the problem has been finally reduced to a question about the rational functions of several variables, we may now apply the fine tools of valuation theory, which is known to form a solid link between algebra and analysis. Let $p \in \mathbb{R}[\boldsymbol{x}]$ be any irreducible polynomial. By a classical result [**10**, Theorem 3.1.2], we extend the p-adic valuation v_p from $\mathbb{R}(\boldsymbol{x})$ to the splitting field of the characteristic polynomial μ_ζ. Using a trick involving Viète's formulas for the characteristic polynomial of $[T_\zeta]^{2s}$, we show that

$$v_p(P_s) = -s v_p(\mu_{\zeta,m-i})/i + O(1)$$

for every irreducible polynomial $p \in \mathbb{R}[\boldsymbol{x}]$, whence $\mu_{\zeta,m-i}$ and $\mu_{\zeta,0}^{i/m}$ are associated, i.e.

$$\mu_{\zeta,m-i} = u_{m-i}\mu_{\zeta,0}^{i/m}$$

for some $u_{m-i} \in \mathbb{R}\setminus\{0\}$. Hence $\mu_{\zeta,0}^{-1/m}\zeta$ is equal to a root $r \in \mathbb{R}$ of a polynomial

$$u_0 + u_1\left(\mu_{\zeta,0}^{-1/m}\zeta\right) + \ldots + u_{m-1}\left(\mu_{\zeta,0}^{-1/m}\zeta\right)^{m-1} + \left(\mu_{\zeta,0}^{-1/m}\zeta\right)^m$$

with constant coefficients, which means that

$$\|\boldsymbol{x}\|_K = \zeta^{-1/2} = (r^m\mu_{\zeta,0})^{-1/(2m)}$$

is a root of order $2m$ of some homogeneous polynomial $(r^m\mu_{\zeta,0})^{-1}$ of degree $2m$.

This is not yet the end, as we want to show that actually $m = 1$. However, it is not as straightforward as it would seem, since the reductions we have made along the way were non-equivalent. Hence to get the desired conclusion, we need to go back to the very beginning of our argument. Slightly abusing the notation, we define $\|\boldsymbol{x}\|_K =: \zeta$ with $\zeta^{2m} =: h(\boldsymbol{x})$ being a homogeneous polynomial of degree $2m$ and observe that $A =: \mathbb{R}(\boldsymbol{x},\zeta)$ may be viewed as a graded algebra

$$A = \bigoplus_{i\in C_{2m}} A_i, \quad A_i =: \mathbb{R}(\boldsymbol{x})\zeta^i,$$

where the index set is the cyclic group C_{2m}. Further, we argue that the Laplace operator Δ defines a graded endomorphism of A, i.e. for every function $f \in A_i$, $i \in C_{2m}$, we have $\Delta f \in A_i$. Finally, we specially devise an infinite family of equations of the form (3.1) such that the left-hand sides clearly belong to either A_1 or A_2, unless $m = 1$. Now, since we otherwise know they are polynomials (i.e. they belong to A_0), they must all be zero. However, recall that only finitely many elements of $\{P_s(\boldsymbol{x})\}_{s\geq\lceil d/2\rceil}$ are zero, unless K is a Euclidean ball.

Most likely the symmetry assumption is superfluous. Unfortunately, exactly as in [**14**], the non-symmetric case is essentially more difficult and requires even more involved algebraic arguments.

The chapter is based on a joint work with V. Yaskin.

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CHAPTER I

On star-convex bodies with rotationally invariant sections

We will prove that an origin-symmetric star-convex body K with sufficiently smooth boundary and such that every hyperplane section of K passing through the origin is a body of affine revolution, is itself a body of affine revolution. This will give a positive answer to the recent question asked by G. Bor, L. Hernández-Lamoneda, V. Jiménez de Santiago, and L. Montejano-Peimbert, though with slightly different prerequisites.

I.1. Introduction

After more than five decades since the seminal works of H. Auerbach, S. Mazur and S. Ulam [**1**], A. Dvoretzky [**5**], M. Gromov [**6**] and V. Milman [**11**], the isometric conjecture of S. Banach again attracted the attention of researchers, launching a whole avalanche of papers by L. Montejano et al. [**3, 4, 14, 15**] and recently also by S. Ivanov, D. Mamaev and A. Nordskova [**8**]. As it was already known that algebraic topology alone would not suffice, more sophisticated methods were developed. For instance, in [**3**] the authors (G. Bor, L. Hernández-Lamoneda, V. Jiménez de Santiago and L. Montejano-Peimbert) showed that under assumptions of the conjecture (namely, that K is a symmetric convex body, all of whose hyperplanar sections are affinely equivalent) supplemented with dimension constraints having its origins in algebraic topology, all the hyperplanar sections of K must be bodies of affine revolution (cf. Definition I.2.2). This observation prompted them to ask the following, somewhat more general question:

Question I.1.1 (cf. [**3**, Remark 2.9]). *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be a convex body containing the origin O in its interior. If every hyperplane section of K passing through O is a body of affine revolution, is K necessarily a body of affine revolution?*

Note that the reverse implication is quite straightforward (cf. [**3**, Lemma 2.4]). Moreover, the authors proved in [**3**, Theorem 1.4] that at least one hyperplane section of such a symmetric convex body must be an ellipsoid, which is an obvious necessary condition. Compared to the initial problem of S. Banach, they decided to keep the assumption that K is convex while forgoing the assumption that K is symmetric. In what follows, we will prove a theorem in the same spirit, but with slightly different prerequisites:

Theorem I.1.2. *Let $K \subset \mathbb{R}^n$, $n \geq 4$, be an origin-symmetric star-convex body. Assume that the boundary ∂K is a submanifold of class C^3. If every hyperplane section of K passing through the origin is a body of affine revolution, then K itself is a body of affine revolution.*

Our argument is rather elementary. It is built mainly upon the tools of differential geometry and linear algebra. Although occasionally we will need to use some more involved facts from other fields like algebraic topology or commutative algebra, they will hide most of the difficulty within themselves. Unlike in [**3**], we forgo the assumption that K is convex while keeping the assumption that K is symmetric. Moreover, to apply our method we need the boundary of K to be sufficiently smooth. Presumably, the superfluous symmetry assumption can be disposed of, but this will significantly complicate any proof along our lines and most likely it will also lose its nice geometric flavor to the intensive computation of general affine differential invariants (cf. Remark I.4). The smoothness assumption seems to be an inherent element of our argument and therefore can not be easily relaxed.

A natural question arises if the assumption $n \geq 4$ is indeed necessary. A compact domain $L \subseteq \mathbb{R}^{n-1}$ is a body of affine revolution if its symmetry group contains a subgroup affinely conjugated to $\mathrm{O}(n-2, \mathbb{R})$ (cf. Definition I.2.2). In dimension $n = 3$, Question I.1.1 has a different flavor because we assume merely that every planar section of K passing through the origin admits an affine reflection, which is satisfied e.g. when K is a cube (every central planar section of a cube is affinely equivalent to either a square or a regular hexagon, both of which are axially symmetric). Therefore the statement is no longer true unless we make some additional assumptions (see e.g. [**13**, §2]). The right counterpart of Question I.1.1 in dimension 3 seems to be an affine version of a similar question asked by K. Bezdek:

QUESTION I.1.3 (cf. [**2**, §10]). *Let $K \subset \mathbb{R}^3$ be a convex body. If every planar section of K [not necessarily passing through the origin ed.] admits an affine reflection, is K necessarily a body of affine revolution?*

To the author's best knowledge, it remains open. Nevertheless, techniques similar to those presented in this paper may be applied also to Question I.1.3, but then they will most likely require higher-order smoothness of the boundary.

I.2. Definitions and basic concepts

We adopt the notation from [**3**].

DEFINITION I.2.1. A compact domain $K \subset \mathbb{R}^n$, $n \geq 1$, is called *star-convex* if there exists $O \in K$ such that for every $x \in K$ the entire line segment from O to x is contained in K. A star-convex body is called *symmetric* if it is centrally symmetric with respect to O.

REMARK. Actually, the same proof of Theorem I.1.2 with minor technical improvements works for general compact domains. However, we will intentionally refrain from these topological considerations, so as not to overshadow the main idea.

DEFINITION I.2.2. A compact domain $K \subset \mathbb{R}^n$, $n \geq 1$, is called *a body of affine k-revolution* if its symmetry group contains a subgroup G affinely conjugated to $\mathrm{O}(n-k, \mathbb{R})$, $0 < k < n$. The ambient space $\mathbb{R}^n$ can be viewed as a direct sum $H \oplus L$ of a linear space H and an affine space L, where H (called *the hyperplane of affine revolution*) is an irreducible representation space of G of dimension $n - k$ and L (called *the hyperaxis of affine revolution*) is a common fixed point subspace of G of dimension k. By body (resp. hyperplane, axis) of affine revolution, we will mean a body (resp. hyperplane, axis) of affine 1-revolution unless expressly stated otherwise.

REMARK. If we additionally assume that K is symmetric, then the center of symmetry O must be a fixed point of any affine symmetry of K. In particular, if K is a star-convex body of affine revolution, the axis of affine revolution must pass through O. Moreover, since every section of K with a hyperplane passing through O is again a star-convex body of affine revolution symmetric with respect to O, the axis of affine revolution of all such hyperplanar sections must likewise pass through O.

REMARK. Note that all these objects are defined (and will be used) in a general affine setting. In particular, the symmetry group of K is a compact subgroup of $\mathrm{GL}(n, \mathbb{R})$, but not necessarily of $\mathrm{O}(n, \mathbb{R})$.

Denote the submanifold ∂K by M^{n-1}. Let $p \in M^{n-1}$ be any point with positive definite second fundamental form of M^{n-1}. After applying a suitable affine map we may assume that $p = \mathbf{0}_{\mathbb{R}^n}$, $O = \mathbf{0}_{\mathbb{R}^n} + \hat{\mathbf{e}}_n$ and $T_pM = \mathbf{0}_{\mathbb{R}^n} + \langle \hat{\mathbf{e}}_n \rangle^{\perp}$, where $\hat{\mathbf{e}}_n$ stands for the n^{th} standard unit vector (fig. I.2.3).

In this coordinate system, we represent the neighborhood of p in M^{n-1} as a graph of some function $f : T_pM^{n-1} \supset U \to \mathbb{R}$ of class C^3, which must be of the form

$$f(\boldsymbol{x}) = O(\|\boldsymbol{x}\|)^2$$

in Big-O notation. Since we assumed that the second fundamental form of M^{n-1} is positive definite at p, after applying a suitable linear change of coordinates in the domain we may further assume that

$$f(\boldsymbol{x}) = \frac{1}{2}\langle \boldsymbol{x}, \boldsymbol{x}\rangle + O(\|\boldsymbol{x}\|)^3.$$

The above will be called the *canonical parametrization* of M^{n-1} at p. Note that it is unique up to an orthogonal change of coordinates in the domain. Moreover, observe that the restriction of f to any codimension 1 hyperplane $H \in \mathrm{Gr}(n-2, T_pM^{n-1})$ is the canonical parametrization of the hyperplanar section $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ at p.

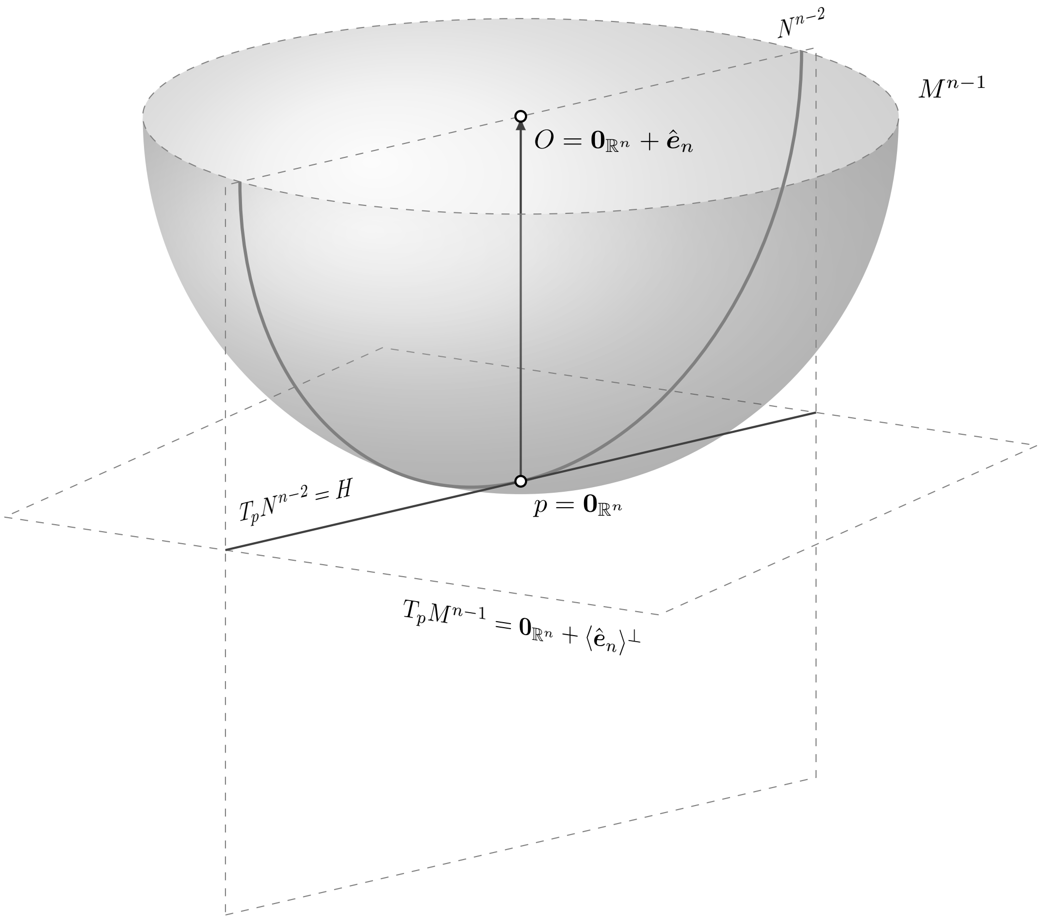

FIGURE I.2.3. The canonical parametrization of M^{n-1} at p

DEFINITION I.2.4. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function of class C^3. The homogeneous part of degree 3 of its series expansion is called the *cubic form* of f and will be denoted by c_f.

REMARK. In the course of the proof, we will consider almost exclusively points with positive definite second fundamental form of M^{n-1}. However, we do not need to assume that M^{n-1} is strongly convex. Indeed, every compact hypersurface contains at least one such point, from which all the local properties will eventually spill over the entire set.

I.3. Hypersurfaces of affine revolution

Although the original hypersurface is $(n-1)$-dimensional, most of the time we will be investigating $(n-2)$-dimensional hypersurfaces of affine revolution since their geometry plays a key role in the proof. Let

$$g : T_pN^{n-2} \supset U \to \mathbb{R}, \quad g(\boldsymbol{x}) = \frac{1}{2}\langle \boldsymbol{x}, \boldsymbol{x}\rangle + O(\|\boldsymbol{x}\|)^3$$

be the canonical parametrization of some hyperplanar section N^{n-2} of M^{n-1} at p (fig. I.2.3), being a hypersurface of affine revolution. From now henceforth, by action of a linear group we always mean the action of its affine matrix representation on a specified affine subspace of $\mathbb{R}^n$, usually clear from the context. By definition, N^{n-2} is invariant under action of $\mathrm{O}(n-2,\mathbb{R})$. Denote by G_p the *isotropy group* of p, i.e. the set of affine symmetries of N^{n-2} which does not change p. If p is already a fixed point of $\mathrm{O}(n-2,\mathbb{R})$ then G_p is affinely conjugated to $\mathrm{O}(n-2,\mathbb{R})$, otherwise G_p is affinely conjugated to $\mathrm{O}(n-3,\mathbb{R})$. Without loss of generality, we may choose U to be invariant under G_p.

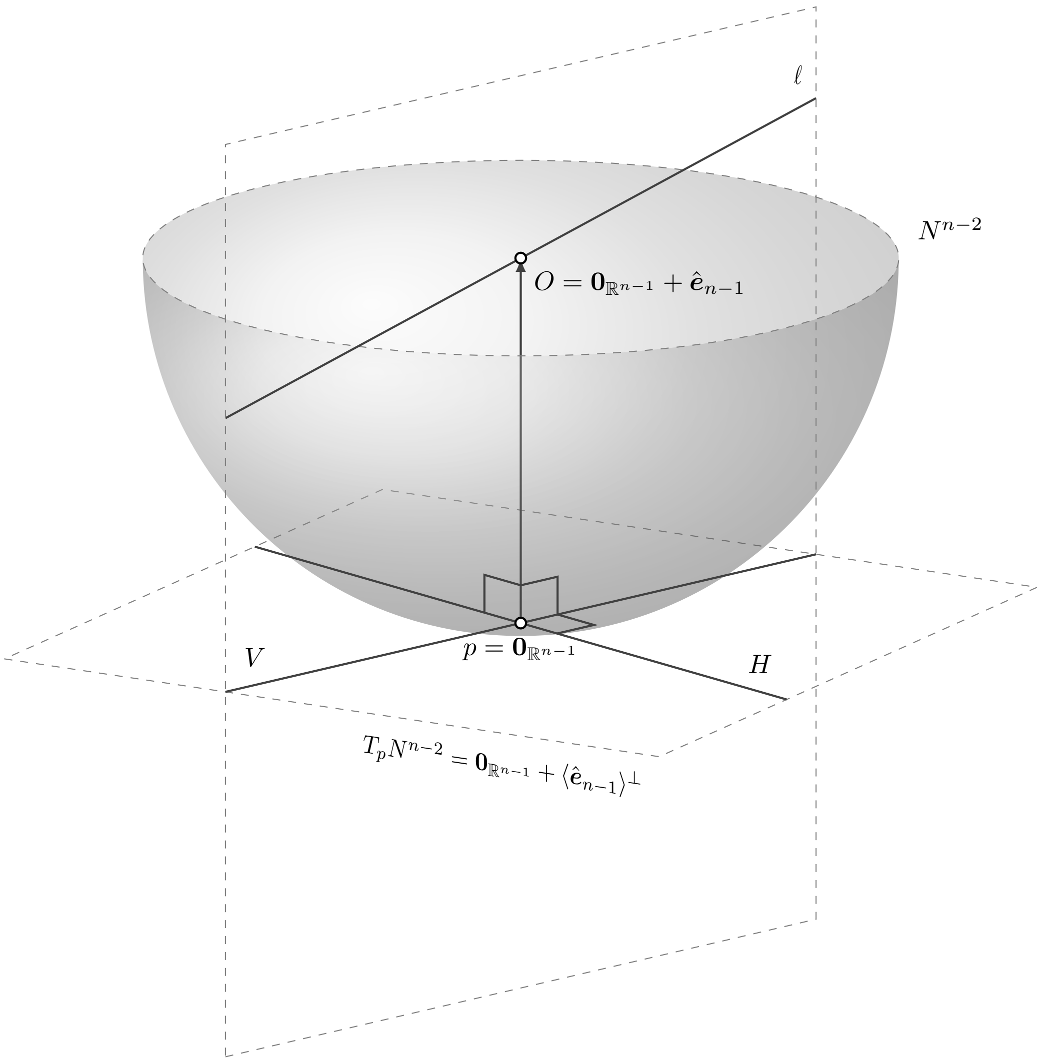

FIGURE I.3.1. The canonical parametrization of N^{n-2} at p

Let $A \in G_p$ be any affine symmetry of N^{n-2} which does not change p. Note that in our coordinate system, A may be regarded as a linear map. Since O is the center of symmetry of N^{n-2}, it must be a fixed point of A. Thus $\hat{\boldsymbol{e}}_{n-1}$ is an eigenvector of A with eigenvalue $+1$. Moreover, the hyperplane $\langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}$ it tangent to N^{n-2} at p and thus it must be an invariant subspace of A. It follows that the matrix representation of A in our canonical coordinate system is of the form

$$[A] = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & [B] & & \vdots \\ & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

for some $B \in \mathrm{GL}(n-2, \mathbb{R})$. Now, for every point $\boldsymbol{x} \in U$ there exists a point $\tilde{\boldsymbol{x}} \in U$ such that

$$[A] \cdot \begin{pmatrix} \boldsymbol{x} \\ g(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{x}} \\ g(\tilde{\boldsymbol{x}}) \end{pmatrix},$$

which reads

$$g(B\boldsymbol{x}) = g(\boldsymbol{x}). \tag{I.3.2}$$

In particular, $[B]$ must preserve the standard quadratic form, in which case it is an orthogonal matrix. Thus A itself must be an orthogonal map, which means that in our coordinate system, G_p is actually a subgroup of $\mathrm{O}(n-2, \mathbb{R})$.

Claim I.3.3. It follows immediately from (I.3.2) that the canonical parametrization g is invariant under action of G_p on its domain $U \subset T_pN^{n-2}$, i.e. $g \circ A|_U \equiv g$ for every $A \in G_p$. $\square$

Claim I.3.4. The tangent space T_pN^{n-2} may be viewed as a $(n-2)$-dimensional representation space of G_p. If $G_p \simeq \mathrm{O}(n-3, \mathbb{R})$, then T_pN^{n-2} admits an orthogonal decomposition $H \oplus V$ into irreducible representations' spaces, where H is a codimension 1 hyperplane of revolution and V is a dimension 1 common fixed-point subspace (fig. I.3.1). In particular, the cubic form c_g vanishes on H. Indeed, $c_g|_H$ must vanish at some direction and by Claim I.3.3 this carries over to all the other directions as well. On the other hand, if $G_p \simeq \mathrm{O}(n-2, \mathbb{R})$, then T_pN^{n-2} is already an irreducible representation's space of G_p and thus c_g vanishes identically (again, by the very same argument). The latter is necessarily the case when N^{n-2} is an ellipsoid. $\square$

Let us recall a simple fact from the original paper [**3**]:

Lemma I.3.5 ([**3**, Lemma 2.3]). *A symmetric body of affine revolution $K \subset \mathbb{R}^m$, $m \geq 3$, admitting two different hyperplanes of affine revolution, is an ellipsoid.*

Now we are ready to prove the following key lemma, which will eventually enable us to figure out the geometry of M^{n-1}:

Lemma I.3.6. *In the above setting, there exists a codimension 1 hyperplane $H \in \mathrm{Gr}(n-2, T_pM^{n-1})$ such that the cubic form $c_f|_H$ is identically zero (i.e. c_f is reducible).*

Interestingly enough, the proof for $n = 4$ and $n \geq 5$ will be essentially different. In the first case, we need an argument from general topology, which holds only in even dimensions n. In the second case, we introduce an argument from algebraic geometry, which holds only in dimensions $n \geq 5$.

Proof of Lemma I.3.6 for $n \geq 5$. Suppose that c_f is irreducible. Theorem of Bertini [**7**, Theorem 17.16] asserts that there exists a codimension 1 hyperplane $H \in \mathrm{Gr}(n-2, T_pM^{n-1})$ such that $c_f|_H$ is again irreducible. However, it follows from Claim I.3.4 that $c_g = c_f|_H$ vanishes on some codimension 1 hyperplane, i.e. admits a factor of degree 1, a contradiction. $\square$

Proof of Lemma I.3.6 for $n = 4$. If there exists a hyperplanar section of M^3 passing through p that admits more than one axis of affine revolution, then by Lemma I.3.5 and Claim I.3.4 we are done. Further, if there exists a hyperplanar section of M^3 passing through p such that its axis of affine revolution also passes through p, then again by Claim I.3.4 we are done. Therefore we may assume that every hyperplanar section of M^3 passing through p admits exactly one axis of affine revolution, which does not pass through p.

In this case, we can define the following distribution on $\mathrm{Gr}(2, T_pM^3)$: for every plane $\pi \in \mathrm{Gr}(2, T_pM^3)$, let $\ell_\pi \subset \pi = T_\pi \mathrm{Gr}(2, T_pM^3)$ be the orthogonal projection of the (unique) axis of affine revolution of $M^3 \cap \mathrm{aff}(\{\pi, O\})$ on π, which we already know is always a 1-dimensional linear subspace of π. Moreover, the map $\pi \mapsto \ell_\pi$ is clearly continuous (cf. [**3**, Lemma 2.8]), which gives rise to a rank-1 subbundle η of $T\mathrm{Gr}(2, T_pM^3)$. Now, its Stiefel-Whitney class $w_1(\eta) \in H^1(\mathrm{Gr}(2, T_pM^3); \mathbb{Z}/2\mathbb{Z}) = \{0\}$ must be 0 and thus η is orientable [**12**, Problem 12-A]. Selecting for each fiber of η the positively oriented unit vector gives rise to a non-vanishing vector field on $\mathrm{Gr}(2, T_pM^3)$, a contradiction. □

Since the canonical parametrization is defined up to an orthogonal change of coordinates in the domain, without loss of generality we may further assume that c_f vanishes on the hyperplane $\langle \hat{\mathbf{e}}_{n-1} \rangle^\perp$, i.e.

$$c_f(\boldsymbol{x}) = x_{n-1} \cdot q_f(\boldsymbol{x}), \tag{I.3.7}$$

where q_f is some quadratic form, not necessarily non-zero.

Claim I.3.8. For every irreducible quadric $Q^{n-2} \subset T_pM^{n-1}$, there exists an open subset $V \subseteq \mathrm{Gr}(n-2, T_pM^{n-1})$ of hyperplanes H such that $Q^{n-2} \cap H$ contains no codimension 1 linear subspace. Indeed, every linear space contained in an irreducible quadric has dimension at most half the dimension of the quadric [**7**, Theorem 22.13]. Therefore if $n \geq 5$, the conclusion is trivial. For $n = 4$, every irreducible quadric is projectively equivalent to either a cone, a straight line, or a single point. In each case, there exists an open subset of planes that intersect Q^2 only at the origin. □

Now, if the quadratic form q_f on the right-hand side of (I.3.7) is irreducible, then from Claim I.3.8 it follows that for every $H \in V$ the zero set of $c_f|_H$ contains exactly one codimension 1 hyperplane, namely $H \cap \langle \hat{\mathbf{e}}_{n-1} \rangle^\perp$. In particular, by Claim I.3.4, $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ is invariant under action of $\mathrm{O}(n-3, \mathbb{R})$ with hyperplane of revolution $H \cap \langle \hat{\mathbf{e}}_{n-1} \rangle^\perp$. On the other hand, if the quadratic form q_f is reducible, then c_f can be decomposed into a product of three linear forms, and hence its zero set is a sum of three (not necessarily different) hyperplanes H_1, H_2, H_3. The same argument shows that for every $H \in \mathrm{Gr}(n-2, T_pM^{n-1}) \setminus \{H_1, H_2, H_3\}$, $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ is invariant under action of $\mathrm{O}(n-3, \mathbb{R})$ with hyperplane of revolution $H \cap H_i$ for some $i \in \{1, 2, 3\}$. Denote by V_i the set of hyperplanes $H \in \mathrm{Gr}(n-2, T_pM^{n-1})$ such that $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ is invariant under action of $\mathrm{O}(n-3, \mathbb{R})$ with hyperplane of revolution $H \cap H_i$, $i = 1, 2, 3$. Since each V_i is closed (cf. [**3**, Lemma 2.7]) and $V_1 \cup V_2 \cup V_3 = \mathrm{Gr}(n-2, T_pM^{n-1})$, at least one of those sets has non-empty interior. After a suitable change of coordinates, we may assume that this is the set corresponding to the plane $\langle \hat{\mathbf{e}}_{n-1} \rangle^\perp$.

Claim I.3.9. In either case, we are eventually in a position where we have an open subset $V \subseteq \mathrm{Gr}(n-2, T_pM^{n-1})$ such that for every $H \in V$, $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ is invariant under the action of $\mathrm{O}(n-3, \mathbb{R})$ with hyperplane of revolution $H \cap \langle \hat{\mathbf{e}}_{n-1} \rangle^\perp$. □

Notation. For any 2-dimensional plane $\pi \in \mathrm{Gr}(2, T_pM^{n-1})$ and any point $\boldsymbol{a} \in T_pM^{n-1}$, denote by $\mathrm{Ref}_\pi(\boldsymbol{a})$ the orthogonal reflection of $\boldsymbol{a}$ across the plane π. Further, for any angle $\alpha \in \mathbb{R}$, denote by $\mathrm{Rot}_\pi^\alpha(\boldsymbol{a})$ the rotation of $\boldsymbol{a}$ around the axis $\pi^\perp$ by the angle α.

Let us define a continuous map

$$\phi : \mathrm{Gr}(1, \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}) \times \mathrm{Gr}(1, \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}) \times (T_pM^{n-1} \setminus \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}) \to \mathrm{Gr}(n-2, T_pM^{n-1}) \times \mathrm{Gr}(n-2, T_pM^{n-1}),$$
$$\phi(\ell_1, \ell_2, \boldsymbol{a}) = (\langle \ell_1^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle, \langle \ell_2^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \mathrm{Ref}_{\langle \ell_1, \hat{\boldsymbol{e}}_{n-1} \rangle}(\boldsymbol{a}) \rangle)$$

(fig. I.3.11) and let $\ell \in \mathrm{Gr}(1, \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp})$, $\boldsymbol{a} \in T_pM^{n-1} \setminus \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}$ be such that $\langle \ell^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle \in V$. Then we have $\langle \ell^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle = \langle \ell^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \mathrm{Ref}_{\langle \ell, \hat{\boldsymbol{e}}_{n-1} \rangle}(\boldsymbol{a}) \rangle$, so

$$\phi(\ell, \ell, \boldsymbol{a}) = (\langle \ell^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle, \langle \ell^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle)$$

is an element of $V \times V$. Since V is open, the preimage $\phi^{-1}(V \times V)$ is an open neighborhood of $(\ell, \ell, \boldsymbol{a})$. Thus it contains contains a product of non-empty open sets $W_1 \times W_2 \times W_3$, where $W_1, W_2 \subseteq \mathrm{Gr}(1, \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp})$ are neighborhoods of ℓ and $W_3 \subseteq (T_pM^{n-1} \setminus \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp})$ is a neighborhood of $\boldsymbol{a}$. Moreover, since $\phi(\ell_1, \ell_2, \boldsymbol{a}) = \phi(\ell_1, \ell_2, \lambda \boldsymbol{a})$ for every $\lambda \neq 0$, we may assume that W_3 is the interior of a generalized cone intersected with U.

Let $\ell_1 \in W_1$, $\ell_2 \in W_2$, $\boldsymbol{a} \in W_3$ and define $\boldsymbol{a}' := \mathrm{Ref}_{\langle \ell_1, \hat{\boldsymbol{e}}_{n-1} \rangle}(\boldsymbol{a})$, $\boldsymbol{a}'' := \mathrm{Ref}_{\langle \ell_2, \hat{\boldsymbol{e}}_{n-1} \rangle}(\boldsymbol{a}')$ (fig. I.3.11). In light of the definition of V, it follows from Claim I.3.3 that $f|_{\langle \ell_1^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a} \rangle}$ is invariant under action of $\mathrm{O}(n-3, \mathbb{R})$ with hyperplane of revolution $\ell_1^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}$. In particular, this group contains the reflection across the common fixed-point subspace, which can be viewed as a restriction of $\mathrm{Ref}_{\langle \ell_1, \hat{\boldsymbol{e}}_{n-1} \rangle}$. Similarly, $f|_{\langle \ell_2^{\perp} \cap \langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}, \boldsymbol{a}' \rangle}$ is invariant under $\mathrm{Ref}_{\langle \ell_2, \hat{\boldsymbol{e}}_{n-1} \rangle}$, which implies

$$f(\boldsymbol{a}'') = f(\boldsymbol{a}') = f(\boldsymbol{a}).$$

Now, observe that

$$\boldsymbol{a}'' = \mathrm{Ref}_{\langle \ell_2, \hat{\boldsymbol{e}}_{n-1} \rangle}(\mathrm{Ref}_{\langle \ell_1, \hat{\boldsymbol{e}}_{n-1} \rangle}(\boldsymbol{a})) = (\mathrm{Ref}_{\langle \ell_2, \hat{\boldsymbol{e}}_{n-1} \rangle} \circ \mathrm{Ref}_{\langle \ell_1, \hat{\boldsymbol{e}}_{n-1} \rangle})(\boldsymbol{a}) = \mathrm{Rot}^{2\angle \ell_1 \ell_2}_{\langle \ell_1, \ell_2 \rangle}(\boldsymbol{a}),$$

which eventually gives us

$$f\left(\mathrm{Rot}^{2\angle \ell_1 \ell_2}_{\langle \ell_1, \ell_2 \rangle}(\boldsymbol{a})\right) = f(\boldsymbol{a})$$

for every $\ell_1 \in W_1$, $\ell_2 \in W_2$, $\boldsymbol{a} \in W_3$. It means that the graph of f (i.e. the surface M^{n-1}) is locally invariant on W_3 under action of $\mathrm{O}(n-2, \mathbb{R})$ with common fixed-point subspace $\langle \hat{\boldsymbol{e}}_{n-1} \rangle^{\perp}$. Indeed, if we fix $\ell_1 = \ell$ and let ℓ_2 vary over W_2, we can rotate $\boldsymbol{a}$ in any direction by any sufficiently small angle. In particular, the series expansion of f at p, as long as it is defined, is invariant under the aforementioned action of $\mathrm{O}(n-2, \mathbb{R})$, which reads

$$q_f(\boldsymbol{x}) = a\langle \boldsymbol{x}, \boldsymbol{x} \rangle + b{x_{n-1}}^2, \quad a, b \in \mathbb{R} \tag{I.3.10}$$

and thus

$$c_f(\boldsymbol{x}) = x_{n-1}(a\langle \boldsymbol{x}, \boldsymbol{x} \rangle + b{x_{n-1}}^2), \quad a, b \in \mathbb{R}.$$

Remark. Our considerations so far show that at every point $p \in M^{n-1}$ with positive definite second fundamental form, the series expansion of M^{n-1}, as long as it is defined, admits a symmetry group $\mathrm{O}(n-2, \mathbb{R})$. Under the additional assumption that M^{n-1} is locally strongly convex, such hypersurfaces have already been classified for $n = 4$ (e.g. in [**9**]). But since they may take a complicated form of warped products, even such a result gives no straightforward solution to our problem, not to mention higher dimensions, where to the author's best knowledge such a classification is still an open problem.

I.4. Proof of the main theorem

With this result at hand, we are ready to prove our main theorem:

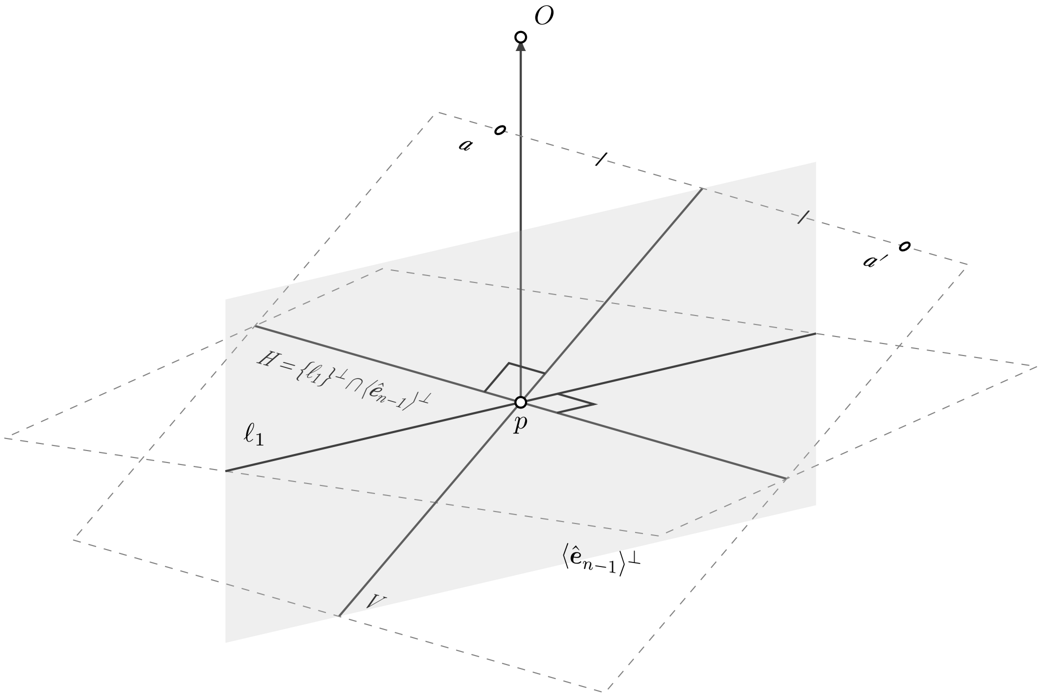

FIGURE I.3.11. The construction of $\boldsymbol{a}'$

PROOF OF THEOREM I.1.2. Denote by $V \subseteq \mathrm{Gr}(n-2, T_pM^{n-1})$ the set of hyperplanes H such that $M^{n-1} \cap \mathrm{aff}(\{H, O\})$ admits an axis of affine revolution ℓ perpendicular to $H \cap \langle \hat{\mathbf{e}}_{n-1} \rangle^{\perp}$. Clearly V is closed (cf. [**3**, Lemma 2.7]) and has non-empty interior (Claim I.3.9).

LEMMA I.4.1. *In the above setting, we have either* $V = \mathrm{Gr}(n-2, T_pM^{n-1})$ *or* $q_f \equiv 0$.

Again, we have to consider separately the special case $n = 4$ and the generic case $n \geq 5$.

PROOF OF LEMMA I.4.1 FOR $n \geq 5$. The projective quadric $Q_f^{n-2} := \{\boldsymbol{x} \in T_pM^{n-1} : q_f(\boldsymbol{x}) = 0\}$ does not contain any linear subspace of dimension $n-3$ unless it is reducible [**7**, Theorem 22.13]. From (I.3.10) it can be readily seen that the latter implies $a = 0$, which reads $c_f(\boldsymbol{x}) = b{x_{n-1}}^3$. Now, if $b = 0$ then $q_f \equiv 0$, and we are done. Otherwise $c_f|_H$ vanishes precisely on $H \cap \langle \hat{\mathbf{e}}_{n-1} \rangle^{\perp}$ for every $H \in \mathrm{Gr}(n-2, T_pM^{n-1})$ and thus $V = \mathrm{Gr}(n-2, T_pM^{n-1})$. ☐

PROOF OF LEMMA I.4.1 FOR $n = 4$. Suppose that $V \neq \mathrm{Gr}(2, T_pM^3)$ and let $\pi \in \partial V$. Then there exists a convergent sequence of planes $\pi_k \to \pi$ such that $M^3 \cap \mathrm{aff}(\{\pi_k, O\})$ admits an axis of affine revolution ℓ_k perpendicular to some line in $\pi_k \cap Q_f^2$. After passing to a subsequence, without loss of generality we may assume that the sequence ℓ_k is convergent to some ℓ_* perpendicular to some line in $\pi \cap Q_f^2$. From (I.3.10) it can be readily seen that the latter is different from $\pi \cap \langle \hat{\mathbf{e}}_3 \rangle^{\perp}$. Moreover, a simple geometric continuity argument shows that ℓ_* is the axis of affine revolution of $M^3 \cap \mathrm{aff}(\{\pi, O\})$ (cf. [**3**, Lemma 2.7]). Now, if $\ell_* \neq \ell$, then $M^3 \cap \mathrm{aff}(\{\pi, O\})$ admits two different axes of affine revolution and hence is an ellipsoid (Lemma I.3.5). In particular, $c_f|_\pi \equiv 0$. On the other hand, if $\ell_* = \ell$, then ℓ is perpendicular to two different lines in the plane π, so it is perpendicular to the plane π itself. Again, it implies $c_f|_\pi \equiv 0$ (Claim I.3.4). Hence $c_f|_\pi \equiv 0$ for every $\pi \in \partial V$. However, $c_f|_\pi$ can vanish for at most 3 different planes π unless $q_f \equiv 0$ and the assertion follows. ☐

DEFINITION I.4.2 ([**17**, II.3]). Let $f : M \to \mathbb{R}^{m+1}$ be a non-degenerate hypersurface immersion. It is well known that there exists a canonical choice of a transversal vector field ξ called the *affine normal* *field* or *Blaschke normal field* [**17**, Definition II.3.1]. The affine normal vector field ξ gives rise to the induced connection ∇, the affine fundamental form h, which is traditionally called the *affine metric*, and the affine shape operator S determined by the formulas

$$D_X Y = \nabla_X Y + h(X,Y)\xi,$$
$$D_X \xi = -SX.$$

We shall call (∇, h, S) the *Blaschke structure* on the hypersurface M [**17**, Definition II.3.2]. From Codazzi equation for h we see that the cubic form

$$C(X,Y,Z) := (\nabla_X h)(Y,Z) \tag{I.4.3}$$

is symmetric in X, Y and Z [**17**, II.4].

Claim I.4.4. It turns out that the condition $c_f \equiv 0$ implies that the cubic form C also vanishes at p. It is by no means obvious, as (I.4.3) can hardly be expressed in the extrinsic coordinate system (cf. [**10**, 1.4.3]). However, we can readily see that C depends only on $J^3_p f$. Indeed, the affine normal field ξ depends only on $J^3_p f$ (cf. [**17**, Example II.3.3]) and the affine metric h depend only on $J^2_p f$ (cf. [**17**, Example II.3.3, Proposition II.2.5]). Hence the covariant derivative

$$(\nabla_X h)(Y,Z) := X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

depends only on $J^3_p f$. In particular, if another function $g : T_p N^{n-1} \supset U \to \mathbb{R}$ satisfies $J^3_p f = J^3_p g$, then the cubic form of M and the cubic form of N coincide at p. Now, since $c_f \equiv 0$, the canonical parametrization of M^{n-1} osculates up to the terms of 3rd order the parametrization of the unit sphere

$$g(\boldsymbol{x}) = 1 - \sqrt{1 - \langle \boldsymbol{x}, \boldsymbol{x} \rangle},$$

for which the cubic form C vanishes identically (cf. [**17**, Corollary II.4.2]). This concludes the argument. $\square$

The following lemma may be considered a counterpart of Lemma I.3.5:

Lemma I.4.5. *A body of affine 2-revolution $K \subset \mathbb{R}^m$, $m \geq 4$, admitting three different codimension 2 hyperplanes of affine revolution, admits a codimension 1 hyperplane of affine revolution (i.e. is a body of affine 1-revolution).*

Proof. Let G be the affine symmetry group of K. Since by [**3**, Lemma 2.2] G is affinely conjugated to a subgroup of $\mathrm{O}(m,\mathbb{R})$, without loss of generality we may assume that $G \subseteq \mathrm{O}(m,\mathbb{R})$. In particular, each codimension 2 hyperplane of affine revolution H_i of K gives rise to a subgroup $G_i \subset G$ isomorphic to $\mathrm{O}(m-2,\mathbb{R})$.

It turns out that the proof of Lemma I.4.5 reduces to a quite simple but tedious linear algebra problem. The key idea is the following: if the hyperplanes H_i were pairwise transversal, then the orbit of a generic point under action of G would be of dimension $m-1$, which means that ∂K would be a sphere. Otherwise i.a. G_2, G_3 share a common representation space $H_2 + H_3$ of dimension $m-1$ and a common fixed point subspace $H_2^\perp \cap H_3^\perp$ of dimension 1, in which case the subgroup $\langle G_2, G_3 \rangle \subseteq G$ generated by G_2, G_3 is by Lemma I.3.5 isomorphic to $\mathrm{O}(m-1,\mathbb{R})$.

Firstly we will show that $\dim H_2 + H_3 = m-1$, unless ∂K is a sphere. Let $p \in \partial K$ be any point on the boundary of K. Since ∂K is invariant under G, we have $T_p(Gp) \subseteq T_p(\partial K)$, where Gp is the orbit of p under action of G. Now, if $\dim T_p(Gp) = m-1 = \dim T_p(\partial K)$ for some $p \in \partial K$, then ∂K is a sphere and we are done. Hence we may assume that for every $p \in \partial K$ we have $\dim T_p(Gp) \leq m-2$. Observe that $T_p(G_i p)$ is a codimension 3 hyperplane parallel to $H_i \cap \langle p\rangle^\perp$, unless $H_i \subset \langle p\rangle^\perp$. Moreover $T_p(G_1p) + T_p(G_2p) + T_p(G_3p) \subseteq T_p(Gp)$. It follows that for every $p \in \partial K$ we have

$$\dim H_1 \cap \langle p\rangle^\perp + H_2 \cap \langle p\rangle^\perp + H_3 \cap \langle p\rangle^\perp \leq \dim T_p(Gp) \leq m-2. \tag{I.4.6}$$

Let L be an arbitrary codimension 3 hyperplane contained in H_1 but not in H_2, H_3. Then $L^\perp$ is a subspace of dimension 3 and $H_2^\perp \cap L^\perp, H_3^\perp \cap L^\perp$ are its subspaces of dimension at most 1. Hence there exists a plane π contained in $L^\perp$ and transversal to $H_2^\perp, H_3^\perp$. Let $p, q \in \partial K$ be its basis. Observe that $H_1 \cap \langle p\rangle^\perp = L = H_1 \cap \langle q\rangle^\perp$, whereas $H_i \cap \langle p\rangle^\perp \neq H_i \cap \langle q\rangle^\perp$, $i = 2, 3$. Indeed, otherwise

$$3 = \operatorname{codim} H_i \cap \langle p\rangle^\perp = \operatorname{codim} H_i \cap \langle p\rangle^\perp \cap \langle q\rangle^\perp = \operatorname{codim} H_i \cap \pi^\perp = \dim H_i^\perp + \pi = 4,$$

a contradiction. Denote

$$H_p := L + H_2 \cap \langle p\rangle^\perp + H_3 \cap \langle p\rangle^\perp, \quad H_q := L + H_2 \cap \langle q\rangle^\perp + H_3 \cap \langle q\rangle^\perp.$$

Clearly, $H_i \cap \langle p\rangle^\perp \subsetneq H_i \cap \langle p\rangle^\perp + H_i \cap \langle q\rangle^\perp \subseteq H_i$, $i = 2, 3$, and since the dimension of the left-hand side and the right-hand side differs by one, the last inclusion must be in fact an equality. Thus

$$H_2 + H_3 = H_2 \cap \langle p\rangle^\perp + H_2 \cap \langle q\rangle^\perp + H_3 \cap \langle p\rangle^\perp + H_3 \cap \langle q\rangle^\perp \subseteq H_p + H_q.$$

Now, by (I.4.6) we have $\dim H_p, \dim H_q \leq m-2$. Moreover, $\dim H_p \cap H_q \geq \dim L = m-3$, which implies

$$\dim H_p + H_q = \dim H_p + \dim H_q - \dim H_p \cap H_q \leq (m-2) + (m-2) - (m-3) = m-1.$$

Comparing the dimensions of the left-hand side and the right-hand side of $H_2 \subsetneq H_2 + H_3 \subseteq H_p + H_q$ yields $\dim H_2 + H_3 = m-1$ and hence also $\dim H_2^\perp \cap H_3^\perp = \dim(H_2+H_3)^\perp = 1$.

Finally, observe that $\mathbb{R}^m$ can be viewed as a direct sum $(H_2 + H_3) \oplus (H_2^\perp \cap H_3^\perp)$ of representation spaces of a subgroup $\langle G_2, G_3\rangle \subseteq G$ generated by G_2, G_3. Indeed,

$$H_2 + H_3 = \bigcup_{v \in H_3} H_2 + v = \bigcup_{v \in H_3} H_2 + \operatorname{proj}_{H_2^\perp}(v)$$

is clearly invariant under G_2 and a similar argument shows that it is also invariant under G_3. Moreover, both G_2 and G_3 act trivially on $H_2^\perp \cap H_3^\perp$. Now, we have

$$\mathrm{SO}(m-2,\mathbb{R}) \simeq (G_2)^0|_{H_2+H_3} \subsetneq \langle G_2, G_3\rangle^0|_{H_2+H_3} \subseteq \mathrm{SO}(m-1,\mathbb{R}),$$

and since $\mathrm{SO}(m-2,\mathbb{R})$ is a maximal connected subgroup of $\mathrm{SO}(m-1,\mathbb{R})$ [**16**, Lemma 4], it follows that $\langle G_2, G_3\rangle^0|_{H_2+H_3} \simeq \mathrm{SO}(m-1,\mathbb{R})$. Therefore $\langle G_2, G_3\rangle \simeq \mathrm{O}(m-1,\mathbb{R})$, which concludes the proof. ☐

Remark. Note that Lemma I.3.5 (without the superfluous symmetry assumption) reads: if the affine symmetry group of a compact domain $K \subset \mathbb{R}^m$, $m \geq 4$, contains two different subgroups affinely conjugated to $\mathrm{O}(m-1,\mathbb{R})$, it contains a subgroup affinely conjugated to $\mathrm{O}(m,\mathbb{R})$. Further, Lemma I.4.5 reads: if the affine symmetry group of a compact domain $K \subset \mathbb{R}^m$, $m \geq 4$, contains three different subgroups affinely conjugated to $\mathrm{O}(m-2,\mathbb{R}^m)$, it contains a subgroup affinely conjugated to $\mathrm{O}(m-1,\mathbb{R})$. Let us state then a more general question:

Question I.4.7. *Does a compact domain $K \subset \mathbb{R}^m$, $m \geq 4$, admitting $k+1$ different codimension k hyperplanes of affine revolution, admit a codimension $k-1$ hyperplane of affine revolution, $0 < k < m$?*

To the author's best knowledge, the answer is an open problem.

Recall Lemma I.4.1 which says that either $V = \mathrm{Gr}(n-2, T_pM^{n-1})$ or $q_f \equiv 0$. In the first case (i.e. $V = \mathrm{Gr}(n-2, T_pM^{n-1})$), we can repeat the geometric argument from §I.3 to show that actually the whole hypersurface M^{n-1} is invariant under action of $\mathrm{O}(n-2,\mathbb{R})$ with common fixed-point space $\mathrm{aff}(\{\langle \hat{\mathbf{e}}_{n-1}\rangle, O\})$. Hence by Lemma I.4.5 all such points $p \in M^{n-1}$ lie on at most two different planes π_1, π_2, unless M^{n-1} is a body of affine revolution.

Finally, we pass to the second case (i.e. $q_f \equiv 0$). Let $p \in M^{n-1}$ be the point attaining the maximal Euclidean distance from the origin. It means that M^{n-1} is contained in some sphere tangent to M^{n-1} at p. In particular, the second fundamental form of M^{n-1} at p majorizes the second fundamental form of the sphere, and thus M^{n-1} is strongly convex on some open neighborhood of p. Let $U \subseteq M^{n-1}$ be a maximal open neighborhood of p where the second fundamental form of M^{n-1} is positive definite. We already know from Claim I.4.4 that the cubic form of M^{n-1} vanishes identically on $U \setminus (\pi_1 \cup \pi_2)$.

Lemma I.4.8 (Maschke, Pick, Berwald [**17**, Theorem II.4.5]). *Let $f : M \to \mathbb{R}^{m+1}$, $m \geq 2$, be a non-degenerate hypersurface with Blaschke structure. If the cubic form* (I.4.3) *vanishes identically, then $f(M)$ is hyperquadric in $\mathbb{R}^{m+1}$.*

It follows from Lemma I.4.8 that U is contained in some hyperquadric Q^{n-1}. Now, suppose that ∂U is non-empty and let $p \in \partial U$. Since Q^{n-1} is locally strongly convex, the second fundamental form of Q^{n-1} at p is positive definite. However, the second fundamental form of M^{n-1} at p is equal to the latter and thus it is also positive definite on some open neighborhood of p, which contradicts the definition of U. It follows that $U = M^{n-1}$, which concludes the proof. □

Remark. In our proof, we used the additional assumption that K is origin-symmetric only to know that all the axes of affine revolution pass through some fixed point, which implies some nice geometric structure of M^{n-1}, determined by its series expansion of order 3. This significantly simplified our argument, which after all required no algebraic computations. Nevertheless, there are e.g. certain partial differential equations of order 5, satisfied whenever g is a local parametrization of a surface of affine revolution. When applied to $f|_\pi$ for every plane $\pi \in \mathrm{Gr}(2, T_pM^3)$, they would yield a system of polynomial equations in partial derivatives of f. However, it is beyond the scope of human to obtain, not to mention to solve. Therefore any approach along those lines would badly need the assistance of a supercomputer.

Acknowledgments

I would like to thank Prof. D. Ryabogin for giving me this problem to consider and for many inspiring discussions, my doctoral advisor Prof. M. Wojciechowski for his help in completing all the details and the exceptional amount of effort spent on editorial work, Prof. A. Weber for helping me understand the algebraic topology behind the proof, Prof. J. Wiśniewski for bringing the theorem of Bertini to my attention and Prof. T. Mostowski for a brief introduction to affine differential geometry.

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CHAPTER II

Differential characterization of quadratic surfaces

Let $f \in W^{3,1}_{\mathrm{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. We will show that its graph is contained in a quadratic surface if and only if f is a weak solution to a certain system of 3^{rd} order partial differential equations unless the Hessian determinant of f is non-positive on the whole Ω. Moreover, we will prove that the system is in some sense the simplest possible in a wide class of differential equations, which will lead to the classification of all polynomial partial differential equations satisfied by parametrizations of generic quadratic surfaces. Although we will mainly use the tools of linear and commutative algebra, the theorem itself is also somehow related to holomorphic functions.

II.1. Introduction

It was already known since the works of Blaschke [**1**, p. 18] that conics are the only planar curves with constant equiaffine curvature. In a special case of a graph of a function of class C^5, this condition is equivalent to a certain 5^{th} order ordinary differential equation, which reads

$$9f''(x)^2 f^{(5)}(x) - 45 f''(x) f^{(3)}(x) f^{(4)}(x) + 40 f^{(3)}(x)^3 = 0. \tag{II.1.1}$$

In higher dimensions, hyperquadrics are characterized by Maschke-Pick-Berwald theorem [**8**, Theorem 4.5] as the only hypersurfaces with vanishing cubic form C defined in [**8**, ch. II, s. 4]. However, the definition implicitely uses the intrinsic Blaschke structure and thus the cubic form C can hardly be expressed in an extrinsic coordinate system. It is also unclear what minimal smoothness we need to assume. Nevertheless, such a result for 2-dimensional surfaces turns out to be a consequence of two relatively simple partial differential equations. The aim of this paper is to prove the following main theorem:

Theorem II.1.2. *Let $f \in W^{3,1}_{\mathrm{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. Suppose that the Hessian determinant of f is not non-positive. Then f is a weak solution to the system of partial differential equations*

$$\begin{aligned} f^{(3,0)} {f^{(0,2)}}^2 - 3 f^{(1,2)} f^{(2,0)} f^{(0,2)} + 2 f^{(0,3)} f^{(1,1)} f^{(2,0)} &= 0, \\ f^{(0,3)} {f^{(2,0)}}^2 - 3 f^{(2,1)} f^{(0,2)} f^{(2,0)} + 2 f^{(3,0)} f^{(1,1)} f^{(0,2)} &= 0 \end{aligned} \tag{II.1.3}$$

if and only if its graph is contained in a quadratic surface.

Therefore Theorem II.1.2 can be considered a 2-dimensional analog of the aforementioned result of Blaschke. Contrary to Maschke-Pick-Berwald, it is formulated in terms of simple, explicit partial differential equations, with weaker smoothness assumption. Moreover, we will show that the system (II.1.3) is minimal in the sense that the left-hand sides form a minimal generating set (viz. a reduced Gröbner basis) of a certain differential ideal.

Such a characterization of quadratic surfaces of positive Gaussian curvature as the only solutions to some partial differential equations without boundary condition may be useful when one wants to prove that some specific convex body is an ellipsoid using e.g. the tools of differential geometry. Such problems arise naturally in convex geometry, especially in various characterizations of Hilbert spaces among all finite-dimensional Banach spaces.

As superfluous as it may seem, the assumption on the Hessian determinant is not purely technical, as the following holds:

Theorem II.1.4. *Let* $f \in W^{3,1}_{\text{loc}}(\Omega)$ *be a function defined on a connected open subset* $\Omega \subseteq \mathbb{R}^2$*. Suppose that the Hessian determinant of* f *is non-positive. Then* f *is a weak solution to the system of partial differential equations* (II.1.3) *if and only if* Ω *contains a countable sum of disjoint open connected subsets* Ω_i *such that:*

(1) *on each* Ω_i *the graph of* f *is contained in either:*
   (a) *a doubly-ruled surface*[1]*, or*
   (b) *a developable surface*[2]*, or*
   (c) *a Catalan surface*[3] *with directrix plane* XZ*, or*
   (d) *a Catalan surface with directrix plane* YZ*,*
(2) *the union* $\bigcup \Omega_i$ *is dense in* Ω*.*

Note that all of the above are particular examples of ruled surfaces[4]. Regrettably, the exact classification of solutions to (II.1.3) seems to be a tedious, technical task and therefore will not be given here, so as not to overshadow the main idea.

To perform lengthy computations, we will employ a widely used technical computing system Wolfram Mathematica [**7**]. Nevertheless, they still could have been done with pen and paper (albeit with some difficulty) and hence the proof remains human-surveyable. A thorough discussion of this aspect can be found in Appendix A. For transparency, all the results obtained with the help of a computer are marked with "Spikey" in the margin:

II.2. Notation and basic concepts

To prove the Theorem II.1.2 we will need some very general facts concerning quadratic surfaces, that are in themselves quite interesting. We begin with rephrasing the problem in the language of commutative algebra.

Definition II.2.1. Let

$$R := \mathbb{R}\left[x, y, \partial^{(0,0)}, \partial^{(0,1)}, \partial^{(1,0)}, \dots\right]$$

be a ring of polynomials in variables x, y and formal partial derivatives $\partial^{(i,j)}$ and let

$$S := \left\langle \partial^{(0,2)}, \partial^{(2,0)}, \partial^{(0,2)}\partial^{(2,0)} - {\partial^{(1,1)}}^2 \right\rangle$$

be a submonoid of the multiplicative monoid of R, with the listed generators. By $S^{-1}R$ we denote a localisation of R at S [**3**, §2.1].

[1]A ruled surface that contains two families of rulings.
[2]A ruled surface having Gaussian curvature $K = 0$ everywhere.
[3]A ruled surface all of whose rulings are parallel to a fixed plane, called the directrix plane of the surface.
[4]A surface that can be swept out by moving a line in space. The straight lines themselves are called rulings. The Gaussian curvature on a ruled regular surface is everywhere non-positive.

The ring $S^{-1}R$ can be viewed as an algebra of a certain type of differential operators T defined for those smooth functions $f : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}$ for which all the expressions

$$f^{(0,2)}(x,y), \quad f^{(2,0)}(x,y), \quad f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2 \tag{II.2.2}$$

do not take a zero value on Ω and thus have reciprocals. We will call such functions *generic*. Examples include but are not limited to functions with positive Hessian determinant, i.e. whose graphs have positive Gaussian curvature.

Notation. Let $\Omega \subseteq \mathbb{R}^2$ be a connected open subset of $\mathbb{R}^2$. Denote by $\mathcal{G}(\Omega)$ the set of generic functions $f : \Omega \to \mathbb{R}$ and by $\mathcal{Q}(\Omega)$ its subset consisting of parametrizations of quadratic surfaces.

Definition II.2.3. Let $D_x, D_y : S^{-1}R \to S^{-1}R$ be *derivations* [**3**, ch. 16], i.e. $\mathbb{R}$-linear endomorphisms of additive group of $S^{-1}R$ satisfying the Leibniz product rule

$$D(r_1 r_2) = D(r_1)r_2 + r_1 D(r_2), \quad r_1, r_2 \in S^{-1}R,$$

and thus uniquely determined by their values on indeterminates:

$$\begin{aligned} D_x(x) &:= 1, \quad D_x(y) := 0, \quad D_x\left(\partial^{(i,j)}\right) := \partial^{(i+1,j)}, \\ D_y(x) &:= 0, \quad D_y(y) := 1, \quad D_y\left(\partial^{(i,j)}\right) := \partial^{(i,j+1)}. \end{aligned}$$

In particular, the well-known formula for differentiating fractions

$$D\left(\frac{r}{s}\right) = \frac{D(r)s - rD(s)}{s^2}$$

follows from the Leibniz product rule. A ring $S^{-1}R$ equipped with derivations D_x, D_y forms a *differential ring*.

Definition II.2.4. A *differential ideal* $\mathfrak{a}$ in a differential ring R is an ideal that is mapped to itself by each derivation.

Definition II.2.5. Let X be a subset of $\mathcal{G}(\Omega)$. The *annihilator* of X in $S^{-1}R$, denoted by $X^\dagger$, is a collection of differential operators $T \in S^{-1}R$ such that $Tf = 0$ for all $f \in X$. The annihilator of any subset is clearly a differential ideal. The annihilator of an empty set is the whole $S^{-1}R$ and the annihilator of the whole $\mathcal{G}(\Omega)$ is just the zero operator.

II.3. Polynomial PDEs satisfied by generic quadratic surfaces

Observe that a graph of a function f is contained in a quadratic surface if and only if its every point satisfies a quadratic equation

$$a_{11}x^2 + a_{12}xy + a_{13}xf + a_{22}y^2 + a_{23}yf + a_{33}f^2 + b_1x + b_2y + b_3f + c = 0 \tag{II.3.1}$$

with constant coefficients a_{ij}, b_k, c. This is equivalent to the fact that the set of functions

$$\left\{x^2, \quad xy, \quad xf(x,y), \quad y^2, \quad yf(x,y), \quad f(x,y)^2, \quad x, \quad y, \quad f(x,y), \quad 1\right\} \tag{II.3.2}$$

is linearly dependent. That is how the concept of generalized Wronskian for functions of several variables enters play. For clarity, we adopt the notation from [**10**].

Definition II.3.3 ([**10**, Definition 1]). A *generalised Wronskian* of $\boldsymbol{\phi} = (\phi_1(t), \ldots, \phi_n(t))$, where $\boldsymbol{t} = (t_1, \ldots, t_m)$, is any determinant of the type

$$\begin{vmatrix} \boldsymbol{\phi} \\ \partial^1 \boldsymbol{\phi} \\ \vdots \\ \partial^{n-1} \boldsymbol{\phi} \end{vmatrix},$$

where $\boldsymbol{\phi}$, $\partial^i \boldsymbol{\phi}$ are row vectors, ∂^i is any partial derivative of order not greater that i and all ∂^i are distinct.

Remark. Note that in the realm of functions in $m \geq 2$ variables a generalized Wronskian of φ is no longer unique, since there are many possible ways of choosing row vectors $\partial^i \boldsymbol{\phi}$ satisfying all the imposed conditions. More precisely, there are

$$\binom{m+i}{m}$$

partial derivatives of order not greater than i and hence there are exactly

$$\prod_{i=0}^{n-1} \left(\binom{m+i}{m} - i \right)$$

generalised Wronskians of n functions in m variables. However, from now henceforth we will identify all generalized Wronskians that differ only in the order of rows as it does not affect the rank of the matrix.

Notation. Denote by $\boldsymbol{\phi}$ the tuple of functions (II.3.2).

Assertion II.3.4. *Each generalized Wronskian of $\boldsymbol{\phi}$ can be viewed as an element of $S^{-1}R$. Moreover, by the very definition, it belongs to $\mathcal{Q}(\Omega)^\dagger$. Indeed, if the set of functions* (II.3.2) *is linearly dependent, then all its generalized Wronskians vanish identically since their columns are themselves linearly dependent.*

The following key proposition characterizes the set of polynomial differential equations satisfied by the parametrization of any generic quadratic surface.

Proposition II.3.5. *Let $\Omega \subseteq \mathbb{R}^2$ be a connected open subset of $\mathbb{R}^2$. Then the annihilator $\mathcal{Q}(\Omega)^\dagger \subseteq S^{-1}R$ is a differential ideal generated by*

$$\begin{aligned} &\partial^{(3,0)} {\partial^{(0,2)}}^2 - 3\partial^{(1,2)}\partial^{(2,0)}\partial^{(0,2)} + 2\partial^{(0,3)}\partial^{(1,1)}\partial^{(2,0)}, \\ &\partial^{(0,3)} {\partial^{(2,0)}}^2 - 3\partial^{(2,1)}\partial^{(0,2)}\partial^{(2,0)} + 2\partial^{(3,0)}\partial^{(1,1)}\partial^{(0,2)}. \end{aligned} \tag{II.3.6}$$

Proof. Clearly $\mathcal{Q}(\Omega)^\dagger$ is a differential ideal in $S^{-1}R$. Denote by $\mathfrak{a}$ the differential ideal generated by (II.3.6). We have to show that $\mathcal{Q}(\Omega)^\dagger = \mathfrak{a}$. We will do this by proving both inclusions.

First, we will show a simpler inclusion $\mathcal{Q}(\Omega)^\dagger \supseteq \mathfrak{a}$. Since both $\mathcal{Q}(\Omega)^\dagger$ and $\mathfrak{a}$ are differential ideals, it is enough to prove that the generators of $\mathfrak{a}$ are contained in $\mathcal{Q}(\Omega)^\dagger$. Let $f \in \mathcal{Q}(\Omega)$ be a parametrization of some generic quadratic surface. By Assertion II.3.4, all the generalized Wronskians of $\boldsymbol{\phi}$ vanish identically on Ω. Denote by $W_{i,j}$ the generalised Wronskian of $\boldsymbol{\phi}$ formed by deleting the row $\boldsymbol{\phi}^{(i,j)}$ from

$$\begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^{(0,1)} \\ \boldsymbol{\phi}^{(1,0)} \\ \boldsymbol{\phi}^{(0,2)} \\ \boldsymbol{\phi}^{(1,1)} \\ \boldsymbol{\phi}^{(2,0)} \\ \boldsymbol{\phi}^{(0,3)} \\ \boldsymbol{\phi}^{(1,2)} \\ \boldsymbol{\phi}^{(2,1)} \\ \boldsymbol{\phi}^{(3,0)} \\ \boldsymbol{\phi}^{(0,4)} \end{pmatrix}.$$

The only non-trivial (i.e. not vanishing algebraically) ones are the following:

$$\begin{aligned}
W_{3,0} &= 24{f^{(0,2)}}^2\Big(3f^{(2,1)}{f^{(0,2)}}^2 - 6f^{(1,1)}f^{(1,2)}f^{(0,2)} - f^{(0,3)}f^{(2,0)}f^{(0,2)} + 4f^{(0,3)}{f^{(1,1)}}^2\Big)\\
W_{2,1} &= 72{f^{(0,2)}}^2\Big(f^{(3,0)}{f^{(0,2)}}^2 - 3f^{(1,2)}f^{(2,0)}f^{(0,2)} + 2f^{(0,3)}f^{(1,1)}f^{(2,0)}\Big)\\
W_{1,2} &= 72{f^{(0,2)}}^2\Big(f^{(0,3)}{f^{(2,0)}}^2 - 3f^{(0,2)}f^{(2,1)}f^{(2,0)} + 2f^{(0,2)}f^{(1,1)}f^{(3,0)}\Big)\\
W_{0,3} &= 24{f^{(0,2)}}^2\Big(4f^{(3,0)}{f^{(1,1)}}^2 - 6f^{(2,0)}f^{(2,1)}f^{(1,1)} + 3f^{(1,2)}{f^{(2,0)}}^2 - f^{(0,2)}f^{(2,0)}f^{(3,0)}\Big)
\end{aligned}$$

Note that although the underlying matrices depend on 4^{th} order partial derivatives, their determinants do not, which is somehow intriguing.

Remark. Since $\boldsymbol{\phi}$ consists of $n = 10$ functions and in $m = 2$ variables there are exactly 10 partial derivatives of at most 3^{rd} order, there is a unique generalized Wronskian of $\boldsymbol{\phi}$ using partial derivatives of at most 3^{rd} order, namely $W_{0,4}$. However, it turns out that $\boldsymbol{\phi}^{(3,0)}, \boldsymbol{\phi}^{(2,1)}, \boldsymbol{\phi}^{(1,2)}, \boldsymbol{\phi}^{(0,3)}$ are always linearly dependent. Indeed, observe that the 4×10 matrix

$$\begin{pmatrix} \boldsymbol{\phi}^{(3,0)} \\ \boldsymbol{\phi}^{(2,1)} \\ \boldsymbol{\phi}^{(1,2)} \\ \boldsymbol{\phi}^{(0,3)} \end{pmatrix}$$

has only 4 non-zero columns corresponding to $xf(x,y)$, $yf(x,y)$, $f(x,y)^2$, $f(x,y)$ and a direct computation shows that the determinant of this only non-trivial 4×4 minor is zero anyway. Thus every generalized Wronskian of $\boldsymbol{\phi}$ vanishes identically unless it is missing some 3^{rd} order partial derivative. In particular, there is no non-trivial generalized Wronskian of $\boldsymbol{\phi}$ using partial derivatives of order at most 3. Moreover, there are (*a priori* at most) only 4 non-trivial generalized Wronskians of $\boldsymbol{\phi}$ using a single partial derivative order greater than 3, since it must replace one of the 4 partial derivatives of order 3.

cf. A.1

Now, since f is assumed to be generic, its 2^{nd} order pure derivative $f^{(0,2)}$ is non-zero. Hence from the vanishing of $W_{2,1}$ and $W_{1,2}$ we obtain that a parametrization of any generic quadratic surface satisfies (II.1.3). This concludes the first part of the proof.

Remark. Note that for any generic function f, if $W_{2,1}$ and $W_{1,2}$ vanish, then the remaining two generalized Wronskians also vanish. Indeed, we have

$$\begin{aligned}
3f^{(2,0)}W_{3,0} &= 2f^{(1,1)}W_{2,1} - f^{(0,2)}W_{1,2},\\
3f^{(0,2)}W_{0,3} &= 2f^{(1,1)}W_{1,2} - f^{(2,0)}W_{2,1},
\end{aligned} \tag{II.3.7}$$

while both $f^{(2,0)}$ and $f^{(0,2)}$ are non-zero. Furthermore, the same holds for any pair of featured generalized Wronskians except for $W_{3,0}$ and $W_{0,3}$, when the above equations (II.3.7) in variables $W_{2,1}$ and $W_{1,2}$ may turn out to be linearly dependent. This is the case exactly when

$$f^{(0,2)}f^{(2,0)} - 4{f^{(1,1)}}^2 = 0,$$

which together with $W_{3,0} = 0$ and $W_{0,3} = 0$ forms a system of partial differential equations. This time, however, apart from parametrizations of certain quadratic surfaces (including degenerate), it admits a single family of exotic solutions of the form

$$f(x,y) = \frac{a}{(x+x_0)(y+y_0)} + b_1x + b_2y + c,$$

which arise as parametrizations of certain cubic surfaces. Moreover, note that all these functions are generic, unless $a = 0$. Therefore the choice of equations (II.1.3) was arbitrary only to some extent.

Remark. Observe that the last factors of $W_{3,0}$ and $W_{0,3}$ as well as $W_{2,1}$ and $W_{1,2}$ are equivalent up to the order of variables. However, the overall symmetry is broken by the common factor ${f^{(0,2)}}^2$, which is the result of choosing $\boldsymbol{\phi}^{(0,4)}$ as a supplementary row. Exactly as we should expect, if we had chosen $\boldsymbol{\phi}^{(4,0)}$, we would have obtained the same set of generalized Wronskians, but this time with common factor ${f^{(2,0)}}^2$ instead of ${f^{(0,2)}}^2$.

Let

$$Q := S^{-1}\mathbb{R}\left[x, y, \partial^{(0,0)}, \partial^{(0,1)}, \partial^{(1,0)}, \partial^{(0,2)}, \partial^{(1,1)}, \partial^{(2,0)}, \partial^{(0,3)}, \partial^{(1,2)}, \partial^{(0,4)}\right]$$

be the localization of a real polynomial ring in selected 11 variables at S. Since localization commutes with adding new external elements, there is a ring isomorphism

$$S^{-1}R \simeq Q\left[\partial^{(2,1)}, \partial^{(3,0)}, \partial^{(1,3)}, \dots\right],$$

where the latter is already a polynomial ring over Q in the remaining infinitely many variables. Let us choose a graded lexicographic order on variables $\partial^{(i,j)}$ and then graded reverse lexicographic order on monomials. We will find a Gröbner basis of $\mathfrak{a}$ with respect to this monomial ordering. For more details on Gröbner bases including definitions and examples we recommend the reader to go through [**2**, ch. 2].

Denote polynomials (II.3.6) respectively by p_1, p_2. Observe that for every $i, j \geq 0$ and $k = 1, 2$, ${D_x}^i {D_y}^j p_k$ is linear in highest order partial derivatives and thus we can write e.g.

(II.3.8)
$$\begin{pmatrix} D_xD_xp_1 \\ D_xD_yp_1 \\ D_yD_yp_1 \\ D_xD_xp_2 \\ D_xD_yp_2 \\ D_yD_yp_2 \end{pmatrix} = \boldsymbol{A}_5 \begin{pmatrix} \partial^{(0,5)} \\ \partial^{(1,4)} \\ \partial^{(2,3)} \\ \partial^{(3,2)} \\ \partial^{(4,1)} \\ \partial^{(5,0)} \end{pmatrix} + \boldsymbol{b}_5,$$

where

$$\boldsymbol{A}_5 := \begin{pmatrix} 0 & 0 & 2f^{(1,1)}f^{(2,0)} & -3f^{(0,2)}f^{(2,0)} & 0 & {f^{(0,2)}}^2 \\ 0 & 2f^{(1,1)}f^{(2,0)} & -3f^{(0,2)}f^{(2,0)} & 0 & {f^{(0,2)}}^2 & 0 \\ 2f^{(1,1)}f^{(2,0)} & -3f^{(0,2)}f^{(2,0)} & 0 & {f^{(0,2)}}^2 & 0 & 0 \\ 0 & 0 & {f^{(2,0)}}^2 & 0 & -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} \\ 0 & {f^{(2,0)}}^2 & 0 & -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} & 0 \\ {f^{(2,0)}}^2 & 0 & -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} & 0 & 0 \end{pmatrix}$$

is a 6×6 matrix and $\boldsymbol{b}_5$ (the definition of which is irrelevant and therefore has been omitted for brevity) is a 6×1 vector over $S^{-1}R$. Moreover, $\boldsymbol{A}_5$ and $\boldsymbol{b}_5$ contain only partial derivatives of order at most 4. One can easily verify that the determinant of $\boldsymbol{A}_5$ is equal to

$$-64{\partial^{(0,2)}}^3{\partial^{(2,0)}}^3\left(\partial^{(0,2)}\partial^{(2,0)}-{\partial^{(1,1)}}^2\right)^3$$

cf. A.2

and thus is a unit in $S^{-1}R$. It follows that $\boldsymbol{A}_5$ is invertible over $S^{-1}R$ and we can rewrite (II.3.8) as

$$\text{(II.3.9)} \qquad {\boldsymbol{A}_5}^{-1}\begin{pmatrix} D_xD_xp_1 \\ D_xD_yp_1 \\ D_yD_yp_1 \\ D_xD_xp_2 \\ D_xD_yp_2 \\ D_yD_yp_2 \end{pmatrix} = \begin{pmatrix} \partial^{(0,5)} \\ \partial^{(1,4)} \\ \partial^{(2,3)} \\ \partial^{(3,2)} \\ \partial^{(4,1)} \\ \partial^{(5,0)} \end{pmatrix} + {\boldsymbol{A}_5}^{-1}\boldsymbol{b}_5.$$

Now, the left-hand side is a vector of elements from $\mathfrak{a}$ and hence so is also the right-hand side. Moreover, since ${\boldsymbol{A}_5}^{-1}\boldsymbol{b}_5$ contains only partial derivatives of order at most 4, the leading term of each polynomial on the right-hand side is a corresponding 5^{th} order partial derivative. By definition, the ideal $\mathfrak{a}$ is closed under derivations and thus by differentiating these polynomials we can obtain an element of $\mathfrak{a}$ with the leading term being any partial derivative of higher order. Using right the same argument we can likewise write

$${\boldsymbol{A}_4}^{-1}\begin{pmatrix} D_xp_1 \\ D_yp_1 \\ D_xp_2 \\ D_yp_2 \end{pmatrix} = \begin{pmatrix} \partial^{(1,3)} \\ \partial^{(2,2)} \\ \partial^{(3,1)} \\ \partial^{(4,0)} \end{pmatrix} + {\boldsymbol{A}_4}^{-1}\boldsymbol{b}_4, \quad {\boldsymbol{A}_3}^{-1}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \partial^{(2,1)} \\ \partial^{(3,0)} \end{pmatrix} + {\boldsymbol{A}_3}^{-1}\boldsymbol{b}_3,$$

since the determinant of

$$\boldsymbol{A}_4 := \begin{pmatrix} 2f^{(1,1)}f^{(2,0)} & -3f^{(0,2)}f^{(2,0)} & 0 & {f^{(0,2)}}^2 \\ -3f^{(0,2)}f^{(2,0)} & 0 & {f^{(0,2)}}^2 & 0 \\ {f^{(2,0)}}^2 & 0 & -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} \\ 0 & -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} & 0 \end{pmatrix}$$

is equal to

$$-24{\partial^{(0,2)}}^4{\partial^{(2,0)}}^2\left(\partial^{(0,2)}\partial^{(2,0)}-{\partial^{(1,1)}}^2\right)$$

cf. A.2

and the determinant of

$$\boldsymbol{A}_3 := \begin{pmatrix} 0 & {f^{(0,2)}}^2 \\ -3f^{(0,2)}f^{(2,0)} & 2f^{(0,2)}f^{(1,1)} \end{pmatrix}$$

is equal to

$$3{\partial^{(0,2)}}^3\partial^{(2,0)}.$$

cf. A.2

Hence all the partial derivatives $\partial^{(2,1)}, \partial^{(3,0)}, \partial^{(1,3)}, \ldots$, which are exactly those not included in the definition of Q, are contained in the ideal of leading terms $\langle \mathrm{LT}(\mathfrak{a})\rangle$ [**2**, Definition 2.5.1].

Remark. It is a mere coincidence that after computing 2^{nd} order partial derivatives of p_1 and p_2 the number of independent equations is equal to the number of 5^{th} order partial derivatives of f and thus the matrix $\boldsymbol{A}_5$ is uniquely determined. The multiplicative monoid $S \subseteq R$ was devised to contain all the prime factors of $\det \boldsymbol{A}_5$. However, to obtain $\boldsymbol{A}_4$ and $\boldsymbol{A}_3$ we had to arbitrarily choose some subset of variables, and this time it was not a coincidence that both $\det \boldsymbol{A}_4$ and $\det \boldsymbol{A}_3$ share the same prime factors as $\det \boldsymbol{A}_5$. Indeed, there are other choices for which it is no longer the case. Thus the set of variables to the polynomial ring Q was carefully selected so that both $\boldsymbol{A}_4$ and $\boldsymbol{A}_5$ are already invertible in $S^{-1}R$.

Denote by G the set of polynomials constructed above, such that every monomial $\partial^{(2,1)}, \partial^{(3,0)}, \partial^{(1,3)}, \ldots$ is a leading term $\mathrm{LT}(g)$ of some polynomial $g \in G$. Suppose that $\mathcal{Q}(\Omega)^\dagger \supsetneq \langle G \rangle$ and let $p \in \mathcal{Q}(\Omega)^\dagger \setminus \langle G \rangle$. After a complete reduction of p by G we obtain a remainder $r \in \mathcal{Q}(\Omega)^\dagger \setminus \langle G \rangle$, which is irreducible by G, i.e. its leading term $\mathrm{LT}(r)$ is not a multiple of any $\mathrm{LT}(g)$, $g \in G$ [**2**, Theorem 2.3.3]. Thus r is an element of the coefficient ring Q and corresponds to some rational function in selected 11 variables. We will prove that $r = 0$, which will eventually give us the desired contradiction. By definition, it vanishes for any tuple consisting of x, y, and relevant partial derivatives of some function parametrizing a generic quadratic surface at (x, y). Since r is rational, it is enough to show that the set of such arguments has a non-empty interior as a subset of $\mathbb{R}^{11}$.

For this, we define an implicit function $\delta : \mathbb{R}^{11} \to \mathbb{R}^{11}$ in the following way. Let f be a parametrization of some quadratic surface satisfying (II.3.1) with $a_{33} = 1$. Then δ maps the tuple of parameters

$$\Big(x, \quad y, \quad a_{11}, \quad a_{12}, \quad a_{13}, \quad a_{22}, \quad a_{23}, \quad b_1, \quad b_2, \quad b_3, \quad c\Big) \tag{II.3.10}$$

to the tuple

$$\Big(x, \quad y, \quad f, \quad f^{(0,1)}, \quad f^{(1,0)}, \quad f^{(0,2)}, \quad f^{(1,1)}, \quad f^{(2,0)}, \quad f^{(0,3)}, \quad f^{(1,2)}, \quad f^{(0,4)}\Big)$$

consisting of x, y and relevant partial derivatives of f at (x, y). We can obtain an explicit formula for δ by symbolically solving the quadratic equation (II.3.1) first and then symbolically differentiating the result. Since in general there are two possible solutions for f, we have to locally select an arbitrary branch of the square root function, so that δ is smooth.

Now, consider a generic function

$$f(x, y) = \sqrt{1 + x^2 + y^2}$$

parametrizing a quadratic surface represented by the tuple of parameters

$$\Big(x, \quad y, \quad -1, \quad 0, \quad 0, \quad -1, \quad 0, \quad 0, \quad 0, \quad 0, \quad -1\Big). \tag{II.3.11}$$

Since (II.2.2) depend continuously on (II.3.10), any point in some open neighborhood U of (II.3.11) also corresponds to a parametrization of some generic quadratic surface and thus $r \circ \delta$ vanishes on U. Hence it is enough to show that $\delta(U)$ has a non-empty interior. Computing the Jacobian determinant of δ at (II.3.11) yields

$$\frac{9\left(x^2+1\right)^4}{128\left(x^2+y^2+1\right)^{11}},$$

cf. A.3

which is non-zero. Hence δ is a local diffeomorphism and so there exists an open subset $V \subseteq U$ such that $\delta|_V : V \to \delta(V)$ is a diffeomorphism. In particular, $\delta(V)$ is open. However, recall that it is contained in the zero set of r, which must therefore be a zero function, a contradiction. It follows that $\mathfrak{a} \subseteq \mathcal{Q}(\Omega)^\dagger = \langle G \rangle \subseteq \mathfrak{a}$, which means that $\mathcal{Q}(\Omega)^\dagger = \mathfrak{a}$ and moreover G is, in fact, a reduced Gröbner basis of $\mathfrak{a}$, which concludes the proof. ☐

Remark. Now we are able to clarify in what sense equations (II.1.3) are minimal. Namely, (II.3.6) form a reduced Gröbner basis of $\mathcal{Q}(\Omega)^\dagger$, while, as it will turn out, we would obtain the same results as in Theorem II.1.2 and Theorem II.1.4 for any generating set of $\mathcal{Q}(\Omega)^\dagger$. Although the elements (II.3.6) seem to be the best choice, the reduced Gröbner basis is by no means unique. Besides, with Proposition II.3.5 at hand, finding other generating sets becomes a purely algorithmic task.

II.4. Smoothing properties and their connection with holomorphicity

At some point in the future, we will want to deduce a linear dependence of a set of functions from the vanishing of their generalized Wronskians. For this, we will use the main result from [**10**], where the necessary and sufficient conditions are established. Although the author roughly requires that all the generalized Wronskians must vanish, in the course of the inherently constructive proof he considers only finitely many generalized Wronskians of bounded order. Nevertheless, our initial assumption that the function f is merely an element of $W^{3,1}_{\mathrm{loc}}(\Omega)$ is too weak for any non-trivial generalized Wronskian to be well-defined. Therefore we need to somehow improve the smoothness of f. As it turns out, the differentiability of class C^5 will be sufficient and thus we will not use the following fact in its full generality.

Lemma II.4.1. *Let $f \in W^{3,1}_{\mathrm{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. Suppose that f is generic and is a weak solution to the system of partial differential equations* (II.1.3)*. Then f is infinitely differentiable.*

Proof. Let $u, v : \Omega \to \mathbb{R}$ be the functions defined as follows:

$$\begin{aligned} u(x,y) &:= \frac{f^{(2,0)}(x,y) - f^{(0,2)}(x,y)}{\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{3/4}}, \\ v(x,y) &:= \frac{2f^{(1,1)}(x,y)}{\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{3/4}}. \end{aligned} \tag{II.4.2}$$

Note that they are well-defined, by the assumption that f is generic. Since they depend only on the 2^{nd} order partial derivatives of f, which are assumed to be elements of $W^{1,1}_{\mathrm{loc}}(\Omega)$, and moreover the Hessian determinant of f is locally bounded away from 0, both functions u, v are elements of $W^{1,1}_{\mathrm{loc}}(\Omega)$. Computing their weak partial derivatives and applying (II.1.3) one can find out that they satisfy the Cauchy-Riemann equations:

$$\begin{aligned} u^{(1,0)}(x,y) - v^{(0,1)}(x,y) &= \pm\frac{\left(3f^{(0,2)}(x,y) + f^{(2,0)}(x,y)\right)p_1 - 2f^{(1,1)}(x,y)p_2}{4f^{(0,2)}(x,y)\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{7/4}} = 0, \\ u^{(0,1)}(x,y) + v^{(1,0)}(x,y) &= \pm\frac{2f^{(1,1)}(x,y)p_1 - \left(3f^{(2,0)}(x,y) + f^{(0,2)}(x,y)\right)p_2}{4f^{(2,0)}(x,y)\left|f^{(0,2)}(x,y)f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2\right|^{7/4}} = 0, \end{aligned}$$ cf. A.4

where p_1, p_2 denote respectively the left-hand sides of (II.1.3) and $\pm$ is the sign of the Hessian determinant. Again, the above formulas are well-defined, by the assumption that f is generic. Thus u, v are analytic on Ω [**4**, Theorem 9]. This is actually a special case of a more general result on the regularity of solutions of hypo-elliptic partial differential equations [**6**].

Now, observe that the 1^{st} order partial derivatives of u, v as well as the left-hand sides of (II.1.3) are linear in 3^{rd} order partial derivatives of f and thus we can write e.g.

$$\begin{pmatrix} u^{(1,0)} \\ u^{(0,1)} \\ p_1 \\ p_2 \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} f^{(0,3)} \\ f^{(1,2)} \\ f^{(2,1)} \\ f^{(3,0)} \end{pmatrix},$$

which allows us to express all the 3^{rd} order partial derivatives of f in terms of the 1^{st} order partial derivatives of u and the 2^{nd} order partial derivatives of f. To do this, we only need to verify that the matrix $\boldsymbol{A}$ is invertible. Indeed, its determinant is equal to

cf. A.4

$$\pm \frac{4 f^{(0,2)}(x,y) f^{(2,0)}(x,y)}{\left| f^{(0,2)}(x,y) f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2 \right|^{1/2}},$$

where $\pm$ is the sign of the Hessian determinant. Therefore we have

$$\begin{pmatrix} f^{(0,3)} \\ f^{(1,2)} \\ f^{(2,1)} \\ f^{(3,0)} \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} u^{(1,0)} \\ u^{(0,1)} \\ p_1 \\ p_2 \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} u^{(1,0)} \\ u^{(0,1)} \\ 0 \\ 0 \end{pmatrix}, \tag{II.4.3}$$

where the right-hand side is linear in 1^{st} order partial derivatives of u and algebraic in 2^{nd} order partial derivatives of f. Since $\boldsymbol{A}^{-1} = (\det \boldsymbol{A})^{-1}(\operatorname{adj} \boldsymbol{A})$, where $\det \boldsymbol{A}$ is locally bounded away from 0 and $\operatorname{adj} \boldsymbol{A}$ is a polynomial in 2^{nd} order partial derivatives of f, we have $\boldsymbol{A}^{-1} \in W^{1,1}_{\text{loc}}(\Omega)$. It follows that the left-hand side (and hence also the right-hand side) is an element of $W^{1,1}_{\text{loc}}(\Omega)$, which means by the very definition that $f \in W^{4,1}_{\text{loc}}(\Omega)$. Now we are able to weakly differentiate the equalities (II.4.3) and iterate the same argument to see that f is indeed infinitely differentiable on Ω. This ends the proof. $\square$

II.5. Proofs of the main theorems

Before we move on to the essential part of this section, we will prove the following lemma, which will play a key role in the proofs of both main theorems:

LEMMA II.5.1. *Let $f \in W^{3,1}_{\text{loc}}(\Omega)$ be a function defined on a connected open subset $\Omega \subseteq \mathbb{R}^2$. Suppose that f is generic. Then f satisfies the system of partial differential equations* (II.1.3) *if and only if its graph is contained in a quadratic surface.*

PROOF. Left implication ($\Longleftarrow$) follows immediately from Proposition II.3.5. A proof of the right implication ($\Longrightarrow$) is not so straightforward, since we want to deduce a linear dependence of a set of functions from vanishing of their generalized Wronskians, which fails to be true in general [**9**] and therefore needs specific arguments.

Again we adopt the notation from [**10**].

DEFINITION II.5.2 ([**10**, Definition 2]). A *critical point* of $\boldsymbol{\phi}$ is a point of the domain at which all generalized Wronskians of $\boldsymbol{\phi}$ vanish.

DEFINITION II.5.3 ([**10**, Definition 3]). An $r \times r$ *generalised sub-Wronskian* of $\boldsymbol{\phi}$, $1 \leq r \leq n$, is a generalised Wronskian of any subsequence of $\boldsymbol{\phi}$.

Note that not every minor of a generalized Wronskian is a generalized sub-Wronskian. Indeed, the above definition requires that it also satisfies the additional condition for orders of partial derivatives.

DEFINITION II.5.4 ([**10**, Definition 4]). The *order* of a critical point $\boldsymbol{t}$ of $\boldsymbol{\phi}$ is the largest positive integer r for which some $r \times r$ generalized sub-Wronskian of $\boldsymbol{\phi}$ is not zero at $\boldsymbol{t}$. Should all sub-Wronskians vanish at $\boldsymbol{t}$, the order is defined to be zero.

We will show that every $\boldsymbol{t} \in \Omega$ is a critical point of $\boldsymbol{\phi}$ of order 9. First, observe that all 10×10 generalized Wronskians of $\boldsymbol{\phi}$ vanish identically on Ω. Indeed, from Lemma II.4.1 we infer that f is smooth and thus all its generalized Wronskians are well-defined. Moreover, by Assertion II.3.4 they belong to $\mathcal{Q}(\Omega)^\dagger$ and hence they vanish identically on Ω since both generators of $\mathcal{Q}(\Omega)^\dagger$ do.

We are left to prove that for every $\boldsymbol{t} \in \Omega$ there exists a 9×9 generalized sub-Wronskian of $\boldsymbol{\phi}$ that is non-zero at $\boldsymbol{t}$. Observe that i.a. every 9×9 minor of the following 9×10 matrix

$$\boldsymbol{W} := \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\phi}^{(0,1)} \\ \boldsymbol{\phi}^{(1,0)} \\ \boldsymbol{\phi}^{(0,2)} \\ \boldsymbol{\phi}^{(1,1)} \\ \boldsymbol{\phi}^{(2,0)} \\ \boldsymbol{\phi}^{(0,3)} \\ \boldsymbol{\phi}^{(1,2)} \\ \boldsymbol{\phi}^{(0,4)} \end{pmatrix}$$

that comprises the first row is a valid sub-Wronskian of $\boldsymbol{\phi}$ and thus it is enough to show that $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{t})$ has full rank at every $\boldsymbol{t} \in \Omega$. Denote by W_i the minor of $\boldsymbol{W}$ obtained by deleting i^{th} column and suppose that all W_i are zero. A direct computation shows that

$$W_6 = 4f^{(0,2)} \left(3f^{(0,2)} f^{(0,4)} - 4{f^{(0,3)}}^2\right),$$

cf. A.1

which implies

$$f^{(0,4)} = \frac{4{f^{(0,3)}}^2}{3f^{(0,2)}}.$$

Applying the above result to the definition of W_5 yields

$$W_5 = -24{f^{(0,2)}}^3 f^{(0,3)}$$

cf. A.1

and consequently

$$f^{(0,3)} = 0.$$

It follows that

$$W_4 = 36{f^{(0,2)}}^5,$$

cf. A.1

which finally gives us the desired contradiction.

We are now at a point where we can apply the following fundamental theorem:

Lemma II.5.5 ([**10**, Theorem 2]). *If G is an open connected set consisting of critical points of the same order $r > 0$, then $\boldsymbol{\phi}$ has a linearly independent subset $S_r = \{\phi_1, \dots, \phi_r\}$, say, which is a basis of* $\operatorname{span}(\boldsymbol{\phi})$*, and consequently $\boldsymbol{\phi}$ is linearly dependent on G.*

By Lemma II.5.5 we know that $\boldsymbol{\phi}$ is linearly dependent on Ω. This concludes the proof. $\square$

Remark. Observe that the system of partial differential equations (II.1.3) is satisfied if and only if a pair of functions (II.4.2) satisfies Cauchy-Riemann equations. Thus the graph of f is contained in a quadratic surface if and only if $u + iv$ is holomorphic. Moreover, if f satisfies (II.3.1), then a direct computation shows that $u + iv$ is simply a quadratic polynomial:

(II.5.6)
$$u + iv = \frac{((Q_{1,1} - Q_{2,2}) - 2iQ_{1,2}) + 2(Q_{1,4} - iQ_{2,4})z + Q_{4,4}z^2}{|\det \boldsymbol{Q}|^{3/4}},$$

cf. A.4

where

$$\text{(II.5.7)} \qquad \boldsymbol{Q} := \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} & \frac{1}{2}b_1 \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} & \frac{1}{2}b_2 \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} & \frac{1}{2}b_3 \\ \frac{1}{2}b_1 & \frac{1}{2}b_2 & \frac{1}{2}b_3 & c \end{pmatrix}$$

is a symmetric matrix defining an affine quadratic form (II.3.1) and $Q_{i,j}$ is the (i,j) minor of $\boldsymbol{Q}$, i.e. the determinant of the submatrix formed by deleting the i^{th} row and j^{th} column.

REMARK. Since a quadratic surface is uniquely determined by 9 parameters and the quadratic polynomial (II.5.6) has only 5 parameters, a natural question arises which functions correspond to the same quadratic polynomial? Note that u, v depend only on 2^{nd} order partial derivatives of f, which means that adding linear terms does not change (II.5.6). For completeness, we still need one more parameter. Careful inspection of (II.5.6) shows e.g. that every function of the form

$$f(x, y) := a\sqrt{1 - ax^2 - ay^2} + bx + cy + d$$

gives rise to the same quadratic polynomial $u + iv = z^2$. Unfortunately, the general answer is far more complicated and will not be given here.

Finally, we still will need one more simple fact, which can be verified by a direct computation:

ASSERTION II.5.8 ([**11**, Theorem 1]). *Let $f : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$ be a function of class C^2 defined on an open subset of $\mathbb{R}^2$ and satisfying a quadratic equation* (II.3.1)*. Then the following formula holds:*

cf. A.4

$$\det \boldsymbol{H}_f(x, y) \cdot \Delta_f(x, y)^2 = -16 \det \boldsymbol{Q},$$

where $\boldsymbol{H}_f$ is the Hessian matrix of f, Δ_f is the discriminant of (II.3.1) *with respect to the variable f and $\boldsymbol{Q}$ is just as defined in* (II.5.7)*.* ■

Now we are ready to prove Theorem II.1.2.

PROOF OF THEOREM II.1.2. Define $U \subseteq \Omega$ to be the subset consisting of those points, where the Hessian determinant of f is positive. Note that U is open, which immediately follows from the continuity of partial derivatives. Moreover, the assumption on f asserts that it is also non-empty.

First, we will show the right implication ($\Longrightarrow$). Since $f|_U$ is generic, by Lemma II.5.1 its graph is contained in a quadratic surface. Let $\boldsymbol{t} \in \Omega$ be a limit point of U, i.e. such that there exists a sequence $\boldsymbol{t}_\bullet$ of points in U whose limit is $\boldsymbol{t}$. Now, if Δ_f vanishes identically on U then f is affine, a contradiction. Thus there exists $\boldsymbol{u} \in U$ such that $\Delta_f(\boldsymbol{u}) > 0$, which implies $-16 \det \boldsymbol{Q} > 0$. Moreover, since Δ_f is a quadratic polynomial, the sequence $\Delta_f(\boldsymbol{t}_\bullet)^2$ is bounded from above. It follows that $\det \boldsymbol{H}_f(\boldsymbol{t}_\bullet) = -16 \det \boldsymbol{Q} \cdot \Delta_f(\boldsymbol{t}_\bullet)^{-2}$ is bounded from below by some positive constant $\varepsilon > 0$. In particular, we have $\det \boldsymbol{H}_f(\boldsymbol{t}) \geq \varepsilon > 0$ and hence $\boldsymbol{t} \in U$, by the very definition. Thus we have shown that U contains all its limit points, which makes it closed in Ω. However, recall that U is also open, in which case we have simply $U = \Omega$. This concludes the first part of the proof.

The remaining left implication ($\Longleftarrow$) follows right the same way. Since the graph of $f|_U$ is by assumption contained in a quadratic surface, we repeat the above limit point argument to see that likewise $U = \Omega$. With this result at hand, we once again apply Lemma II.5.1 to conclude the proof. □

Finally, we are in a position to clear out the assumption on the Hessian determinant. However, it turns out to be important, since (II.1.3) is also satisfied by parametrizations of some ruled surfaces, the Hessian determinant of which is non-positive.

Proof of Theorem II.1.4. Define the following open sets:

$$\begin{aligned}
\Omega_{\mathrm{a}} &:= \left\{ f^{(0,2)} \neq 0, f^{(2,0)} \neq 0, f^{(0,2)} f^{(2,0)} - {f^{(1,1)}}^2 < 0 \right\}, \\
\Omega_{\mathrm{b}} &:= \operatorname{Int} \left\{ f^{(0,2)} f^{(2,0)} - {f^{(1,1)}}^2 = 0 \right\}, \\
\Omega_{\mathrm{c}} &:= \operatorname{Int} \left\{ f^{(2,0)} = 0 \right\}, \\
\Omega_{\mathrm{d}} &:= \operatorname{Int} \left\{ f^{(0,2)} = 0 \right\}.
\end{aligned}$$

Clearly their sum $\Omega_{\mathrm{a}} \cup \Omega_{\mathrm{b}} \cup \Omega_{\mathrm{c}} \cup \Omega_{\mathrm{d}}$ is dense in Ω. By definition, on each connected component of Ω_{b} the graph of f is contained in a developable surface. Moreover, on each connected component of Ω_{c} (respectively: Ω_{d}) f is linear along every straight line parallel to the OX (respectively: OY) axis and thus, again by definition, its graph is contained in a Catalan surface with directrix plane XZ (respectively: YZ). Hence to prove the right implication ($\Longrightarrow$) it remains for us to show that on every connected component of Ω_{a} the graph of f satisfying (II.1.3) is contained in a doubly-ruled surface, which readily follows from Lemma II.5.1. Indeed, we immediately obtain that the graph of f is contained in a quadratic surface of negative Gaussian curvature. The only two are hyperbolic paraboloid and single-sheeted hyperboloid, both of which are doubly-ruled [**5**, p. 15]. This concludes the first part of the proof.

On the other hand, observe that $f|_{\Omega_{\mathrm{b}}}$ automatically satisfies (II.1.3). Indeed, denote

$$H_f(x,y) := f^{(0,2)}(x,y) f^{(2,0)}(x,y) - f^{(1,1)}(x,y)^2$$

and observe that

$$\begin{aligned}
p_1 &= -4 f^{(1,2)} H_f + f^{(0,2)} \frac{\partial H_f}{\partial x} + 2 f^{(1,1)} \frac{\partial H_f}{\partial y} = 0, \\
p_2 &= -4 f^{(2,1)} H_f + f^{(2,0)} \frac{\partial H_f}{\partial y} + 2 f^{(1,1)} \frac{\partial H_f}{\partial x} = 0,
\end{aligned}$$

where p_1, p_2 again stand for the left-hand sides of (II.1.3), respectively. Moreover, $f|_{\Omega_{\mathrm{c}}}$ (respectively: $f|_{\Omega_{\mathrm{d}}}$) satisfies $f^{(2,0)} \equiv 0$ and consequently $f^{(3,0)} \equiv 0$ (respectively: $f^{(0,2)} \equiv 0$ and consequently $f^{(0,3)} \equiv 0$), in which case a simple check shows that it satisfies (II.1.3) as well. Hence to prove the left implication ($\Longleftarrow$) it remains for us to show that for each Ω_i, $f|_{\Omega_i \cap \Omega_{\mathrm{a}}}$ satisfies (II.1.3). However, $\Omega_i \cap \Omega_{\mathrm{a}}$ is non-empty if and only if the graph of $f|_{\Omega_i}$ is contained in a doubly-ruled surface of negative Gaussian curvature. The only two are hyperbolic paraboloid and single-sheeted hyperboloid [**5**, p. 15], both of which are quadratic. The assertion follows from Lemma II.5.1, which concludes the proof. □

Remark. Denote by

$$f(\boldsymbol{t} + \boldsymbol{h}) =: \sum_{k=0}^{3} \frac{1}{k!} f_k(\boldsymbol{t})[\boldsymbol{h}] + o(\|\boldsymbol{h}\|)^3$$

the series expansion of f at $\boldsymbol{t} \in \Omega$, where $f_k(\boldsymbol{t})$ stands for its k^{th} order homogeneous Taylor polynomial in $\boldsymbol{h}$. Generally, any 2^{nd} order homogeneous polynomial vanishes on at most two lines in $\mathbb{RP}^1$. Therefore if the graph of f is contained in a doubly-ruled surface, $f_2(\boldsymbol{t})$ vanishes exactly on the two rulings that pass through $\boldsymbol{t}$. Moreover, $f_3(\boldsymbol{t})$ likewise must vanish on the same two rulings. In particular, it follows that $f_2(\boldsymbol{t})$ divides $f_3(\boldsymbol{t})$ as a polynomial. So it should come as no surprise to us that, for a generic function

f, equations (II.1.3) are satisfied if and only if $f_2(\boldsymbol{t})$ divides $f_3(\boldsymbol{t})$. Indeed,

$$\begin{aligned} f_3(\boldsymbol{t})[h_1,h_2] = &\left(\frac{f^{(3,0)}(\boldsymbol{t})}{f^{(2,0)}(\boldsymbol{t})}h_1 + \frac{f^{(0,3)}(\boldsymbol{t})}{f^{(0,2)}(\boldsymbol{t})}h_2\right) f_2(\boldsymbol{t})[h_1,h_2] \\ &- h_1h_2\left(\frac{p_2(\boldsymbol{t})}{f^{(2,0)}(\boldsymbol{t})f^{(0,2)}(\boldsymbol{t})}h_1 + \frac{p_1(\boldsymbol{t})}{f^{(2,0)}(\boldsymbol{t})f^{(0,2)}(\boldsymbol{t})}h_2\right), \end{aligned}$$

where p_1, p_2 again stand for the left-hand sides of (II.1.3), respectively. Observe that the remainder is a product of h_1, h_2 and some linear homogeneous polynomial, whereas $f_2(\boldsymbol{t})[h_1,h_2]$ is divisible by neither h_1 nor h_2, which means it can not divide the remainder unless the latter is zero. Thus we have found yet another way of looking on (II.1.3): for, in generic case, it arises as generalized Wronskians of a certain set of functions, as Cauchy-Riemann equations for a certain pair of functions and now as coefficients of a certain remainder from dividing $f_3(\boldsymbol{t})$ by $f_2(\boldsymbol{t})$.

Remark. Since the proofs of both theorems were mainly algebraic, the same results hold also in a complex setting, if we assume $f : \mathbb{C}^2 \supseteq \Omega \to \mathbb{C}$ to be holomorphic. Although the author did not point it out, the same applies likewise to the cited work [**10**] concerning generalized Wronskians, which allows us to apply the results in an analogous manner. Only smoothing Lemma II.4.1 ceases to make sense, but actually it is not needed anymore.

Let us conclude our considerations with an alternative proof of a well-known corollary from the aforementioned theorem of Maschke-Pick-Berwald [**8**, Theorem 4.5]:

Corollary II.5.9. *Let $S \subset \mathbb{R}^3$ be a convex surface of class C^3 such that for every $\boldsymbol{x} \in S$ there is a quadratic surface having 3^{rd} order contact with S at $\boldsymbol{x}$. Then S is itself a quadratic surface.*

Proof. Define $f : \mathbb{R}^2 \supseteq U \to \mathbb{R}$ to be a function such that its graph contains an open subset of S. Fix $\boldsymbol{x} \in U$ and define $g : \mathbb{R}^2 \supseteq V \to \mathbb{R}$ to be a parametrization of a quadratic surface having 3^{rd} order contact with S at $\boldsymbol{x}$. It follows from the 'if' part of Theorem II.1.2 that g satisfies (II.1.3) at $\boldsymbol{x}$. Moreover, by assumption, we have the equality of jets $J^3_{\boldsymbol{x}}g = J^3_{\boldsymbol{x}}f$ and hence f likewise satisfies (II.1.3) at $\boldsymbol{x}$. Now, since $\boldsymbol{x}$ was arbitrary, it means that f satisfies (II.1.3) on the whole domain U and finally from the 'only if' part of Theorem II.1.2 we obtain that its graph is contained in a quadratic surface. This concludes the proof. □

The above differential characterization of quadratic surfaces is expressed in the language of differential geometry rather than differential equations. However, unlike Theorem II.1.2, the assumption of Corollary II.5.9 is clearly invariant under affine change of coordinate system, which is a highly desirable property.

Acknowledgements

I would like to thank my doctoral advisor Prof. Michał Wojciechowski for his breakthrough ideas, insistence on giving a profound background to all the intermediate results, help in completing all the details and especially high amount of effort spent on editorial work.

Data availability. The data that support the findings of this study are available from a public institutional repository, `https://cloud.impan.pl/s/1kIXqE87mHVn7nT`.

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CHAPTER III

On separably integrable symmetric convex bodies

An infinitely smooth symmetric convex body $K \subset \mathbb{R}^d$ is called k-separably integrable, $1 \leq k < d$, if its k-dimensional isotropic volume function $V_{K,H}(t) = \mathcal{H}^d(\{\boldsymbol{x} \in K : \operatorname{dist}(\boldsymbol{x}, H^{\perp}) \leq t\})$ can be written as a finite sum of products in which the dependence on $H \in \operatorname{Gr}(k, \mathbb{R}^d)$ and $t \in \mathbb{R}$ is separated. In this paper, we will obtain a complete classification of such bodies. Namely, we will prove that if $d-k$ is even, then K is an ellipsoid, and if $d-k$ is odd, then K is a Euclidean ball. This generalizes the recent classification of polynomially integrable convex bodies in the symmetric case.

III.1. Introduction

Newton argued in *Principia* that the areas of caps of a planar convex body with infinitely smooth boundary are not expressible in terms of algebraic equations:

Theorem III.1.1 ([**15**, §VI, Lemma XXVIII]). *There is no oval figure whose area, cut off by right lines at pleasure, can be universally found by means of equations of any number of finite terms and dimensions.*

On the other hand, it was already known since the remarkable result of Archimedes that the volume cut off by a plane from a Euclidean ball in $\mathbb{R}^3$ depends algebraically on the plane. Further, it can be easily verified that the latter is true for any ellipsoid in any odd-dimensional space. It is, therefore, natural to ask to what extent can Newton's result be generalized.

Several related questions of this type were brought up by Arnold [**6**, 1987-14, 1988-13, 1990-27] in his famous seminar at Moscow State University. In 2015, Vassiliev [**21**] solved the problem 1988-13 by showing that if $K \subset \mathbb{R}^d$ is a bounded domain in an even-dimensional space, then the volumes $V^{\pm}_{K,H}(t)$ cut off by a hyperplane parallel to $H \in \operatorname{Gr}(d-1, \mathbb{R}^d)$ at distance $t \in \mathbb{R}$ from the origin are not algebraic functions of H and t. Two years later, Agranovsky [**1**] suggested a new direction for further research, introducing the concept of polynomial integrability.

Definition III.1.2 ([**11**, §2.1]). Let $K \subset \mathbb{R}^d$ be a convex body. For $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ we define the *parallel section function* of K by

$$A_{K,\boldsymbol{\xi}}(t) := \mathcal{H}^{d-1}(K \cap \{\langle \boldsymbol{\xi} \rangle^{\perp} + t\boldsymbol{\xi}\}),$$

where $\{\langle \boldsymbol{\xi} \rangle^{\perp} + t\boldsymbol{\xi}\}$ is the hyperplane perpendicular to $\boldsymbol{\xi}$ at distance t from the origin.

Definition III.1.3 ([**12**, Definition 1.1]). Let K be a convex body in $\mathbb{R}^d$. Then K is called *polynomially integrable* if its parallel section function $A_{K,\boldsymbol{\xi}}(t)$ is a polynomial in t on its support, i.e.,

$$A_{K,\boldsymbol{\xi}}(t) = \sum_{i=0}^{n} a_i(\boldsymbol{\xi}) t^i, \quad a_i : \mathbb{S}^{d-1} \to \mathbb{R}.$$

Agranovsky showed that if $K \subset \mathbb{R}^d$ is a domain with a smooth boundary in an even-dimensional space (the smoothness assumption was already known to be necessary), then it is not polynomially integrable [**1**, Theorem 2]. Equivalently, the volume $V^{\pm}_{K,H}(t)$ is not a polynomial in t. Since, on the one hand, the assumption of polynomial integrability imposes additional constraints on the dependence of $V^{\pm}_{K,H}(t)$ on t but, on the other, removes all constraints on the dependence of $V^{\pm}_{K,H}(t)$ on H, his result has a slightly different flavor. Agranovsky also showed that in an odd-dimensional space, all polynomially integrable bodies must be convex [**1**, Theorem 5]. Their classification was completed shortly afterward by Koldobsky, Merkurjev, and Yaskin [**12**], who proved the following theorem:

THEOREM III.1.4 ([**12**, Theorem 3.7]). *Let d be an odd positive integer. If K is an infinitely smooth polynomially integrable convex body in $\mathbb{R}^d$, then K is an ellipsoid.*

In a recent work, Agranovsky, Koldobsky, Ryabogin, and Yaskin [**5**] established a similar result in even-dimensional spaces, assuming that the parallel section function $A_{K,\boldsymbol{\xi}}(t)$ can be expressed in the form

$$A_{K,\boldsymbol{\xi}}(t) = P(\boldsymbol{\xi}, t)\sqrt{Q(\boldsymbol{\xi}, t)},$$

where P, Q are polynomials in t and $\deg Q = 2$ [**5**, Theorem 1.4].

For other developments on the problem, the reader is referred to [**2, 3, 4, 7, 8, 22, 23, 24**].

III.2. Statement of the result

In this paper, we are going to significantly weaken the polynomial integrability condition. Before we do this, however, we need to introduce the notion of separable integrability.

DEFINITION III.2.1. Suppose that X, Y are Hausdorff spaces and A, B are subalgebras of $C(X,\mathbb{R}), C(Y,\mathbb{R})$, respectively. The *algebraic tensor product* $A \otimes B$ of A, B is a subalgebra of $C(X \times Y, \mathbb{R})$ generated by *pure products* of the form

$$(a \otimes b)(x, y) := a(x)b(y), \quad a \in A, \ b \in B.$$

DEFINITION III.2.2. A function in $C(X \times Y, \mathbb{R})$ is called *separable* if it is a finite sum of pure products (i.e., an element of the algebraic tensor product $C(X,\mathbb{R}) \otimes C(Y,\mathbb{R})$). A function is called *entangled* if it is not separable.

Denote by $\mathrm{Gr}(k, \mathbb{R}^d)$ the *Grassmann manifold* of all k-dimensional linear subspaces of $\mathbb{R}^d$.

DEFINITION III.2.3. Let $K \subset \mathbb{R}^d$ be a convex body. For a k-dimensional linear subspace $H \in \mathrm{Gr}(k, \mathbb{R}^d)$, $1 \le k < d$, we define the *k-dimensional isotropic volume function* of K by

$$V_{K,H}(t) = \mathcal{H}^d(\{\boldsymbol{x} \in K : \mathrm{dist}(\boldsymbol{x}, H^{\perp}) \le t\}),$$

where $\{\boldsymbol{x} \in K : \mathrm{dist}(\boldsymbol{x}, H^{\perp}) \le t\}$ is the intersection of K with a k-dimensional right circular hypercylinder with base space H, axis $H^{\perp}$ and radius t.

Being careful in making delicate decisions about the domain, we will intentionally define the key concept of separable integrability only locally.

DEFINITION III.2.4. A convex body $K \subset \mathbb{R}^d$ is called *locally k-separably integrable*, $1 \le k < d$, if its k-dimensional isotropic volume function $V_{K,H}(t) : \mathrm{Gr}(k, \mathbb{R}^d) \times [0,\infty) \to \mathbb{R}$ is separable in some open neighborhood U of $\mathrm{Gr}(k, \mathbb{R}^d) \times \{0\}$, i.e.,

$$V_{K,H}(t) = \sum_{i=0}^{n} a_i(H) b_i(t), \quad a_i : \mathrm{Gr}(k, \mathbb{R}^d) \to \mathbb{R}, \ b_i : [0,\infty) \to \mathbb{R} \tag{III.2.5}$$

for every $(H, t) \in U$.

Remark. By definition, if a convex body $K \subseteq \mathbb{R}^d$ is polynomially integrable, then it is also locally 1-separably integrable. Indeed, for t such that the interval $[-t, +t]$ is contained in the support of $A_{K,\boldsymbol{\xi}}$ we have

$$V_{K,\langle\boldsymbol{\xi}\rangle}(t) = \int_{[-t,+t]} A_{K,\boldsymbol{\xi}}(r)\ \mathrm{d}r = \sum_{i=0}^{n} a_i(\boldsymbol{\xi}) \left(\int_{[-t,+t]} r^i\ \mathrm{d}r \right).$$

Remark. If $d-k$ is even and $K \subseteq \mathbb{R}^d$ is an ellipsoid, then it is locally k-separably integrable. Indeed, for t such that the ball $\mathbb{B}^d(t) \cap H$ is contained in the projection $K \mid H$ we have

$$\begin{aligned}
V_{K,H}(t) &= \int_{\mathbb{B}^d(t)\cap H} \mathcal{H}^{d-k}(K \cap \{H^\perp + \boldsymbol{u}\})\ \mathrm{d}\boldsymbol{u} \\
&= \int_{\mathbb{S}^{d-1}\cap H} \int_{[0,t]} r^{k-1} \mathcal{H}^{d-k}(K \cap \{H^\perp + r\boldsymbol{\theta}\})\ \mathrm{d}r\ \mathrm{d}\boldsymbol{\theta} \\
&= \int_{\mathbb{S}^{d-1}\cap H} \int_{[0,t]} r^{k-1} A_{K\cap\langle H^\perp,\boldsymbol{\theta}\rangle,\boldsymbol{\theta}}(r)\ \mathrm{d}r\ \mathrm{d}\boldsymbol{\theta}.
\end{aligned}$$

Now, since $K \cap \langle H^\perp, \boldsymbol{\theta}\rangle$ is an ellipsoid in an odd-dimensional space $\langle H^\perp, \boldsymbol{\theta}\rangle$, it is polynomially integrable. It follows that

$$V_{K,H}(t) = \int_{\mathbb{S}^{d-1}\cap H} \int_{[0,t]} r^{k-1} \sum_{i=0}^{d-k} a_{H,i}(\boldsymbol{\theta}) r^i\ \mathrm{d}r\ \mathrm{d}\boldsymbol{\theta} = \sum_{i=0}^{d-k} \left(\int_{\mathbb{S}^{d-1}\cap H} a_{H,i}(\boldsymbol{\theta})\ \mathrm{d}\boldsymbol{\theta} \right) \left(\int_{[0,t]} r^{k+i-1}\ \mathrm{d}r \right).$$

Our main result is the following theorem:

Theorem III.2.6. *Let $K \subseteq \mathbb{R}^d$ be an origin-symmetric convex body with infinitely smooth boundary ∂K. If K is locally k-separably integrable, then $d-k$ is even and K is an ellipsoid or $d-k$ is odd and K is a Euclidean ball.*

Note that none of the results mentioned in §III.1 requires K to be symmetric. Therefore Theorem III.2.6 generalizes Theorem III.1.4 only under this additional assumption. Unfortunately, exactly as in [**12**], the non-symmetric case is essentially more difficult and requires more involved algebraic arguments. Nevertheless, Theorem III.2.6 seems to indicate the crux of polynomial integrability. Namely, it is not so much the rigidity of polynomials that makes Theorem III.1.4 true as the fact that the linear space of polynomials of fixed degree is finite-dimensional. Interestingly enough, [**5**, Theorem 1.4] generalizes Theorem III.1.4 in a completely different way than Theorem III.2.6. On the one hand, it still needs the rigidity of polynomials, but on the other, it is more flexible in terms of the linear structure. This phenomenon prompts us to ask the following question:

Question III.2.7. *Let K be a bounded domain in $\mathbb{R}^d$ with an infinitely smooth boundary ∂K. If the k-dimensional isotropic volume function $V_{K,H}(t)$ can be locally expressed in the form*

$$V_{K,H}(t) = \Phi(a_1(H), a_2(H), \ldots, a_m(H), b_1(t), b_2(t), \ldots, b_n(t))$$

on some open neighborhood of $\mathrm{Gr}(k, \mathbb{R}^d) \times \{0\}$, where $\Phi : \mathbb{R}^{m+n} \to \mathbb{R}$ is algebraic and $a_i : \mathrm{Gr}(k, \mathbb{R}^d) \to \mathbb{R}$, $b_i : [0, \infty) \to \mathbb{R}$ are smooth, is K necessarily an ellipsoid?

It contains all the aforementioned results, including ours. To the authors' best knowledge, no counterexample is known so far.

Remark. By the superposition theorem of Kolmogorov [**13**], there always exist monotonic increasing functions $a_i \in C(\mathrm{Gr}(k, \mathbb{R}^d), \mathbb{R})$, $b_i \in C([0, \infty), \mathbb{R})$ with the property that each continuous function $V_{K,H}(t) \in C(\mathrm{Gr}(k, \mathbb{R}^d) \times [0, \infty), \mathbb{R})$ can be (locally) represented in the form

$$V_{K,H}(t) = \sum_{i=1}^{5} \phi_i(a_i(H) + b_i(t))$$

with functions $\phi_i \in C(\mathbb{R}, \mathbb{R})$. Therefore the question is not really about the separability of variables or even finiteness of the representation, but rather if the individual functions in such a representation can be made infinitely smooth or even algebraic. This type of question is already much more delicate, as in Kolmogorov's proof, there is an overt rivalry between the smoothness of a_i and b_i and the smoothness of ϕ_i. This also indicates why the initial smoothness assumption was crucial. However, since we do not insist that all the functions ϕ_i should be one-parameter, we seem to avoid the basic difficulty (cf. [**25**]). After all, our question may be considered as yet another (local) variant of the superposition problem for a particular class of functions arising as k-dimensional isotropic volumes of smooth convex bodies.

III.3. Definitions and basic concepts

We will begin with a brief reminder of the basic concepts and definitions that we will frequently use in the rest of the work.

NOTATION. Throughout the text, we will use the multi-index notation. A d-dimensional *multi-index* is a d-tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ of non-negative integers. For multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^d$ and a vector $\boldsymbol{x} \in \mathbb{R}^d$ we define the partial order

$$\boldsymbol{\alpha} \leq \boldsymbol{\beta} \iff \alpha_i \leq \beta_i \ \forall i \in \{1, 2, \ldots, d\},$$

the absolute value

$$|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \ldots + \alpha_d,$$

the power

$$\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

and the high-order partial derivative

$$\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}.$$

III.3.1. Fourier analysis. We adopt the notation and definitions from [**11**].

DEFINITION III.3.1 ([**11**, §2.2, §2.1]). A closed compact set $K \subset \mathbb{R}^d$ with a non-empty interior is called a *convex body* if it contains the line segment connecting any two of its points. If a convex body K is origin-symmetric, then its *Minkowski functional* defined by

$$\|\boldsymbol{x}\|_K := \min\{a \geq 0 \mid \boldsymbol{x} \in aK\}$$

is a norm on $\mathbb{R}^d$.

It is easy to see that the Minkowski functional is a homogeneous function of degree 1 on $\mathbb{R}^d$ and that

$$K = \{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\|_K \leq 1\}. \tag{III.3.2}$$

Also, it follows from the definition that the origin is an interior point of every symmetric convex body, so the Minkowski functional is strictly positive outside the origin.

NOTATION ([**11**, §2.5]). We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of complex-valued functions $\phi \in C^\infty(\mathbb{R}^d)$ converging to zero at infinity together with all their derivatives faster than any negative power of $\|\cdot\|_2$. Elements of the space $\mathcal{S}(\mathbb{R}^d)$ will be called *test functions*. As usual, we denote by $\mathcal{S}(\mathbb{R}^d)'$ the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^d)$, which we call *distributions* over $\mathcal{S}(\mathbb{R}^d)$.

DEFINITION III.3.3 ([**11**, §2.5]). We define the *Fourier transform* of a function $\phi \in L_1(\mathbb{R}^d)$ by

$$\mathcal{F}\phi(\boldsymbol{\xi}) := \hat{\phi}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \phi(\boldsymbol{x}) e^{-i(\boldsymbol{x}, \boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^d.$$

Further, we define the action of a complex-valued function $f \in L_1(\mathbb{R}^d)$ on a test function ϕ as

$$\langle f, \phi \rangle := \int_{\mathbb{R}^d} f(\boldsymbol{x})\phi(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

and finally, we define the Fourier transform of a distribution f by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle.$$

For any multi-index $\boldsymbol{\alpha} \in \mathbb{N}^d$, the derivative of the order $\boldsymbol{\alpha}$ of a distribution f is defined by

$$\langle \partial^{\boldsymbol{\alpha}} f, \phi \rangle = (-1)^{|\boldsymbol{\alpha}|} \langle f, \partial^{\boldsymbol{\alpha}} \phi \rangle.$$

The Fourier transform is related to differentiation as follows:

$$(\partial^{\boldsymbol{\alpha}} f)^{\wedge} = i^{|\boldsymbol{\alpha}|} \boldsymbol{x}^{\boldsymbol{\alpha}} f^{\wedge}. \tag{III.3.4}$$

Denote by $\mathrm{St}(k, \mathbb{R}^d)$ the *Stiefel manifold* of all orthonormal k-frames in $\mathbb{R}^d$ (i.e., the set of ordered orthonormal k-tuples of vectors in $\mathbb{R}^d$).

Definition III.3.5 ([**11**, §3.5]). Let $K \subset \mathbb{R}^d$ be a convex body. For an orthonormal k-frame $\Xi \in \mathrm{St}(k, \mathbb{R}^d)$ we define the $(d-k)$-*dimensional parallel section function* of K by

$$A_{K,\Xi}(\boldsymbol{t}) = \mathcal{H}^{d-k}(K \cap \{\langle \Xi \rangle^{\perp} + t_1 \boldsymbol{\xi}_1 + t_2 \boldsymbol{\xi}_2 + \ldots + t_k \boldsymbol{\xi}_k\}), \quad \boldsymbol{t} \in \mathbb{R}^k.$$

The following result expresses the derivatives of the parallel section function $A_{K,\Xi}$ in terms of the Fourier transform of powers of the Minkowski functional.

Lemma III.3.6 ([**11**, Theorem 3.26]). *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^d$, $1 \leq k < d$. Then for every orthonormal k-frame Ξ in $\mathbb{R}^d$ and every $s \in \mathbb{N}$, $s \neq (d-k)/2$,*

$$\Delta^s A_{K,\Xi}(\boldsymbol{0}) = \frac{(-1)^s}{2^k \pi^k (d-2s-k)} \int_{\mathbb{S}^{d-1} \cap \langle \Xi \rangle} (\| \cdot \|_K^{-d+2s+k})^{\wedge}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta},$$

where $\Delta := \sum_{i=1}^d \partial^2 / \partial x_i^2$ is the Laplace operator on $\mathbb{R}^d$.

III.3.2. Field theory. We adopt the notation and definitions from [**17**]. Let $F[x]$ denote the ring of polynomials in a single variable x over a field F.

Definition III.3.7 ([**17**, §1.4]). If a polynomial $f(x) \in F[x]$ factors into linear factors

$$f(x) = a(x - \zeta_1)(x - \zeta_2) \cdots (x - \zeta_n)$$

in an extension field E, that is, if $\zeta_1, \zeta_2, \ldots, \zeta_n \in E$, we say that $f(x)$ *splits* in E.

Definition III.3.8 ([**17**, §1.4]). Let $\mathcal{F} = \{f_i(x)\}_{i \in I}$ be family of polynomials over a field F. A *splitting field* for $\mathcal{F}$ is the smallest extension field E of F such that each $f_i(x) \in \mathcal{F}$ splits over E.

Theorem III.3.9 ([**17**, Theorem 1.4.1]). *Every finite family of polynomials over a field F has a splitting field.*

Definition III.3.10 ([**17**, §1.5]). Let E/F be a field extension. An element $\zeta \in E$ is said to be *algebraic* over F if ζ is a root of some polynomial over F. An element that is not algebraic over F is said to be *transcendental* over F.

Definition III.3.11 ([**17**, §1.5]). If ζ is algebraic over F, the set of all polynomials with a root at ζ

$$\mathcal{I}_{\zeta} := \{f(x) \in F[x] \mid f(\zeta) = 0\}$$

is a non-zero ideal in $F[x]$ and is therefore generated by a unique monic polynomial, called the *minimal polynomial* of ζ over F and denoted by $\mu_{\zeta}(x)$.

The following theorem characterizes minimal polynomials in a variety of useful ways.

THEOREM III.3.12 ([**17**, Theorem 1.5.1]). *Let E/F be a field extension and let $\zeta \in E$ be algebraic over F. Then among all polynomials in $F[x]$, the minimal polynomial $\mu_\zeta(x)$ is:*

(1) the unique monic irreducible polynomial $\mu(x)$ for which $\mu(\zeta) = 0$;
(2) the unique monic polynomial $\mu(x)$ of smallest degree for which $\mu(\zeta) = 0$;
(3) the unique monic polynomial $\mu(x)$ with the property that for $f(x) \in F[x]$, $f(\zeta) = 0$ if and only if $\mu(x) \mid f(x)$.

In other words, $\mu_\zeta(x)$ is the unique monic generator of the ideal $\mathcal{I}_\zeta$.

III.3.3. Valued fields. We adopt the notation and definitions from [**10**, §2].

DEFINITION III.3.13. Let F be a field. A *valuation* on F is a map $v : F \to \mathbb{R} \cup \{\infty\}$ satisfying the following axioms for all $x, y \in F$:

(1) $v(x) = \infty \iff x = 0$;
(2) $v(xy) = v(x) + v(y)$;
(3) $v(x + y) \geq \min(v(x), v(y))$.

As a consequence, we obtain for all $x, y \in F$:

(4) $v(1) = 0$;
(5) $v(x^{-1}) = -v(x)$;
(6) $v(-x) = v(x)$;
(7) $v(x) < v(y) \implies v(x + y) = v(x)$.

An example of a non-trivial valuation is the p-adic valuation on the rational function field $F(\boldsymbol{x})$, where p is any irreducible polynomial from $F[\boldsymbol{x}]$, F being an arbitrary field.

DEFINITION III.3.14. Let F be a field. For every irreducible polynomial $p \in F[\boldsymbol{x}]$, the *p-adic valuation* on the rational function field $F(\boldsymbol{x})$ is defined by

$$v_p\left(p^\nu \frac{f}{g}\right) = \nu,$$

where $\nu \in \mathbb{Z}$ and $f, g \in F[x] \setminus \{0\}$ are not divisible by p.

Note that v_p restricted to F is trivial. There is one more interesting valuation on $F(\boldsymbol{x})$, trivial on F.

DEFINITION III.3.15. Let F be a field. The *degree valuation* on the rational function field $F(\boldsymbol{x})$ is defined by

$$v_\infty\left(\frac{f}{g}\right) = \deg g - \deg f,$$

where $f, g \in F[\boldsymbol{x}] \setminus \{0\}$.

Interestingly enough, there are no valuations on $F[\boldsymbol{x}]$ other than the ones just mentioned, assuming triviality on F (cf. [**10**, Theorem 2.1.4]).

DEFINITION III.3.16. Let F be a field. A subring $\mathcal{O}$ of F satisfying $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$ for all $x \in F \setminus \{0\}$ is called a *valuation ring* of F.

The following is a direct consequence of Chevalley's Theorem [**10**, Theorem 3.1.1]:

THEOREM III.3.17 ([**10**, Theorem 3.1.2]). *Let F_2/F_1 be a field extension and let $\mathcal{O}_1 \subseteq F_1$ be a valuation ring. Then there is an extension $\mathcal{O}_2$ of $\mathcal{O}_1$ in F_2.*

In particular, it means that any valuation v on a field F always admits at least one extension to every field E containing F.

III.4. Proof of main theorem

Let us begin with the following regularity lemma:

LEMMA III.4.1. *Let $K \subseteq \mathbb{R}^d$ be a convex body with an infinitely smooth boundary ∂K. If the k-dimensional isotropic volume function of K satisfies* (III.2.5) *with $a_1, a_2, \dots, a_n$ being linearly independent, then $b_1, b_2, \dots, b_n$ are infinitely smooth in some neighborhood of $t = 0$.*

PROOF. Since $a_1, a_2, \dots, a_n$ are linearly independent, there exist $H_1, H_2, \dots, H_n \in \mathrm{Gr}(k, \mathbb{R}^d)$ such that the alternant matrix

$$A := \begin{pmatrix} a_1(H_1) & a_2(H_1) & \cdots & a_n(H_1) \\ a_1(H_2) & a_2(H_2) & \cdots & a_n(H_2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(H_n) & a_2(H_n) & \cdots & a_n(H_n) \end{pmatrix}$$

is invertible. By definition, for every $t \in [0, \infty)$ we have $\boldsymbol{v}(t) = A \cdot \boldsymbol{b}(t)$, where

$$\boldsymbol{v}(t) := \begin{pmatrix} V_{K,H_1}(t) \\ V_{K,H_2}(t) \\ \vdots \\ V_{K,H_n}(t) \end{pmatrix}, \quad \boldsymbol{b}(t) := \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

Now, it follows that $\boldsymbol{b}(t) = A^{-1} \cdot \boldsymbol{v}(t)$ is infinitely smooth in some neighborhood of $t = 0$ because so is $\boldsymbol{v}(t)$. This concludes the proof. □

Let us also rephrase Lemma III.3.6 in an equivalent, coordinate-free way:

PROPOSITION III.4.2. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^d$, $1 \leq k < d$. Then for every k-dimensional linear subspace $H \in \mathrm{Gr}(k, \mathbb{R}^d)$ and every $s \in \mathbb{N}$, $s \neq (d-k)/2$,*

$$V_{K,H}^{(2s+k)}(0) = C(d,s,k) \int_{\mathbb{S}^{d-1} \cap H} (\|\cdot\|_K^{-d+2s+k})^\wedge(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta},$$

where $C(d,s,k)$ is a non-zero constant.

PROOF. Let $\Xi \in \mathrm{St}(k, \mathbb{R}^d)$ be an orthonormal basis of H. Clearly, we have

$$V_{K,H}(t) = \int_{\mathbb{B}^k(t)} A_{K,\Xi}(\boldsymbol{u}) \, \mathrm{d}\boldsymbol{u},$$

so by [**16**, Theorem 3] the k-dimensional isotropic volume function admits the series expansion of the form

$$V_{K,H}(t) = \omega_k \sum_{i=0}^{s} \frac{\Delta^i A_{K,\Xi}(\mathbf{0})}{2^i i! \prod_{j=1}^{i}(2j+k)} t^{2i+k} + o(t^{2s+k}),$$

where ω_k denotes the volume of the unit ball in k dimensions. In particular, we get

$$V_{K,H}^{(2s+k)}(0) = \omega_k \frac{\Delta^s A_{K,\Xi}(\mathbf{0})}{2^s s! \prod_{j=1}^{s}(2j+k)} (2s+k)!,$$

which further by Lemma III.3.6 equals

$$\omega_k \frac{1}{2^s s! \prod_{j=1}^{s}(2j+k)} (2s+k)! \frac{(-1)^s}{2^k \pi^k (d-2s-k)} \int_{\mathbb{S}^{d-1} \cap H} (\|\cdot\|_K^{-d+2s+k})^\wedge(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta}$$

for every $s \in \mathbb{N}$, $s \neq (d-k)/2$. This concludes the proof. □

Finally, we are ready to prove the main theorem.

PROOF OF THEOREM III.2.6. The proof will consist of three clearly outlined parts. Firstly, using simple linear algebra, we will reduce the problem to solving an abstract system of polynomial equations. Secondly, using more sophisticated tools of valuation theory, we will eventually characterize its solutions. Finally, we will check the solutions by plugging them into the original problem.

III.4.1. Constructing the system of polynomial equations. Suppose that K is locally k-separably integrable. Without loss of generality, we may assume that the functions $a_1, a_2, \dots, a_n$ are linearly independent. In light of Lemma III.4.1, differentiating (III.2.5) with respect to t yields

$$V_{K,H}^{(2s+k)}(0) = \sum_{i=1}^{n} a_i(H) b_i^{(2s+k)}(0) \tag{III.4.3}$$

for every $s \in \mathbb{N}$. Observe that the right-hand sides of (III.4.3) span a finite-dimensional subspace of $C(\mathrm{Gr}(k, \mathbb{R}^d), \mathbb{R})$ of dimension not greater than n. Indeed, they are linear combinations of a finite set of functions $a_1, a_2, \dots, a_n$. Hence also the left-hand sides of (III.4.3) for all $s \in \mathbb{N}$ span a finite-dimensional subspace of $C(\mathrm{Gr}(k, \mathbb{R}^d), \mathbb{R})$. It follows that for every $s \in \mathbb{N}$ there exist scalars $c_{s,0}, c_{s,1}, \dots, c_{s,n}$, not all zero, such that

$$\sum_{i=0}^{n} c_{s,i} V_{K,H}^{(2s+2i+k)}(0) = 0. \tag{III.4.4}$$

By virtue of Proposition III.4.2, for every $s \geq \lceil d/2 \rceil$ this reads

$$\int_{\mathbb{S}^{d-1} \cap H} \sum_{i=0}^{n} \tilde{c}_{s,i} (-1)^i (\| \cdot \|_K^{-d+2s+2i+k})^\wedge(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{\theta} = 0,$$

where

$$\tilde{c}_{s,i} := (-1)^i c_{s,i} C(d, s+i, k).$$

It means precisely that the k-dimensional spherical Radon transform (cf. [**11**, §2.3]) of the integrand is zero for every $H \in \mathrm{Gr}(k, \mathbb{R}^d)$. Since K is origin-symmetric, the integrand is an even function, whence

$$\sum_{i=0}^{n} \tilde{c}_{s,i} (-1)^i (\| \cdot \|_K^{-d+2s+2i+k})^\wedge(\boldsymbol{\theta}) = 0 \tag{III.4.5}$$

for every $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (cf. [**11**, Corollary 3.10]). Further, using the simple fact that

$$(\| \cdot \|_K^{-d+2s+2i+k})^\wedge(\boldsymbol{\theta}) = t^d (\| t \cdot \|_K^{-d+2s+2i+k})^\wedge(t\boldsymbol{\theta}) = t^{2s+2i+k} (\| \cdot \|_K^{-d+2s+2i+k})^\wedge(t\boldsymbol{\theta})$$

(cf. [**11**, Lemma 2.21]) we can rewrite (III.4.5) in the form

$$\sum_{i=0}^{n} \tilde{c}_{s,i} (-1)^i t^{2s+2i+k} (\| \cdot \|_K^{-d+2s+2i+k})^\wedge(t\boldsymbol{\theta}) = 0.$$

Dividing both sides by t^{2s+k} and using $\|\boldsymbol{\theta}\|_2 = 1$ yields

$$\sum_{i=0}^{n} \tilde{c}_{s,i} (-1)^i \| t\boldsymbol{\theta} \|_2^{2i} (\| \cdot \|_K^{-d+2s+2i+k})^\wedge(t\boldsymbol{\theta}) = 0.$$

By the differentiation property of the Fourier transform (III.3.4), we get

$$\sum_{i=0}^{n} \tilde{c}_{s,i} (\Delta^i \| \cdot \|_K^{-d+2s+2i+k})^\wedge(t\boldsymbol{\theta}) = 0,$$

It follows that the Fourier transform of the distribution

$$\sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^i \|\cdot\|_K^{-d+2s+2i+k} \tag{III.4.6}$$

is supported at the origin, in which case it is a polynomial (cf. [**18**, §7.16]). Denote this polynomial by P_s and observe that it is homogeneous of degree $-d+2s+k$.

Claim III.4.7. For any multi-index $\boldsymbol{\alpha} \in \mathbb{N}^d$, $m \in \mathbb{Z}$ and any infinitely smooth function $f : \mathbb{R}^d \to \mathbb{R}$ we have

$$\partial^{\boldsymbol{\alpha}} f^m = Q^m_{\boldsymbol{\alpha}}(\{\partial^{\boldsymbol{\beta}} f : \boldsymbol{\beta} \le \boldsymbol{\alpha}\}) f^{m-|\boldsymbol{\alpha}|},$$

where $Q^m_{\boldsymbol{\alpha}}$ is a polynomial depending on m only through its coefficients. Moreover,

$$Q^m_{\boldsymbol{\alpha}}(\{\partial^{\boldsymbol{\beta}} f : \boldsymbol{\beta} \le \boldsymbol{\alpha}\}) = m^{|\boldsymbol{\alpha}|} (\nabla f)^{\boldsymbol{\alpha}} + O(m^{|\boldsymbol{\alpha}|-1}),$$

where $(\nabla f)^{\boldsymbol{\alpha}}$ denotes a multi-index power of the vector $\nabla f \in \mathbb{R}^d$. In particular, we have

$$\Delta^i f^m = \tilde{Q}^m_i(\{\partial^{\boldsymbol{\beta}} f : |\boldsymbol{\beta}| \le 2i\}) f^{m-2i}$$

and

$$\tilde{Q}^m_i(\{\partial^{\boldsymbol{\beta}} f : |\boldsymbol{\beta}| \le 2i\}) = m^{2i} \|\nabla f\|_2^{2i} + O(m^{2i-1}),$$

where $\tilde{Q}^m_i$ is again a polynomial depending on m only through its coefficients and $\|\nabla f\|_2$ denotes the Euclidean norm of the vector $\nabla f \in \mathbb{R}^d$. ■

Since the proof is a tedious but conceptually straightforward induction on $|\boldsymbol{\alpha}|$, we leave it to the reader.

Applying Claim III.4.7 to (III.4.6) yields

$$\sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^i \|\cdot\|_K^{-d+2s+2i+k} = \sum_{i=0}^{n} \tilde{c}_{s,i} \tilde{Q}_i^{-d+2s+2i+k}(\{\partial^{\boldsymbol{\beta}} \|\boldsymbol{x}\|_K : |\beta| \le 2i\}) \|\boldsymbol{x}\|_K^{-d+2s+k}.$$

Thus finally for every $s \ge \lceil d/2 \rceil$ and $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ we obtain

$$\sum_{i=0}^{n} \tilde{c}_{s,i} \tilde{Q}_i^{-d+2s+2i+k}(\{\partial^{\boldsymbol{\beta}} \|\boldsymbol{x}\|_K : |\beta| \le 2i\}) = P_s(\boldsymbol{x}) \|\boldsymbol{x}\|_K^{d-2s-k}, \tag{III.4.8}$$

where $\tilde{c}_{s,i}$ are constants, $\tilde{Q}_i^{-d+2s+2i+k}$ are polynomials that depend on s only through their coefficients and P_s is a homogeneous polynomial of degree $-d+2s+k$.

Claim III.4.9. All but finitely many polynomials P_s, $s \ge \lceil d/2 \rceil$, are non-zero unless K is a Euclidean ball. Indeed, suppose that there exists an increasing sequence $(s_j)_{j\in\mathbb{N}}$ such that $P_{s_j} = 0$ for every $j \in \mathbb{N}$. Then (III.4.8) reads

$$\sum_{i=0}^{n} \tilde{c}_{s_j,i} \tilde{Q}_i^{-d+2s_j+2i+k}(\{\partial^{\boldsymbol{\beta}} \|\boldsymbol{x}\|_K : |\beta| \le 2i\}) = 0.$$

It follows immediately from Claim III.4.7 that

$$\sum_{i=0}^{n} \tilde{c}_{s_j,i} s_j^{2i} \left(\big\|\nabla \|\boldsymbol{x}\|_K \big\|_2^{2i} + O(s_j^{-1}) \right) = 0. \tag{III.4.10}$$

Denote by $\gamma_j := \left[\tilde{c}_{s_j,0} : \tilde{c}_{s_j,1} s_j^2 : \ldots : \tilde{c}_{s_j,n} s_j^{2n}\right] \in \mathbb{RP}^n$ the homogeneous vector of coefficients on the left-hand side of (III.4.10). Since the projective space is compact, after passing to a subsequence, we may assume without loss of generality that γ_j converges to some $\gamma_* \in \mathbb{RP}^n$ as j goes to infinity. That being so, the limiting case of (III.4.10) yields

$$\sum_{i=0}^{n} \gamma_{*,i} \big\|\nabla\|\boldsymbol{x}\|_K\big\|_2^{2i} = 0. \tag{III.4.11}$$

Now, as $\big\|\nabla\|\boldsymbol{x}\|_K\big\|_2$ is continuous and satisfies a polynomial equation with constant coefficients, it must itself be constant, which gives rise to an eikonal equation of the form $\big\|\nabla\|\boldsymbol{x}\|_K\big\|_2 = r^{-1}$, $\boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, where $r > 0$ (cf. [**19**, Proposition 2.1]). However, in this special case, for $\boldsymbol{x} \in \partial K$ we may simply write

$$\nabla\|\boldsymbol{x}\|_K = h_K(u_K(\boldsymbol{x}))^{-1} u_K(\boldsymbol{x}), \tag{[20, (1.39)]}$$

where $u_K : \partial K \to \mathbb{S}^{d-1}$ is the spherical image map of K [**20**, §2.5] and $h_K : \mathbb{R}^d \to \mathbb{R}$ is the support function of K [**20**, §1.7.1]. Hence $h_K(\boldsymbol{u}) = r$ for all outer unit normal vectors $\boldsymbol{u}$ in the spherical image of K. But since u_K is surjective, it follows that

$$K = \{\boldsymbol{x} \in \mathbb{R}^d \mid \langle \boldsymbol{x}, \boldsymbol{u}\rangle \le h_K(\boldsymbol{u}) \text{ for all } \boldsymbol{u} \in \mathbb{S}^{d-1}\} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \langle \boldsymbol{x}, \boldsymbol{u}\rangle \le r \text{ for all } \boldsymbol{u} \in \mathbb{S}^{d-1}\} = \mathbb{B}^d(r),$$

whence K is indeed a Euclidean ball. □

Remark. By repeating essentially the same argument, we may see that the left-hand side of (III.4.11) restricted to ∂K is a uniform limit of a certain sequence of homogeneous polynomials. However, since multivariate homogeneous polynomials are dense in the family of continuous even functions on ∂K [**14**], this observation does not yield any further constraints.

Note that the set of monomials (i.e., power products) depending on $\boldsymbol{x}$ that appear on the left-hand sides of (III.4.8) is finite. In particular, the left-hand sides of (III.4.8) span a finite-dimensional subspace of $C(\mathbb{R}^d, \mathbb{R})$ and thus so do also the right-hand sides. The set of their non-trivial zero linear combinations forms an infinite system of polynomial equations for $\|\boldsymbol{x}\|_K^{-2}$ with polynomial coefficients. We will investigate it in the next section.

III.4.2. Solving the system of polynomial equations. Denote by $N \in \mathbb{N}$ the dimension of the subspace of $C(\mathbb{R}^d, \mathbb{R})$ spanned by $\{P_s(\boldsymbol{x})\|\boldsymbol{x}\|_K^{d-2s-k}\}_{s \ge \lceil d/2 \rceil}$ and let $\|\boldsymbol{x}\|_K^{-2} := \zeta$. Then for any tuple of indices $(s_i)_{0 \le i \le N}$ there exists a tuple of coefficients $(c_i)_{0 \le i \le N}$ such that

$$\sum_{i=0}^{N} c_i P_{s_i}(\boldsymbol{x}) \|\boldsymbol{x}\|_K^{d-2s_i-k} = 0.$$

After dividing both sides by $\|\boldsymbol{x}\|_K^{d-k}$, the above equation reads

$$\sum_{i=0}^{N} c_i P_{s_i}(\boldsymbol{x}) \zeta^{s_i} = 0, \tag{III.4.12}$$

which is an example of a polynomial equation for ζ with coefficients in the ring $\mathbb{R}[\boldsymbol{x}]$ of polynomials in $\boldsymbol{x} \in \mathbb{R}^d$. However, it is usually easier to consider polynomial equations over a field, of which we will soon take advantage.

In particular, ζ is algebraic over the field $\mathbb{R}(\boldsymbol{x})$ of rational functions in $\boldsymbol{x} \in \mathbb{R}^d$ and thus it has a minimal polynomial of the form

$$\mu_\zeta(\lambda) = \mu_{\zeta,0} + \mu_{\zeta,1}\lambda + \ldots + \mu_{\zeta,m-1}\lambda^{m-1} + \lambda^m,$$

where $\mu_{\zeta,i} \in \mathbb{R}(\boldsymbol{x})$ and $m = [\mathbb{R}(\boldsymbol{x}, \zeta) : \mathbb{R}(\boldsymbol{x})]$ is the degree of a field extension. Recall that the simple algebraic extension $\mathbb{R}(\boldsymbol{x}, \zeta)/\mathbb{R}(\boldsymbol{x})$ is a finite-dimensional vector space over $\mathbb{R}(\boldsymbol{x})$. In fact, the set $\mathcal{B} = \{1, \zeta, \ldots, \zeta^{m-1}\}$ is a vector space basis for $\mathbb{R}(\boldsymbol{x}, \zeta)$ over $\mathbb{R}(\boldsymbol{x})$ (cf. [**17**, Theorem 2.4.1]). The multiplication map $T_\zeta : \mathbb{R}(\boldsymbol{x}, \zeta) \to \mathbb{R}(\boldsymbol{x}, \zeta)$ is an $\mathbb{R}(\boldsymbol{x})$-linear operator on $\mathbb{R}(\boldsymbol{x}, \zeta)$ defined by $T_\zeta(\alpha) = \zeta\alpha$. The matrix of T_ζ with respect to the ordered basis $\mathcal{B}$ has the form

$$[T_\zeta]_\mathcal{B} = \begin{pmatrix} & & & & -\mu_{\zeta,0} \\ 1 & & & & -\mu_{\zeta,1} \\ & 1 & & & -\mu_{\zeta,2} \\ & & \ddots & & \vdots \\ & & & 1 & -\mu_{\zeta,m-1} \end{pmatrix}$$

and the characteristic polynomial of $[T_\zeta]_\mathcal{B}$ is precisely the minimal polynomial μ_ζ (cf. [**17**, Theorem 8.1.1]). The well-known Cayley-Hamilton theorem implies that $\mu_\zeta([T_\zeta]_\mathcal{B}) = 0$ and therefore $f([T_\zeta]_\mathcal{B}) = 0$ for any polynomial $f \in \mathcal{I}_\zeta$, which follows from Theorem III.3.12. In particular, for every linear combination (III.4.12) we have

$$\sum_{i=0}^{N} c_i P_{s_i} [T_\zeta]_\mathcal{B}^{s_i} = 0.$$

LEMMA III.4.13. *Let X, Y be real vector spaces and let $\{\boldsymbol{x}_i\}_{i\in I} \subseteq X, \{\boldsymbol{y}_i\}_{i\in I} \subseteq Y$ be sets of vectors. Then there is a linear map $f : X \to Y$ such that $f(\boldsymbol{x}_i) = \boldsymbol{y}_i$ for each $i \in I$ if and only if for all finite subsets $J \subseteq I$ and sets of scalars $\{a_j\}_{j\in J} \subseteq \mathbb{R}$ such that $\sum_{j\in J} a_j \boldsymbol{x}_j = \boldsymbol{0}$ the equality $\sum_{j\in J} a_j \boldsymbol{y}_j = \boldsymbol{0}$ holds.*

Since the proof is an easy exercise from linear algebra, we leave it to the reader.

Now, it follows from Lemma III.4.13 that there exists an $\mathbb{R}$-linear map $f : C(\mathbb{R}^d, \mathbb{R}) \to \mathbb{R}(\boldsymbol{x})^{m\times m}$ such that $f(P_s\zeta^s) = P_s[T_\zeta]_\mathcal{B}^s$ for each $s \geq \lceil d/2 \rceil$. In particular, the rank-nullity theorem implies

$$\dim(\operatorname{span}(\{P_s[T_\zeta]_\mathcal{B}^s\}_{s\geq\lceil d/2\rceil})) = \dim(\operatorname{span}(\{f(P_s\zeta^s)\}_{s\geq\lceil d/2\rceil})) \leq \dim(\operatorname{span}(\{P_s\zeta^s\}_{s\geq\lceil d/2\rceil})) = N,$$

whence $\{P_s[T_\zeta]_\mathcal{B}^s\}_{s\geq\lceil d/2\rceil}$ span a finite-dimensional subspace of $\mathbb{R}(\boldsymbol{x})^{m\times m}$. Therefore the problem has been reduced to a question about rational functions, to which we can now apply the theory of valued fields.

CLAIM III.4.14. Let L/K be a field extension and let $v : L \to \mathbb{R} \cup \{\infty\}$ be a valuation on L, trivial on K. Suppose that $\{x_i\}_{i\in I} \subseteq L$ is a subset of L such that $\{v(x_i)\}_{i\in I} \subseteq \mathbb{R}$ are pairwise different. Then $\{x_i\}_{i\in I}$ is K-linearly independent. For suppose that there exists a finite subset $J \subseteq I$ and a set of scalars $\{a_j\}_{j\in J} \subseteq K$ such that $\sum_{j\in J} a_j x_j = 0$. Since $v(a_j x_j) = v(a_j) + v(x_j) = v(x_j)$ are again pairwise different, we have

$$\infty = v(0) = v\left(\sum_{j\in J} a_j x_j\right) = \min_{j\in J} v(x_j),$$

which reads $v(x_j) = \infty$ for all $j \in J$, a contradiction. In particular, if $\{x_i\}_{i\in I}$ span a finite-dimensional K-linear space, then the set $\{v(x_i)\}_{i\in I}$ is necessarily finite. $\square$

Let $p \in \mathbb{R}[\boldsymbol{x}]$ be any irreducible polynomial and denote by $\hat{v}_p$ some extension of the p-adic valuation on $\mathbb{R}(\boldsymbol{x})$ to the splitting field of μ_ζ, i.e., the smallest extension of $\mathbb{R}(\boldsymbol{x})$ containing all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $[T_\zeta]_\mathcal{B}$. Now, observe that at least one of the coefficients $\mu_{\zeta,m-i}$, $i = 1, 2, \ldots, m$ is non-zero. Otherwise, the minimal polynomial μ_ζ would be reducible, a contradiction. Let $\mu_{\zeta,m-i}$ be some non-zero coefficient of μ_ζ. By Viète's formulas for μ_ζ we have

$$\mu_{\zeta,m-i} = (-1)^i \sum_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_i\leq m} \lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_i}.$$

In particular,

$$\begin{aligned} v_p(\mu_{\zeta,m-i}) &= \hat{v}_p\left((-1)^i \sum_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_i\leq m} \lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_i}\right) \\ &\geq \min_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_i\leq m} \hat{v}_p(\lambda_{\alpha_1}\lambda_{\alpha_2}\cdots\lambda_{\alpha_i}) \\ &= \min_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_i\leq m} \hat{v}_p(\lambda_{\alpha_1}) + \hat{v}_p(\lambda_{\alpha_2}) + \ldots + \hat{v}_p(\lambda_{\alpha_i}), \end{aligned}$$

whence there exists an eigenvalue λ of $[T_\zeta]_{\mathcal{B}}$ such that $\hat{v}_p(\lambda) \leq v_p(\mu_{\zeta,m-i})/i$. Without loss of generality we may reorder the eigenvalues so that $\hat{v}_p(\lambda_1), \hat{v}_p(\lambda_2), \ldots, \hat{v}_p(\lambda_j) \leq \hat{v}_p(\lambda)$ and $\hat{v}_p(\lambda_{j+1}), \hat{v}_p(\lambda_{j+2}), \ldots, \hat{v}_p(\lambda_m) > \hat{v}_p(\lambda)$ for some $1 \leq j \leq m$. Denote by

$$\chi_{[T_\zeta]^s_{\mathcal{B}}}(\lambda) := \det(\lambda I - [T_\zeta]^s_{\mathcal{B}})$$

the characteristic polynomial of $[T_\zeta]^s_{\mathcal{B}}$. Since the eigenvalues of $[T_\zeta]^s_{\mathcal{B}}$ are precisely $\lambda_1^s, \lambda_2^s, \ldots, \lambda_m^s$, again by Viète's formulas for $\chi_{[T_\zeta]^s_{\mathcal{B}}}$ we have

$$\chi_{[T_\zeta]^s_{\mathcal{B}},m-j} = (-1)^j \sum_{1\leq\alpha_1<\alpha_2<\ldots<\alpha_j\leq m} \lambda^s_{\alpha_1}\lambda^s_{\alpha_2}\cdots\lambda^s_{\alpha_j}, \tag{III.4.15}$$

where $\chi_{[T_\zeta]^s_{\mathcal{B}},m-j}$ stands for the coefficient of λ^{m-j}. This time we know, however, that $\lambda_1^s\lambda_2^s\cdots\lambda_j^s$ attains the smallest valuation among all summands on the right-hand side of (III.4.15), whence

$$v_p(\chi_{[T_\zeta]^s_{\mathcal{B}},m-j}) = \hat{v}_p(\lambda_1^s\lambda_2^s\cdots\lambda_j^s) = s(\hat{v}_p(\lambda_1) + \hat{v}_p(\lambda_2) + \ldots + \hat{v}_p(\lambda_j)) \leq jsv_p(\mu_{\zeta,m-i})/i.$$

On the other hand, using the fact that $\chi_{[T_\zeta]^s_{\mathcal{B}},m-j}$ may be computed as the sum of all principal minors of $[T_\zeta]^s_{\mathcal{B}}$ of size j and each of those minors may itself be computed as the sum of products of certain entries of the matrix $[T_\zeta]^s_{\mathcal{B}}$ of length j, it follows that

$$v_p(\chi_{[T_\zeta]^s_{\mathcal{B}},m-j}) \geq jv_p([T_\zeta]^s_{\mathcal{B}}),$$

where $v_p(M)$ denotes the minimum of valuations taken over all entries of a matrix M. Chaining those two inequalities yields

$$v_p([T_\zeta]^s_{\mathcal{B}}) \leq sv_p(\mu_{\zeta,m-i})/i$$

and consequently

$$v_p(P_s[T_\zeta]^s_{\mathcal{B}}) \leq v_p(P_s) + sv_p(\mu_{\zeta,m-i})/i.$$

Finally, by Claim III.4.14 we have $v_p(P_s[T_\zeta]^s_{\mathcal{B}}) = O(1)$, which implies

$$v_p(P_s) \geq -sv_p(\mu_{\zeta,m-i})/i + O(1). \tag{III.4.16}$$

Claim III.4.17. Since ζ is homogeneous of degree -2, the coefficient $\mu_{\zeta,m-i}$ is homogeneous of degree $-2i$ for every $1 \leq i \leq m$. Indeed, for every $t \neq 0$ and $x \in \mathbb{R}^d$ we have

$$\begin{aligned} 0 &= t^{-2m}\mu_\zeta(\zeta(\boldsymbol{x})) \\ &= t^{-2m}\mu_\zeta(t^2\zeta(t\boldsymbol{x})) \\ &= t^{-2m}\mu_{\zeta,0}(x) + t^{-2m+2}\mu_{\zeta,1}(\boldsymbol{x})\zeta(t\boldsymbol{x}) + \ldots + t^{-2}\mu_{\zeta,m-1}(\boldsymbol{x})\zeta(t\boldsymbol{x})^{m-1} + \zeta(t\boldsymbol{x})^m \\ &= t^{-2m}\mu_{\zeta,0}(t^{-1}\boldsymbol{y}) + t^{-2m+2}\mu_{\zeta,1}(t^{-1}\boldsymbol{y})\zeta(\boldsymbol{y}) + \ldots + t^{-2}\mu_{\zeta,m-1}(t^{-1}\boldsymbol{y})\zeta(\boldsymbol{y})^{m-1} + \zeta(\boldsymbol{y})^m, \end{aligned}$$

where $\boldsymbol{y} := t\boldsymbol{x}$. Recall that μ_ζ is the unique monic polynomial of degree m with root at ζ, which implies

$$t^{-2i}\mu_{\zeta,m-i}(t^{-1}\boldsymbol{y}) = \mu_{\zeta,m-i}(\boldsymbol{y})$$

for every $1 \leq i \leq m$. ☐

Since $\mathbb{R}(\boldsymbol{x})$ is the field of fractions of a unique factorization domain, there is a system of irreducible polynomials $\mathcal{P} \subset \mathbb{R}[\boldsymbol{x}]$ such that every non-zero element $f \in \mathbb{R}(\boldsymbol{x})$ admits a unique representation

$$f = u \prod_{p \in \mathcal{P}} p^{v_p(f)}, \tag{III.4.18}$$

where $u \in \mathbb{R} \setminus \{0\}$ is invertible and the integral exponents $v_p(f) \in \mathbb{Z}$ are non-zero for only a finite number of elements $p \in \mathcal{P}$. This representation may be viewed as an analog of the *product formula* for rational numbers. Computing the degree valuation of both sides of (III.4.18) yields

$$v_\infty(f) = -\sum_{p \in \mathcal{P}} v_p(f) \deg p. \tag{III.4.19}$$

Denote by $\mathcal{P}_{m-i} \subset \mathcal{P}$ the finite subset of all irreducible polynomials $p \in \mathcal{P}$ such that $v_p(\mu_{\zeta,m-i}) \neq 0$. From our considerations so far, it follows that

$$\begin{aligned}
-d + 2s + k &= -v_\infty(P_s) \\
&\overset{\text{(III.4.19)}}{=} \sum_{p \in \mathcal{P}} v_p(P_s) \deg p \\
&\geq \sum_{p \in \mathcal{P}_{m-i}} v_p(P_s) \deg p \\
&\overset{\text{(III.4.16)}}{\geq} \sum_{p \in \mathcal{P}_{m-i}} -s v_p(\mu_{\zeta,m-i}) \deg p / i + O(1) \\
&\overset{\text{(III.4.19)}}{=} s v_\infty(\mu_{\zeta,m-i})/i + O(1) \\
&\overset{\text{Claim III.4.17}}{=} 2s + O(1)
\end{aligned}$$

for every non-zero $\mu_{\zeta,m-i}$. Hence all the inequalities used above actually must be equalities up to some bounded error term. Thus

$$v_p(P_s) = -s v_p(\mu_{\zeta,m-i})/i + O(1) \tag{III.4.20}$$

for every $p \in \mathcal{P}$. Now, observe that $\mu_{\zeta,0} \neq 0$, because otherwise μ_ζ would be reducible. That being so, for every $p \in \mathcal{P}$ and every non-zero $\mu_{\zeta,m-i}$ we have

$$-s v_p(\mu_{\zeta,m-i})/i + O(1) = v_p(P_s) = -s v_p(\mu_{\zeta,0})/m + O(1),$$

whence asymptotically (as s goes to infinity) we get

$$v_p(\mu_{\zeta,m-i})/i = v_p(\mu_{\zeta,0})/m.$$

By the product formula (III.4.18), it means that $\mu_{\zeta,m-i}$ and $\mu_{\zeta,0}^{i/m}$ are associated, i.e.,

$$\mu_{\zeta,m-i} = u_{m-i} \mu_{\zeta,0}^{i/m}$$

for some unit $u_{m-i} \in \mathbb{R} \setminus \{0\}$. On the other hand, if $\mu_{\zeta,m-i} = 0$, the same equality holds if we simply put $u_{m-i} := 0$. Thus

$$\begin{aligned}
0 &= \mu_{\zeta,0}^{-1} \mu_\zeta(\zeta) \\
&= u_0 + u_1 \mu_{\zeta,0}^{-1/m} \zeta + \ldots + u_{m-1} \mu_{\zeta,0}^{-(m-1)/m} \zeta^{m-1} + \mu_{\zeta,0}^{-1} \zeta^m \\
&= u_0 + u_1 \left(\mu_{\zeta,0}^{-1/m} \zeta\right) + \ldots + u_{m-1} \left(\mu_{\zeta,0}^{-1/m} \zeta\right)^{m-1} + \left(\mu_{\zeta,0}^{-1/m} \zeta\right)^m
\end{aligned}$$

is a polynomial equation with constant coefficients satisfied by $\mu_{\zeta,0}^{-1/m} \zeta$, which therefore must itself be constant, equal to some $r \in \mathbb{R}$. Finally, since $v_p(P_s) \geq 0$ for every $p \in \mathcal{P}$, it follows from (III.4.20) that

$v_p(\mu_{\zeta,0}) \leq 0$, which means that

$$\|\boldsymbol{x}\|_K = \zeta^{-1/2} = (r^m \mu_{\zeta,0})^{-1/(2m)} \tag{III.4.21}$$

is a root of order $2m$ of some homogeneous polynomial $(r^m\mu_{\zeta,0})^{-1}$ of degree $2m$.

III.4.3. Filtering out the incidental solutions. Although (III.4.21) already imposes a rigid structure on the Minkowski functional $\|\boldsymbol{x}\|_K$, it does not solve the problem immediately. Indeed, this condition is satisfied, e.g., when K is the unit ball in ℓ^d_{2m}. Moreover, for every Minkowski functional satisfying (III.4.21) we can easily find a sequence of polynomials P_s such that $\{P_s(\boldsymbol{x})\|\boldsymbol{x}\|_K^{d-2s-k}\}_{k\in\mathbb{N}}$ span a finite-dimensional subspace of $C(\mathbb{R}^d,\mathbb{R})$. Therefore we need to go back to the very beginning of our argument to get the desired contradiction.

Slightly abusing the notation, let $\|\boldsymbol{x}\|_K := \zeta$ with $\zeta^{2m} := h(\boldsymbol{x})$ being a homogeneous polynomial of degree $2m$. Without loss of generality, we may assume that h is not a perfect power. Then $A := \mathbb{R}(\boldsymbol{x},\zeta)$ may be viewed as a graded algebra

$$A = \bigoplus_{i\in C_{2m}} A_i, \quad A_i := \mathbb{R}(\boldsymbol{x})\zeta^i,$$

where the index set is the cyclic group C_{2m}.

Claim III.4.22. The Laplace operator Δ defines a graded endomorphism of A, i.e., for every function $f \in A_i$, $i \in C_{2m}$, we have $\Delta f \in A_i$. Indeed, for $f := gh^\nu$, where $g \in \mathbb{R}(\boldsymbol{x})$ and $\nu := \frac{i}{2m}$, we have

$$\begin{aligned}\Delta f &= \Delta g h^\nu + 2\nu\nabla g\cdot\nabla h h^{\nu-1} + \nu g\Delta h h^{\nu-1} + \nu(\nu-1)g\nabla h\cdot\nabla h h^{\nu-2}\\ &= \left(\Delta g + 2\nu\nabla g\cdot\nabla h h^{-1} + \nu g\Delta h h^{-1} + \nu(\nu-1)g\nabla h\cdot\nabla h h^{-2}\right)h^\nu.\end{aligned} \tag{III.4.23}$$

Clearly the expression in parentheses is again an element of $\mathbb{R}(\boldsymbol{x})$. ☐

Remark. Furthermore, it follows from (III.4.23) that $v_p(\Delta f) = v_p(f) - 2$ for any generic irreducible polynomial $p \in \mathbb{R}[\boldsymbol{x}]$, unless $v_p(f) \in \{0,1\}$. Indeed, for $f := gp^\nu$, where $g \in \mathbb{R}(\boldsymbol{x})$ is not divisible by p and $\nu \in \mathbb{Z}$, we have

$$\Delta f = \left(\Delta g p^2 + 2\nu\nabla g\cdot\nabla p p + \nu g\Delta p p + \nu(\nu-1)g\nabla p\cdot\nabla p\right)p^{\nu-2}.$$

Now, the expression in parentheses is generally not divisible by p unless $p \mid \nabla p\cdot\nabla p$. In other words, the dual variety of the projective hypersurface defined by p is contained in the standard hyperquadric. In particular, for $d \leq 3$ there are no such polynomials p with $\deg p > 2$, but already for $d = 4$ we have e.g.

$$\begin{aligned}&4x_0^8 + 28x_0^6x_1^2 + 16x_0^6x_2^2 - 20x_0^6x_3^2 + 73x_0^4x_1^4 + 124x_0^4x_1^2x_2^2 - 90x_0^4x_1^2x_3^2 - 8x_0^4x_2^4 + 60x_0^4x_2^2x_3^2 + 33x_0^4x_3^4\\ &+ 84x_0^2x_1^6 + 270x_0^2x_1^4x_2^2 - 124x_0^2x_1^4x_3^2 + 180x_0^2x_1^2x_2^4 + 140x_0^2x_1^2x_2^2x_3^2 + 60x_0^2x_1^2x_3^4 - 48x_0^2x_2^6\\ &- 124x_0^2x_2^4x_3^2 - 90x_0^2x_2^2x_3^4 - 20x_0^2x_3^6 + 36x_1^8 + 180x_1^6x_2^2 - 48x_1^6x_3^2 + 297x_1^4x_2^4 + 180x_1^4x_2^2x_3^2 - 8x_1^4x_3^4\\ &+ 180x_1^2x_2^6 + 270x_1^2x_2^4x_3^2 + 124x_1^2x_2^2x_3^4 + 16x_1^2x_3^6 + 36x_2^8 + 84x_2^6x_3^2 + 73x_2^4x_3^4 + 28x_2^2x_3^6 + 4x_3^8,\end{aligned} \tag{III.4.24}$$

obtained as the dual variety to the complete intersection of two hyperquadrics defined by $x_0^2+x_1^2+x_2^2+x_3^2$ and $x_1^2 + 2x_2^2 + 3x_3^2$. The above example is fairly complicated, and unfortunately, this will always be the case as long as we believe in the celebrated Hartshorne's conjecture (cf. [**9**]). Nevertheless, we showed that $v_p(\Delta f) = v_p(f) - 2$ unless $p \mid \nabla p\cdot\nabla p$, when $v_p(\Delta f) \geq v_p(f) - 1$. The question of whether inequality may be replaced with equality seems to be hard, and likely the answer in that generality will be negative.

Having said all that, we are ready to finish the proof. Firstly, suppose that $d-k$ is odd. Going back as far as (III.4.4), for every $s \in \mathbb{N}$ there exist scalars $c_{s,0}, c_{s,1}, \ldots, c_{s,n}$, not all zero, such that

$$\sum_{i=0}^{n} c_{s,i} V_{K,H}^{(2(ms+\frac{d-k+1}{2})+2mi+k)}(0) = 0,$$

whence the distribution

$$\sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} \|\boldsymbol{x}\|_K^{2ms+2mi+1} = \sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} h^{s+i+\frac{1}{2m}} = \sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} [h^{s+i}\zeta]$$

is a polynomial (cf. (III.4.6)), i.e., the element of A_0. On the other hand, by Claim III.4.22, it is also an element of $A_1 \neq A_0$, in which case it must be zero. It follows from Claim III.4.9 that K is a Euclidean ball.

Secondly, suppose that $d-k$ is even and $m > 1$. Then for every $s \in \mathbb{N}$ there exist scalars $c_{s,0}, c_{s,1}, \ldots, c_{s,n}$, not all zero, such that

$$\sum_{i=0}^{n} c_{s,i} V_{K,H}^{(2(ms+\frac{d-k+2}{2})+2mi+k)}(0) = 0,$$

whence the distribution

$$\sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} \|\boldsymbol{x}\|_K^{2ms+2mi+2} = \sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} h^{s+i+\frac{2}{2m}} = \sum_{i=0}^{n} \tilde{c}_{s,i} \Delta^{mi} [h^{s+i}\zeta^2]$$

is a polynomial, i.e., the element of A_0. On the other hand, this time it is also an element of $A_2 \neq A_0$, in which case it must be zero anyway. Again, it follows that K is a Euclidean ball.

Finally, suppose that $d-k$ is even and $m = 1$. But then K defined by (III.3.2) is a hyperquadric, which concludes the proof. □

Remark. Note that, in fact, we proved a much stronger theorem. Indeed, we did not take full advantage of assumption (III.2.5) (formulated in terms of the space of germs), but instead, we used only its infinitesimal version (III.4.3) (formulated in terms of the space of jets). The latter, in general, contains less information unless we restrict ourselves to analytic functions when the two coincide. However, in order to avoid further complicating an already complicated assumption and thus overshadowing the main idea, we have deliberately abandoned the formally weaker formulation in favor of a much simpler and more intuitive one.

Acknowledgements

We would like to thank Prof. M. Wojciechowski for offering many valuable comments, including the remark on the superposition theorem of Kolmogorov, and Prof. J. Buczyński for pointing out the counterexample (III.4.24) and his eagerness to discuss the algebraic geometry behind the proof.

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APPENDIX A

Computer assistance in symbolic computations

All functions were implemented in Wolfram Mathematica 11.0.0. The code itself can be found online at `https://cloud.impan.pl/s/1kIXqE87mHVn7nT`. Computations were performed on a Linux x86 (64-bit) machine with a single Intel® Xeon® CPU E5-2697 v3 processor and 64GB memory. The total execution time was negligible.

A.1. Notebook-1.nb

In the beginning, we use symbolic differentiation D to obtain the Wronskian matrix of (II.3.2). Afterward, we use Minors to compute 210 symbolic determinants of order 4 and thus find out that the four rows corresponding to 3^{rd} order partial derivatives are indeed linearly dependent. Then we again use Minors to compute 11 symbolic determinants of order 10, among which the only non-trivial ones are $W_{3,0}$, $W_{2,1}$, $W_{1,2}$ and $W_{0,3}$. Finally, we use Minors to compute 10 symbolic determinants of order 9 and select the simplest-looking ones. Based on them, we solve some simple linear equations to find out that all the featured minors can not vanish simultaneously. Performing the same calculations with pen and paper would be tedious, however possible. Although we sometimes applied Factor to factorize the results, in all cases the factorization turned out to be trivial.

A.2. Notebook-2.nb

In the beginning, we use symbolic differentiation D to compute the left-hand side of (II.3.8). Afterward, we use CoefficientArrays to extract the explicit form of the matrix $\mathbf{A}_5$ of order 6. Then we apply Det and Factor to obtain its determinant in the form from which we can readily see that it is an element of the multiplicative submonoid S. Finally, we repeat the same steps for $\mathbf{A}_4$ of order 5 and $\mathbf{A}_3$ of order 4. The same calculations could well be done with pen and paper, though it is pointless.

A.3. Notebook-3.nb

At the beginning, we solve the quadratic equation (II.3.1) for f, assuming previously that $a_{33} = 1$. Afterward, we use symbolic differentiation to obtain the explicit formula for δ. Then we use Grad to compute the Jacobian matrix of δ with respect to the 11-dimensional vector (II.3.10). Now, since its symbolic determinant is difficult to compute even for a supercomputer, we instantiate the matrix at (II.3.11) and only then we apply Det and Factor to obtain its determinant of order 11 in the simplest form. The content of this notebook is by far the most demanding computational task because, in addition to the heavy workload, it also requires manipulating algebraic expressions containing square roots.

A.4. Notebook-4.nb

In the beginning, we define p_1, p_2, u, v and verify that u, v satisfy (II.4.2), which requires symbolic differentiation and manipulating algebraic expressions containing square roots. Afterward, we use CoefficientArrays to extract the explicit form of the matrix $\mathbf{A}$ of order 4. Then we apply Det and Together to obtain its determinant in the simplest form. Further, we solve the quadratic equation (II.3.1) for f and then put the result into the formula for $u + iv$. We apply Together and PowerExpand to bring the result to a simpler form. Finally, we define the matrix $\mathbf{Q}$ of order 4 and verify the formula (II.5.6), using Minors to compute 16 symbolic determinants of order 3 along the way. Then we apply Together to force the expansion of the underlying expression. At the very end, we verify Assertion II.5.8, using symbolic differentiation D composed with Det to obtain the Hessian determinant of f and Discriminant to compute the discriminant of (II.3.1) with respect to the variable f. Again, we apply Together to force the expansion of the underlying expression. The same calculations could well be done with pen and paper, though it is pointless.