Summary of the PhD dissertation of Jacek Krajczok

The PhD dissertation titled "Modular properties of locally compact quantum groups" is concerned with certain aspects of the theory of locally compact quantum groups. Such objects are generalisations of topological groups which are locally compact[1]. One of the motivations to introduce them, was a desire to extend the Pontryagin duality: if G is an abelian locally compact group, then the set $\widehat{G}$ of (strongly continuous) irreducible representations carries a structure of an abelian locally compact group. Since all irreducible representations of G are one dimensional, we can define the multiplication on $\widehat{G}$ via $(\chi\eta)(g) = \chi(g)\eta(g)\,(\chi, \eta \in \widehat{G}, g \in G)$. Such constructed group $\widehat{G}$ is called a group dual to G in the sense of Pontryagin. This construction can be repeated for $\widehat{G}$ – we will end up with the original group. To be more precise, the group $\widehat{\widehat{G}}$ is canonically isomorphic with G, where $g \in G$ corresponds to a representation of $\widehat{G}$ given by $\widehat{G} \ni \chi \mapsto \chi(g) \in \mathbb{T}$. The most famous dual groups are $\widehat{\mathbb{R}} \simeq \mathbb{R}$ and $\widehat{\mathbb{T}} \simeq \mathbb{Z}$.

In this work we use the definition of a locally compact quantum group $\mathbb{G}$ given by Kustermans and Vaes [4]. According to their definition, every locally compact quantum group $\mathbb{G}$ is described by a (possibly non-commutative) von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$ called the algebra of bounded functions on $\mathbb{G}$, comultiplication $\Delta\colon \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})\bar{\otimes}\mathrm{L}^\infty(\mathbb{G})$ and left, right Haar integrals φ, ψ – these are weights on $\mathrm{L}^\infty(\mathbb{G})$. A particular class of locally compact quantum groups is formed by compact or discrete quantum groups – their definition was introduced by Woronowicz in [7, 5]. Pontryagin duality can be extended to the class of locally compact quantum groups: with every such group $\mathbb{G}$ we can associate its dual quantum group $\widehat{\mathbb{G}}$, wheras $\widehat{\widehat{\mathbb{G}}}$ is canonically isomorphic with $\mathbb{G}$. As in the classical theory, $\mathbb{G}$ is compact if, and only if $\widehat{\mathbb{G}}$ is discrete. Furthermore, these objects do extend the class of locally compact groups: every classical group G can be treated as a locally compact quantum group in the following way: its von Neumann algebra of bounded functions is $\mathrm{L}^\infty(G)$, the commutative algebra of (equivalence classes of) measurable, bounded functions, comultiplication Δ is the map dual to multiplication and weights φ, ψ are given by integration with respect to the Haar measures. Its dual quantum group $\widehat{G}$ is related to the operator algebras studied in the abstract harmonic analysis: group C*-algebras $\mathrm{C}^*_r(G), \mathrm{C}^*(G)$ and the group von Neumann algebra $\mathrm{L}(G)$.

An exceptionally interesting phenomenon that appears in the theory of quantum groups is non-traciality of Haar integrals; let us denote by h the Haar integral on $\mathrm{SU}_q(2)\,(0 < q < 1)$, the compact quantum group introduced by Woronowicz in [6]. We can find elements $a, b \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ for which $h(ab) \neq h(ba)$. Non-traciality of Haar integrals is a

[1] We will use the convention according to which locally compact spaces are Hausdorff.

source of many intriguing problems which are at the heart of this dissertation. Using the Tomita–Takesaki theory we can define groups of modular automorphisms $(\sigma_t^{\varphi})_{t\in\mathbb{R}}, (\sigma_t^{\psi})_{t\in\mathbb{R}}$ associated with the left φ and the right ψ Haar integral. Besides those, there is also a third group of automorphisms – the scaling group $(\tau_t)_{t\in\mathbb{R}}$. These maps act on the von Neumann algebra $\mathrm{L}^{\infty}(\mathbb{G})$. We do not see these groups of automorphisms in the classical case (they are trivial). However, groups of modular automorphisms appear already in the case of quantum groups dual to classical ones. If G is a classical locally compact group, then the Haar integral $\widehat{\varphi}$ on $\widehat{G}$ is tracial (equivalently: the modular automorphisms $(\sigma_t^{\widehat{\varphi}})_{t\in\mathbb{R}}$ are trivial) if, and only if G is unimodular. We can see in this example, that there is a relation between unimodularity of a quantum group, and traciality of Haar integrals on its dual. In the general case this connection is more complicated and this kind of relations are among the problems we study in the dissertation.

An important part of our work is concerned with the class of type I locally compact quantum groups. These are quantum groups which full group C*-algebra (which we can identify with the universal version of the C*-algebra of C_0 function on a dual quantum group $\widehat{\mathbb{G}}$) is of type I. A seminal result concerning these quantum groups is a theorem of Desmedt [1], which establish an existence of the Plancherel measure and associated objects. In particular, it gives us a unitary operator $\mathcal{Q}_L\colon \mathrm{L}^2(\mathbb{G}) \to \int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{HS}(\mathsf{H}_\pi)\,\mathrm{d}\mu(\pi)$ (where $\mathrm{L}^2(\mathbb{G})$ is the Hilbert space coming from the GNS representation for φ and $\mathrm{Irr}(\mathbb{G})$ is the space of irreducible representations of $\mathbb{G}$), which maps the von Neumann algebra $\mathrm{L}^{\infty}(\widehat{\mathbb{G}})$ onto the direct integral $\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} \mathrm{B}(\mathsf{H}_\pi)\otimes \mathbb{1}_{\overline{\mathsf{H}_\pi}}\,\mathrm{d}\mu(\pi)$. The operator $\mathcal{Q}_L$ allows us to express the left Haar integral $\widehat{\varphi}$ on $\widehat{\mathbb{G}}$ using a measurable field of strictly positive, self-adjoint operators $(D_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$ (an analogous result holds for the right Haar integral $\widehat{\psi}$ – it is associated with operators $(E_\pi)_{\pi\in\mathrm{Irr}(\mathbb{G})}$). In the dissertation we have established equations which express operators canonically associated with $\mathbb{G}$ and $\widehat{\mathbb{G}}$ on the level of direct integrals, for example:

$$\nabla_{\widehat{\varphi}}^{it} = \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} D_\pi^{-2it}\otimes (D_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L,$$
$$\nabla_{\widehat{\psi}}^{it} = \mathcal{Q}_L^*\big(\int_{\mathrm{Irr}(\mathbb{G})}^{\oplus} E_\pi^{-2it}\otimes (E_\pi^{2it})^{\top}\,\mathrm{d}\mu(\pi)\big)\mathcal{Q}_L,$$

for $t\in\mathbb{R}$, where $\nabla_{\widehat{\psi}}, \nabla_{\widehat{\varphi}}$ are modular operators associated with integrals $\widehat{\psi}, \widehat{\varphi}$. For type I quantum groups we are able to express properties such as unimodularity (the equality of integrals φ and ψ) or traciality of Haar integrals in terms of operators D_π, E_π ($\pi\in\mathrm{Irr}(\mathbb{G})$).

Using results concerning type I quantum groups, together with Piotr Sołtan we were able to prove a result stating that the quantum disc (described by the Toeplitz algebra) does not admit a structure of a compact quantum group [2].

Another work included in the PhD dissertation are results obtained together with Mateusz Wasilewski in [3]. The problem we were studying is a question whether the von Neumann algebra $\mathscr{C}_{O_F^+}$ generated by characters is maximal abelian in $\mathrm{L}^{\infty}(O_F^+)$, the algebra of bounded functions on the quantum orthogonal group O_F^+, in the non-Kac case (i.e. when the Haar integral is not tracial). We obtained a negative answer. Our techniques allowed us to obtain also an interesting result concerning the von Neumann algebra $\mathrm{L}^\infty(U_F^+)$ of bounded functions on a quantum unitary group U_F^+: (under some conditions) we showed that the relative commutant $\mathscr{C}'_{U_F^+} \cap \mathrm{L}^\infty(U_F^+)$ is not contained in $\mathscr{C}_{U_F^+}$. These results were obtained using the notion of a quasi-split inclusion $\mathscr{C}_{\mathbb{G}} \subseteq \mathrm{L}^\infty(\mathbb{G})$. In this part of the dissertation we also present a construction of a compact quantum group $\mathbb{H}$, which appears as the bicrossed product $\mathbb{H} = \mathrm{SU}_q(2) \bowtie \mathbb{Q}$. It has interesting properties: some of its scaling automorphisms are inner, and its von Neumann algebra $\mathrm{L}^\infty(\mathbb{H})$ is the injective type II_∞ factor.

In our work we also present results connecting approximation properties of a (usually discrete) quantum group $\mathbb{G}$ and the von Neumann algebra $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. We focus on amenability for the quantum group $\mathbb{G}$ and w*-completely positive approximation property (w*-CPAP) for $\mathrm{L}^\infty(\widehat{\mathbb{G}})$. Connections like this are present in the literature in the case when $\mathbb{G}$ has tracial Haar integrals (that is, when $\widehat{\mathbb{G}}$ is of Kac type), but in the general case of discrete quantum groups, the equivalence between amenability of $\mathbb{G}$ and w*-CPAP of $\mathrm{L}^\infty(\widehat{\mathbb{G}})$ is an open problem. We obtained a partial result: equivalence of this type is true, provided we modify the w*-CPAP in a such way that it takes into consideration also the von Neumann algebra $\ell^\infty(\mathbb{G})$.

References

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