Singular limits and rough behavior in evolutionary equations arising in physics and biology

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I hereby declare that this thesis is my own work.

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Abstract

The thesis discusses two topics in the theory of evolutionary equations: analysis of singular limits in PDEs and analysis of PDEs with non-standard growth where the growth changes irregularly over time. All are motivated by applications.

In the first part, we study singular limits of several PDEs from mathematical biology and physics. We begin with the fast reaction limit for a reaction-diffusion-ODE system with a nonmonotone fast reaction as motivated by applications in neuroscience. Conversely to what was observed so far for this type of problem, in the limit, we observe fast oscillations which we analyse precisely with the theory of Young measures.

Next, we study the hydrodynamic limit of the Vlasov-type equation with the appropriately chosen force so that in the limit we obtain the Cahn-Hilliard equation, a fourth-order PDE used in materials science and tumor growth. This is the first result aiming at the rigorous derivation of this macroscopic equation from the microscopic one, motivated by formal computations of Takata and Noguchi (*J. Stat. Phys.*, 2018).

Subsequently, we prove the convergence of the nonlocal Cahn-Hilliard equation to the local one. This problem has been extensively studied in recent years. Our work is the first one to consider degenerate mobilities as motivated by applications to tumor growth and cell adhesion. It can be viewed as a completion of the Giacomin-Lebowitz derivation of the Cahn-Hilliard equation from particle processes on the lattice (*J. Stat. Phys.*, 1997): they derived the nonlocal equation and left it open to prove its convergence to the local equation. This is the gap we fill with our results.

Finally, we discuss the convergence of the Euler-Korteweg equation to the Cahn-Hilliard equation in the so-called high-friction limit. This problem was studied recently by Lattanzio and Tzavaras (*Comm. PDEs*, 2017) who proved, using the rela-

tive entropy method, the convergence under the assumption that the limiting system admits a smooth positive solution. However, there is no theory guaranteeing that. Therefore, we propose to study the nonlocal Euler-Korteweg equation. Then, the limiting system is the nonlocal Cahn-Hilliard equation which has the desired properties and we can conclude by the relative entropy method. Furthermore, using the result of nonlocal-to-local convergence for the Cahn-Hilliard equation, we obtain convergence of the nonlocal Euler-Korteweg equation to the local Cahn-Hilliard equation.

The second part of the thesis is concerned with parabolic PDEs of non-standard growth where the growth is changing discontinuously in time. The classical example is the $p(t,x)$-Laplace equation with p strictly separated from 1 and $+\infty$. We prove existence and uniqueness of solutions for p discontinuous in t and log-Hölder continuous in x. This is the first result of this type as all the papers so far assumed log-Hölder continuity in both t and x. The proof is based on a simple observation that mollification in space of a solution to a parabolic equation is already regular in time.

Subsequently, we extend the existence result to the case of non-Newtonian fluids with stress tensor which is discontinuous in time. This is well-motivated by behavior of electrorheological fluid (a fluid composed of charged particles) moving in the electric field which drastically changes in time.

Finally, we briefly report on our result on double phase functionals, that is functionals switching their growth between p and q, depending on the point of the space. This is a thoroughly studied topic since the groundbreaking work of Mingione and Colombo (*ARMA*, 2014). Using methods we developed, we improve so-far known range of exponents p, q such that the minimizers can be approximated in a nice way (so-called lack of Lavrentiev phenomenon). In the case $p \leq d$ (d is the dimension), we obtain the first sharp result concerning the range of exponents (A. K. Balci et al, *Calc. Var. PDE*, 2020). This is important as it is usually the first step to prove the smoothness of the minimizers. In applications, such minimizers describe the optimal configuration of a composite material under an external force.

Streszczenie

Praca dotyczy dwóch zagadnień z ewolucyjnych równań cząstkowych: analizy singularnych granic oraz analizy równań z niestandardowym wzrostem, gdzie wzrost zmienia się nieregularnie w czasie. Wszystkie są umotywowane zastosowaniami.

W pierwszej części pracy badamy granice osobliwe kilku równań z biologii i fizyki matematycznej. Zaczynamy od granicy *szybkiej reakcji* dla układu równań reakcji-dyfuzji z niemonotoniczną szybką reakcją, co jest umotywowane zastosowaniami w neuronauce. Po raz pierwszy dla tego typu problemu, w granicy obserwujemy szybkie oscylacje, które dokładnie analizujemy za pomocą teorii miar Younga.

Następnie badamy granicę hydrodynamiczną równania Vlasova z odpowiednio dobraną siłą tak, aby w granicy otrzymać równanie Cahna-Hilliarda, tzn. równanie czwartego rzędu stosowane w naukach o materiałach i modelowaniu nowotworów. Jest to pierwsze rygorystyczne wyprowadzenie tego równania z opisu mikroskopowego, motywowane formalnymi obliczeniami Takaty i Noguchi (*J. Stat. Phys.*, 2018).

Następnie dowodzimy zbieżności nielokalnego równania Cahna-Hilliarda do równania lokalnego. Problem ten był szeroko badany w ostatnich latach. Nasza praca jest pierwszą, która rozważa przypadek zdegenerowanej mobilności, co jest motywowane przez zastosowania do modelowania wzrostu nowotworów. Nasze wyniki mogą być postrzegane jako dokończenie programu Giacomina-Lebowitza wyprowadzenia równania Cahna-Hilliarda z układów cząstek (*J. Stat. Phys.*, 1997), którzy wyprowadzili równanie jedynie w formie nielokalnej.

Na koniec tej części omawiamy zbieżność równania Eulera-Kortewega do równania Cahna-Hilliarda w tzw. granicy wysokiego tarcia (ang. *high-friction limit*). Problem ten był badany przez Lattanzio i Tzavarasa (*Comm. PDEs*, 2017), którzy udowodnili

zbieżność metodą relatywnej entropii przy założeniu, że układ graniczny ma gładkie i dodatnie rozwiązanie. Nie ma jednak teorii, która by to gwarantowała. Dlatego proponujemy badanie nielokalnego równania Eulera-Kortewega. Wówczas, układem granicznym jest nielokalne równanie Cahna-Hilliarda, który ma wymagane własności. Ponadto, używając wyniku z poprzedniego rozdziału, pokazujemy zbieżność nielokalnego równania Eulera-Kortewega do lokalnego równania Cahna-Hilliarda.

Druga część rozprawy dotyczy równań parabolicznych o niestandardowym wzroście, gdzie wzrost zmienia się nieregularnie, powiedzmy, nieciągle w czasie. Klasycznym przykładem jest równanie $p(t,x)$-Laplace'a z p ściśle oddzielonym od 1 i $+\infty$. Dowodzimy istnienia i jednoznaczności rozwiązań dla p nieciągłych po zmiennej czasowej i log-Hölderowsko ciągłych po zmiennej przestrzennej. Jest to pierwszy wynik tego typu, gdyż wszystkie dotychczasowe prace zakładały ciągłość wykładnika p. Dowód oparty jest na prostej obserwacji, że wygładzenie po zmiennej przestrzennej rozwiązania równania parabolicznego jest już regularne po zmiennej czasowej.

Następnie uzyskujemy wynik istnienia dla płynów nienewtonowskich z tensorem naprężeń, który jest nieciągły po zmiennej czasowej. Jest to problem motywowany zachowaniem płynu elektroreologicznego (złożonego z naładowanych cząstek) poruszającego się w polu elektrycznym, które drastycznie zmienia się w czasie.

Ostatni rozdział dotyczy funkcjonałów dwufazowych, czyli funkcjonałów, których wzrost zmienia się z p na q, w zależności od punktu przestrzeni. Są one dokładnie badane od czasu przełomowej pracy Mingione i Colombo (*ARMA*, 2014). Używając nowych metod aproksymacji, poprawiamy dotychczas znane zakresy wykładników p, q, że funkcje minimalizujące mogą być aproksymowane w *dobry* sposób (tzw. brak zjawiska Lavrentieva). W przypadku $p \leq d$ (d jest wymiarem przestrzeni), otrzymujemy jako pierwsi optymalny zakres takich wykładników (A. K. Balci et al, *Calc. Var. PDE*, 2020). Jest to ważne, bo jest to pierwszy krok przy dowodzeniu gładkości tych funkcji. W zastosowaniach opisują one konfiguracje kompozytów składających się z dwóch materiałów w reakcji na siłę zewnętrzną.

List of papers

The thesis is based on 9 papers, 5 of them are already published:

[1] B. Perthame, J. Skrzeczkowski. *Fast reaction limit with nonmonotone reaction function.* Communications on Pure and Applied Mathematics, published online, doi: 10.1002/cpa.22042, cited as [230].

[2] J. Skrzeczkowski. *Fast reaction limit and forward-backward diffusion: a Radon-Nikodym approach.* Comptes Rendus Mathématique, 360, p. 189-203, 2022, cited as [249].

[3] C. Elbar, M. Mason, B. Perthame, J. Skrzeczkowski. *From Vlasov equation to degenerate nonlocal Cahn-Hilliard equation.* Communications in Mathematical Physics, published online, doi: 10.1007/s00220-023-04663-3, cited as [121].

[4] C. Elbar, J. Skrzeczkowski. *Degenerate Cahn-Hilliard equation: From nonlocal to local.* Available at arXiv:2208.08955, cited as [123].

[5] J. A. Carrillo, C. Elbar, J. Skrzeczkowski. *Degenerate Cahn-Hilliard systems: from nonlocal to local.* In preparation, cited as [69].

[6] C. Elbar, P. Gwiazda, J. Skrzeczkowski, A. Świerczewska–Gwiazda. *From nonlocal Euler-Korteweg to local Cahn-Hilliard.* In preparation, cited as [120].

[7] M. Bulíček, P. Gwiazda, J. Skrzeczkowski. *Parabolic equations in Musielak – Orlicz spaces with discontinuous in time N-function.* Journal of Differential Equations, 290, 17-56, 2021, cited as [56].

[8] M. Bulíček, P. Gwiazda, J. Skrzeczkowski, J. Woźnicki. *Non-Newtonian fluids with discontinuous-in-time stress tensor.* Available at arXiv:2209.10695, cited as [54].

[9] M. Bulíček, P. Gwiazda, J. Skrzeczkowski. *On a range of exponents for absence of Lavrentiev phenomenon for double phase functionals.* Archive for Rational Mechanics and Analysis, 246, 209–240, 2022, cited as [57].

Contents

Chapter 1

Introduction and notation

1.1 Overview of Part I: Singular limits

The first part of the thesis is dedicated to the analysis of singular limits in several evolutionary PDEs arising in physics and biology. In all of the discussed problems, we consider a sequence of solutions, say $\{u_\varepsilon\}$, to a given PDE which depends on the parameter ε where $\varepsilon \to 0$. Our goal is to obtain enough compactness of the sequence $\{u_\varepsilon\}$ so that, up to a subsequence, $u_\varepsilon \to u$ in some topology and we can write a PDE for the limit u.

Studying such limits have several motivations. First, the singular limits provide a connection between various PDEs of different natures. The classical example here is the well-studied incompressible limit [51, 160, 178, 229] which links two widely studied models of tumor growth: cell density models (where the density of tumor cells solves a porous medium equation) [60, 229] and the free boundary models [59] (where the domain is itself a part of the solution and is interpreted as a tumor). Another example is a passage from the Boltzmann equation (describing the density of the gas and taking into account interactions between particles like collisions) to the Navier-Stokes equation (describing the flow of the fluids in terms of macroscopic quantities) [158, 194]. The latter is strongly related to Hilbert's sixth problem. Understanding connections between equations is important as it assures us that physics (or mathematical biology) is simply correct.

Second, the singular limits allow the reduction of the complexity of the system. As a classical example, we can mention here the mean-field limit. One can view the gas as a collection of a big number of particles N and model it as a system of ODEs. Nevertheless, the gas is composed of a large number of such particles and in particular, nobody is interested in the position or velocity of an individual particle. It turns out that via appropriate limit $N \to \infty$, one can derive macroscopic equations which can be for instance an aggregation equation or Vlasov equation [66, 173, 176] which are much simpler to study and analyze. Similar problems arise in mathematical biology, for instance in the modelling of cell-cell adhesion [70].

Third, singular limits can be used to regularize systems. Here, a classical example could be the regularization of hyperbolic conservation laws via the vanishing viscosity method. More precisely, there is currently no theory of global-in-time solutions for the system of hyperbolic conservation law in several space dimensions

$$\boldsymbol{u}_t + \operatorname{div} F(\boldsymbol{u}) = 0,$$

where $\boldsymbol{u} = (u_1, ..., u_d)$. However, the following equation

$$(\boldsymbol{u}_\varepsilon)_t + \operatorname{div} F(\boldsymbol{u}_\varepsilon) = \varepsilon \Delta \boldsymbol{u}_\varepsilon$$

is already well-posed as a system of advection-diffusion equations [34]. A similar regularization is presented in Chapter 5 for the degenerate Cahn-Hilliard equation.

Chapter 2: Compactness methods

Most of the considered equations are nonlinear and thus, we need strong compactness of the considered sequence $\{u_\varepsilon\}$ to pass to the limit in the nonlinear terms. This is the main challenge for us. Chapter 2 contains basic tools that will be used throughout Part I. This includes the theory of Young measures, the theory of compensated compactness, velocity averaging lemmas, several variants of the Riesz-Kolmogorov-Fréchet theorem and its adaptation to the nonlocal problems developed by Bourgain, Brézis, Mironescu and Ponce. The chapter is not original at all and all of the presented results are well-known. For the results that one can find easily in the literature, we provide precise citations and a sketch of the proofs. If this is not the case, in particular, it concerns some variants of the Riesz-Kolmogorov-Fréchet theorem presented in Section 2.5, we provide detailed proofs here.

Chapter 3: Fast reaction limit with nonmonotone reaction

This chapter follows the articles [230] and [249]. We consider a coupled system of a reaction-diffusion equation and an ODE:

$$
\begin{aligned}
\partial_t u^\varepsilon &= \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon},\\
\partial_t v^\varepsilon &= \Delta v^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}.
\end{aligned}
$$

Such systems can be viewed as a toy model to understand fast reversible reactions that occur in neurons when the electric signal propagates. Here, u^ε and v^ε are concentrations of two chemicals undergoing a reaction between themselves. Concerning nonlinearity F, we assume that it is not monotone (see Assumption 3.2.2) as we want to guarantee that constant steady states are nonstable (lack of stability is a typical feature of neurons) which is possible only if F is not monotone [221]. With these assumptions, our target is to understand the asymptotic behaviour $\varepsilon \to 0$, hoping for a simplified system description. Let us remark that in this model only one component is allowed to diffuse but this is a simplifying mathematical assumption as otherwise, we are not able to prove anything, see Section 3.6 for the related open problem.

From the mathematical point of view, the problem of sending $\varepsilon \to 0$ is called the fast reaction limit. If F is strictly increasing, it is fairly classical [40, 171] and one obtains in the limit porous medium equation, see Example 3.1.1 which also illustrates why lack of monotonicity creates difficulties. However, up to now, there has been no result in the literature about the fast reaction limit for nonmonotone F. To see why it is not easy, let us note that in the limit $\varepsilon \to 0$, one expects $v^\varepsilon \approx F(u^\varepsilon)$. Since F is not invertible, this does not give any information on u^ε. While we cannot give any limiting PDE for this problem, we prove that $\{v^\varepsilon\}$ is strongly compact in

L^2 which easily implies that u^ε accumulates on the set $F^{-1}(v)$ where v is the limit of $\{v^\varepsilon\}$. As long as the set $F^{-1}(v)$ has at least two elements, it does not allow for the strong compactness of $\{u^\varepsilon\}$. This is illustrated in Figure 3.1 and we make it more precise in terms of the Young measures (see Section 2.1).

The method of the proof of strong compactness follows the ideas of Murat and Tartar on compensated compactness [222, 257] adapted by Plotnikov to the regularization of ill-posed porous media equation $u_t = \Delta F(u)$ [232]. In our first paper [230], we adjust Plotnikov's method to our setting. In our second paper [249], we extend its applicability to the bigger class of nonlinearities F, allowing for some piecewise affine nonlinearities which were excluded by the setting of Plotnikov.

Chapter 4: Kinetic derivation of degenerate Cahn-Hilliard

This chapter is based on [121] and it introduces the topic of degenerate Cahn-Hilliard equation, a fourth-order PDE of the form

$$\begin{aligned} \partial_t u &= \operatorname{div}(u \nabla \mu), \\ \mu &= -\Delta u + F'(u), \end{aligned} \tag{1.1.1}$$

that will be continued in Chapters 5 and 6. Here, μ is called a chemical potential and F is called a potential. Equation (1.1.1) is equipped with an appropriate boundary condition if considered on a bounded domain.

The adjective *degenerate* refers to the presence of u under the divergence because when u is approaching 0, the term $\operatorname{div}(u\nabla\mu)$ vanish. More generally, one can consider the general equation

$$\begin{aligned} \partial_t u &= \operatorname{div}(m(u) \nabla \mu), \\ \mu &= -\Delta u + F'(u), \end{aligned} \tag{1.1.2}$$

where $m(u)$ is called mobility. When $m(u) > c > 0$ for some constant C we say that (1.1.2) is a non-degenerate Cahn-Hilliard equation and otherwise, we call it a degenerate Cahn-Hilliard equation. Variants of both of them are nowadays widely studied, see [143, 155] and [87, 99, 142] respectively, but let us point out two things.

First, the non-degenerate equation is easier to study because, by multiplying (1.1.2) with μ, one can easily obtain estimates on $\{\nabla u\}$ and $\{\nabla \mu\}$ which gives a lot of information. Therefore, one can establish several properties for the non-degenerate equation, including the well-posedness in the class of classical solutions, the so-called separation property and long-time asymptotics [218]. Second, the degenerate equation, particularly with $m(u) = u$, seems to be more justified by physics. Indeed, one can obtain it as a limit of interacting particle systems [151, 152], as a high-friction limit of the Euler-Korteweg equation (Chapter 6) or from the appropriate Vlasov equation as described in this chapter. This is the reason why we focus on the degenerate equation in Chapters 4, 5 and 6.

Equation (1.1.1) was introduced by Cahn and Hilliard [64,65] to model the dynamics of phase separation in binary mixtures (see Section 4.2 for a sketch of the derivation in [65]). Currently, it is applied in numerous fields, including mathematical biology [147, 148, 198]. For the mathematical theory of the Cahn-Hilliard equation we refer to [218] and [124].

Our target is to derive (1.1.1) from PDEs at the microscopic level. Such derivations have been obtained for several equations of mathematical physics [119,159,234,241]. To this end, following the formal approach of Takata and Noguchi [256], we consider the Vlasov-type PDE

$$\begin{gathered}\varepsilon^2 \partial_t f_\varepsilon + \varepsilon\, \xi \cdot \nabla_x f_\varepsilon + \varepsilon\, \mathcal{F}_\varepsilon \cdot \nabla_\xi f_\varepsilon = \varrho_\varepsilon(t,x) M(\xi) - f_\varepsilon, \\ \varrho_\varepsilon(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,\xi)\, \mathrm{d}\xi,\end{gathered} \tag{1.1.3}$$

where $f_\varepsilon(t,x,\xi)$ denotes the density of particles in time t, position x and velocity ξ. Furthermore, $\mathcal{F}_\varepsilon$ is a suitably chosen force and M is a Maxwellian (centered Gaussian distribution). We prove that ϱ_ε converges to a certain nonlocal version of (1.1.1) which is the first rigorous result aiming at the derivation of (1.1.1) from a microscopic equation.

To see, why (1.1.3) should converge to the Cahn-Hilliard equation, we observe that

(RHS) of (1.1.3) suggests that in the limit $\varepsilon \to 0$, we have $f(t,x,\xi) = \varrho(t,x)\, M(\xi)$. By integrating (1.1.3) with respect to ξ

$$\partial_t \varrho_\varepsilon(t,x) + \operatorname{div} J_\varepsilon(t,x) = 0, \qquad J_\varepsilon(t,x) := \int_{\mathbb{R}^d} \frac{\xi}{\varepsilon} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi.$$

Similarly, integrating (1.1.3) against ξ, we obtain the flux equation

$$\varepsilon^2 \partial_t J_\varepsilon(t,x) + \nabla_x \cdot \int_{\mathbb{R}^d} \xi \otimes \xi f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi - \mathcal{F}_\varepsilon \varrho_\varepsilon = -J_\varepsilon(t,x),$$

which allows identifying the limit of $\{J_\varepsilon\}$ and suggests the choice of the force $\mathcal{F}(\rho) \approx -\nabla(\Delta\rho)$. Of course, there are many difficulties to overcome; for instance (1.1.3) with such force is not well-posed.

Chapter 5: Degenerate Cahn-Hilliard: nonlocal to local

This chapter is based on [123] and [69]. We consider the nonlocal regularization of the Cahn-Hilliard equation:

$$\begin{aligned} \partial_t u_\varepsilon &= \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon), \quad \text{in} \quad (0,+\infty) \times \mathbb{T}^d, \\ \mu_\varepsilon &= B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in} \quad (0,+\infty) \times \mathbb{T}^d, \end{aligned} \tag{1.1.4}$$

where $B_\varepsilon[u]$ is a nonlocal operator defined with

$$B_\varepsilon[u] = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^d} \omega_\varepsilon(y)(u(x) - u(x-y))\,\mathrm{d}y$$

where $\{\omega_\varepsilon\}_\varepsilon$ is a usual mollification kernel. When $\varepsilon \to 0$, $B_\varepsilon[u_\varepsilon] \to -\Delta u$ in the sense of distributions, so that the sequence of solutions to (1.1.4) converge to a solution of degenerate Cahn-Hilliard equation. The target of this chapter is to prove this rigorously which is the first such result for the degenerate Cahn-Hilliard equation.

Let us first point out a substantial difficulty related to the degeneracy of the equation. There are several results [90, 91, 92, 215] on the nonlocal-to-local convergence for the non-degenerate system:

$$\begin{aligned} \partial_t u_\varepsilon &= \operatorname{div}(\nabla \mu_\varepsilon), \quad \text{in} \quad (0,+\infty) \times \mathbb{T}^d, \\ \mu_\varepsilon &= B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in} \quad (0,+\infty) \times \mathbb{T}^d, \end{aligned} \tag{1.1.5}$$

where $m(u) > c > 0$ for some constant C. If $F \in C^2$, this case is substantially simpler because the energy estimates (i.e. multiplying (1.1.5) by μ_ε) easily imply that $\{u_\varepsilon\}$ is strongly compact in $L^2((0,T)\times\mathbb{T}^d)$ and $\{B_\varepsilon[u_\varepsilon]\}$ is bounded in $L^2((0,T)\times\mathbb{T}^d)$ so that, up to a subsequence, it converges to $-\Delta u$ in $L^2((0,T)\times\mathbb{T}^d)$. It follows that one can pass to the limit in (1.1.5). In our case, we focus on the degenerate problem, where the estimate on $\{B_\varepsilon[u_\varepsilon]\}$ is not available.

There are many motivations to study the problem of nonlocal-to-local convergence. First, as already mentioned, there are several recent results for the non-degenerate equation [90, 91, 92, 215], but up to now, there was no such result for the degenerate mobility case. Second, there exists a derivation of the Cahn-Hilliard equation from interacting particle systems by Giacomin and Lebowitz [151, 152]. However, they arrived in fact at (1.1.4) and did not discuss whether it is possible to rigorously send $\varepsilon \to 0$. Third, there are papers on modelling biological phenomena [32, 133] where such a limit is stated without a rigorous argument.

The main tool we use is the energy and entropy for this system which gives us uniform (in $\varepsilon > 0$) bounds on

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\omega_\varepsilon(y)}{\varepsilon^2}|u_\varepsilon(t,x)-u_\varepsilon(t,x-y)|^2\,\mathrm{d}x\,\mathrm{d}y \le C,$$

$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\omega_\varepsilon(y)}{\varepsilon^2}|\nabla u_\varepsilon(t,x)-\nabla u_\varepsilon(t,x-y)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le C.$$

Together with information on $\partial_t u_\varepsilon$, they give strong compactness of $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ in $L^2((0,T)\times\mathbb{T}^d)$ by the results of Bourgain-Brézis-Mironescu [41] and Ponce [233], reviewed in Section 2.6. Then, our strategy is to formulate a weak formulation of (1.1.1) only in terms of u and ∇u (that is, not using any higher-order derivatives) so that strong compactness of $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ is sufficient to pass to the limit.

After discussing the result for (1.1.4) following [123], we also discuss how to adapt the result for the following aggregation-diffusion system used to model cell-cell adhesion as in [69] (a process by which biological cells interact with each other and attach to neighbouring cells):

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= \nabla \cdot \left(\rho \nabla \left(\kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma\rho - \beta\eta\right)\right), && \text{in} \quad (0,+\infty)\times\mathbb{T}^d,\\
\frac{\partial \eta}{\partial t} &= \nabla \cdot \left(\eta \nabla \left(\alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta\rho - \eta\right)\right), && \text{in} \quad (0,+\infty)\times\mathbb{T}^d
\end{aligned}
$$

In this case, the nonlocal model is motivated by the mean-field limit (derivation via particles) [70] while the local model allows computing explicitly some stationary states [133]. Thus, a rigorous convergence result is necessary to link these two models.

This is the first result of the nonlocal-to-local type for degenerate systems.

Chapter 6: High friction limit for the Euler-Korteweg equation

This chapter is based on the paper [120]. Our target is to link two PDEs: the Euler-Korteweg equation and the Cahn-Hilliard equation. The first one describes the flow of a fluid which is a liquid-vapor mixture (it is composed of one substance having two different phases). The second models the dynamics of phase separation in such mixtures. The idea is that if the fluid experiences sufficient damping, it slows down and the process that is mostly observed is the phase separation between liquid and vapor.

More rigorously, we consider the nonlocal Euler-Korteweg system re-scaled in time $t \to \frac{t}{\varepsilon}$ and with high friction coefficient $\frac{1}{\varepsilon}$

$$
\begin{aligned}
&\partial_t \rho + \frac{1}{\varepsilon}\operatorname{div}(\rho\mathbf{u}) = 0,\\
&\partial_t(\rho\mathbf{u}) + \frac{1}{\varepsilon}\operatorname{div}\left(\rho\mathbf{u}\otimes\mathbf{u}\right) = -\frac{1}{\varepsilon^2}\rho\mathbf{u} - \frac{1}{\varepsilon}\rho\nabla(F'(\rho) + B_\eta[\rho]),
\end{aligned}
$$

considered on $(0,+\infty)\times\mathbb{T}^d$. This equation models the long-time asymptotics of the motion of a compressible fluid with density ρ, velocity $\mathbf{u}$ which is in fact a liquid-vapor mixture. The fluid experiences high friction (due to the term $-\frac{1}{\varepsilon^2}\rho\mathbf{u}$) and additional capillary effects in the transition zone between liquid and vapour (due to the term $-\frac{1}{\varepsilon}\rho\nabla(F'(\rho)+B_\eta[\rho])$ as proposed by Korteweg [182]). As ε, $\eta \to 0$ in some scaling to be determined, we prove that densities converge to a solution of the local degenerate Cahn-Hilliard equation.

Several recent papers considered the high-friction limit for systems arising in fluid mechanics [80, 172, 186, 187]. As observed by Lattanzio and Tzavaras [187], for the Euler-Korteweg equation, the limit yields the degenerate Cahn-Hilliard equation. However, their proof, based on the relative entropy method, assumes the global existence of classical solutions bounded away from the vacuum (that is, $\rho > c > 0$ for some constant c) which is not guaranteed by the current theory (one can construct such solutions only on a very small interval of time). To overcome this problem, we propose to consider the nonlocal Euler-Korteweg equation. Then, the high friction limit yields the nonlocal Cahn-Hilliard equation which has much better properties. In the end, using the result of Chapter 5, we obtain convergence to the local Cahn-Hilliard equation.

Our proof is formulated for dissipative measure-valued solutions of the Euler-Korteweg equation. The dissipativity means that solutions satisfy the energy inequality (which means that the energy is non-increasing). Concerning the measure-valued solutions, their formulation is quite abstract and it involves the theory of Young measures reviewed in Sections 2.1 and 2.2. Roughly speaking, one can think about them as limits of smooth solutions to some approximating problems. Their great advantage is that they are known to exist on arbitrary intervals of time. This is not the case for weak solutions to most of the equations coming from fluid mechanics. Dissipative measure-valued solutions became important after they were found to satisfy the weak-strong uniqueness property: they coincide with the strong solution whenever the latter exists. Since the weak-strong uniqueness property was observed by Brenier, De Lellis and Székelyhidi in [46], measure-valued solutions were studied for several systems [68, 101, 146, 162, 165].

For the convergence proof, we use the relative entropy method. The method is based on introducing a functional called relative entropy (or energy), which measures the dissipation between two solutions of the system. In fact, the same method is used to prove the aforementioned weak-strong uniqueness when the relative entropy measures the distance between weak (measure-valued) and strong solutions.

More generally, the relative entropy method can be used to compare two solutions where at least one has to be classical (because we can test the weak formulation only with the classical solution). This strategy has been applied to numerous singular limits [71, 72, 172, 187] and we also refer to the excellent review on weak-strong uniqueness [261].

Let us remark that application of the relative entropy method enforces us to make an additional assumption on the initial data. More precisely, we assume that as $\varepsilon \to 0$, the initial velocity $\mathbf{u}_0$ (which depends on ε) converges to 0 (for instance, in $L^2(\mathbb{T}^d)$, cf. (6.2.1) and (6.2.2)) so that the kinetic energy at time 0 is very small. This guarantees that the relative entropy at time $t = 0$, denoted by $\Theta(0)$, converges to 0 so that $\Theta(t) \to 0$ for all $t > 0$, cf. (6.8.5), which implies the main result. In this case, we say that the initial data are *well-prepared.* It is a fairly restrictive assumption but it is often made for studying singular limits in fluid dynamics [88, 197]. Usually, much less can be proved for *ill-prepared data.* In this case, the usual strategy for studying the high-friction limits is based on compactness arguments. As they require quite strong estimates, they are mostly applicable in some particular cases like equations in one spatial dimension [202] (so that one can use the div-curl lemma) or equations with a viscosity [136]. In general, for well-prepared data much more can be proved than in the case of ill-prepared initial conditions [13].

Let us also stress that the dissipativity of solutions is a crucial feature for weak-strong uniqueness and application of the relative entropy method. This is illustrated for the 3D compressible Euler equations by the solutions obtained via the convex integration method: for a smooth initial condition there are infinitely many solutions [96] but they do not coincide with a local smooth solution which exists by [24]. The problem is that they are not dissipative: the energy has an instantaneous jump at time zero [252]. Nevertheless, one can construct dissipating solutions to the compressible Euler equations via the convex integration method as in [97,98], however only for the initial condition that is not sufficiently smooth so that there is no classical solution in this case.

1.2 Overview of Part II: Rough behavior

The second part of the thesis is devoted to the analysis of PDEs in the non-standard growth setting. The main example we have in mind is

$$u_t = \mathrm{div}(\nabla u |\nabla u|^{p(t,x)-2}) + f \tag{1.2.1}$$

with $1 < p_- \leq p(t,x) \leq p_+ < \infty$. This is the $p(t,x)$-Laplace equation and it is a generalization of the heat equation (think about $p(t,x) = 2$). Several models in materials science and fluids mechanics (electrorheological fluids) have this type of structure [3, 28, 33, 240]. Furthermore, they attract a lot of mathematical interest [78, 79, 104].

Equations like (1.2.1) need an appropriate functional analytic setting. Assuming that one has a solution to (1.2.1), we expect that

$$\int_0^T \int_\Omega |\nabla u|^{p(t,x)} \,\mathrm{d}x \,\mathrm{d}t < \infty,$$

suggesting that one needs a certain generalization of L^p spaces which takes into account different growth of the function in each point of the time-space domain. Such spaces are called Orlicz and Musielak–Orlicz spaces and they can be thought of as the space of functions such that

$$\int_0^T \int_\Omega M(t,x,u) \,\mathrm{d}x \,\mathrm{d}t < \infty$$

where M has to satisfy certain conditions (think about $M(t,x,u) = |u|^{p(t,x)}$). We refer to [77] for an excellent review of the theory of these spaces.

Musielak–Orlicz spaces arise in important problems of calculus of variations, for instance in the theory of regularity of minimizers to the double–phase functionals

$$\mathcal{G}(u) = \int_\Omega |\nabla u|^p + a(x)\, |\nabla u|^q \,\mathrm{d}x, \tag{1.2.2}$$

where a is a continuous function vanishing on some part of Ω. Such minimizers are important in materials science where they describe the optimal configuration of hyperelastic materials under external force [81,236]. The topic is thoroughly studied [20, 25, 26, 61, 63] since the papers of Colombo and Mingione [82, 83]. The utility of the Musielak–Orlicz spaces comes from the fact that they provide suitable function spaces to study this type of problems. We will also study double-phase problems in Chapter 10.

Chapter 7: Non-standard growth spaces

This is an introductory chapter, briefly describing the theory of Orlicz and Musielak–Orlicz spaces. It is mostly based on [77]. Rigorously, they are defined as the space of functions ξ on $(0,T)\times\Omega$ such that there is $\lambda>0$ such that

$$\int_{\Omega_T} M\left(t,x,\frac{\xi(t,x)}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t<\infty,$$

where M is called an N-function satisfying certain conditions (see Definition 7.1.2). For example, one can consider $M(t,x,\xi)=|\xi|^{p(t,x)}$ or $M(t,x,\xi)=|\xi|^p+a(t,x)|\xi|^q$, the latter corresponds to the functional (1.2.2).

After outlining the general theory, we focus briefly on two particular cases. The first is when the N-function satisfies the so-called Δ_2 condition: for some constant C we have

$$M(t,x,2\xi)\le C\,M(t,x,\xi).$$

The condition substantially simplifies the theory and this will be useful in Chapter 10. The second special case corresponds to $M(t,x,\xi)=|\xi|^{p(t,x)}$ so that the resulting space is the variable exponent space. We briefly discuss their properties which will be used in Chapter 9.

Chapter 8: Parabolic equations with roughly changing growth

This chapter is based on [56]. We focus here on the abstract parabolic equation of the form

$$u_t(t,x)=\mathrm{div}A(t,x,\nabla u(t,x))+f(t,x)\ \text{in}\ (0,T)\times\Omega.$$

This abstract form includes many equations studied in the literature: $p(t,x)$-Laplace equation (1.2.1) [7,16,39,62,191] and parabolic double-phase problems with

$$A(t,x,\nabla u) = \nabla u|\nabla u|^{p-2} + a(t,x)\,\nabla u|\nabla u|^{q-2},$$

see [37,38,93]. It is also a prototype of the PDE modelling flow of the electrorheological fluid which will be discussed in the next chapter.

To explain our result, let us restrict our attention to the $p(t,x)$-Laplace equation (1.2.1). So far, all of the papers aiming at well-posedness of (1.2.1) assumed log-Hölder continuity of the exponent $p(t,x)$ with respect to time and space [77,78,79]:

$$|p(t,x) - p(s,y)| \leq -\frac{C}{\log(|x-y| + |t-s|)}. \tag{1.2.3}$$

Our work relaxes this assumption by requiring (1.2.3) only in space:

$$|p(t,x) - p(t,y)| \leq -\frac{C}{\log(|x-y|)}. \tag{1.2.4}$$

In the particular case of the $p(t)$-Laplace equation, there is no continuity assumption needed (but we still have to assume $1 < p_- \leq p(t) \leq p_+$ for some p_-, p_+). The result is very surprising and unexpected as the space $L^{p(t)}(\Omega_T)$ can change discontinuously in time but we can still guarantee the complete well-posedness theory.

Condition (1.2.3) is necessary for mollifications to be well-defined in the Musielak–Orlicz space $L^{p(t,x)}(\Omega_T)$ so that smooth functions are dense in $L^{p(t,x)}(\Omega_T)$, see (8.1.5) for a simple explanation. Our main idea is that to establish the well-posedness of (1.2.1), one does not need to approximate every function in $L^{p(t,x)}$ but only a distributional solution to (1.2.1). Then, if one mollifies (1.2.1) in the spatial variable only, we have

$$(u_\varepsilon)_t(t,x) = \operatorname{div}(\nabla u|\nabla u|^{p(t,x)-2})_\varepsilon + f_\varepsilon(t,x) \text{ in } (0,T)\times\Omega', \tag{1.2.5}$$

for all Ω' compactly contained in Ω. It follows that mollification in space has a locally–Sobolev derivative in time and this is sufficient to conclude all the proofs. This observation dates back to the work of DiPerna and Lions on the renormalized solution to transport equation [112].

Chapter 9: Non-Newtonian fluids with discontinuous-in-time stress tensor

This chapter is based on [54]. It presents applications of the abstract theory developed in Chapter 8 to the particular system arising in physics. We consider here the PDE

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p(t,x) = \operatorname{div} S(t,x,Du) + f(t,x) \\ \operatorname{div} u = 0 \end{cases} \tag{1.2.6}$$

describing the flow of incompressible, homogeneous, non-Newtonian fluid. Here, u is the velocity, p is the pressure and S is the Cauchy stress tensor which depends on the symmetric gradient. The typical situation we have in mind is

$$S(t,x,Du) = Du\,|Du|^{s(t,x)-2},$$

which corresponds, for example, to the electrorheological fluid [3,28,240]. This is the fluid composed of charged particles moving in the electric field $\boldsymbol{E}$ where $\boldsymbol{E}$ solves certain Maxwell equation. In this case, the exponent $s(t,x)$ can be assumed to be a smooth function of $|\boldsymbol{E}|^2$ cf. [240, eq. (4.10)–(4.12)] so that $s(t,x)$ has to depend on t and x.

Our target is to establish the existence of weak solutions with $s(t,x)$ satisfying log-Hölder continuity condition in the spatial variable only (1.2.4). This is motivated by the aforementioned electrorheological fluids. Recall that in this case, $s(t,x)$ is a smooth function of $|\boldsymbol{E}|^2$. While $\boldsymbol{E}$ can be assumed to be regular (smooth) in space because it solves Maxwell equation which is elliptic, there is no reason to assume that it is continuous in time.

Our analysis is based on our methods developed in Chapter 8. The main difficulty compared to the analysis of (1.2.1) is that (1.2.6) can be tested only with divergence-free functions (because we want the pressure to vanish). However, our methods in Chapter 8 are based on considering equations locally in space, cf. (1.2.5). This requires testing equations with cutoff functions which are certainly not divergence-free.

Therefore, we need to recover the weak formulation of (1.2.6) that can be tested with arbitrary smooth functions. This is achieved by a careful adaptation of the method of harmonic pressure cf. [107, 139, 262] to the variable-exponent case.

In the end, we obtain the existence of weak solutions to (1.2.6) under the continuity assumption (1.2.4) and the classical lower-bound

$$s(t,x) \geq \frac{3d+2}{d+2},$$

which is required to handle the advection term $\operatorname{div}(u \otimes u)$.

Chapter 10: New results on the absence of Lavrentiev phenomenon for double phase functionals

This chapter is based on [57]. While the presented result belongs in fact to the field of calculus of variations, we wanted to include it in the thesis to present the wider applicability of approximation methods developed in Chapter 8. We are concerned with the regularity of minimizers to the double-phase functionals of the form (1.2.2). In general, power-growth functionals are used to model the configuration of the hyperelastic material under external stress [81, 236]. The exponent is related to the hardening properties of the material. Therefore, minimizers of (1.2.2) can be thought of as optimal configurations of composites consisting of two materials with different hardening properties. Understanding their regularity is important in applications.

One of the interesting features of functional $\mathcal{G}$ is the so-called Lavrentiev phenomenon. Let $1 \leq p < q$, d be the dimension and let $a \in C^{\alpha}(\Omega)$ (α-Hölder continuous functions). Then, when $q - p > \alpha \max\left(1, \frac{p-1}{d-1}\right)$, we have for some function $a \in C^{\alpha}(\Omega)$

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{G}(u) < \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{G}(u),$$

where $u_0 \in W^{1,q}(\Omega)$ represents the boundary datum, see [20]. This is called the Lavrentiev phenomenon and it implies that the minimizers are not even in $W^{1,q}(\Omega)$. On the other hand, if $q - p \leq \frac{p\alpha}{d}$ (d is the dimension), the Lavrentiev phenomenon does not occur and minimizers of $\mathcal{G}$ belong to $C^{1,\beta}_{loc}(\Omega)$ [83]. The result can be improved if one assumes that the minimizers are bounded to $q - p \leq \alpha$ [82]. Let us point out that the absence of Lavrentiev phenomenon is usually the first step to prove regularity of minimizers of $\mathcal{G}$, see [82, 83].

Our result states that the Lavrientiev phenomenon does not occur when $q - p \leq \alpha \max\left(1, \frac{p}{d}\right)$ without any a priori assumption on the boundedness of minimizers. In the case $p \leq d$, in view of the counterexamples in [20], this is the first optimal result (it cannot be improved). Our methods are surprisingly elementary. We observe that the absence of the Lavrentiev phenomenon is equivalent to the density of smooth functions in an appropriate Musielak–Orlicz space. In this space, bounded functions are dense (see Lemma 10.2.2), therefore one can assume that the minimizer is bounded which allows for a better range of exponents.

We remark that the analysis of the Lavrentiev phenomenon is in fact a long and deep research program. In the most general form, given two linear (or affine) spaces $X \subset Y$ with X dense in Y and functional $\mathcal{H} : Y \to \mathbb{R} \cup \{+\infty\}$, Lavrentiev phenomenon occurs if

$$\inf_{u \in X} \mathcal{H}(u) > \inf_{u \in Y} \mathcal{H}(u).$$

Since its discovery by Lavrentiev in 1926 [188], more examples were proposed by Ball and Mizel [23], Mania [200]. Further examples were given in the context of variable exponent first by Zhikov [269, 271], which were generalized later to cover broader classes of functionals [20, 127]. However, these examples are based on the fact that X is not dense in Y where $X = C_c^\infty(\Omega)$ and Y is a certain Musielak–Orlicz space.

The example of Mania is concerned with the minimization of

$$\mathcal{H}(u) = \int_0^1 (u(t)^3 - t)^2 \, u'(t)^6 \, \mathrm{d}t$$

over the functions satisfying $u(0) = 0$ and $u(1) = 1$. It turns out that

$$\inf_{W^{1,1}(0,1)} \mathcal{H}(u) < \inf_{W^{1,\infty}(0,1)} \mathcal{H}(u)$$

where the infimum is taken over the functions satisfying $u(0) = 0$ and $u(1) = 1$. This example is important as it shows that the Lavrentiev phenomenon is not just an academic problem: from the numerical point of view it shows that to compute the minimizer, one cannot use simply piecewise affine finite element approximations (which are in $W^{1,\infty}$) and more sophisticated methods are necessary [23, Section 2.4].

1.3 Notation

The arguments and domain. Usually, d denotes the dimension of the space. We write $\mathbb{R}^d$ for the d-dimensional space of vectors $(x_1, ..., x_d)$ with $x_i \in \mathbb{R}$ and $\mathbb{T}^d$ for the d-dimensional torus. Given two sets A, B we write $A \subset B$ for an inclusion of set A into B and $A \Subset B$ for a compact inclusion (that is, there is a compact set C such that $A \subset C \subset B$ and $C \neq B$. By $\Omega \subset \mathbb{R}^d$ we mean a (usually) Lipschitz domain. In the chapters concerning kinetic theory (Chapters 2.4 and 4) we use additionally ξ for the variable denoting velocity. For the evolutionary problems, we consider functions of space (variable x) and time (variable t). In this case, $T > 0$ denotes the length of time interest. The corresponding parabolic domain will be denoted by $\Omega_T := (0,T) \times \Omega$ and its specific subdomains as $\Omega_t = (0,t) \times \Omega$. When the time-space structure does not play any role, we use Q for an arbitrary set.

Vector/matrix operations. For any vectors $a, b \in \mathbb{R}^d$ we write $a \cdot b$ for the standard scalar product of a and b. The dot will be sometimes omitted when it is clear that we mean the scalar product. Next, the space $\mathbb{R}^{d\times d}_{\text{sym}}$ denotes the space of symmetric $d \times d$ matrices and for any $A, B \in \mathbb{R}^{d\times d}_{\text{sym}}$ we denote the scalar product by $A : B$. In addition, the symbol $\otimes$ is reserved for the tensorial product, i.e., for $a, b \in \mathbb{R}^d$ we denote $a \otimes b \in \mathbb{R}^{d\times d}_{\text{sym}}$ as $(a \otimes b)_{ij} := a_i b_j$ for $i, j = 1, \ldots, d$. On the space of matrices we always use the usual Frobenius norm: for $A = (a_{i,j})_{i,j}$ we define

$$|A| = \left(\sum_{i,j=1}^{d} a_{i,j}^2 \right)^{1/2}.$$

Exponents. For the exponent $p \in [1, \infty]$, we denote by p' its Hölder conjugate defined by the equation $\frac{1}{p} + \frac{1}{p'} = 1$.

Sequences. We denote sequences using usual notation with curly brackets, for example, $\{x_n\}_{n\in\mathbb{N}}$. When there is only one parameter indexing the sequence and its range is clear, we often skip the lower index and write $\{x_n\}$. In particular, this applies to:

- ε, δ and θ which are always small parameters in the interval $(0,1)$; for example, $\{u^\varepsilon\} = \{u^\varepsilon\}_{\varepsilon\in(0,1)}$.
- letters j, k, l, n, m where, if not stated otherwise, range over natural numbers $\mathbb{N}$; for example, $\{f_n\} = \{f_n\}_{n\in\mathbb{N}}$.

Special functions and operators. We list below several functions and operators that will be used throughout the text:

- for a given set A, $\mathbb{1}_A$ is a characteristic function of set A,
- given $f, g : \mathbb{R}^d \to \mathbb{R}$ with $f, g \in L^1(\mathbb{R}^d)$ we define convolution of f and g by

$$f * g = \int_{\mathbb{R}^d} f(y)\, g(x-y) \,\mathrm{d}y = \int_{\mathbb{R}^d} f(x-y)\, g(y) \,\mathrm{d}y;$$

  similar definition works in the case of $\mathbb{T}^d$.

Differential operators. For a function of one variable f, we denote its derivatives by $f', f'', f^{(3)}, f^{(4)}$ and so on. Concerning the functions of several variables, we write ∂_t for the derivative with respect to the time variable and ∇ for the gradient with respect to the spatial variable. We also use the lower index to denote derivatives, for instance u_t, u_{x_i} etc. Symbol Du denotes the symmetric part of the gradient, i.e. $Du = \left(\nabla u + (\nabla u)^\intercal\right)/2$ where u is the vector-valued function (so that ∇u is the matrix-valued function). By $\operatorname{div} u$, where $u = (u_1, ..., u_d)$ is a vector field, we mean $\operatorname{div} u = u_{1,\,x_1} + ... + u_{d,\,x_d}$. Finally, for a scalar-valued function u, $\Delta u = \operatorname{div} \nabla u$.

Measures. We assume that the Reader is familiar with a definition of signed measure on a Polish metric space (X, d) (that is, complete and separable) (see [138]). Any signed measure μ can be decomposed as $\mu = \mu^+ - \mu^-$ where both measures μ^+, μ^- are nonnegative. We say that μ is finite if $\mu^+(X), \mu^-(X) < \infty$. In this case, we define the total variation of μ as

$$\|\mu\|_{TV} = \mu^+(X) + \mu^-(X).$$

We write $\mathcal{M}(X)$ for the space of Radon measures on X (here, the word *Radon* does not bring anything new as on Polish metric space X, any finite measure is Radon, see [116, Appendix F.2]). Its subspace consisting of nonnegative measures is denoted by $\mathcal{M}^+(X)$ and its subspace consisting of probability measures (nonnegative with mass 1) is denoted by $\mathcal{P}(X)$. Finally, consider two metric spaces (X_1, d_1) and (X_2, d_2). Let μ be a measure on (X_1, d_1) and $F : X_1 \to X_2$ be a measurable map. Then, we can define the push-forward of μ along map F:

$$F^{\#}\mu(A) = \mu(F^{-1}(A))$$

which is a measure on (X_2, d_2).

Spaces of continuous functions. We write $C_c(\Omega)$ for the space of continuous functions with compact support in Ω and $C_c^\infty(\Omega)$ for its subspace consisting of smooth functions. By $C_0(\mathbb{R}^d)$, we mean the space of continuous functions vanishing at infinity. The dual of $C_0(\mathbb{R}^d)$ (equipped with the supremum norm) is the space $\mathcal{M}(\mathbb{R}^d)$ with the total variation norm.

Lebesgue and Sobolev spaces. We use the standard notation for Sobolev and Lebesgue function spaces and frequently do not distinguish between scalar-, vector-, or matrix-valued functions. It will be always clear from the context what we have in mind. The norm in these spaces is denoted by $\|\cdot\|$ and we specify the space in the lower index of the norm symbol: for instance, $\|f\|_{L^p_x}$ is the norm of f in $L^p(\Omega)$, $\|f\|_{L^\infty_t L^2_x}$ is the norm of f in $L^\infty(0,T; L^2(\Omega)$, etc. The domain will be always clear from the context. Less standard function are listed below:

- $L^{s(t,x)}(\Omega_T)$ is the variable exponent space defined in Chapter 9,
- $L^2_{0,\mathrm{div}}(\Omega)$ is defined as a closure of the set $\{u \in C_c^\infty(\Omega; \mathbb{R}^d), \operatorname{div} u = 0\}$ in $L^2(\Omega)$.

Functions with values in a Banach spaces. When $(X, \|\cdot\|_X)$ is a Banach space and X^* is its dual, we write $L^p(Q;X)$ for the space of strongly measurable functions $f : Q \to X$ such that $\int_Q \|f(y)\|^p \,\mathrm{d}y < \infty$. Strong measurability means here that there exists a sequence of simple measurable functions $\{f_k\}$ such that $f_k \to f$ a.e. on Q and

$$\int_Q \|f_k(y) - f_j(y)\|_X^p \,\mathrm{d}y \to 0 \text{ as } k, j \to \infty.$$

Sinilarly, we can define the spaces $L^p_w(Q;X)$ of functions $f : Q \to X$ which are weakly measurable (i.e. for all $\varphi \in X^*$, $y \mapsto \varphi(f(x))$ is measurable) and $L^p_{w^*}(Q;X^*)$ of functions $f : Q \to X'$ which are weakly* measurable. With this notation, we have the representation of the dual space

$$L^p(Q;X)^* = L^{p'}_{w^*}(Q;X^*),$$

see [227, Chap. 6.7]. We will use it in the particular case $(L^1(Q;C_0(\mathbb{R}^d)))^* = L^\infty_{w^*}(Q;\mathcal{M}(\mathbb{R}^d))$.

Convergence. By $\to$ we will always mean strong convergence which is strong with respect to the topology that is always clear from the context. Otherwise, we always make it more precise by specifying whether this is convergence in measure, almost everywhere or in the sense of distributions. The symbols $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ denote weak and weak* convergence respectively. When we want to represent the weak (weak*) limit of a given sequence, say $\{u_n\}$, we write $\text{w-lim}\, u_n$ (respectively $\text{w}^*\text{-lim}\, u_n$).

Part I

Singular limits

Chapter 2

Compactness results

We collect here several compactness results which will be used throughout the thesis. We assume the Reader is familiar with notions of weak/strong convergence as well as basic results on this topic, including theorems of Arzela-Ascoli, Reillich-Kondrachov, Banach-Alaoglu, Lions-Aubin and Riesz-Kolmogorov.

2.1 Young measures

We introduce the framework of Young measures introduced by Young [267, 268] and recalled in the seminal paper of Ball [22]. The theory is helpful with passing to the weak limits under nonlinearities. Roughly speaking, if $z_j \rightharpoonup z$ weakly (say, in L^2), it is not true that $f(z_j) \rightharpoonup f(z)$ even weakly. The most standard example is $z_j = \sin(2\pi j x)$ so that $z_j \rightharpoonup 0$ in $L^2(0,1)$ but $z_j^2 \rightharpoonup \frac{1}{2}$ in $L^2(0,1)$. This poses a lot of problems in nonlinear PDEs as one cannot use weak compactness so smoothly as in the linear theory.

The theory of Young measures allows at least to represent weak limits in terms of one fixed family of probability measures, independent of the nonlinearity. More precisely, we have the following theorem ([227, Theorem 6.2]):

Theorem 2.1.1 (Fundamental Theorem of Young Measures)**.** *Let $Q \subset \mathbb{R}^n$ be a measurable set and let $z_j : Q \to \mathbb{R}^m$ be measurable functions such that*

$$\sup_{j\in\mathbb{N}} \int_Q g(|z_j(y)|)\, \mathrm{d}y < +\infty$$

for some continuous, nondecreasing function $g : [0,+\infty) \to [0,+\infty)$ such that $\lim_{t\to+\infty} g(t) = +\infty$. Then, there exists a subsequence (not relabeled) and a weakly star measurable family of probability measures $\nu = \{\nu_y\}_{y\in\Omega}$ with the property that whenever the sequence $\{\psi(y, z_j(y))\}$ is weakly compact in $L^1(Q)$ for a Carathéodory function (measurable in the first argument and continuous in the second argument) $\psi : Q \times \mathbb{R}^m \to \mathbb{R}$, we have

$$\psi(y, z_j(y)) \rightharpoonup \int_{\mathbb{R}^m} \psi(y,\lambda)\, \mathrm{d}\nu_y(\lambda) \quad \textit{in } L^1(\Omega).$$

We say that the sequence $\{z_j\}$ generates the sequence of Young measures $\{\nu_y\}_{y\in\Omega}$.

The proof uses the duality $(L^1(\Omega; C_0(\mathbb{R}^d)))^* = L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^d))$, Banach-Alaoglu theorem and representation:

$$\psi(y, z_j(y)) = \int_{\mathbb{R}^m} \psi(y,\lambda)\, \mathrm{d}\delta_{z_j(y)}.$$

To gain more intuition, let us study two examples:

1. Let $z(x) = \mathbb{1}_{[0,1/2]}(x) - \mathbb{1}_{[1/2,1]}(x)$ and let us extend this function periodically from $[0,1]$ to $\mathbb{R}$. We define $z_j(x) = z(jx)$ and we consider $Q = (0,1)$. The Young measure of this sequence is (see [236, Example 4.8])

   $$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.$$

   This has the following interpretation. It means that the values of this sequence concentrates in two points: 1 and -1. Moreover, it spends equal amount of time in both of them.

2. Let $z_j(x) = \sin(2\pi jx)$. The Young measure of this sequence is in fact absolutely continuous with respect to the Lebesgue measure on $[0,1]$. It density reads (see [236, Example 4.9]) $\frac{1}{\pi\sqrt{1-y^2}}$. This time the sequence does not concentrate in the finite amount on points. Nevertheless, it is easy to see that the sequence spends more time around $y=0$ than $y=1$.

We now recall the result which allows to upgrade weak converence of a sequence to the strong one.

Lemma 2.1.2. *Let $\{z_j\}$ be a sequence in $L^p(Q)$ $(1 \le p < \infty)$ such that $\{|z_j|^p\}$ is weakly convergent in $L^1(Q)$. Then, $z_j \to z$ strongly in $L^p(Q)$ if and only if $\nu_y = \delta_{z(y)}$ for a.e. $y \in Q$.*

In a typical situation, one knows only weak compactness and has to use Young measures to pass to the limit. But then, using some additional information, one can prove that the Young measure is a Dirac mass. This immediately upgrades weak compactness to the strong one.

We recall also a result which will be useful in Chapter 3.

Lemma 2.1.3. *Under the notation of Theorem 2.1.1, the following hold true.*

(A) Suppose that $\{u_j\}$, $\{w_j\}$ are two sequences bounded in $L^p(Q)$. Assume that

$$|\{x \in Q : u_j \neq w_j\}| \to 0 \text{ as } j \to \infty.$$

Then, the Young measure generated by $\{u_j\}$ and $\{w_j\}$ is the same. In particular, this is true if $\|u_j - w_j\|_p \to 0$.

(B) If $\{u_n\}$ is a sequence bounded in $L^\infty(Q)$ and $F : \mathbb{R} \to \mathbb{R}$ is continuous, the sequence $\{F(u_n)\}$ generates Young measure $F^{\#}\mu_{t,x}$ (i.e. push-forward $\mu_{t,x} \circ F^{-1}$ given by $\mu_{t,x} \circ F^{-1}(A) = \mu_{t,x}(F^{-1}(A))$ where $F^{-1}(A)$ is the preimage of the set A) .

Proof. For (A) we refer to [227, Lemma 6.3]. For (B) it is sufficient to write

$$G(F(u_n)) \rightharpoonup \int_{\mathbb{R}} G(F(\lambda)) \, \mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}} G(\lambda) \, \mathrm{d}(\mu_{t,x} \circ F^{-1})(\lambda).$$

and use the uniqueness of Young measure. □

Remark 2.1.4 (kinetic formulation)**.** There exists an equivalent approach to analyze weak limits by the so-called kinetic function. Following [228], we define the kinetic function $\chi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\chi(\xi; u) = \mathbb{1}_{0<\xi<u} - \mathbb{1}_{u<\xi<0}.$$

Then, for all functions $S : \mathbb{R} \to \mathbb{R}$ such that S' is locally bounded, we have

$$S(u) = S(0) + \int_{\mathbb{R}} S'(\xi)\, \chi(\xi, u)\, \mathrm{d}\xi.$$

By introducing an additional variable ξ, this identity allows to identify the weak limit of the sequence $\{S(u_j)\}$. Indeed, let $\{u_j\}$ where $u_j : Q \to \mathbb{R}$ be a weakly compact sequence in $L^1(Q)$ and $S' \in L^\infty(\mathbb{R})$. Then, the weak limit of $\{S(u_j)\}$ equals

$$S(0) + \int_{\mathbb{R}} S'(\xi)\, f(y, \xi)\, \mathrm{d}\xi.$$

where $f(y, \xi)$ is the weak* limit of $\chi(\xi, u_\varepsilon(y))$ in $L^\infty(Q \times \mathbb{R})$. Note that the function f is independent of nonlinearity S. Furthermore, if $\{\mu_y\}_{y \in Q}$ is the Young measure of sequence $\{u_j\}$, we have

$$\partial_\xi f(y, \xi) = \delta_0(\xi) - \mu_y(\xi),$$

in the sense of distributions, see [228, Section 2.6]. In this sense, the approach of kinetic functions and Young measures is equivalent. Similar properties as for Young measures, can be established for kinetic functions. For instance, when f is again of the form of χ, that is $f(y, \xi) = \chi(\xi; u(y))$, the convergence is strong which is an equivalent version of Lemma 2.1.2. The concept of kinetic formulation is extremely useful to rewrite various PDEs (degenerate parabolic, hyperbolic conservation laws) as kinetic equations [150,195,231,255] and use velocity averaging lemmas (see Section 2.4) for them. $\square$

Finally, as in Chapter 3, we characterize the Young measure pointwisely, we recall the definition of the support of a measure on $\mathbb{R}^d$ [243, Definition 1.14]. For this, let $B(x, r)$ denote a ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$.

Definition 2.1.5. *Let μ be a nonnegative measure on $\mathbb{R}^d$. We say that $x \in \operatorname{supp} \mu$ if and only if $\mu(B(x, r)) > 0$ for all $r > 0$.*

Remark 2.1.6. When a given property (like an equation) is satisfied for almost every x (with respect to μ) one may worry that it is not true for the particularly chosen value of x. This is not the problem if one takes $x \in \operatorname{supp} \mu$ because in each neighbourhood of x there is $y \in \operatorname{supp} \mu$ such that the considered property has to be satisfied as the measure of each neighbourhood is nonzero.

2.2 Young measures with concentration effects

Note that in Theorem 2.1.1, we require that the sequence $\{\psi(y, z_j(y))\}$ is weakly compact in $L^1(Q)$. This prevents the concentration effect to appear (think about the family of standard mollifiers). When we don't have weak compactness, we use the following proposition which follows from the Banach-Alaoglu theorem. We formulate it with a distinguishment between time and space variables (that is, $Q = (0,T) \times \Omega$, $y = (t,x)$ with $t \in (0,T)$ and $x \in \Omega$) as usually in applications one have better integrability in time which results in better characterization of the resulting measure. The following proposition is a consequence of Banach-Alaoglu theorem and Radon-Nikodym theorem, see [46].

Proposition 2.2.1. *Let f be a continuous function and a sequence $\{f(t, x, z_j(t, x))\}$ be bounded in $L^p(0, T; L^1(\Omega))$ with $p \geq 1$. Let $\{\nu_{t,x}\}_{t,x}$ be the Young measure generated by $\{z_j\}$. Then there exists a measure m^f such that (up to a subsequence not relabelled)*

$$f(t, x, z_j(t, x)) - \langle \nu_{t,x}, f \rangle \overset{*}{\rightharpoonup} m^f \quad \text{in } L^p(0, T; \mathcal{M}(\Omega)) \text{ if } p > 1,$$

$$f(t, x, z_j(t, x)) - \langle \nu_{t,x}, f \rangle \overset{*}{\rightharpoonup} m^f \quad \text{in } \mathcal{M}((0, T) \times \Omega) \text{ if } p = 1,$$

Moreover, if $p > 1$, the measure m^f is absolutely continuous with respect to time: for a.e. $t \in (0, T)$, there exists measure $m^f(t, \cdot)$ on Ω such that

$$\int_{(0,T)\times\Omega} \psi(t, x) \, \mathrm{d}m^f(t, x) = \int_0^T \int_\Omega \psi(t, x) \, m^f(t, \mathrm{d}x) \, \mathrm{d}t.$$

Proof. The first part follows by the Banach-Alaoglu theorem. For the second part, let $\sigma^f(A) = m^f(A \times \Omega)$ be the projection measure. By the disintegration theorem (cf. [131, Theorem 1.45]), for σ^f-a.e. $t \in (0, T)$, there exists a probability measure $n^f(t, \cdot)$ on Ω such that

$$\int_{(0,T)\times\Omega} \psi(t, x) \, \mathrm{d}m^f(t, x) = \int_0^T \int_\Omega \psi(t, x) \, n^f(t, \mathrm{d}x) \, \mathrm{d}\sigma(t). \tag{2.2.1}$$

We consider $\psi(t, x) = \mathbb{1}_A(t)$ where $A \subset (0, T)$ (some care is necessary when Ω is not bounded because then, it is not an admissible test function in the sense that it does not vanish at infinity). Then, the (RHS) of (2.2.1) equals $\sigma(A)$. As m^f belongs to $L^p(0,T;\mathcal{M}(\Omega))$, we have

$$\left|\int_{(0,T)\times\Omega} \mathbb{1}_A(t)\,\mathrm{d}m^f(t,x)\right| \leq \|\mathbb{1}_A\|_{L^{p'}(0,T)}\,\|m^f\|_{L^p(0,T;\mathcal{M}(\Omega))}.$$

It follows that σ is absolutely continuous with respect to the Lebesgue measure (we use here that $p>1$ as if $p=1$ then $\|\mathbb{1}_A\|_{L^{p'}(0,T)}=1$) so that by the Radon-Nikodym theorem, σ has density with respect to the Lebesgue measure: $\mathrm{d}\sigma(t)=\sigma(t)\,\mathrm{d}t$. The conclusion follows by defining $m^f(t,\mathrm{d}x)=\sigma(t)\,n^f(t,\mathrm{d}x)$. □

Let us remark that by the fundamental theorem, we have $m^f=0$ when the sequence $\{f(z_j)\}$ is weakly compact in $L^1((0,T)\times\Omega)$. We use the notation:

$$\overline{f} = \langle f(\lambda), \nu_{t,x}\rangle + m^f \tag{2.2.2}$$

to represent weak limit of $f(t,x,z_j(t,x))$. We also need the following result which allows to compare two concentration measures m^{f_1} and m^{f_2} for two different nonlinearities f_1, f_2.

Proposition 2.2.2. *Let $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ be a Young measure generated by a sequence $\{z_j\}$. If two continuous functions f_1 and $f_2\geq 0$ satisfy $|f_1(z)|\leq f_2(z)$ for every z, and if $\{f_2(z_j)\}$ is uniformly bounded in $L^1((0,T)\times\Omega)$, then we have*

$$|m^{f_1}|(A) \leq m^{f_2}(A),$$

for any Borel set $A\subset(0,T)\times\Omega$.

Here, $|\mu|$ is the total variation measure defined as $|\mu|(A)=\mu^+(A)-\mu^-(A)$ where μ^+, μ^- are positive and negative parts of μ.

Proof of Proposition 2.2.2. We follow [134, Lemma 2.1]. We consider only the case of scalar-valued f as the case of vector-valued f is similar. We prove the following formula for the measure m^f: for all test functions $\varphi:(0,T)\times\Omega\to\mathbb{R}$ we have

$$\int_{(0,T)\times\Omega}\varphi(t,x)\,\mathrm{d}m^f(t,x) = \lim_{M\to\infty}\lim_{j\to\infty}\int_{(0,T)\times\Omega}\mathbb{1}_{|z_j|>M}\,\varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x. \tag{2.2.3}$$

The formula (2.2.3) immediately implies the result. To prove (2.2.3), we write

$$\begin{aligned}\int_{(0,T)\times\Omega} \varphi(t,x)\,\overline{f}(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \lim_{j\to\infty}\int_{(0,T)\times\Omega} \varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x = \\ &= \lim_{j\to\infty}\int_{(0,T)\times\Omega} \mathbb{1}_{|z_j|\le M}\,\varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x \\ &\quad+ \lim_{j\to\infty}\int_{(0,T)\times\Omega} \mathbb{1}_{|z_j|> M}\,\varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

We can split the limit above because the limit of the both terms exists. Indeed, for fixed M, the sequence $\{\mathbb{1}_{|z_j|\le M}\,f(z_j(t,x))\}_{j\in\mathbb{N}}$ is weakly compact in $L^1(\Omega_T)$ so that applying Theorem 2.1.1,

$$\lim_{j\to\infty}\int_{(0,T)\times\Omega} \mathbb{1}_{|z_j|\le M}\,\varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x = \langle \mathbb{1}_{|\cdot|\le M}\,f, \nu_{t,x}\rangle.$$

Applying dominated convergence theorem (it is allowed because function f is bounded),

$$\lim_{M\to\infty}\lim_{j\to\infty}\int_{(0,T)\times\Omega} \mathbb{1}_{|z_j|\le M}\,\varphi(t,x)\,f(z_j(t,x))\,\mathrm{d}t\,\mathrm{d}x = \langle f, \nu_{t,x}\rangle.$$

By comparison with (2.2.2), we conclude the proof of (2.2.3). □

Let us conclude with few comments about the measure m^f which captures concentration effects. One can describe it more precisely. The first attempts to do so by some generalizations of the Young measures were initiated by DiPerna and Majda in the case of the incompressible Euler equations [114]. Then, Alibert and Bouchitté extended the result to more general class of nonlinearities in [9]. They proved that there exists a subsequence (not relabeled) as well as a parametrised probability measure $\nu \in L^\infty_w(Q;\mathcal{P}(\mathbb{R}^n))$ (which is identical with the "classical" Young measure), a non-negative measure $m \in \mathcal{M}^+(Q)$, and a parametrized probability measure $\nu^\infty \in L^\infty_w(Q,m;\mathcal{P}(\mathbb{S}^{n-1}))$ such that for any Carathéodory function f such that $f(x,z)/(1+|z|)$ is bounded and uniformly continuous with respect to z,

$$f(y,z_j(y)) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^d} f(y,\lambda)d\nu_y(\lambda) + \int_{\mathbb{S}^{n-1}} f^\infty(y,\beta)\,\mathrm{d}\nu^\infty_y(\beta)m(y)$$

weakly* in the sense of measures. Here,

$$f^\infty(y,\beta) := \lim_{s\to\infty}\frac{f(y,t\beta)}{t}.$$

Their result was also extended to the case when f has different growth with respect to different variables, see for instance [165].

2.3 Compensated compactness

In general, if two sequences converge weakly, it is not true that their product converges weakly to the product of their limits. However, sometimes there is some additional information which allows to conclude so. The most classical result in this direction is the celebrated div-curl lemma [129, Theorem 4, p.54]. Recall that if $w \in L^2(Q;\mathbb{R}^d)$ is a given vector field, we define $\operatorname{curl} w \in W^{-1,2}(Q;\mathbb{R}^{d\times d})$ as the matrix with entries given by

$$(\operatorname{curl} w)_{i,j} = w^i_{x_j} - w^j_{x_i}.$$

Theorem 2.3.1. *Assume that $\{v_k\}$, $\{w_k\}$ are two sequences bounded in $L^2(Q;\mathbb{R}^d)$ such that*

- $\{\operatorname{div} v_k\}$ *lies in a compact subset of* $W^{-1,2}(Q)$*;*
- $\{\operatorname{curl} w_k\}$ *lies in a compact subset of* $W^{-1,2}(Q;\mathbb{R}^{d\times d})$*.*

Suppose that $v_k \rightharpoonup v$ and $w_k \rightharpoonup w$ in $L^2(Q;\mathbb{R}^d)$. Then,

$$v_k \cdot w_k \to v \cdot w$$

in the sense of distribution.

In the thesis, we will use the following lemma formulated on the time-space domain $(0,T)\times\Omega$. For the proof we refer to [220, Proposition 1].

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Suppose that*

- $\{a_n\}$ *is uniformly bounded in* $L^2(0,T;H^1(\Omega))$*,*
- $\{b_n\}$ *is uniformly bounded in* $L^2(0,T;L^2(\Omega))$*,*
- $\{\partial_t b_n\}$ *is uniformly bounded in* $C(0,T;H^m(\Omega))^*$ *for some* $m \in \mathbb{N}$*.*

Then, if $a_n \to a$, $b_n \to b$ and $a_n\, b_n \to c$ in the sense of distributions, we have $c = a\, b$.

Of course, if the product $a_n\, b_n$ has better integrability, the resulting weak convergence of the product will be in some better sense.

To give some idea about the proof of Lemma 2.3.2, let us assume that $\{\partial_t b_n\}$ is uniformly bounded in $L^2((0,T);H^{-1}(\Omega))$. Defining operator $R = \Delta^{-1}$ with Dirichlet boundary conditions, we let $c_n = R(b_n)$. We claim that $\{\nabla c_n\}$ is strongly convergent in $L^2((0,T)\times\Omega)$. Indeed, compactness in space follows from the fact that $R : L^2(\Omega) \to H^2(\Omega)$ and Reillich-Kondrachov theorem while for the compactness in time we compute for a smooth and compactly supported test function $\varphi(t,x)$:

$$\int_{\Omega_T} \nabla c_n\, \partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \nabla c_n\, \Delta R(\partial_t\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} b_n\, \mathrm{div}\, R(\partial_t\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

As R commutes with the time derivative, the (RHS) can be estimated by

$$\|b_n\|_{L^2_t H^{-1}_x} \|R\varphi\|_{L^2_t H^2_x} \le \|b_n\|_{L^2_t H^{-1}_x} \|R\| \, \|\varphi\|_{L^2_t L^2_x}.$$

By duality, $\{\partial_t \nabla c_n\}$ is uniformly bounded in $L^2((0,T)\times\Omega)$ so that by Lions-Aubin lemma, $\{\nabla c_n\}$ is compact in $L^2((0,T)\times\Omega)$. Moreover, its limit equals to ∇c where $c = R(b)$ thanks to uniqueness of solutions to Poisson equation. Now, we write

$$\int_{\Omega_T} a_n\, b_n\, \varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} a_n\, \Delta c_n\, \varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \nabla a_n\, \nabla c_n\, \varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a_n\, \nabla c_n\, \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.$$

Under both integrals we deal with a product of a weakly and strongly converging sequences. Therefore,

$$\lim_{n\to\infty} \int_{\Omega_T} a_n\, b_n\, \varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \nabla a\, \nabla c\, \varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a\, \nabla c\, \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} a\, b\, \varphi\,\mathrm{d}x\,\mathrm{d}t.$$

The conclusion follows.

2.4 Velocity averaging lemmas

We now recall several lemmas on solutions to transport equations and kinetic equations called velocity averaging lemmas. To illustrate them, let $f = f(t,x,\xi)$ be a distributional solution to the following transport equation

$$\partial_t f + \xi \cdot \nabla_x f = S \tag{2.4.1}$$

where $S = S(t,x,\xi)$ with t, x, ξ being time, space and velocity, respectively. Velocity averaging lemmas assert that the velocity average $\int_{\mathbb{R}^d} f(t,x,\xi)\,\varphi(\xi)\,\mathrm{d}\xi$ is a more regular function than f itself. The most basic result in this spirit reads:

Theorem 2.4.1. *Let $f, S \in L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ and let f be the distributional solution to* (2.4.1)*. Then, for each compactly supported and bounded $\varphi : \mathbb{R}^d \to \mathbb{R}$ we have*

$$\left\| \int_{\mathbb{R}^d} f(t,x,\xi)\,\varphi(\xi)\,\mathrm{d}\xi \right\|_{\dot{H}^{1/2}_{t,x}} \leq C\,\|f\|^{1/2}_{L^2_{t,x,\xi}}\,\|S\|^{1/2}_{L^2_{t,x,\xi}}, \tag{2.4.2}$$

where C can deppend only on $\|\varphi\|_\infty$ and the size of the support of φ.

Here, $\dot{H}^{1/2}_{t,x}$ denotes usual fractional Sobolev space. Together with the L^2 estimate on f, inequality (2.4.2) implies (local) compactness in $L^2((0,\infty)\times\mathbb{R}^d)$.

This regularizing effect was observed for the first time in [156, 157]. Then, it has been extended to cover more general situations and obtain better regularity of the average (usually in the language of fractional Sobolev and Besov spaces) [102, 113, 174, 175, 260]. Finally, there is also a long history in applying velocity averaging lemmas to study regularity of solutions to hyperbolic conservation laws [228] and degenerate parabolic equations [255], in particular optimal regularity to the porous media equation [150].

Proof of Theorem 2.4.1. We follow closely the presentation from the lecture notes [244]. Let $\widehat{f}$ be the Fourier transform of f in time and space (t,x) variables and let (τ,ζ) be the respective variables in the phase space. Applying the Fourier transform to (2.4.1) we obtain

$$i\,(\tau + \xi\cdot\zeta)\,\widehat{f} = \widehat{S}.$$

Therefore, we estimate the integral by splitting the domain of integration (with respect to ξ) for $A := \{\xi : |\tau+\xi\cdot\zeta| \leq \alpha\}$ and $B := \{\xi : |\tau+\xi\cdot\zeta| > \alpha\}$:

$$\begin{aligned}\left| \int_{\mathbb{R}^d} \widehat{f}(\tau,\zeta,\xi)\,\varphi(\xi)\,\mathrm{d}\xi \right| &\leq \left| \int_A \widehat{f}(\tau,\zeta,\xi)\,\varphi(\xi)\,\mathrm{d}\xi \right| + \left| \int_B \widehat{f}(\tau,\zeta,\xi)\,\varphi(\xi)\,\mathrm{d}\xi \right| \\ &\leq \|\widehat{f}\|_{L^2_\xi} \left(\int_A |\varphi(\xi)|^2\,\mathrm{d}\xi \right)^{1/2} + \|\widehat{S}\|_{L^2_\xi} \left(\int_B \frac{|\varphi(\xi)|^2}{|\tau+\xi\cdot\zeta|^2}\,\mathrm{d}\xi \right)^{1/2}.\end{aligned}$$

We will prove that for some constant C we have

$$\int_A |\varphi(\xi)|^2\,\mathrm{d}\xi \leq \frac{C\,\alpha}{\sqrt{\tau^2+|\zeta|^2}}, \tag{2.4.3}$$

$$\int_B \frac{|\varphi(\xi)|^2}{|\tau + \xi \cdot \zeta|^2} \, \mathrm{d}\xi \leq \frac{C}{\alpha \sqrt{\tau^2 + |\zeta|^2}}. \tag{2.4.4}$$

Then, integrating with respect to τ and ζ, and then choosing $\alpha^2 = \frac{\|S\|_{L^2_{t,x,\xi}}}{\|f\|_{L^2_{t,x,\xi}}}$ we conclude the proof of Theorem 2.4.1.

It remains to prove (2.4.3)–(2.4.4). First, we introduce normalized variables $\rho = \sqrt{\tau^2 + |\zeta|^2}$, $\tau_0 = \frac{\tau}{\rho}$, $\zeta_0 = \frac{\zeta}{\rho}$. Then, we decompose ξ as follows

$$\xi = \left(\xi \cdot \frac{\zeta_0}{|\zeta_0|} + \frac{\tau_0}{|\zeta_0|} - \frac{\tau_0}{|\zeta_0|} \right) \frac{\zeta_0}{|\zeta_0|} + \left(\xi - \xi \cdot \zeta_0 \, \frac{\zeta_0}{|\zeta_0|^2} \right)$$

This is, in fact, an orthogonal decomposition. Introducing $y = \xi \cdot \frac{\zeta_0}{|\zeta_0|} + \frac{\tau_0}{|\zeta_0|}$ and $\xi^\perp = \xi - \xi \cdot \zeta_0 \, \frac{\zeta_0}{|\zeta_0|^2}$ we have

$$|\xi|^2 = \left| y - \frac{\tau_0}{|\zeta_0|} \right|^2 + |\xi^\perp|^2, \qquad \xi \cdot \zeta_0 + \tau_0 = y \, |\zeta_0|.$$

After rotation and translation, we can consider new variables $(y, \xi^\perp)$ instead of ξ. Note also, that since φ is compactly supported, we can assume that $|v| \leq R$ for some $R > 0$.

Integral (2.4.3). If $\frac{\alpha}{\rho} \geq \frac{1}{4}$ we can estimate the integral brutally by $\|\varphi\|^2_{L^2_\xi} \leq 4 \, \|\varphi\|^2_{L^2_\xi} \, \frac{\alpha}{\rho}$. Therefore, we only need to study the case $\frac{\alpha}{\rho} \leq \frac{1}{4}$. We note that the the contraint $|\tau + \xi \cdot \zeta| \leq \alpha$ is equivalent to $|y| \, |\zeta_0| \leq \frac{\alpha}{\rho}$ so that $|y| \leq \frac{1}{4 \, |\zeta_0|}$. In this case we can simply integrate

$$\int_A |\varphi(\xi)|^2 \, \mathrm{d}\xi \leq C \int_{|y| \, |\zeta_0| \leq \frac{\alpha}{\rho}} \mathrm{d}y \leq C \, \frac{\alpha}{\rho \, |\zeta_0|}.$$

The only thing we need to prove is that ζ_0 can be assumed to be bounded from below. We claim that

$$|\zeta_0| > C_A := \min \left(\frac{1}{2R}, \frac{\sqrt{7}}{4} \right).$$

For if not, $|\tau_0| \geq \frac{3}{4}$ (by $|\tau_0|^2 + |\zeta_0|^2 = 1$) and then we can estimate

$$\frac{\tau_0}{|\zeta_0|} - y \geq \frac{3}{4 \, |\zeta_0|} - \frac{1}{4 \, |\zeta_0|} = \frac{1}{2 \, |\zeta_0|} > R$$

so that $|\xi| > R$ raising contradiction.

Integral (2.4.4). Thanks to the support of φ and the identity $\xi \cdot \zeta_0 + \tau_0 = y\,|\zeta_0|$, we can integrate on $B' = \left\{ \left| y - \frac{\tau_0}{|\zeta_0|} \right| \le R \right\} \cap \left\{ |y| > \frac{\alpha}{|\zeta_0|\,\rho} \right\}$ while the integral can be written as (after appropriate change of variables)

$$\int_B \frac{|\varphi(\xi)|^2}{|\tau + \xi \cdot \zeta|^2}\, \mathrm{d}\xi \le \frac{C}{\rho^2\,|\zeta_0|^2} \int_{B'} \frac{1}{|y|^2}\, \mathrm{d}y.$$

The difficulty is to remove $|\zeta_0|$ from the denominator. We consider two cases. First, if $\frac{|\tau_0|}{|\zeta_0|} - R > \frac{\alpha}{\rho\,|\zeta_0|}$ we can simply integrate

$$\frac{C}{\rho^2\,|\zeta_0|^2} \int_{B'} \frac{1}{|y|^2}\, \mathrm{d}y \le \frac{C}{\rho^2\,|\zeta_0|^2} \int_{\frac{|\tau_0|}{|\zeta_0|} - R}^{\frac{|\tau_0|}{|\zeta_0|} + R} \frac{1}{y^2}\, \mathrm{d}y \le \frac{C}{\rho^2\,|\zeta_0|^2} \frac{2R}{\left(\frac{|\tau_0|}{|\zeta_0|} - R\right)\left(\frac{|\tau_0|}{|\zeta_0|} + R\right)}$$
$$\le \frac{1}{\alpha\,\rho} \frac{2\,R\,C}{|\tau_0| + R\,|\zeta_0|} \le \frac{1}{\alpha\,\rho} \frac{2\,R\,C}{\min(1,R)}$$

where in the last step we used that $|\tau_0|^2 + |\zeta_0|^2 = 1$. In the second case, that is when $\frac{|\tau_0|}{|\zeta_0|} - R \le \frac{\alpha}{\rho\,|\zeta_0|}$ we integrate in fact between $\frac{\alpha}{|\zeta_0|\,\rho}$ and $\frac{|\tau_0|}{|\zeta_0|} + R$:

$$\frac{C}{\rho^2\,|\zeta_0|^2} \int_{B'} \frac{1}{|y|^2}\, \mathrm{d}y \le \frac{C}{\rho^2\,|\zeta_0|^2} \int_{\frac{\alpha}{|\zeta_0|\,\rho}}^{\frac{|\tau_0|}{|\zeta_0|} + R} \frac{1}{y^2}\, \mathrm{d}y = \left(1 - \frac{\alpha/\rho}{|\tau_0| + R\,|\zeta_0|}\right) \frac{C}{\alpha\,\rho\,|\zeta_0|} \le$$
$$\le \left(1 - \frac{|\tau_0| - R\,|\zeta_0|}{|\tau_0| + R\,|\zeta_0|}\right) \frac{C}{\alpha\,\rho\,|\zeta_0|} \le \frac{2\,R\,C}{\min(1,R)} \frac{1}{\alpha\,\rho}.$$

□

Below, we cite another variant of velocity averaging lemma from [213, Lemma 4.2] that will be used in Chapter 4.

Lemma 2.4.2. *Assume that $\{h^\varepsilon\}$ is bounded in $L^2((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$, $\{h_0^\varepsilon\}$ and $\{h_1^\varepsilon\}$ are bounded in $L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$. Moreover, suppose that*

$$\varepsilon \partial_t h^\varepsilon + \xi \cdot \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_\xi \cdot h_1^\varepsilon.$$

Then, for all $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\left\| \int_{\mathbb{R}^d} \left(h^\varepsilon(t, x + y, \xi) - h^\varepsilon(t, x, \xi)\right) \psi(\xi)\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \to 0,$$

when $y \to 0$ uniformly in ε.

Note that this version does not yield compactness in time because of the parameter ε in front of the time derivative. Nevertheless, in applications, compactness in time can be usually deduced from compactness in space and equation on the macroscopic quantity, see Lemma 2.5.2.

Finally, we recall the following renormalization trick which allows to apply Lemma 2.4.2 when L^2 bound is not available. This is a usual situation in the kinetic theory when only L^1 estimates are easily available. The following family of functions will be important: $\beta_\nu(f) = \frac{f}{1+\nu f}$, $\nu > 0$.

Lemma 2.4.3 (Compactness of $\beta_\nu(f_n)$ implies compactness of f_n)**.** *Let $\{f_n(t,x,\xi)\}$ be a sequence such that $\{f_n\}$ and $\{f_n \log f_n\}$ are bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Let $\psi(\xi) \in C_c^\infty(\mathbb{R}^d)$. Suppose that for all $\nu > 0$ and all $\varepsilon > 0$, there exists $\delta(\nu,\varepsilon)$ such that if $|y| \le \delta(\nu,\varepsilon)$ then*

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_n(t,x+y,\xi)) - \beta_\nu(f_n(t,x,\xi)))\, \psi(\xi) \,\mathrm{d}\xi \right\|_{L^1_{t,x}} \le \varepsilon.$$

Then, for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|y| \le \delta(\varepsilon)$ then

$$\left\| \int_{\mathbb{R}^d} (f_n(t,x+y,\xi) - f_n(t,x,\xi))\, \psi(\xi) \,\mathrm{d}\xi \right\|_{L^1_{t,x}} \le \varepsilon.$$

Proof. First, we observe that

$$|\beta_\nu(s) - s| \le \left| \frac{s}{1+s\,\nu} - s \right| = \frac{\nu s^2}{1+\nu\, s} \le \min(\nu\, s^2, s).$$

Therefore, for M and ν to be chosen later

$$\begin{aligned}
\left\| \int_{\mathbb{R}^d} (f_n(t,x+y,\xi) - \beta_\nu(f_n(t,x+y,\xi)))\, \psi(\xi) \,\mathrm{d}\xi \right\|_{L^1_{t,x}} &\le \\
&\le \|\psi\|_\infty\, \nu \int_{f_n(t,x+y,\xi)\le M} f_n^2(t,x+y,\xi) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \|\psi\|_\infty \int_{f_n(t,x+y,\xi)\ge M} f_n(t,x+y,\xi) \,\mathrm{d}\xi \,\mathrm{d}x \,\mathrm{d}t \\
&\le \|\psi\|_\infty\, \nu\, M\, \|f_n\|_1 + \|\psi\|_\infty \frac{\|f_n \log f_n\|_1}{\log M}.
\end{aligned}$$

Similarly,

$$\left\| \int_{\mathbb{R}^d} (f_n(t,x,\xi) - \beta_\nu(f_n(t,x,\xi)))\, \psi(\xi) \,\mathrm{d}\xi \right\|_{L^1_{t,x}} \le \|\psi\|_\infty\, \nu\, M\, \|f_n\|_1 + \|\psi\|_\infty \frac{\|f_n \log f_n\|_1}{\log M}.$$

Let $\varepsilon > 0$. First, we choose ν and M such that

$$\|\psi\|_\infty \, \nu \, M \, \|f_n\|_1 + \|\psi\|_\infty \, \frac{\|f_n \log f_n\|_1}{\log M} \le \frac{\varepsilon}{3}.$$

Then, we take $\delta(\nu, \varepsilon/3)$ such that

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_n(t, x+y, \xi)) - \beta_\nu(f_n(t, x, \xi))) \, \psi(\xi) \, \mathrm{d}\xi \right\|_{L^1_{t,x}} \le \varepsilon/3$$

when $|y| \le \delta(\nu, \varepsilon/3)$. The conclusion follows by the triangle inequality. $\square$

Remark 2.4.4. Of course, in the statement of Lemma 2.4.3, it is sufficient to assume that $\{\varphi(f_n)\}$ is bounded in $L^1((0,T) \times \mathbb{R}^d \times \mathbb{R}^d)$ where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a nonnegative, superlinear function, i.e. $\lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty$.

2.5 Riesz-Kolmogorov-Fréchet theorems

We present here several classical results which are useful to get compactness using information on derivatives. As all of them will be used either on $\mathbb{R}^d$ or $\mathbb{T}^d$ (d-dimensional torus), we restrict ourselves only to these cases. Nevertheless, most of the results can be easily extended to the case of bounded domain Ω.

The most classical is the following:

Theorem 2.5.1 (Riesz-Kolmogorov-Fréchet)**.** *Let* $1 \le p < \infty$.

- **Case** $\mathbb{R}^d$. *Suppose that* $\{\varrho_\varepsilon\}$ *is a sequence bounded in* $L^p((0,T) \times \mathbb{R}^d)$ *such that*

$$\lim_{|y| \to 0} \int_0^T \int_{\mathbb{R}^d} |\varrho_\varepsilon(t, x+y) - \varrho_\varepsilon(t, x)|^p \, \mathrm{d}x \, \mathrm{d}t = 0 \quad \textit{uniformly in } \varepsilon, \tag{2.5.1}$$

$$\lim_{|h| \to 0} \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t, x)|^p \, \mathrm{d}x \, \mathrm{d}t = 0 \quad \textit{uniformly in } \varepsilon, \tag{2.5.2}$$

$$\lim_{|h| \to 0} \int_h^T \int_{\mathbb{R}^d} |\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t-h, x)|^p \, \mathrm{d}x \, \mathrm{d}t = 0 \quad \textit{uniformly in } \varepsilon. \tag{2.5.3}$$

Moreover, suppose that $\{\varrho_\varepsilon\}$ *satisfies the tightness condition on* L^p*:*

$$\textit{for all } \kappa > 0 \textit{ there is compact } K \subset \mathbb{R}^d \textit{ with } \int_0^T \int_{\mathbb{R}^d \setminus K} |\varrho_\varepsilon|^p \, \mathrm{d}x \, \mathrm{d}t \le \kappa. \tag{2.5.4}$$

Then, $\{\varrho_\varepsilon\}$ *has a subsequence converging strongly in* $L^p((0,T) \times \mathbb{R}^d)$.

- **Case $\mathbb{T}^d$.** *If $\mathbb{R}^d$ is replaced with $\mathbb{T}^d$ above, the same conclusion follows without the tightness condition.*

We will not prove Theorem 2.5.1 but below, we will prove a variant of Theorem 2.5.1 and it will be clear that the same argument works to prove Theorem 2.5.1.

Lemma 2.5.2. *Let $1 \leq p < \infty$. Let $\{\varrho_\varepsilon\}$ be a sequence of functions $\varrho_\varepsilon = \varrho_\varepsilon(t,x)$ bounded in $L^p((0,T)\times\mathbb{R}^d)$ and compact in the spatial variable as in* (2.5.1)*. Moreover, suppose that $\{\partial_t \varrho_\varepsilon\}$ satisfies one of the following:*

- *(case $p = 1$) $\partial_t \varrho_\varepsilon = \nabla^k J_\varepsilon$ where $\{J_\varepsilon\}$ is bounded in $L^1((0,T)\times\mathbb{R}^d)$ and ∇^k is any differential operator of order k,*
- *(case $1 < p < \infty$) $\{\partial_t \varrho_\varepsilon\}$ is bounded in $L^p(0,T;(W^{k,q}(\mathbb{R}^d))^*)$ for some $k \in \mathbb{N}$ and $q \geq p'$.*

Then, $\{\varrho_\varepsilon\}$ is compact in the time variable, i.e. it satisfies (2.5.2) *and* (2.5.3)*. The same conclusion follows if $\mathbb{R}^d$ is replaced with $\mathbb{T}^d$.*

Remark 2.5.3. (A) It will be clear from the proof that $\mathbb{R}^d$ can be replaced without any difficulty with the torus $\mathbb{T}^d$. As $\mathbb{T}^d$ is a bounded domain, in the case $1 < p < \infty$, one can simply require that $\{\partial_t \varrho_\varepsilon\}$ is bounded in $L^p(0,T;(W^{k,q}(\mathbb{T}^d))^*)$ for *any* $q \geq 1$ (this is because of the natural embedding $W^{k,q_1}(\mathbb{T}^d) \subset W^{k,q_2}(\mathbb{T}^d)$ for $q_1 \geq q_2$).
(B) The distinguishment between cases $p = 1$ and $p > 1$ is necessary because for $p = 1$ there is no good characterization of the dual space of $L^{p'}(\mathbb{R}^d)$. In this case, we need the representation of the time derivative to be given directly by some equation. Of course, in most applications in PDEs that one can have in mind, the representation will be given by the considered PDE. □

The proof of Lemma 2.5.2 exploits a family of mollifiers $\{\varphi_\delta\}$, i.e. $\varphi_\delta(x) = \frac{1}{\delta^d}\varphi\left(\frac{x}{\delta}\right)$ with φ smooth, nonnegative, compactly supported and $\int_{\mathbb{R}^d}\varphi = 1$. A simple computation shows that for any differential operator ∇^k of order $k \in \mathbb{N}$ we have

$$\|\nabla^k \varphi_\delta\|_{L^1(\mathbb{R}^d)} \leq \frac{C}{\delta^k}.$$

Proof of Lemma 2.5.2. Using the mollifiers with δ depending on h to be specified later on, we first notice that

$$\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t+h,x)-\varrho_\varepsilon(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t \le C(p)\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t$$
$$+C(p)\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t+h,x)-\varrho_\varepsilon(t+h,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t$$
$$+C(p)\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t+h,\cdot)*\varphi_\delta(x)-\varrho_\varepsilon(t,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t.$$

For the first and second terms, the computations are the same, hence, we only present it for the first term. Using the properties of the mollifiers and the compactness of $\{\varrho_\varepsilon\}$ in space, we want to prove that

$$\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t\le\theta(\delta).$$

where $\theta(\delta)\to 0$ when $\delta\to 0$ uniformly in ε. We write

$$\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t=$$
$$=\int_0^{T-h}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\varphi(y)(\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,x-\delta y))\,\mathrm{d}y\right|^p\mathrm{d}x\,\mathrm{d}t.$$

Then we use Fubini's theorem and the fact that φ is compactly supported in some compact set K to obtain

$$\int_0^{T-h}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\varphi(y)(\varrho_\varepsilon(t,x)-\varrho_\varepsilon(t,x-\delta y))\,\mathrm{d}y\right|^p\mathrm{d}x\,\mathrm{d}t\le\int_K\|\tau_{\delta y}\varrho_\varepsilon-\varrho_\varepsilon\|^p_{L^p((0,T)\times\mathbb{R}^d)}\,\mathrm{d}y.$$

where τ_x is the translation operator in x variable. Now we use the compactness in space, so that

$$\int_K\|\tau_{\delta y}\varrho_\varepsilon-\varrho_\varepsilon\|^p_{L^p((0,T)\times\mathbb{R}^d)}\,\mathrm{d}y\le|K|\sup_{y\in K}\|\tau_{\delta y}\varrho_\varepsilon-\varrho_\varepsilon\|^p_{L^p((0,T)\times\mathbb{R}^d)}\le\theta(\delta).$$

Therefore the first and the second term are bounded by $\theta(\delta)$ where $\theta(\delta)\to 0$ when $\delta\to 0$ uniformly in ε. It remains to study the third term. The third term reads

$$\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_\varepsilon(t+h,\cdot)*\varphi_\delta(x)-\varrho_\varepsilon(t,\cdot)*\varphi_\delta(x)|^p\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^{T-h}\int_{\mathbb{R}^d}\left|\int_t^{t+h}\partial_t\varrho_\varepsilon*\varphi_\delta(s,x)\,\mathrm{d}s\right|^p\mathrm{d}x\,\mathrm{d}t.$$

We perform the change of variables $v = \frac{s-t}{h}$ and use Jensen's inequality to obtain

$$\int_0^{T-h} \int_{\mathbb{R}^d} \left| \int_t^{t+h} \partial_t \varrho_\varepsilon * \varphi_\delta(s,x) \, \mathrm{d}s \right|^p \mathrm{d}x \, \mathrm{d}t =$$
$$= h^p \int_0^1 \int_0^{T-h} \int_{\mathbb{R}^d} |\partial_t \varrho_\varepsilon * \varphi_\delta(t+vh,x)|^p \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v.$$

Then we use the change of variables $\tau = vh + t$ and obtain

$$h^p \int_0^1 \int_0^{T-h} \int_{\mathbb{R}^d} |\partial_t \varrho_\varepsilon * \varphi_\delta(t+vh,x)|^p \, \mathrm{d}t \, \mathrm{d}x \, \mathrm{d}v \leq$$
$$\leq h^p \int_0^T \int_{\mathbb{R}^d} |\partial_t \varrho_\varepsilon * \varphi_\delta(\tau,x)|^p \, \mathrm{d}x \, \mathrm{d}\tau = h^p \left\| \partial_t \varrho_\varepsilon * \varphi_\delta \right\|^p_{L^p_{t,x}}.$$

It remains to estimate the L^p norm of $\partial_t \varrho_\varepsilon * \varphi_\delta(\tau, x)$.

Case $p = 1$. We have $\partial_t \varrho_\varepsilon * \varphi_\delta(\tau,x) = J_\varepsilon * \nabla^k \varphi_\delta$ so that by Young's convolutional inequality

$$h \left\| \partial_t \varrho_\varepsilon * \varphi_\delta \right\|_{L^1_{t,x}} \leq h \left\| J_\varepsilon \right\|_{L^1_{t,x}} \left\| \nabla^k \varphi_\delta \right\|_{L^1} \leq C \frac{h}{\delta^k}$$

so that choosing $h = \delta^{k+1}$ we conclude the proof.

Case $1 < p < \infty$. We fix $t \in (0,T)$. By Riesz theorem in Sobolev spaces [5, Theorem 3.9], for fixed value of t and for each multiindex α (with $|\alpha| \leq k$), there exists unique function $v_\alpha^\varepsilon \in L^{q'}_x$ such that action of the functional $\partial_t \varrho_\varepsilon$ can be represented as

$$(\partial_t \varrho_\varepsilon)(\varphi) = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^d} v_\alpha^\varepsilon(x) \, D^\alpha \varphi(x) \, \mathrm{d}x, \qquad \sum_{|\alpha| \leq k} \left\| v_\alpha^\varepsilon \right\|_{L^{q'}} \leq C \left\| \partial_t \varrho_\varepsilon \right\|_{(W^{k,q})^*}$$

Therefore, using definition of mollification of a distribution, we have

$$\left\| \partial_t \varrho_\varepsilon * \varphi_\delta \right\|^p_{L^p_x} \leq C(k,p) \sum_{|\alpha| \leq k} \left\| v_\alpha^\varepsilon * D^\alpha \varphi_\delta \right\|^p_{L^p_x} \leq C(k,p) \sum_{|\alpha| \leq k} \left\| v_\alpha^\varepsilon \right\|^p_{L^{q'}_x} \left\| D^\alpha \varphi_\delta \right\|^p_{L^r},$$

where we used Young's convolution inequality with $r = \frac{pq'}{pq'+q'-p}$ (here, we use that $q' \leq p$). For small δ, $\left\| D^\alpha \varphi_\delta \right\|^p_{L^r_x} \leq \frac{C}{\delta^{p(k+d/r')}}$ and so,

$$h^p \left\| \partial_t \varrho_\varepsilon * \varphi_\delta \right\|^p_{L^p_{t,x}} \leq C(k,p) \, \frac{h^p}{\delta^{p(k+d/r')}} \left\| \partial_t \varrho_\varepsilon \right\|^p_{L^p_t (W^{k,q}_x)^*}.$$

The conclusion follows by choosing $h = \delta^{(k+d/r')+1}$. □

An immediate consequence of Theorem 2.5.1 and Lemma 2.5.2 is the following result:

Theorem 2.5.4. *Suppose all assumptions of Lemma 2.5.2. In the case of* $\mathbb{R}^d$, *suppose additionally that* $\{\varrho_\varepsilon\}$ *satisfies tightness condition* (2.5.4). *Then,* $\{\varrho_\varepsilon\}$ *has a subsequence converging strongly in* $L^p((0,T)\times\mathbb{R}^d)$ *(or in* $L^p((0,T)\times\mathbb{T}^d)$ *in the case of* $\mathbb{T}^d$*).*

We conclude with a variant of Theorem 2.5.1 that will be important for compactness arguments for nonlocal equations presented in Section 2.6.

Theorem 2.5.5. *Let* $\{\varrho_k\}$ *be a countable sequence satisfying all assumptions of Lemma 2.5.2 except* (2.5.1) *which is replaced by the following:*

$$\limsup_{\delta\to 0}\limsup_{k\to\infty}\int_0^T\int_{\mathbb{T}^d}|\varrho_k*\varphi_\delta(t,x)-\varrho_k(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t=0, \tag{2.5.5}$$

where $\{\varphi_\delta\}$ *is a usual sequence of mollifiers (i.e.* $\varphi_\delta(x)=\frac{1}{\delta^d}\varphi\left(\frac{x}{\delta}\right)$ *with* φ *smooth, nonnegative, compactly supported and* $\int_{\mathbb{R}^d}\varphi=1$*). Then,*

- *In the case of* $\mathbb{R}^d$, *if* $\{\varrho_k\}$ *satisfies tightness condition* (2.5.4), *then* $\{\varrho_k\}$ *has a subsequence converging strongly in* $L^p((0,T)\times\mathbb{R}^d)$.
- *In the case of* $\mathbb{T}^d$, *the same conclusion follows without any additional assumption.*

Proof. We define two *mollifiers* in the time variable:

- $\psi_\delta(t)$ is smooth, compactly supported on $[-\delta,0]$ and $\int_{-\delta}^0\psi_\delta(t)\,\mathrm{d}t=1$,
- $\eta_\delta(t)$ is smooth, compactly supported on $[0,\delta]$ and $\int_0^\delta\eta_\delta(t)\,\mathrm{d}t=1$.

Then, the proof of Lemma 2.5.2 implies

$$\limsup_{h\to 0^+}\limsup_{k\to\infty}\int_0^{T-h}\int_{\mathbb{R}^d}|\varrho_k*\psi_\delta(t,x)-\varrho_k(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t=0 \tag{2.5.6}$$

$$\limsup_{h\to 0^+}\limsup_{k\to\infty}\int_h^T\int_{\mathbb{R}^d}|\varrho_k*\eta_\delta(t,x)-\varrho_k(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t=0 \tag{2.5.7}$$

and this gives compactness in $L^p((0,T)\times\mathbb{R}^d)$. To see the latter, we first prove compactness in $L^p((0,T/2)\times\mathbb{R}^d)$ using (2.5.6). We estimate

$$\begin{aligned}\|\varrho_k*\psi_\delta*\varphi_\delta-\varrho_k\|^p_{L^p_{t,x}}&\le\|\varrho_k*\psi_\delta*\varphi_\delta-\varrho_k*\varphi_\delta\|^p_{L^p_{t,x}}+\|\varrho_k*\varphi_\delta-\varrho_k\|^p_{L^p_{t,x}}\\&\le\|\varrho_k*\psi_\delta-\varrho_k\|^p_{L^p_{t,x}}+\|\varrho_k*\varphi_\delta-\varrho_k\|^p_{L^p_{t,x}}.\end{aligned}$$

Therefore,

$$\limsup_{\delta\to 0}\limsup_{k\to\infty}\int_0^{T/2}\int_{\mathbb{R}^d}|\varrho_k*\psi_\delta*\varphi_\delta(t,x)-\varrho_k(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t=0.$$

which can be written as

$$\forall_\gamma\,\exists_{\delta=\delta(\gamma)}\,\exists_{K=K(\delta)}\,\forall_{k\geq K}\|\varrho_k*\psi_\delta*\varphi_\delta-\varrho_k\|_{L^p((0,T/2)\times\Omega)}\leq\gamma \tag{2.5.8}$$

As $L^p((0,T/2)\times\mathbb{R}^d)$ is complete, it is sufficient to prove that $\{\varrho_k\}$ is totally bounded (i.e. for given $r>0$, it can be covered by finite number of balls of radius r). First, by the tightness condition, we find Ω such that $\|\varrho_k\|_{L^p((0,T/2)\times\mathbb{R}^d\setminus\Omega)}\leq r/4$. Second, we choose $\gamma=r/4$ in (2.5.8) which fixes some δ and K. For this value of δ, the sequence $\{\varrho_k*\psi_\delta*\varphi_\delta\}_{k\in\mathbb{N}}$ satisfies assumptions of Arzela-Ascoli theorem on $(0,T/2)\times\Omega$ so that it is compact in $L^p((0,T/2)\times\Omega)$. It follows that there exists a finite number of functions g_1, ..., g_N such that

$$\forall k\,\exists i\,\|\varrho_k*\psi_\delta*\varphi_\delta-g_i\|_{L^p((0,T/2)\times\Omega)}\leq r/4. \tag{2.5.9}$$

We claim that $\{\varrho_k\}_{k\geq K}\subset\cup_{i=1}^N B(\overline{g_i},r)$ where $\overline{g_i}$ is extension of g_i (which is defined only on $(0,T/2)\times\Omega$) with 0. Indeed,

$$\begin{aligned}\|\varrho_k-\overline{g_i}\|_{L^p((0,T/2)\times\mathbb{R}^d)}\leq\|\varrho_k-g_i\|_{L^p((0,T/2)\times\Omega)}+\|\varrho_k\|_{L^p((0,T/2)\times\mathbb{R}^d\setminus\Omega)}\leq&\\ \leq\|\varrho_k-g_i\|_{L^p((0,T/2)\times\Omega)}+\frac{r}{4}&.\end{aligned}$$

Now, using (2.5.8) and (2.5.9) we can estimate

$$\begin{aligned}\|\varrho_k-g_i\|_{L^p((0,T/2)\times\Omega)}\leq\|\varrho_k-\varrho_k*\psi_\delta*\varphi_\delta\|_{L^p((0,T/2)\times\Omega)}+\|g_i-\varrho_k*\psi_\delta*\varphi_\delta\|_{L^p((0,T/2)\times\Omega)}&\\ \leq\frac{r}{4}+\frac{r}{4}&\end{aligned}$$

and this proves that $\{\varrho_k\}_{k\geq K}\subset\cup_{i=1}^N B(\overline{g_i},r)$. To cover the whole sequence, we just add a finite number of balls: $\{\varrho_k\}\subset\cup_{i=1}^N B(\overline{g_i},r)\cup\cup_{i=1}^K B(\varrho,r)$. This way, we obtain compactness in $L^p((0,T/2)\times\Omega)$. To deduce compactness in $L^p((T/2,T)\times\Omega)$ it is sufficient to use (2.5.7) instead of (2.5.6). $\square$

2.6 Bourgain-Brézis-Mironescu and Ponce compactness result

We now present a variant of Rellich–Kondrachov theorem adapted to evolutionary nonlocal equations. To motivate, recall that if a sequence $\{f_\varepsilon\}$ is bounded in $W^{1,p}(\mathbb{T}^d)$ then it has a subsequence converging strongly in $L^p(\mathbb{T}^d)$. Now, consider sequence of radial functions $\{\rho_\varepsilon\}$ such that $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}^d} \rho_\varepsilon = 1$ and

$$\lim_{\varepsilon \to 0} \int_{|x|>\delta} \rho_\varepsilon(x)\, \mathrm{d}x = 0 \text{ for all } \delta > 0.$$

Then, the following was proven in [233, Theorem 1.2] and [41, Theorem 4]:

Proposition 2.6.1. *Let $d \geq 2$, $1 \leq p < \infty$ and let $\{f_\varepsilon\}$ be a sequence bounded in $L^p(\mathbb{T}^d)$. Suppose that*

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)\, \mathrm{d}x\, \mathrm{d}y \leq C \tag{2.6.1}$$

for some constant C. Then, $\{f_\varepsilon\}$ is strongly compact in $L^p(\mathbb{T}^d)$ and the limit $f \in W^{1,p}(\mathbb{T}^d)$ (or $f \in BV(\mathbb{T}^d)$ if $p = 1$).

Remark 2.6.2. **(A)** Let us comment the difference between [233, Theorem 1.2] and [41, Theorem 4]. In [41, Theorem 4] the result is obtained under additional assumption on radial monotonicity of ρ_ε. In [233, Theorem 1.2], there is no additional assumption on ρ_ε.

(B) The assumption on the dimension $d \geq 2$ is necessary. In [41, Counterexample 2], Authors construct an example (quite pathological) of sequence $\{\rho_\varepsilon\}$ in one dimension for which Proposition 2.6.1 does not hold. Still, one can prove Proposition 2.6.1 under additional assumptions on $\{\rho_\varepsilon\}$, see [233, Theorem 1.3]. $\square$

Now, we formulate Proposition 2.6.1 adapted to evolutionary problems.

Theorem 2.6.3. *Let $d \geq 2$. Let $\{f_\varepsilon\}$ be a sequence bounded in $L^p((0,T) \times \mathbb{T}^d)$. Suppose that there exists a sequence $\{\rho_\varepsilon\}$ as above such that*

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(t,x) - f_\varepsilon(t,y)|^p}{|x-y|^p} \rho_\varepsilon(|x-y|)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \leq C \tag{2.6.2}$$

for some constant C. Then, $\{f_\varepsilon\}$ is compact in the spatial variable in $L^p((0,T)\times\mathbb{T}^d)$, i.e.

$$\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^d}|f_\varepsilon*\varphi_\delta(t,x)-f_\varepsilon(t,x)|^p\,\mathrm{d}x\,\mathrm{d}t=0, \tag{2.6.3}$$

where $\{\varphi_\delta\}\subset C_c^\infty(\mathbb{R}^d)$ is a sequence of standard mollifiers such that $\varphi_\delta(x)=\frac{1}{\delta^d}\varphi(\frac{x}{\delta})$ with φ of mass 1 and compactly supported.

Remark 2.6.4. Let $\omega:\mathbb{R}^d\to\mathbb{R}$ be a smooth function, supported in the unit ball such that $\int_{\mathbb{R}^d}\omega(x)\,\mathrm{d}x=1$. Consider $\omega_\varepsilon=\frac{1}{\varepsilon^d}\omega\left(\frac{x}{\varepsilon}\right)$. Suppose that

$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|f_\varepsilon(x)-f_\varepsilon(y)|^p}{\varepsilon^p}\omega_\varepsilon(|x-y|)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\le\widetilde{C}.$$

Then, (2.6.2) is satisfied. Indeed, we consider

$$\rho_\varepsilon(x)=\frac{\omega_\varepsilon(|x|)\,|x|^p}{\varepsilon^p\int_{\mathbb{R}^d}\omega(y)|y|^p\,\mathrm{d}y} \tag{2.6.4}$$

so that (2.6.2) holds true with $\frac{\widetilde{C}}{\int_{\mathbb{R}^d}\omega(y)|y|^p\,\mathrm{d}y}$.

Remark 2.6.5. The definition of *compactness in the spatial variable* used in Theorem 2.6.3 is motivated by condition (2.5.5) in Theorem 2.5.5.

Proof of Theorem 2.6.3. The result for sequences that do not depend on time has been obtained in [41, 233]. To demonstrate that it is sufficient to *integrate in time* the reasoning mentioned above, we will show how to adapt the proof presented in [41, Theorem 4] which makes an additional assumption that for every ε, ρ_ε is a nonincreasing function. For the general case, one has to adapt the proof presented in [233, Theorem 1.2]. We define

$$\begin{aligned}F_\varepsilon(s)&:=\int_0^T\int_{|y|=1}\int_{\mathbb{T}^d}|f_\varepsilon(t,x+sy)-f_\varepsilon(t,x)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\&=\frac{1}{s^{d-1}}\int_0^T\int_{|y|=s}\int_{\mathbb{T}^d}|f_\varepsilon(t,x+y)-f_\varepsilon(t,x)|^p\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

By virtue of the computation above, we can express the assumption 2.6.2 using function F_ε as follows

$$\int_0^\delta s^{d-1}\frac{F_\varepsilon(s)\,\rho_\varepsilon(s)}{s^p}\,\mathrm{d}s\le C. \tag{2.6.5}$$

Using the triangle inequality

$$|f_\varepsilon(t,x+2sy)-f_\varepsilon(t,x)| \le |f_\varepsilon(t,x+2sy)-f_\varepsilon(t,x+sy)|+|f_\varepsilon(t,x+sy)-f_\varepsilon(t,x)|$$

and change of variables we obtain

$$F_\varepsilon(2s) \le 2^p F_\varepsilon(s), \qquad \frac{F_\varepsilon(2s)}{(2s)^p} \le \frac{F_\varepsilon(s)}{s^p}. \tag{2.6.6}$$

We estimate by Jensen's inequality

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta - f_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t &\le \frac{C}{\delta^d}\int_0^T\int_{\mathbb{T}^d}\int_{|x-y|\le\delta} |f_\varepsilon(x)-f_\varepsilon(y)|^p \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{C}{\delta^d}\int_0^T\int_{\mathbb{T}^d}\int_{|h|\le\delta} |f_\varepsilon(x+h)-f_\varepsilon(x)|^p \,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{C}{\delta^d}\int_0^T\int_{\mathbb{T}^d}\int_0^\delta s^{d-1}\int_{|h|=s} |f_\varepsilon(x+h)-f_\varepsilon(x)|^p \,\mathrm{d}h\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{C}{\delta^d}\int_0^\delta s^{d-1}F_\varepsilon(s)\,\mathrm{d}s.
\end{aligned} \tag{2.6.7}$$

Now, we use functional inequality (which requires monotonicity assumption on $\{\rho_\varepsilon\}$ and doubling condition (2.6.6), cf. [41, Eq. (24)])

$$\delta^{-d}\int_0^\delta s^{d-1}\frac{F_\varepsilon(s)}{s^p}\,\mathrm{d}s \le C(d)\,\frac{\int_0^\delta s^{d-1}\frac{F_\varepsilon(s)\,\rho_\varepsilon(s)}{s^p}\,\mathrm{d}s}{\int_{|x|<\delta}\rho_\varepsilon(x)\,\mathrm{d}x} \tag{2.6.8}$$

For each $\delta > 0$, there exists $\varepsilon(\delta)$ such that for all $\varepsilon < \varepsilon(\delta)$ we have $\int_{|x|<\delta}\rho_\varepsilon(x)\,\mathrm{d}x = 1$. In particular, for $\varepsilon < \varepsilon(\delta)$ we have by (2.6.8) and (2.6.5)

$$\delta^{-d}\int_0^\delta s^{d-1}\,F_\varepsilon(s)\,\mathrm{d}s \le C(d)\,\delta^p.$$

In view of (2.6.7), the proof is concluded.

□

Chapter 3

Fast reaction limit with nonmonotone reaction

The results in this chapter have been published in:

- B. Perthame, J. Skrzeczkowski. *Fast reaction limit with nonmonotone reaction function.* Communications on Pure and Applied Mathematics, published online, doi: 10.1002/cpa.22042, cited as [230]
- J. Skrzeczkowski. *Fast reaction limit and forward-backward diffusion: a Radon-Nikodym approach.* Comptes Rendus Mathématique, tome 360, p. 189-203, 2022, cited as [249].

The first paper used the kinetic formulation (see Remark 2.1.4) while the second paper exploited the concept of Young measures (see Definition 2.1.1). As this concepts are equivalent for bounded sequences, we decided to formulate all the results using exclusively Young measures.

Concerning the content of the articles, the second paper is a generalization of the first one. More precisely, the first one contains the proof of Theorem 3.2.5 while the second one contains the proofs of Theorems 3.2.4, 3.2.5, 3.2.6 and 3.2.7 so it contains the result of the first paper. This is why, our presentation below follows mostly the second paper.

3.1 Introduction and the main results

We begin with the results concerning the limiting behavior (as $\varepsilon \to 0$) for the reaction-diffusion system

$$\begin{aligned} \partial_t u^\varepsilon &= \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}, \\ \partial_t v^\varepsilon &= \Delta v^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}, \end{aligned} \tag{3.1.1}$$

equipped with initial conditions u_0, v_0 and usual Neumann boundary conditions. Here, $\Omega \subset \mathbb{R}^d$ and F is a nonmonotone nonlinearity (this will be made rigorous in Assumption 3.2.2 but Reader may have a look at Figure 3.2). We point out that under suitable assumption, (3.1.1) admits unique, global-in-time, classical solutions cf. Lemma 3.3.2. Equations of the form (3.1.1) are widely studied in the literature, for instance from the point of view of stability theory [85, 86, 208, 209].

As $\varepsilon \to 0$, equation (3.1.1) becomes an interesting toy model for studying oscillations in reaction-diffusion systems as they are known to occur in their steady states [221] when F is not monotone. Furthermore, the limit $\varepsilon \to 0$ is well-motivated biologically. Indeed, it corresponds to the fast reactions in neuroscience where neurotransmitters bind to the channels in order to allow for the flow of the electric signal through the neuron. Moreover, the oscillations that we expect are also well-motivated as they correspond to the lack of stability which is a typical feature of neural network. Therefore, by understanding the limit $\varepsilon \to 0$, we hope to find a generic structure of oscillations observed in such systems. Finally, as for small values of ε it is hard to solve (3.1.1) numerically, fast reaction limit can provide a limiting system which can be easier to simulate.

Let us remark that for monotone F the problem is fairly classical and has been studied for a great variety of reaction-diffusion systems, also with more than two components, see an excellent review article [171] and references therein. In the limit $\varepsilon \to 0$, one obtains widely studied cross-diffusion systems where the gradient of one quantity induces a flux of another one. Finally, for non-monotone F, the only available results were established recently by B. Perthame and the Author [230,249]. They will be presented in this chapter.

Example 3.1.1 (Limit $\varepsilon \to 0$ with $F' > 0$)**.** We briefly explain how to pass to the limit in (3.1.1) under assumption that $F' > 0$. First, standard energy estimates (see Lemma 3.3.2) give us the following uniform bounds

$$\begin{aligned} &\{u^\varepsilon\}, \{v^\varepsilon\} \text{ in } L^\infty(\Omega_T), && \{\nabla v^\varepsilon\} \text{ in } L^2(\Omega_T), \\ &\left\{\frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}\right\} \text{ in } L^2(\Omega_T), && \{\partial_t(u^\varepsilon + v^\varepsilon)\} \text{ in } L^2(0,T; H^{-1}(\Omega)). \end{aligned}$$

Now, up to a subsequence, $\text{w}^*\text{-lim}\, u^\varepsilon = u$, $\text{w}^*\text{-lim}\, v^\varepsilon = v$. Applying Lemma 2.3.2 we have

$$\text{w}^*\text{-lim}\ [(u^\varepsilon + v^\varepsilon)\, v^\varepsilon] = \text{w}^*\text{-lim}\, (u^\varepsilon + v^\varepsilon)\ \text{w}^*\text{-lim}\, v^\varepsilon.$$

In this identity, v^ε can be replaced with $F(u^\varepsilon)$ because $v^\varepsilon - F(u^\varepsilon) \to 0$ strongly in $L^2(\Omega_T)$. Therefore,

$$\text{w}^*\text{-lim}\ [(u^\varepsilon + F(u^\varepsilon))\, F(u^\varepsilon)] = \text{w}^*\text{-lim}\, (u^\varepsilon + F(u^\varepsilon))\ \text{w}^*\text{-lim}\, F(u^\varepsilon). \tag{3.1.2}$$

Let $\{\mu_{t,x}\}$ be the Young measure generated by sequence $\{u^\varepsilon\}$, cf. Theorem 2.1.1. In terms of $\{\mu_{t,x}\}$, (3.1.2) can be written as

$$\int_{\mathbb{R}} (\lambda + F(\lambda))\, F(\lambda)\, \mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}} (\lambda + F(\lambda))\, \mathrm{d}\mu_{t,x}(\lambda) \int_{\mathbb{R}} F(\lambda)\, \mathrm{d}\mu_{t,x}(\lambda).$$

This can be rewritten as

$$\int_{\mathbb{R}} (\lambda + F(\lambda) - \tau - F(\tau))\, (F(\lambda) - F(\tau))\, \mathrm{d}\mu_{t,x}(\lambda)\, \mathrm{d}\mu_{t,x}(\tau) = 0.$$

The integrand is always positive except the case $\lambda = \tau$. It follows that $\mu_{t,x}$ is a Dirac mass which implies strong convergence of $u^\varepsilon \to u$ (say, in $L^2(\Omega_T)$), cf. Lemma 2.1.2. As $v^\varepsilon - F(u^\varepsilon) \to 0$, we obtain $v^\varepsilon \to v = F(u)$ in $L^2(\Omega_T)$. The limiting PDE reads

$$\partial_t(u + F(u)) = \Delta F(u).$$

Introducing an auxillary function $I(u) = u + F(u)$ which is invertible we have $\partial_t u = \Delta F \circ I^{-1}(u)$, that is a porous medium equation. Similar argument were originally formulated for hyperbolic conservation laws, see [110, 253, 254]. □

Example 3.1.1 shows that things get complicated when F is not monotone. Multiplying (3.1.1) by ε and testing against a smooth test function, we observe that in the limit we shall expect "$v = F(u)$". As F is not monotone, its inverse has at least two branches and u can jump between these branches. This is indeed the case, see Figure 3.1. In fact, we will not have sufficient compactness to pass to the limit in the term $F(u^\varepsilon)$ and so, the equality "$v = F(u)$" will only hold in the Young measures sense.

Under additional structural assumptions on F, we will show that the following surprising phenomenon holds: as $\varepsilon \to 0$, $F(u^\varepsilon) \to v$ and $v^\varepsilon \to v$ converge strongly without any known a priori estimates allowing to conclude so (we do not have compactness in time variable for v^ε). As a consequence, u^ε converges weakly to

$$u(t,x) = \lambda_1(t,x)\, S_1(v(t,x)) + \lambda_2(t,x)\, S_2(v(t,x)) + \lambda_3(t,x)\, S_3(v(t,x)) \qquad (3.1.3)$$

where $\sum_{i=1}^{3} \lambda_i(t,x) = 1$ and $\{S_i\}_{i=1,2,3}$ are the branches of F^{-1}, see Figure 3.2. More precisely, if $\{\mu_{t,x}\}_{t,x}$ is the Young measure generated by $\{u^\varepsilon\}$ (see Theorem 2.1.1), we have

$$\mu_{t,x} = \lambda_1(t,x)\, \delta_{S_1(v(t,x))} + \lambda_2(t,x)\, \delta_{S_2(v(t,x))} + \lambda_3(t,x)\, \delta_{S_3(v(t,x))}$$

which represents oscillations between phases $S_1(v(t,x))$, $S_2(v(t,x))$ and $S_3(v(t,x))$, see Figure 3.1.

The proof exploits a family of energies which brings information on Young measures in the spirit of Murat and Tartar's work on conservation laws and compensated compactness [222, 257]. Using the family of energies, we will prove that the Young measure generated by $\{v^\varepsilon\}$ is a Dirac mass which implies strong convergence $v^\varepsilon \to v$. Then, as $F(u^\varepsilon) \approx v^\varepsilon$, we deduce representation formula (3.1.3).

Let us explain a little bit how energy identities can be useful in proving strong compactness. Our method is in fact a generalization of Example 3.1.1. Energy identities

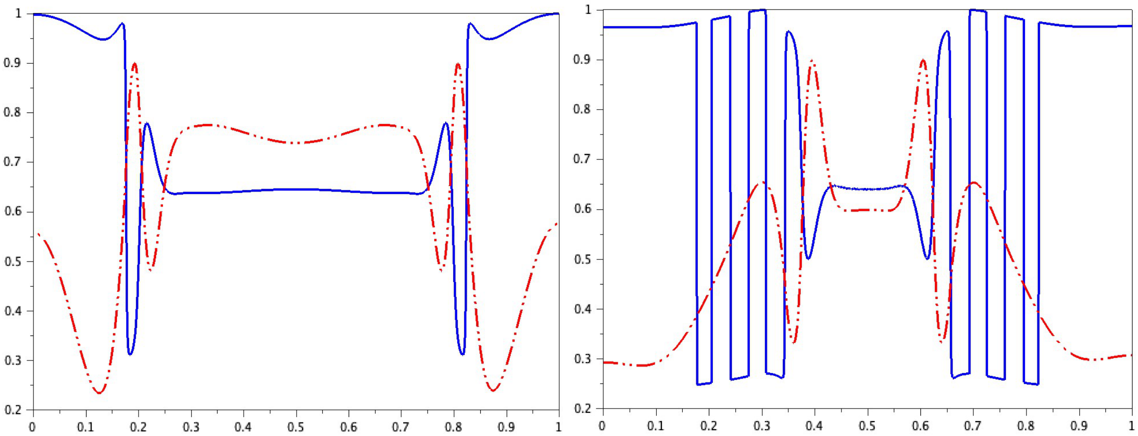

Figure 3.1: Evolution of u^ε (continuous line) and v^ε (dash-dotted line) solving (3.1.1) in one space dimension with fixed and small value of $\varepsilon > 0$. Two time shots are presented to show dependence between oscillations of u^ε and v^ε. When u^ε oscillates, v^ε also exhibits oscillatory behaviour. However, when the weights in the equation (3.1.3) stabilize and only one of them is not vanishing, oscillations of v^ε disappear. The simulation was performed with $\varepsilon = 5 \cdot 10^{-5}$, $F(u) = \frac{1}{3}u^3 - Au^2 + (A^2 - B^2)u$, $u_0(x) = 20(x-B)^4 + A$, $v_0(x) = F(u_0(x))$ where $A = 1.5$, $B = 0.5$.

(Lemma 3.3.1) and compensated compactness (Lemma 2.3.2) give us

$$\operatorname*{w^*\text{-}lim}_{\varepsilon\to 0} \left(\Psi(u^\varepsilon) + \Phi(v^\varepsilon)\right) \varphi(v^\varepsilon) = \operatorname*{w^*\text{-}lim}_{\varepsilon\to 0} \left(\Psi(u^\varepsilon) + \Phi(v^\varepsilon)\right) \operatorname*{w^*\text{-}lim}_{\varepsilon\to 0} \varphi(v^\varepsilon)$$

for functions Φ, Ψ defined in (3.3.1) for arbitrary ϕ and φ. As $v^\varepsilon - F(u^\varepsilon) \to 0$ in $L^2(\Omega_T)$, this can be written in terms of Young measure $\{\mu_{t,x}\}$ of the sequence $\{u^\varepsilon\}$:

$$\begin{aligned} &\int_{\mathbb{R}^+} \left(\Psi(\lambda) + \Phi(F(\lambda))\right) \varphi(F(\lambda))\, \mathrm{d}\mu_{t,x}(\lambda) = \\ &\qquad = \int_{\mathbb{R}^+} \left(\Psi(\lambda) + \Phi(F(\lambda))\right) \mathrm{d}\mu_{t,x}(\lambda) \int_{\mathbb{R}^+} \varphi(F(\lambda))\, \mathrm{d}\mu_{t,x}(\lambda). \end{aligned} \tag{3.1.4}$$

Now, one could use the property of the push-forward measure $F^{\#}\mu_{t,x}$

$$\int_{\mathbb{R}^+} \varphi(F(\lambda))\, \mathrm{d}\mu_{t,x}(\lambda) = \int_{\mathbb{R}^+} \varphi(\lambda)\, \mathrm{d}F^{\#}\mu_{t,x}(\lambda)$$

to get an integral identity for the measure $F^{\#}\mu_{t,x}$ which turns out to be the Young measure of the sequence $\{v^\varepsilon\}$ (cf. Lemma 2.1.3). The difficulty is that in the integral identity (3.1.4), there is term $\Psi(\lambda)$ which cannot be written as $\Psi(F^{-1}(F(\lambda)))$

because F is not invertible. Therefore, as in (3.2.1), we split the measure $\mu_{t,x}$ for three intervals so that F is invertible on each of them. Then, one can localize identity (3.1.4) to get a pointwise identity for $F^{\#}\mu_{t,x}(\lambda)$ valid for $F^{\#}\mu_{t,x}$-a.e. λ (here, t and x are fixed) presented in Theorem 3.2.4. Because of the splitting, analysis of the pointwise identity is difficult and requires some structural assumptions on F: in Theorems 3.2.5–3.2.7 we present three different conditions that imply that $F^{\#}\mu_{t,x}$ has to be a Dirac mass for a.e. (t, x) which implies strong convergence by Lemma 2.1.2.

Finally, let us remark that one can study a similar problem with two reaction-diffusion equations but it is very difficult, see Section 3.6.

3.2 Rigorous formulation of the main results

We start with rigorous formulation of the assumptions. The initial conditions u_0, v_0 and the nonlinearity F satisfy the following.

Assumption 3.2.1 (Initial data for (3.1.1)). Functions $u_0(x)$, $v_0(x)$ satisfy

1. (nonnegativity) $u_0, v_0 \geq 0$.
2. (regularity) $u_0, v_0 \in C^{2+\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.
3. (boundary condition) u_0, v_0 satisfy the Neumann boundary condition.

Assumption 3.2.2 (Reaction function F). We assume that the function $F(u)$ satisfies:

1. (nonnegativity) $F(0) = 0$ and $F \geq 0$.
2. (piecewise monotonicity) There are $\alpha_- < \alpha_+ < \beta_- < \beta_+$ such that $F(\beta_-) = F(\alpha_-)$, $F(\alpha_+) = F(\beta_+)$, F is strictly increasing on $(-\infty, \alpha_+) \cup (\beta_-, \infty)$ and strictly decreasing on (α_+, β_-) (see Fig. 3.2). Moreover, $\lim_{u\to\infty} F(u) = \infty$.
3. (regularity) F is Lipschitz continuous. Moreover, it is continuously differentiable on each of the intervals $(-\infty, \alpha_+)$, (α_+, β_-) and (β_-, ∞).

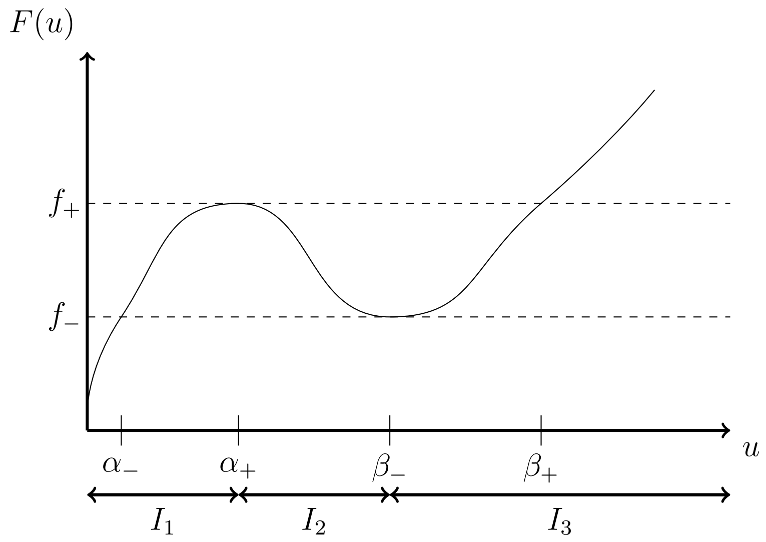

Figure 3.2: Plot of a typical function F. It is strictly increasing in the intervals $I_1 := (-\infty, \alpha_+]$, $I_3 := [\beta_-, \infty)$ and strictly decreasing in $I_2 := (\alpha_+, \beta_-)$. For $r \in [f_-, f_+]$, the function F is not invertible and equation $F(u) = r$ has three roots $u = S_1(r) \leq S_2(r) \leq S_3(r)$.

In what follows, it will be crucial to introduce a notation related to the inverses of function F.

Notation 3.2.3. Let $S_1(\lambda) \leq S_2(\lambda) \leq S_3(\lambda)$ be the solutions of equation $F(S_i(\lambda)) = \lambda$ (see Fig. 3.2). These are inverses of F satisfying

$$S_1 : (-\infty, f_+] \to (-\infty, \alpha_+], \qquad S_2 : (f_-, f_+) \to (\alpha_+, \beta_-), \qquad S_3 : [f_-, \infty) \to [\beta_-, \infty).$$

Their role is to focus our analysis on parts of the plot of F where monotonicity of F does not change. By a small abuse of notation, we extend functions S_i by a constant value to the whole of $\mathbb{R}$. We usually write

$$I_1 = (-\infty, \alpha_+], \qquad I_2 = (\alpha_+, \beta_-), \qquad I_3 = [\beta_-, \infty),$$

$$J_1 = (-\infty, f_+], \qquad J_2 = (f_-, f_+), \qquad J_3 = [f_-, \infty).$$

for images of functions S_1, S_2, S_3 and for their domains.

Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^\varepsilon\}$ solving (3.1.1), i.e. for any bounded function $G : \mathbb{R} \to \mathbb{R}$ we have (up to a subsequence and for a.e.

$(t,x) \in (0,T) \times \Omega)$

$$G(u^\varepsilon) \xrightharpoonup{*} \int_{\mathbb{R}} G(\lambda)\, \mathrm{d}\mu_{t,x}(\lambda),$$

see Section 2.1 if necessary. This is well-defined because the sequence $\{u^\varepsilon\}$ is bounded in $L^\infty((0,T)\times\Omega)$ cf. Lemma 3.3.2. To analyze the amount of $\mu_{t,x}$ on intervals I_1, I_2 and I_3, see Fig. 3.2, we introduce restrictions

$$\mu^{(1)}_{t,x} := \mu_{t,x}\, \mathbb{1}_{I_1}, \qquad \mu^{(2)}_{t,x} := \mu_{t,x}\, \mathbb{1}_{I_2}, \qquad \mu^{(3)}_{t,x} := \mu_{t,x}\, \mathbb{1}_{I_3}. \tag{3.2.1}$$

The reason we introduce these measures is that in the sequel, we will gain information only about measure $F^\#\mu_{t,x}$, i.e. a push-forward (image) of $\mu_{t,x}$ along F defined as

$$F^\#\mu_{t,x} = \mu_{t,x}(F^{-1}(A)), \qquad A \subset \mathbb{R}^+.$$

Observe that for all $i = 1,2,3$, measures $F^\#\mu^{(i)}_{t,x}$ are absolutely continuous with respect to $F^\#\mu_{t,x}$. Therefore, the Radon-Nikodym theorem implies that there exist densities $g^{(1)}(\lambda)$, $g^{(2)}(\lambda)$ and $g^{(3)}(\lambda)$ such that

$$F^\#\mu^{(i)}_{t,x}(A) = \int_A g^{(i)}(\lambda)\, \mathrm{d}F^\#\mu_{t,x}(\lambda), \qquad i = 1,2,3. \tag{3.2.2}$$

We also note that for all $A \subset \mathbb{R}^+$

$$\sum_{i=1}^{3} F^\#\mu^{(i)}_{t,x}(A) = \sum_{i=1}^{3} \mu_{t,x}(F^{-1}(A) \cap I_i) = \mu_{t,x}(F^{-1}(A)) = F^\#\mu_{t,x}(A). \tag{3.2.3}$$

In particular, from (3.2.2) and (3.2.3) we deduce that for $F^\#\mu_{t,x}$-a.e. λ we have

$$\sum_{i=1}^{3} g_i(\lambda) = 1. \tag{3.2.4}$$

The first main result gives the identity characterizing the Young measure $\mu_{t,x}$. It will turn out that from this identity we can conclude its precise form.

Theorem 3.2.4. *Let $\{\mu_{t,x}\}_{t,x}$ be the Young measure generated by sequence $\{u^\varepsilon\}$ solving* (3.1.1)*. Then, for almost all λ_0 (with respect to $F^\#\mu_{t,x}$) and all $\tau_0 \neq f_-, f_+$ we have*

$$\sum_{i=1}^{3} (S_i'(\tau_0)+1)\, \Big[\mathbb{1}_{\lambda_0>\tau_0}\, g_i(\lambda_0) - F^\#\mu^{(i)}_{t,x}(\tau_0,\infty)\Big] +$$
$$+ (S_1'(\tau_0) - S_2'(\tau_0))\, (F^\#\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(\lambda_0)) = 0.$$

where S_i are the inverses of F as in Notation 3.2.3 and g_i are the Radon-Nikodym densities as in (3.2.2)*. Moreover, for $\lambda_0 \neq f_-, f_+$ we have*

$$\left(1 - F^{\#}\mu_{t,x}\left\{\lambda_0\right\}\right) \sum_{i=1}^{3} (S_i'(\lambda_0) + 1)\, g_i(\lambda_0) \; = 0. \tag{3.2.5}$$

As $F^{\#}\mu_{t,x}$ turns out to be the Young measure generated by $\{v^{\varepsilon}\}$ cf. Corollary 3.3.3, strong convergence $v^{\varepsilon} \to v$ can be deduced if one proves that $F^{\#}\mu_{t,x}$ is the Dirac measure cf. Lemma 2.1.2. For instance, Equation (3.2.5) shows that the latter follows if one finds λ_0 in the support such that the sum $\sum_{i=1}^{3}(S_i'(\lambda_0) + 1)\, g_i(\lambda_0)$ does not vanish (some additional care is needed when $\lambda_0 = f_-, f_+$, cf. Lemma 3.5.1).

First, we show that the form presented in Theorem 3.2.4 can be used to obtain the result for the non-degenerate functions F, that is satisfying for all intervals $R \subset (f_-, f_+)$

$$\sum_{i=1}^{3} a_i \left(S_i'(r) + 1\right) = 0 \text{ for } r \in R \implies a_1 + a_2 + a_3 = 0. \tag{3.2.6}$$

While it is fairly classical for this type of problems [12, 224, 232], it is hard to be verified for a given nonlinearity F. Moreover, the non-degeneracy condition excludes piecewise affine functions used in more explicit computations as in [212].

Theorem 3.2.5. *Suppose that non-degeneracy condition* (3.2.6) *is satisfied. Then, up to a subsequence, $v^{\varepsilon} \to v$ strongly in $L^2((0,T) \times \Omega)$. Moreover, there are nonnegative numbers $\lambda_1(t,x)$, $\lambda_2(t,x)$, $\lambda_3(t,x)$ such that $\sum_{i=1}^{3} \lambda_i(t,x) = 1$ and*

$$\mu_{t,x} = \lambda_1(t,x)\, \delta_{S_1(v(t,x))} + \lambda_2(t,x)\, \delta_{S_2(v(t,x))} + \lambda_3(t,x)\, \delta_{S_3(v(t,x))}.$$

Now, we move to further results that easily follow from Theorem 3.2.4. The first one asserts that if one knows a priori that the Young measure $\{\mu_{t,x}\}_{t,x}$ is not supported in the interval I_2 where F is decreasing, the strong convergence occurs. The fact concerning the support of $\{\mu_{t,x}\}_{t,x}$ was observed in the numerical simulations [132] and so, the next theorem may serve as a tool to prove strong convergence without the non-degeneracy condition.

Theorem 3.2.6. *Suppose that:*

- *there exists* $\tau_0 \in (f_-, f_+)$ *such that* $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$,
- *Young measure* $\{\mu_{t,x}\}_{t,x}$ *is not supported in the interval* I_2 *(see Fig. 3.2).*

Then, $v^\varepsilon \to v$ *strongly in* $L^2((0,T) \times \Omega)$. *Moreover, there are nonnegative numbers* $\lambda_1(t,x)$, $\lambda_3(t,x)$ *such that* $\lambda_1(t,x) + \lambda_3(t,x) = 1$ *and*

$$\mu_{t,x} = \lambda_1(t,x)\, \delta_{S_1(v(t,x))} + \lambda_3(t,x)\, \delta_{S_3(v(t,x))}.$$

The next result shows that we can establish a simple condition on F implying strong convergence of $v^\varepsilon \to v$ that does not exclude piecewise affine functions as in the case of non-degeneracy condition (3.2.6).

Theorem 3.2.7. *Let* $\{\mu_{t,x}\}_{t,x}$ *be the Young measure generated by sequence* $\{u^\varepsilon\}$ *solving* (3.1.1). *Suppose that:*

- *there exists* $\tau_0 \in (f_-, f_+)$ *such that* $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$,
- $S_2'(\lambda) + 1 > 0$ *for all* $\lambda \in (f_-, f_+)$.

Then, $v^\varepsilon \to v$ *strongly in* $L^2((0,T) \times \Omega)$. *Moreover, there are nonnegative numbers* $\lambda_1(t,x)$, $\lambda_2(t,x)$, $\lambda_3(t,x)$ *such that* $\sum_{i=1}^3 \lambda_i(t,x) = 1$ *and*

$$\mu_{t,x} = \lambda_1(t,x)\, \delta_{S_1(v(t,x))} + \lambda_2(t,x)\, \delta_{S_2(v(t,x))} + \lambda_3(t,x)\, \delta_{S_3(v(t,x))}.$$

As an example, the following function F satisfies assumptions of Theorem 3.2.7:

$$F(\lambda) = \begin{cases} 2\lambda & \text{if } \lambda \in [0,1], \\ 3 - 2\lambda & \text{if } \lambda \in \left[1, \frac{5}{4}\right], \\ 4\lambda - \frac{9}{2} & \text{if } \lambda \in \left[\frac{5}{4}, \infty\right). \end{cases}$$

Then, $S_1'(\lambda) = \frac{1}{2}$, $S_2'(\lambda) = -\frac{1}{2}$ and $S_3'(\lambda) = \frac{1}{4}$ so that $S_1'(\lambda) - S_3'(\lambda) = \frac{1}{4} \neq 0$ and $S_2'(\lambda) + 1 = \frac{1}{2} > 0$. Note again that F does not satisfy non-degeneracy condition (3.2.6).

The proofs of Theorem 3.2.6 and 3.2.7 are based on equation (3.2.5), namely one uses $g_1(\lambda_0) + g_2(\lambda_0) + g_3(\lambda_0) = 1$ to show that for any $\lambda_0 \in \operatorname{supp} F$ we have $F^{\#}\mu_{t,x}\{\lambda_0\} =$

1. Note that (3.2.5) is not valid for $\lambda_0 = f_-, f_+$ so some additional care is needed if the support of measure $F^{\#}\mu_{t,x}$ accumulates only in these points. This is studied in Lemma 3.5.1 and it requires an additional assumption that $S_1'(\tau) - S_3'(\tau)$ does not vanish at least for one value of τ, see also Remark 3.5.2.

3.3 Properties of the system (3.1.1)

We begin with the energy equality and the well-posedness result.

Lemma 3.3.1 (energy equality)**.** *Given a smooth test function $\phi : \mathbb{R} \to \mathbb{R}$, we define*

$$\Psi(\lambda) := \int_0^\lambda \phi(F(\tau))\,\mathrm{d}\tau, \qquad \Phi(\lambda) := \int_0^\lambda \phi(\tau)\,\mathrm{d}\tau. \tag{3.3.1}$$

Then, if $(u^\varepsilon, v^\varepsilon)$ solve (3.1.1)*, it holds*

$$\partial_t \Psi(u^\varepsilon) + \partial_t \Phi(v^\varepsilon) = \Delta\Phi(v^\varepsilon) - \phi'(v^\varepsilon)\,|\nabla v^\varepsilon|^2 - \frac{\big(v^\varepsilon - F(u^\varepsilon)\big)\,\big(\phi(v^\varepsilon) - \phi(F(u^\varepsilon))\big)}{\varepsilon}. \tag{3.3.2}$$

Proof. Multiplying equation for u^ε in (3.1.1) with $\phi(F(u^\varepsilon))$ and equation for v^ε with $\phi(v^\varepsilon)$ we obtain

$$\begin{aligned}
\partial_t \Psi(u^\varepsilon) &= \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}\,\phi(F(u^\varepsilon)),\\
\partial_t \Phi(v^\varepsilon) &= \Delta\Phi(v^\varepsilon) - \phi'(v^\varepsilon)\,|\nabla v^\varepsilon|^2 + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon}\,\phi(v^\varepsilon).
\end{aligned}$$

Summing up these equations we deduce (3.3.2). □

Lemma 3.3.2. *There exists the unique classical solution $u^\varepsilon, v^\varepsilon : [0,\infty) \times \Omega \to \mathbb{R}$ of* (3.1.1) *which is nonnegative and has regularity*

$$u^\varepsilon \in C^{\alpha, 1+\alpha/2}\left([0,\infty) \times \overline{\Omega}\right), \qquad v^\varepsilon \in C^{2+\alpha, 1+\alpha/2}\left([0,\infty) \times \overline{\Omega}\right).$$

Moreover, we have

1. $0 \le u^\varepsilon \le M$, $0 \le v^\varepsilon \le M$ *with* $M = \max(\|F(u_0)\|_\infty,\, \|u_0\|_\infty,\, \|v_0\|_\infty,\, f_+,\, \beta_+)$,
2. $\{\nabla v^\varepsilon\}$, $\left\{\frac{F(u^\varepsilon) - v^\varepsilon}{\sqrt{\varepsilon}}\right\}$ *and* $\{\sqrt{\varepsilon}\,\Delta v^\varepsilon\}$ *are uniformly bounded in* $L^2((0,\infty) \times \Omega)$,
3. $\{\partial_t u^\varepsilon + \partial_t v^\varepsilon\}$ *is uniformly bounded in* $L^2(0,T; H^{-1}(\Omega))$,

4. *for all smooth functions* $\varphi : \mathbb{R} \to \mathbb{R}$, $\{\nabla\varphi(v^\varepsilon)\}$ *is uniformly bounded in* $L^2((0,\infty)\times\Omega)$,

5. *for all smooth functions* $\phi : \mathbb{R} \to \mathbb{R}$, $\{\partial_t \Psi(u^\varepsilon) + \partial_t \Phi(v^\varepsilon)\}$ *is uniformly bounded in* $(C(0,T;H^k(\Omega)))^*$ *for sufficiently large* $k \in \mathbb{N}$.

Proof. First, local well-posedness and nonnegativity follows from the classical theory [238]. To extend these results to an arbitrary interval of time, we need to prove a priori estimates as in (1). To this end, we note that thanks to (3.3.2), the nonnegative map

$$t \mapsto \int_\Omega \big[\Psi(u^\varepsilon(t,x)) + \Phi(v^\varepsilon(t,x))\big]\,\mathrm{d}x$$

is nonincreasing whenever $\phi' \geq 0$. Choosing ϕ vanishing on $(0,M)$ and stricly increasing for (M,∞) we obtain (1) and the global well-posedness. Then, (2) follows from (3.3.2) with $\phi(v) = v$. Furthemore, (3) follows from the equality $\partial_t u^\varepsilon + \partial_t v^\varepsilon = \Delta v^\varepsilon$ and property (2) while (4) follows from the chain rule for Sobolev functions, boundedness of v^ε from (1) and (2). Finally, to see (5) we choose $k \geq d$ so that $H^k(\Omega)$ embedds continuously into $L^\infty(\Omega)$. Let $\varphi \in C(0,T;H^k(\Omega))$. Note that there is a constant C such that

$$\|\varphi\|_\infty \leq C\,\|\varphi\|_{C(0,T;H^k(\Omega))}, \qquad \|\varphi\|_{L^2(0,T;H^1(\Omega))} \leq C\,\|\varphi\|_{C(0,T;H^k(\Omega))}. \tag{3.3.3}$$

Thanks to (3.3.2) we have

$$\int_{(0,T)\times\Omega} (\partial_t\Psi(u^\varepsilon) + \partial_t\Phi(v^\varepsilon))\,\varphi\,\mathrm{d}t\,\mathrm{d}x - \int_{(0,T)\times\Omega} \nabla\Phi(v^\varepsilon)\cdot\nabla\varphi\,\mathrm{d}t\,\mathrm{d}x =$$
$$= -\int_{(0,T)\times\Omega} \phi'(v^\varepsilon)\,|\nabla v^\varepsilon|^2\,\varphi\,\mathrm{d}t\,\mathrm{d}x - \int_{(0,T)\times\Omega} \frac{\big(v^\varepsilon - F(u^\varepsilon)\big)\,\big(\phi(v^\varepsilon) - \phi(F(u^\varepsilon))\big)}{\varepsilon}\,\varphi\,\mathrm{d}t\,\mathrm{d}x.$$

As $|\phi'(v^\varepsilon)| \leq C$ and $|\phi(v^\varepsilon) - \phi(F(u^\varepsilon)| \leq C\,|v^\varepsilon - F(u^\varepsilon)|$ we use bounds (3.3.3) together with points (2) to deduce for some possibly larger constant C (independent of ε)

$$\left|\int_{(0,T)\times\Omega} (\partial_t\Psi(u^\varepsilon) + \partial_t\Phi(v^\varepsilon))\,\varphi\,\mathrm{d}t\,\mathrm{d}x\right| \leq C\,\|\varphi\|_{C(0,T;H^k(\Omega))}.$$

□

Corollary 3.3.3. Let $\{\mu_{t,x}\}_{t,x}$ and $\{\nu_{t,x}\}_{t,x}$ be the Young measures generated by sequences $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ respectively. Combining Lemma 3.3.2 (2) and Lemma 2.1.3 we obtain that $F^\#\mu_{t,x} = \nu_{t,x}$.

3.4 Proof of Theorem 3.2.4

We begin by formulating the entropy equality.

Lemma 3.4.1 (entropy equality). *Let Ψ and Φ be defined with* (3.3.1), *$\{\mu_{t,x}\}$ be the Young measure generated by sequence $\{u^\varepsilon\}$ solving* (3.1.1) *and g_i be the densities given by* (3.2.2). *Then, for almost all λ_0 (with respect to $F^\# \mu_{t,x}$) we have*

$$\sum_{i=1}^{3} (\Psi(S_i(\lambda_0)) + \Phi(\lambda_0))\, g_i(\lambda_0) = \sum_{i=1}^{3} \int_{\mathbb{R}^+} (\Psi(S_i(\lambda)) + \Phi(\lambda))\, g_i(\lambda) \,\mathrm{d}F^\# \mu_{t,x}(\lambda), \quad (3.4.1)$$

where S_i are the inverses of F as in Notation 3.2.3.

Proof. Thanks to Lemma 3.3.2 (5), for all smooth $\phi : \mathbb{R} \to \mathbb{R}$, $\{\partial_t \Psi(u^\varepsilon) + \partial_t \Phi(v^\varepsilon)\}$ is uniformly bounded in $(C(0,T; H^k(\Omega)))^*$. Similarly, for all smooth $\varphi : \mathbb{R} \to \mathbb{R}$, $\{\nabla \varphi(v^\varepsilon)\}$ is uniformly bounded in $L^2((0,\infty) \times \Omega)$. Hence, Lemma 2.3.2 implies

$$\operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} (\Psi(u^\varepsilon) + \Phi(v^\varepsilon))\, \varphi(v^\varepsilon) = \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} (\Psi(u^\varepsilon) + \Phi(v^\varepsilon)) \ \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} \ \varphi(v^\varepsilon).$$

As $v^\varepsilon - F(u^\varepsilon) \to 0$ cf. Lemma 3.3.2 (2), we may replace v^ε with $F(u^\varepsilon)$ in the identity above to obtain

$$\operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} (\Psi(u^\varepsilon) + \Phi(F(u^\varepsilon)))\, \varphi(F(u^\varepsilon)) = \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} (\Psi(u^\varepsilon) + \Phi(F(u^\varepsilon))) \ \operatorname*{w^*\text{-}lim}_{\varepsilon \to 0} \ \varphi(F(u^\varepsilon)).$$

In the language of Young measures, this identity reads

$$\int_{\mathbb{R}^+} (\Psi(\lambda) + \Phi(F(\lambda)))\, \varphi(F(\lambda)) \,\mathrm{d}\mu_{t,x}(\lambda) = \\ = \int_{\mathbb{R}^+} (\Psi(\lambda) + \Phi(F(\lambda))) \,\mathrm{d}\mu_{t,x}(\lambda) \int_{\mathbb{R}^+} \varphi(F(\lambda)) \,\mathrm{d}\mu_{t,x}(\lambda).$$

We observe that $\lambda = \sum_{i=1}^{3} S_i(F(\lambda))\, \mathbb{1}_{\lambda \in I_i}$. Hence, we may use the concept of push-forward measure to write

$$\sum_{i=1}^{3} \int_{\mathbb{R}^+} (\Psi(S_i(\lambda)) + \Phi(\lambda))\, \varphi(\lambda) \,\mathrm{d}F^\# \mu_{t,x}^{(i)}(\lambda) = \\ = \sum_{i=1}^{3} \int_{\mathbb{R}^+} (\Psi(S_i(\lambda)) + \Phi(\lambda)) \,\mathrm{d}F^\# \mu_{t,x}^{(i)}(\lambda) \int_{\mathbb{R}^+} \varphi(\lambda) \,\mathrm{d}F^\# \mu_{t,x}(\lambda).$$

Using (3.2.2) with densities $g_1(\lambda)$, $g_2(\lambda)$ and $g_3(\lambda)$ we obtain

$$\sum_{i=1}^{3}\int_{\mathbb{R}^+}(\Psi(S_i(\lambda))+\Phi(\lambda))\,\varphi(\lambda)\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda)=$$
$$=\sum_{i=1}^{3}\int_{\mathbb{R}^+}(\Psi(S_i(\lambda))+\Phi(\lambda))\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda)\int_{\mathbb{R}^+}\varphi(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

Hence, when λ_0 belongs to the support of measure $F^{\#}\mu_{t,x}$, we obtain

$$\sum_{i=1}^{3}(\Psi(S_i(\lambda_0))+\Phi(\lambda_0))\,g_i(\lambda_0)=\sum_{i=1}^{3}\int_{\mathbb{R}^+}(\Psi(S_i(\lambda))+\Phi(\lambda))\,g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).$$

□

To analyze the entropy inequality, we need to deal with integrals of the form $\int_0^{S_i(\lambda)}\phi(F(\tau))\,\mathrm{d}\tau$. This is the content of the next lemma.

Lemma 3.4.2. *We have*

$$\Psi(S_i(\lambda_0))=\int_0^{S_i(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=\int_0^{\lambda_0}\phi(\tau)\,S_i'(\tau)\,\mathrm{d}\tau+C_i(\phi)$$

where $C_1(\phi)=0$ *and* $C_2(\phi)=C_3(\phi)=\int_0^{f_+}\phi(\tau)\,(S_1'(\tau)-S_2'(\tau))\,\mathrm{d}\tau$.

Proof. For $i=1$ we note that F is invertible on $(0,S_1(\lambda))$ so that a simple change of variables implies

$$\Psi(S_1(\lambda_0))=\int_0^{S_1(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=\int_0^{\lambda_0}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau.$$

For $i=2$ we first split the integral for two intervals $(0,\alpha_+)$, (α_+,λ_0) cf. Notation 3.2.3. On each of them, F is invertible so we can apply a change of variables again:

$$\Psi(S_2(\lambda_0))=\int_0^{\alpha_+}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\alpha_+}^{S_2(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=$$
$$=\int_0^{f_+}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau-\int_{\lambda_0}^{f_+}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau=C_2(\phi)+\int_0^{\lambda_0}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau.$$

For $i=3$ we split the integral for three intervals and apply a change of variables again:

$$\Psi(S_3(\lambda_0))=\int_0^{\alpha_+}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\alpha_+}^{\beta_-}\phi(F(\tau))\,\mathrm{d}\tau+\int_{\beta_-}^{S_3(\lambda_0)}\phi(F(\tau))\,\mathrm{d}\tau=$$
$$=\int_0^{f_+}\phi(\tau)\,S_1'(\tau)\,\mathrm{d}\tau-\int_{f_-}^{f_+}\phi(\tau)\,S_2'(\tau)\,\mathrm{d}\tau+\int_{f_-}^{\lambda_0}\phi(\tau)\,S_3'(\tau)\,\mathrm{d}\tau.$$

As $S_2'(\tau)=0$ and $S_3'(\tau)=0$ for $\tau\in(0,f_-)$, the proof is concluded. □

Lemma 3.4.3. *Consider function*

$$\mathcal{F}(\tau_0) = \sum_{i=1}^{3} (S_i'(\tau_0) + 1)\, F^{\#}\mu_{t,x}^{(i)}((\tau_0, \infty)) + (S_1'(\tau_0) - S_2'(\tau_0))\,(1 - F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+)).$$

Then, for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) and $\tau_0 \neq f_-, f_+$ we have

$$\mathbb{1}_{\lambda_0 > \tau_0} \sum_{i=1}^{3} (S_i'(\tau_0) + 1)\, g_i(\lambda_0) + (S_1'(\tau_0) - S_2'(\tau_0))\,(1 - g_1(\lambda_0)) = \mathcal{F}(\tau_0).$$

Proof. We consider $\phi(\tau) = \phi^\delta(\tau) = \frac{1}{\delta}\,\mathbb{1}_{[\tau_0, \tau_0+\delta]}$ and send $\delta \to 0$ so that $\Phi(\lambda_0) = \int_0^{\lambda_0} \phi^\delta(\tau)\,\mathrm{d}\tau \to \mathbb{1}_{\lambda > \tau_0}$. Moreover, $\int_0^{\lambda_0} \phi^\delta(\tau)\, S_i'(\tau)\,\mathrm{d}\tau \to S_i'(\tau_0)\,\mathbb{1}_{\lambda_0 > \tau_0}$. Therefore, from Lemmas 3.4.1 and 3.4.2 we deduce

$$\begin{aligned}
&\sum_{i=1}^{3} \Big(\mathbb{1}_{\lambda_0 > \tau_0}\,(S_i'(\tau_0) + 1) + (S_1'(\tau_0) - S_2'(\tau_0))\,\mathbb{1}_{i=2,3} \Big)\, g_i(\lambda_0) = \\
&\qquad = \sum_{i=1}^{3} \int_{\mathbb{R}^+} \Big(\mathbb{1}_{\lambda > \tau_0}\,(S_i'(\tau_0) + 1) + (S_1'(\tau_0) - S_2'(\tau_0))\,\mathbb{1}_{i=2,3} \Big)\, g_i(\lambda)\,\mathrm{d}F^{\#}\mu_{t,x}(\lambda).
\end{aligned}$$

Using identities from (3.2.3) and (3.2.4)

$$1 - g_1(\lambda_0) = g_2(\lambda_0) + g_3(\lambda_0), \qquad 1 - F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+) = F^{\#}\mu_{t,x}^{(2)}(\mathbb{R}^+) + F^{\#}\mu_{t,x}^{(3)}(\mathbb{R}^+),$$

we conclude the proof. □

Proof of Theorem 3.2.4. The first part of Theorem 3.2.4 is proved in Lemma 3.4.3. To see the second one, fix $\lambda_0 \neq f_-, f_+$. For $\tau_0 := \eta > \lambda_0$ we obtain

$$\sum_{i=1}^{3} (S_i'(\eta) + 1)\, F^{\#}\mu_{t,x}^{(i)}((\eta, \infty)) + (S_1'(\eta) - S_2'(\eta))\,(F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+) - g_1(\lambda_0)) = 0$$

while for $\tau_0 := \xi < \lambda_0$ we deduce

$$\sum_{i=1}^{3} (S_i'(\xi) + 1)\,(g_i(\lambda_0) - F^{\#}\mu_{t,x}^{(i)}((\xi, \infty))) + (S_1'(\xi) - S_2'(\xi))\,(F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+) - g_1(\lambda_0)) = 0.$$

Sending $\xi, \eta \to \lambda_0$ and using continuity of $\lambda \mapsto S_i'(\lambda)$ at $\lambda \neq f_-, f_+$ we obtain

$$\sum_{i=1}^{3} (S_i'(\lambda_0) + 1)\, g_i(\lambda_0) = \sum_{i=1}^{3} (S_i'(\lambda_0) + 1)\, F^{\#}\mu_{t,x}^{(i)}\{\lambda_0\}.$$

Finally, we note that for almost all λ_0 (with respect to $F^{\#}\mu_{t,x}$) $F^{\#}\mu_{t,x}^{(i)}\{\lambda_0\} = g_i(\lambda_0)\, F^{\#}\mu_{t,x}\{\lambda_0\}$ and this concludes the proof. □

3.5 Proofs of Theorems 3.2.5, 3.2.6 and 3.2.7

Proof of Theorem 3.2.5. Suppose that $\operatorname{supp} F^{\#}\mu_{t,x} \cap (0, f_-)$ is nonempty. Let $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (0, f_-)$. Note that $S_2'(\lambda_0) = S_3'(\lambda_0) = 0$. Moreover, (3.2.5) in Theorem 3.2.4 implies

$$\left(1 - F^{\#}\mu_{t,x}\{\lambda_0\}\right)\left(S_1'(\lambda_0) + 1\right) g_1(\lambda_0) = 0.$$

For almost all $\lambda_0 \in (0, f_-)$ we have $g_1(\lambda_0) = 1$ so we conclude $F^{\#}\mu_{t,x}\{\lambda_0\} = 1$. A similar argument works in the case $\lambda_0 \in (f_+, \infty)$.

Now, let $\lambda_0 \in [f_-, f_+] \cap \operatorname{supp} F^{\#}\mu_{t,x}$. If $\operatorname{supp} F^{\#}\mu_{t,x} = \{\lambda_0\}$, we conclude $F^{\#}\mu_{t,x} = \delta_{\lambda_0}$. Otherwise, there are $\lambda_1, \lambda_2 \in \operatorname{supp} F^{\#}\mu_{t,x}$ such that $f_- \le \lambda_1 < \lambda_2 \le f_+$. For any $\tau_0 \in (\lambda_1, \lambda_2)$ we use Theorem 3.2.4 with $\lambda_0 = \lambda_1, \lambda_2$ to obtain two equations:

$$\sum_{i=1}^{3}(S_i'(\tau_0)+1)\left[g_i(\lambda_2) - F^{\#}\mu_{t,x}^{(i)}(\tau_0,\infty)\right] + (S_1'(\tau_0) - S_2'(\tau_0))\left(F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+) - g_1(\lambda_2)\right) = 0,$$

$$-\sum_{i=1}^{3}(S_i'(\tau_0)+1)\,F^{\#}\mu_{t,x}^{(i)}(\tau_0,\infty) + (S_1'(\tau_0) - S_2'(\tau_0))\left(F^{\#}\mu_{t,x}^{(1)}(\mathbb{R}^+) - g_1(\lambda_1)\right) = 0.$$

Hence, $\sum_{i=1}^{3}(S_i'(\tau_0)+1)\,g_i(\lambda_2) + (S_1'(\tau_0) - S_2'(\tau_0))\,(g_1(\lambda_1) - g_1(\lambda_2)) = 0$. But then, non-degeneracy condition (3.2.6) implies that $\sum_{i=1}^{3} g_i(\lambda_2) = 0 \neq 1$ raising contradiction.

It follows that $F^{\#}\mu_{t,x}$ is the Dirac measure. From Corollary 3.3.3 we deduce that the Young measure $\{\nu_{t,x}\}_{t,x}$ generated by $\{v^{\varepsilon}\}$ is also the Dirac measure so $v^{\varepsilon} \to v$ strongly and $\nu_{t,x} = \delta_{v(t,x)}$, cf. Lemma 2.1.2. The representation formula for $\mu_{t,x}$ follows from $F^{\#}\mu_{t,x} = \delta_{v(t,x)}$. $\square$

Before proceeding to the proofs of Theorems 3.2.6 and 3.2.7, we will state a simple lemma concerning the case when $F^{\#}\mu_{t,x}$ is supported only at f_- and f_+. This needs some care as functions S_1', S_2' and S_3' are not continuous at these points.

Lemma 3.5.1 (Accumulation at the interface)**.** *Suppose that there exists $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$. Assume that* $\operatorname{supp} F^{\#}\mu_{t,x} \subset \{f_-, f_+\}$. *Then, $F^{\#}\mu_{t,x} = \delta_{f_-}$ or $F^{\#}\mu_{t,x} = \delta_{f_+}$.*

Proof. Aiming at contradiction, we assume that $F^{\#}\mu_{t,x}\{f_+\} > 0$ and $F^{\#}\mu_{t,x}\{f_-\} > 0$. Note that $F^{-1}(f_+) \notin I_2$ so that

$$0 = \mu^{(2)}_{t,x}(F^{-1}(f_+) \cap I_2) = F^{\#}\mu^{(2)}_{t,x}\{f_+\} = g_2(f_+)\, F^{\#}\mu_{t,x}\{f_+\}.$$

It follows that $g_2(f_+) = 0$ and similarly $g_2(f_-) = 0$. Applying Theorem 3.2.4 with $\tau_0 \in (f_-, f_+)$ and $\lambda_0 \in \{f_-, f_+\}$ we obtain

$$\sum_{i=1}^{3}(S_i'(\tau_0) + 1)\, \left[\mathbb{1}_{\lambda_0 > \tau_0}\, g_i(\lambda_0) - F^{\#}\mu^{(i)}_{t,x}(\tau_0, \infty)\right] +$$
$$+ (S_1'(\tau_0) - S_2'(\tau_0))\, (F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(\lambda_0)) = 0.$$

As $\tau_0 \in (f_-, f_+)$, we have

$$F^{\#}\mu^{(i)}_{t,x}(\tau_0, \infty) = F^{\#}\mu^{(i)}_{t,x}\{f_+\} = g_i(f_+)\, F^{\#}\mu_{t,x}\{f_+\}.$$

But this implies

$$\left(\mathbb{1}_{\lambda_0 > \tau_0} - F^{\#}\mu_{t,x}\{f_+\}\right) \sum_{i=1,3}(S_i'(\tau_0) + 1)\, g_i(\lambda_0) +$$
$$+ (S_1'(\tau_0) - S_2'(\tau_0))\, (F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(\lambda_0)) = 0.$$

Considering $\lambda_0 = f_+, f_-$ and using $1 - F^{\#}\mu_{t,x}\{f_+\} = F^{\#}\mu_{t,x}\{f_-\}$ we obtain two equations:

$$F^{\#}\mu_{t,x}\{f_-\} \sum_{i=1,3}(S_i'(\tau_0) + 1)\, g_i(f_+) + (S_1'(\tau_0) - S_2'(\tau_0))\, (F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_+)) = 0, \tag{3.5.1}$$

$$-F^{\#}\mu_{t,x}\{f_+\} \sum_{i=1,3}(S_i'(\tau_0) + 1)\, g_i(f_-) + (S_1'(\tau_0) - S_2'(\tau_0))\, (F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_-)) = 0. \tag{3.5.2}$$

Using $1 - F^{\#}\mu_{t,x}\{f_+\} = F^{\#}\mu_{t,x}\{f_-\}$ once again we obtain

$$F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_+) = g_1(f_+)\, F^{\#}\mu_{t,x}\{f_+\} + g_1(f_-)\, F^{\#}\mu_{t,x}\{f_-\} - g_1(f_+) =$$
$$= (g_1(f_-) - g_1(f_+))\, F^{\#}\mu_{t,x}\{f_-\}$$

and similarly for $F^{\#}\mu^{(1)}_{t,x}(\mathbb{R}^+) - g_1(f_-)$. As we assume that $F^{\#}\mu_{t,x}\{f_-\}, F^{\#}\mu_{t,x}\{f_+\} > 0$, we may simplify (3.5.1)–(3.5.2) to obtain

$$\sum_{i=1,3} (S_i'(\tau_0) + 1)\, g_i(f_+) + (S_1'(\tau_0) - S_2'(\tau_0))\, (g_1(f_-) - g_1(f_+)) = 0, \tag{3.5.3}$$

$$-\sum_{i=1,3} (S_i'(\tau_0) + 1)\, g_i(f_-) + (S_1'(\tau_0) - S_2'(\tau_0))\, (g_1(f_+) - g_1(f_-)) = 0. \tag{3.5.4}$$

We observe further that $g_1(\lambda_0) + g_3(\lambda_0) = 1$, cf. (3.2.4), so that

$$\sum_{i=1,3} (S_i'(\tau_0) + 1)\, g_i(\lambda_0) = (S_1'(\tau_0) - S_3'(\tau_0))\, g_1(\lambda_0) + (S_3'(\tau_0) + 1).$$

Hence, we may further simplify (3.5.3)–(3.5.4) to get

$$(S_1'(\tau_0) - S_3'(\tau_0))\, g_1(f_+) + (S_3'(\tau_0) + 1) + (S_1'(\tau_0) - S_2'(\tau_0))\, (g_1(f_-) - g_1(f_+)) = 0, \tag{3.5.5}$$

$$-(S_1'(\tau_0) - S_3'(\tau_0))\, g_1(f_-) - (S_3'(\tau_0) + 1) + (S_1'(\tau_0) - S_2'(\tau_0))\, (g_1(f_+) - g_1(f_-)) = 0. \tag{3.5.6}$$

By assumption, there is $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$. Using (3.5.5)–(3.5.6) for such τ_0 we see that $g_1(f_+) = g_1(f_-)$. But then, coming back to (3.5.3)–(3.5.4), we deduce that

$$\sum_{i=1,3} (S_i'(\tau_0) + 1)\, g_i(f_+) = 0, \qquad \sum_{i=1,3} (S_i'(\tau_0) + 1)\, g_i(f_-) = 0.$$

As S_1, S_3 are increasing, this implies $g_1(f_-) = g_3(f_-) = g_1(f_+) = g_3(f_+) = 0$ raising contradiction with $g_1(f_-) + g_3(f_-) = 1$ and $g_1(f_+) + g_3(f_+) = 1$. □

Remark 3.5.2. Without the assumption that there is $\tau_0 \in (f_-, f_+)$ such that $S_1'(\tau_0) - S_3'(\tau_0) \neq 0$ we observe that (3.5.5)–(3.5.6) degenerate to the same equation:

$$g_1(f_+) - g_1(f_-) = \frac{1 + S_3'(\tau_0)}{S_1'(\tau_0) - S_2'(\tau_0)}$$

valid for all $\tau_0 \in (f_-, f_+)$. Hence, it the function $\tau_0 \mapsto \frac{1+S_3'(\tau_0)}{S_1'(\tau_0)-S_2'(\tau_0)}$ is not constant, we may also obtain contradiction. Nevertheless, we believe that the assumption on $S_1'(\tau_0) - S_3'(\tau_0)$ is easier to formulate.

Proof of Theorem 3.2.6. As in the proof of Theorem 3.2.5, we may assume without loss of generality that $\operatorname{supp} F^{\#}\mu_{t,x} \subset [f_-, f_+]$ (this did not use the non-degeneracy condition!). By assumption of the theorem, for any set $A \subset \mathbb{R}^+$

$$0 = \mu_{t,x}(F^{-1}(A) \cap I_2) = F^{\#}\mu_{t,x}^{(2)}(A) = \int_A g_2(\lambda)\, \mathrm{d}F^{\#}\mu_{t,x}(\lambda)$$

so $g_2(\lambda) = 0$ for almost all λ. Hence, when $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$, the sum

$$\sum_{i=1}^{3} (S_i'(\lambda_0) + 1)\, g_i(\lambda_0) \geq \min(S_1'(\lambda_0) + 1, S_3'(\lambda_0) + 1) > 0$$

because $g_1(\lambda_0) + g_3(\lambda_0) = 1$ and S_1, S_3 are strictly increasing. It follows from Theorem 3.2.4 that $F^{\#}\mu_{t,x}\{\lambda_0\} = 1$, i.e. $F^{\#}\mu_{t,x} = \delta_{\lambda_0}$. Finally, if there is no such $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$, we apply Lemma 3.5.1.

It follows that $F^{\#}\mu_{t,x}$ is the Dirac measure so that we can conclude as in Theorem 3.2.5. □

Proof of Theorem 3.2.7. Following the lines of the proof of Theorem 3.2.6, we take $\lambda_0 \in \operatorname{supp} F^{\#}\mu_{t,x} \cap (f_-, f_+)$ and we observe that the sum

$$\sum_{i=1}^{3} (S_i'(\lambda_0) + 1)\, g_i(\lambda_0) \geq \min(1, \delta(\lambda_0)) \sum_{i=1}^{3} g_i(\lambda_0) = \min(1, \delta(\lambda_0)) > 0$$

where $\delta(\lambda_0)$ is such that $S_2'(\lambda_0) + 1 > \delta(\lambda_0) > 0$. We conclude as in the proof of Theorem 3.2.6. □

3.6 Open problems

System (3.1.1) studied in this paper is a special case of

$$\begin{aligned} \partial_t u^\varepsilon &= d_1\, \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}, \\ \partial_t v^\varepsilon &= d_2\, \Delta v^\varepsilon + \frac{F(u^\varepsilon) - v^\varepsilon}{\varepsilon} \end{aligned} \tag{3.6.1}$$

for some $d_1, d_2 \geq 0$. Mimicking the method for $d_1 = 0$, we obtain

$$\partial_t \int_\Omega \widetilde{F}(u^\varepsilon) + (v^\varepsilon)^2/2 = -d_1 \int_\Omega F'(u^\varepsilon)\, |\nabla u^\varepsilon|^2 - d_2 \int_\Omega |\nabla v^\varepsilon|^2 - \int_\Omega \frac{(v^\varepsilon - F(u^\varepsilon))^2}{\varepsilon}$$

where $\widetilde{F}(u) = \int_0^u F(\tau)\,\mathrm{d}\tau$. We see that when F' is not strictly positive, we cannot conclude. Using refined energy estimates from [219] (they are based on multiplying the first equation by Δu^ε), fast reaction limit was established in [221, Theorem 2.9] for two special cases

$$d_2 \geq d_1,\; F'(u) + \frac{d_1}{d_2} > 0, \qquad \text{or} \qquad d_1 > d_2,\; F'(u) + \frac{d_2}{d_1} > 0. \tag{3.6.2}$$

More precisely, it was proved that $w^\varepsilon := u^\varepsilon + v^\varepsilon$ converges strongly to the solution of

$$\partial_t w - \Delta A(w) = 0, \qquad \frac{\partial}{\partial \mathbf{n}} w = 0 \tag{3.6.3}$$

where

$$A(w) = d_1 u + d_2 F(u) \quad \text{with} \quad w = u + F(u).$$

Function A is well-defined because conditions (3.6.2) imply that $F'(u) > -1$. Limiting equation (3.6.3) is a consequence of summing up (3.6.1) together with a priori estimates that gives strong convergence of $u^\varepsilon + v^\varepsilon \to u + v$ and $v^\varepsilon - F(u^\varepsilon) \to 0$. However, if one only assumes $F'(u) > -1$ without (3.6.2), the only available energy estimate is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\widetilde{F}(u^\varepsilon) + \frac{1}{2}(v^\varepsilon)^2 + \varepsilon\, d_1 |\nabla u^\varepsilon|^2 + \frac{d_1^2 + d_1\, d_2}{2\,(d_2^2 - d_1^2)}\,(w^\varepsilon)^2\right] \mathrm{d}x =$$
$$= -\varepsilon \int_\Omega \left(d_1\, \Delta u^\varepsilon + \frac{v^\varepsilon - F(u^\varepsilon)}{\varepsilon}\right)^2 \mathrm{d}x - \frac{1}{d_2 - d_1}\int_\Omega |d_1 \nabla u^\varepsilon + d_2 \nabla v^\varepsilon|^2\,\mathrm{d}x$$

where $\widetilde{F}$ is a primitive function of F. This equality is too weak to deduce any strong convergence. The only result we can prove in that case is that, seting $w^\varepsilon = u^\varepsilon + v^\varepsilon$ and $z^\varepsilon = d_1 u^\varepsilon + d_2 v^\varepsilon$, we have

$$w^\varepsilon \overset{*}{\rightharpoonup} w := u + v, \qquad z^\varepsilon \overset{*}{\rightharpoonup} z := d_1 u + d_2 v, \qquad v^\varepsilon - F(u^\varepsilon) \overset{*}{\rightharpoonup} 0, \qquad w_t = \Delta z$$

but it is not clear at all what is the coupling between functions w and z.

Our conjecture is that one can prove strong compactness of the sequence $\{z^\varepsilon\}$. However, the method exploited for the case $d_1 = 0$ cannot be applied as we do not have a family of energy identities which could be used to identify the Young measure. Nevertheless, let us comment that the cases $d_1 = 0$ and $d_1 \neq 0$ are somehow similar:

we have estimate on the spatial gradient and we lack any information on the time derivative.

Concerning the case $d_1 = 0$, let us also point out once again that strong convergence $v^\varepsilon \to v$ in our work is rather unavailable to be obtained from a priori estimates. It is a consequence of careful analysis of Young measure and an additional structural assumption on F: either nondegeneracy condition (3.2.6) or the one in Theorem 3.2.7. Both of them seems to be technical and we would like to know whether they can be waived.

Chapter 4

Kinetic derivation of degenerate Cahn-Hilliard

The results in this chapter have been submitted for publication as the following preprint:

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4.1 Introduction

The target of this chapter is to derive the Cahn-Hilliard equation with degenerate mobility

$$\partial_t \varrho = \operatorname{div}\left(\varrho \nabla \left(F(\varrho) - \delta \Delta \varrho\right)\right) \to \begin{cases} \partial_t \varrho & = \operatorname{div}\left(\varrho \, \nabla \mu\right), \\ \mu & = -\delta \Delta \varrho + F(\varrho), \end{cases} \tag{4.1.1}$$

which is a macroscopic equation via the so-called hydrodynamic limit of Vlasov type equation describing the matter at the level of particles. Such derivation was recently achieved by Takata and Noguchi in a formal way [256]. Our target is to make it mathematically rigorous.

Equation (4.1.1) will be studied throughout Chapters 5 and 6. Its analysis is fairly difficult because it degenerates whenever ϱ is approaching 0. Up to now, there is no satisfactory well-posedness theory of classical solutions (but see [124], [87] for the theory of weak solutions). Many authors consider a variant of Cahn-Hilliard equation with a non-degenerate mobility, i.e.

$$\begin{cases} \partial_t \varrho &= \operatorname{div}\left(b(\varrho)\, \nabla \mu\right), \\ \mu &= -\delta \Delta \varrho + F(\varrho), \end{cases}$$

where the mobility b satisfies $b(\varrho) > c > 0$ [143, 144, 155]. Such assumption allows to obtain an L^2 estimate for $\nabla \mu$ which simplifies mathematical analysis of the system and brings new insights. Nevertheless, several limiting procedures aiming at a derivation of the Cahn-Hilliard equation, including the one presented in this chapter, the high-friction limit in Chapter 6 as well as the limit of interacting particle systems in [151, 152] shows that it is the equation with degenerate mobility that is more physically relevant and so, it deserves mathematical studies despite difficulties.

Let us be more precise concerning the result of this chapter. We consider the following Vlasov-Cahn-Hilliard equation (VCH in short)

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon + \varepsilon\, \xi \cdot \nabla_x f_\varepsilon + \varepsilon \mathcal{F}_\varepsilon \cdot \nabla_\xi f_\varepsilon = \varrho_\varepsilon(t,x) M(\xi) - f_\varepsilon, \\ \varrho_\varepsilon(t,x) = \int_{\mathbb{R}^d} f_\varepsilon(t,x,\xi)\, \mathrm{d}\xi, \end{cases} \tag{4.1.2}$$

where $t \geq 0$ is time, x is position and ξ is velocity. Equation (4.1.2) is equipped with an initial data $f_\varepsilon(0,x,\xi) = f^0(x,\xi) \geq 0$. The unknown is the function

$$f_\varepsilon \equiv f_\varepsilon(t,x,\xi), \quad t \in (0,T),\ x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d,$$

such that, for every infinitesimal volume $\mathrm{d}x\, \mathrm{d}\xi$ around the point (x, ξ) in the phase space, the quantity $f_\varepsilon(t,x,\xi)\, \mathrm{d}x\, \mathrm{d}\xi$ is the number of particles which have position x and velocity ξ at fixed time t. The small parameter $\varepsilon > 0$ arises from physical dimensions of the system and we are interested in the limit when it tends to 0.

To derive Cahn-Hilliard equation from (4.1.2), one has to choose the force field $\mathcal{F}_\varepsilon$ in the appropriate way. Following [256], the force field $\mathcal{F}_\varepsilon(t,x)$ is decomposed as long-range attractive and short-range repulsive

$$\mathcal{F}_\varepsilon = \mathcal{F}_\varepsilon^L + \mathcal{F}_\varepsilon^S, \qquad \mathcal{F}_\varepsilon^{L,S}(t,x) = -\nabla \Phi_\varepsilon^{L,S}(t,x). \tag{4.1.3}$$

With the notation $f * g = \int_{\mathbb{R}^d} f(y)g(x-y)\,\mathrm{d}y$, we set

$$\Phi^S_{\alpha,\varepsilon}(t,x) = \frac{1}{\alpha^2}\omega^S * \omega^S * \varrho_\varepsilon,$$

where $\omega^S \geq 0$ is a usual mollification kernel, i.e.

$$\int_{\mathbb{R}^d} \omega^S(y)\,\mathrm{d}y = 1, \quad \int_{\mathbb{R}^d} y\,\omega^S(y)\,\mathrm{d}y = 0, \quad \int_{\mathbb{R}^d} |y|^2\omega^S(y)\,\mathrm{d}y < \infty. \tag{4.1.4}$$

We use a double convolution in order to enforce positivity of the corresponding operator as it appears in energy considerations. It should be noted that the right choice for $\Phi^S_{\alpha,\varepsilon}(t,x)$ is simply ϱ_ε but with such choice of the potential, equation (4.1.2) is not well-posed. Thus, the main role of ω^S is to have well-posedness theory for (4.1.2).

The long-range potential is of the form

$$\Phi^L_{\alpha,\varepsilon}(t,x) = -\frac{1}{\alpha^2}\omega^L_\alpha * \omega^S * \omega^S * \varrho_\varepsilon, \tag{4.1.5}$$

where $\omega^L_\alpha(x) = \frac{1}{\alpha^d}\,\omega^L\left(\frac{x}{\alpha}\right)$ may be thought of as a high temperature Gaussian and ω^L is a smooth, nonnegative, symmetric, compactly supported function such that, for some $\delta > 0$,

$$\int_{\mathbb{R}^d} \omega^L(y)\,\mathrm{d}y = 1, \int_{\mathbb{R}^d} y\omega^L(y)\,\mathrm{d}y = 0, \int_{\mathbb{R}^d} y_i y_j \omega^L\,\mathrm{d}y = \delta_{i,j}\,\delta, \int_{\mathbb{R}^d} \omega^L(y)|y|^3\,\mathrm{d}y < \infty. \tag{4.1.6}$$

The equilibrium distribution $M(\xi) \geq 0$ is a Maxwellian that we normalize as

$$M(\xi) := \left(\frac{1}{2\pi D}\right)^{d/2} \exp\left(-\frac{|\xi|^2}{2D}\right), \tag{4.1.7}$$

and we have, for $i = 1, \ldots, d$,

$$\int_{\mathbb{R}^d} M(\xi)\,\mathrm{d}\xi = 1, \qquad \int_{\mathbb{R}^d} \xi_i M(\xi)\,\mathrm{d}\xi = 0, \qquad \int_{\mathbb{R}^d} \xi_i^2 M(\xi)\,\mathrm{d}\xi = D < \infty, \tag{4.1.8}$$

so that D can be interpreted as the diffusion coefficient.

Let us see what is the limit of (4.1.2) formally. The right-hand side of (4.1.2) suggests that

$$f_\varepsilon(t,x,\xi) \to \varrho(t,x)M(\xi), \quad \text{as } \varepsilon \to 0. \tag{4.1.9}$$

The mass conservation equation on ϱ_ε is obtained by integrating (4.1.2) with respect to ξ against 1,

$$\partial_t \varrho_\varepsilon(t,x) + \mathrm{div} J_\varepsilon(t,x) = 0, \qquad J_\varepsilon(t,x) = \int_{\mathbb{R}^d} \frac{\xi}{\varepsilon} f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi. \tag{4.1.10}$$

Then, integrating against ξ, we obtain the flux equation

$$\varepsilon^2 \partial_t J_\varepsilon(t,x) + \nabla_x \cdot \int_{\mathbb{R}^d} \xi \otimes \xi f_\varepsilon(t,x,\xi)\,\mathrm{d}\xi - \mathcal{F}_\varepsilon \varrho_\varepsilon = -J_\varepsilon(t,x). \tag{4.1.11}$$

Combined with (4.1.9), this flux equation allows us to identify the limit of $\{J_\varepsilon\}$ and to prove that as $\varepsilon, \alpha \to 0$, the macroscopic densities tend to a solution of a degenerate nonlocal Cahn-Hilliard equation type. More precisely, we have the

Theorem 4.1.1 (Limit $\varepsilon \to 0$)**.** *With the assumptions and notations* (4.1.3)–(4.1.7), *let $\alpha = \varepsilon$. Let f^0 be a non-negative distribution that satisfies* (4.3.1)-(4.3.2) *and let f_ε be a solution of* (4.1.2) *with initial condition f^0. Then, we can extract a subsequence (not relabelled) such that $\varrho_\varepsilon \to \varrho$ in $L^p(0,T;L^1(\mathbb{R}^d))$ strongly for $1 \le p < \infty$ where ϱ solves in the distributional sense the equation*

$$\partial_t \varrho - D\Delta\varrho - \mathrm{div}(\varrho \nabla \Phi) = 0, \qquad \Phi = -\delta \Delta[\omega^S * \omega^S * \varrho], \tag{4.1.12}$$

with initial data $\varrho^0 = \int_{\mathbb{R}^d} f^0(x,\xi)\,\mathrm{d}\xi$.

Let us stress that this is the first rigorous result aiming at a derivation of (4.1.1) from a microscopic equation (4.1.2), following a formal approach presented in [256]. Let us make a few remarks.

Remark 4.1.2.
- Writing formally $\Delta\varrho = \mathrm{div}(\varrho\nabla\log(\varrho))$, the term $\Delta\varrho$ can be included in the potential term.
- Different scalings between α and ε can be considered. The case α fixed is also possible.

- In fact, our derivation is not complete as we obtain nonlocality in Φ. It is an open problem to rigorously send $\omega^S \overset{*}{\rightharpoonup} \delta_0$, see Section 4.5. It seems that one can achieve this in dimension $d = 2$ which is a work in progress.

- Equation (4.1.12) can be referred to as a nonlocal Cahn-Hilliard equation. Nevertheless, it is a different nonlocal Cahn-Hilliard equation then the one studied in Chapters 5 and 6 (see (5.1.3)–(5.1.4)). The nonlocal effect in (4.1.12) analyzed in this chapter is artificial: it was introduced only to guarantee well-posedness of the Vlasov equation (4.1.2). On the other hand, the nonlocality in (5.1.3)–(5.1.4) studied in Chapters 5 and 6 is motivated by the derivation from interacting particle systems [151, 152].

- In fact, Takata and Noguchi [256] can formally obtain a Cahn-Hilliard equation with a potential which we cannot prove at the moment (see Section 4.5).

4.2 Kinetic theory and Cahn-Hilliard equation

Kinetic theory. The main purpose of kinetic theory is to provide a description of the evolution of a gas or plasma, and more generally a many-particle system made up of N similar individual elements, in the limit when N tends to infinity which corresponds to the so-called thermodynamical limit.

In the kinetic theory, the density of particles is described with the probability measure

$$f \equiv f(t, x, \xi), \quad t \geq 0,\ x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d,$$

such that, for every infinitesimal volume $\mathrm{d}x\,\mathrm{d}\xi$ around the point (x, ξ) in the phase space, the quantity $f(t, x, \xi)\,\mathrm{d}x\,\mathrm{d}\xi$ is the number of particles which have position x and velocity ξ at fixed time t. For this reason, f is a nonnegative function and integrable in both space and velocity variables, but it is not directly observable. Nevertheless, at each point of the domain it provides all measurable macroscopic quantities which can be expressed in terms of microscopic averages:

$$\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)\,\mathrm{d}\xi \qquad \text{(macroscopic density)},$$
$$J(t,x) = \int_{\mathbb{R}^d} \xi f(t,x,\xi)\,\mathrm{d}\xi \qquad \text{(flux)}.$$

It is clear that such a statistical description makes sense only with a very large number of particles, and as a consequence, all kinetic equations are expected to approximate the true dynamics of gases just in the thermodynamical limit. The procedure of rescaling the time and space with a parameter ε, *i.e.* $t \to \varepsilon^2 t$, $x \to \varepsilon x$ and sending $\varepsilon \to 0$ is called the hydrodynamic limit. It allows us to find a rigorous derivation of macroscopic models from a microscopic description of matter. There are many systems that can be derived this way, see [241] for passage from Boltzmann to Navier-Stokes/Euler equations and [119,159,234] for passage from Vlasov-Poisson to Smoluchowski equation.

The Cahn-Hilliard equation. Equation (4.1.12) is an example of a Cahn-Hilliard type equation that is widely used nowadays to represent phase transitions in fluids and living tissues [122, 147, 148, 198]. It was introduced by Cahn and Hilliard in the context of material science to model the dynamics of phase separation in a binary mixture spontaneously separating into two domains. Currently, it is applied in numerous fields, including complex fluids, polymer science, and mathematical biology. For the mathematical theory of Cahn-Hilliard equation we refer to [218] and [124].

Cahn-Hilliard equation is a fourth order PDE that takes the form of

$$\partial_t \varrho = \operatorname{div}\left(b(\varrho)\nabla\left(F'(\varrho) - \delta\Delta\varrho\right)\right) \to \begin{cases} \partial_t \varrho & = \operatorname{div}\left(b(\varrho)\nabla\mu\right), \\ \mu & = -\delta\Delta\varrho + F'(\varrho), \end{cases} \tag{4.2.1}$$

where ϱ represents the relative density of one component $\varrho = \varrho_1/(\varrho_1+\varrho_2)$, $b(\varrho)$ is the mobility, F is the interaction potential and μ is so-called chemical potential.

The case of $b(\varrho) = \varrho$ is referred to as an equation with degenerate mobility and it is particularly interesting. From the derivation presented in this chapter (see, for instance, (4.1.10)–(4.1.11)) it is the only one that can be derived from the Vlasov-type equation. Similarly, several other limits from different physical systems (see [151, 152] and Chapter 6) leads to the Cahn-Hilliard equation with degenerate mobility $b(\varrho) = \varrho$. On the other hand, the case $b(\varrho) = \varrho$ is more difficult than when b is bounded from below $b(\varrho) > c > 0$. Indeed, when b can touch zero, we loose an important estimate on $\nabla \mu$ (which implies that ϱ has weak derivaties of order three).

Let us briefly explain the physical derivation of (4.2.1) with $b(\varrho) = \varrho$ as in [65, 189]. We consider a binary mixture Ω consisting of two materials and we let ϱ to be a relative density of one of them. The underlying assumption made by Cahn and Hilliard is that the total free energy of the system is a function f which can depend on ϱ, $\nabla \varrho$, $\nabla^2 \varrho$. To determine its precise form, one expands f in Taylor series around $(\varrho, 0, 0, ...)$. As the energy should be invariant under translations and rotations, one postulates its form

$$f(\varrho) = F(\varrho) + \kappa_1 \, \Delta \varrho + \frac{\kappa_2}{2} \, |\nabla \varrho|^2, \tag{4.2.2}$$

where the terms of order bigger than 2 were neglected and

$$F(\varrho) = f(\varrho, 0, 0, ...), \qquad \kappa_1 = \frac{\partial f}{\partial \varrho_{x_i, x_i}}(\varrho, 0, 0, ...), \qquad \kappa_2 = \frac{\partial^2 f}{\partial (\varrho_{x_i})^2}(\varrho, 0, 0, ...)$$

(note that by the invariance, derivatives with respect to each direction should be equal). The second term in (4.2.2) will vanish after integrating over Ω so that the total energy equals $\mathcal{E} := \int_\Omega F(\varrho) + \frac{\kappa_2}{2} |\nabla \varrho|^2$. Assuming that κ_2 is a constant, its first variation equals $\frac{\delta \mathcal{E}}{\delta \varrho} = F'(\varrho) - \kappa_2 \Delta \varrho$. Assuming that the phase separation occurs to minimize the energy, we can model it is a gradient flow of $\mathcal{E}$, we arrive at the Cahn-Hilliard equation

$$\partial_t \varrho = \operatorname{div}\left(\varrho \nabla \frac{\delta \mathcal{E}}{\delta \varrho}\right) = \operatorname{div}\left(\varrho \, \nabla (F'(\varrho) - \kappa_2 \Delta \varrho)\right).$$

In this work we obtain a nonlocal version of the Cahn-Hilliard equation. The nonlocality comes from the convolution of the Laplace operator with a smooth kernel ω^S concentrated around the origin. There is a different possibility to approximate this operator nonlocally and we refer for instance to [92, 215] and Chapter 5, where one approximates Laplace operator with a nonlocal operator.

4.3 Entropy, energy, and uniform estimates

The analysis relies on various uniform bounds in ε which use an initial data that satisfies

$$\int_{\mathbb{R}^{2d}} (1+|x|+|\xi|^2+|\log f^0|) f^0(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi < +\infty, \tag{4.3.1}$$

$$\sup_{\alpha\le 1} \frac{1}{\alpha^2}\int_{\mathbb{R}^{2d}} \omega_\alpha^L(y)[\varrho^0 * \omega^S(x) - \varrho^0 * \omega^S(x-y)]^2\,\mathrm{d}x\,\mathrm{d}y < +\infty. \tag{4.3.2}$$

Then, we begin with proving the uniform bounds:

Theorem 4.1 (Uniform estimates)**.** *With the assumptions* (4.3.1) *and* (4.3.2)*, the following uniform estimates hold for* $\varepsilon \in (0,1)$*:*

(A) $\{f_\varepsilon\}$ *in* $L^\infty(0,T;L^1(\mathbb{R}^d\times\mathbb{R}^d))$ *and* $\{\varrho_\varepsilon\}$ *in* $L^\infty(0,T;L^1(\mathbb{R}^d))$*,*

(B) $\{f_\varepsilon|\log(f_\varepsilon)|\}$ *and* $\{f_\varepsilon\,|\xi|^2\}$ *in* $L^\infty(0,T;L^1(\mathbb{R}^d\times\mathbb{R}^d))$*,*

(C) $\{\varrho_\varepsilon|\log(\varrho_\varepsilon)|\}$ *in* $L^\infty(0,T;L^1(\mathbb{R}^d))$*,*

(D) $\left\{\frac{(\varrho_\varepsilon M - f_\varepsilon)\,(\log(\varrho_\varepsilon M)-\log(f_\varepsilon))}{\varepsilon^2}\right\}$ *in* $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$*,*

(E) $\left\{\frac{\varrho_\varepsilon M - f_\varepsilon}{\varepsilon}\right\}$ *in* $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$*,*

(F) $\{J_\varepsilon\}$ *and* $\{J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e)\}$ *in* $L^1((0,T)\times\mathbb{R}^d)$*,*

(G) $\{f_\varepsilon|x|\}$*,* $\{\varrho_\varepsilon\,|x|\}$ *in* $L^\infty(0,T;L^1(\mathbb{R}^d\times\mathbb{R}^d))$ *and* $L^\infty(0,T;L^1(\mathbb{R}^d))$ *respectively.*

Moreover, $\{\varrho_\varepsilon\}$ *and* $\{J_\varepsilon\}$ *are weakly compact in* $L^1((0,T)\times\mathbb{R}^d)$*.*

The proof of these estimates uses a fundamental property of energy dissipation. To show that, we define the energy (kinetic+potential) and the Helmholtz free energy respectively as

$$\mathcal{E}(t) := \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi + \frac{1}{2\alpha^2}\int_{\mathbb{R}^{2d}} \omega_\alpha^L(y)[\varrho_\varepsilon * \omega^S(t,x) - \varrho_\varepsilon * \omega^S(t,x-y)]^2\,\mathrm{d}x\,\mathrm{d}y, \tag{4.3.3}$$

$$\begin{aligned}\mathcal{G}(t) := \int_{\mathbb{R}^{2d}} &[2D\log(f_\varepsilon) + |\xi|^2] f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi \\ &+ \frac{1}{2\alpha^2}\int_{\mathbb{R}^{2d}} \omega_\alpha^L(y)[\varrho_\varepsilon * \omega^S(t,x) - \varrho_\varepsilon * \omega^S(t,x-y)]^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.3.4}$$

The Helmholtz free energy satisfies the

Theorem 4.2 (Free energy dissipation)**.** *The free energy* $\mathcal{G}(t)$ *is dissipated as*

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) = -\frac{2D}{\varepsilon^2}\int_{\mathbb{R}^{2d}} [f_\varepsilon - \varrho_\varepsilon M(\xi)]\,[\log f_\varepsilon - \log(\varrho_\varepsilon M(\xi))]\,\mathrm{d}x\,\mathrm{d}\xi = \\ = -2D\int_{\mathbb{R}^{2d}} \mathcal{D}_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi,\end{aligned} \tag{4.3.5}$$

where the dissipation term is defined as

$$\mathcal{D}_\varepsilon(t,x,\xi) := \frac{1}{\varepsilon^2}\,[f_\varepsilon - \varrho_\varepsilon M(\xi)]\,[\log f_\varepsilon - \log(\varrho_\varepsilon M(\xi))] \geq 0. \tag{4.3.6}$$

This theorem can be seen as a combination of relations for both the total energy and the entropy of the system.

Proposition 4.3.1 (Total energy dissipation)**.** *The total energy* $\mathcal{E}(t)$ *is dissipated as*

$$\frac{d}{dt}\mathcal{E}(t) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^{2d}} |\xi|^2\,[\varrho_\varepsilon M(\xi) - f_\varepsilon]\,\mathrm{d}x\,\mathrm{d}\xi. \tag{4.3.7}$$

Proof. By multiplying (4.1.2) by $|\xi|^2$ and taking the integrals with respect to x and ξ we obtain

$$\begin{aligned}\varepsilon^2\int_{\mathbb{R}^{2d}} |\xi|^2\partial_t f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi + \varepsilon\int_{\mathbb{R}^{2d}} |\xi|^2\xi\cdot\nabla_x f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi + \varepsilon\int_{\mathbb{R}^{2d}} |\xi|^2\mathcal{F}_\varepsilon\nabla_\xi f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi \\ = \int_{\mathbb{R}^{2d}} |\xi|^2[\varrho_\varepsilon M(\xi) - f_\varepsilon]\,\mathrm{d}x\,\mathrm{d}\xi.\end{aligned} \tag{4.3.8}$$

For integrable solutions, the second term on the left-hand side vanishes. Furthermore, with integration by parts, the above equation reduces to

$$\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi - 2\varepsilon\int_{\mathbb{R}^{2d}} \xi\mathcal{F}_\varepsilon f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^{2d}} |\xi|^2[\varrho_\varepsilon M(\xi) - f_\varepsilon]\,\mathrm{d}x\,\mathrm{d}\xi. \tag{4.3.9}$$

By recalling (4.1.10), the second term can be rewritten as

$$-2\varepsilon\int_{\mathbb{R}^{2d}} \xi\mathcal{F}_\varepsilon f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi = -2\varepsilon^2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}\,\mathrm{div}J_\varepsilon\,\mathrm{d}x = 2\varepsilon^2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}\,\partial_t\varrho_\varepsilon\,\mathrm{d}x.$$

We now want to prove that

$$2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}\partial_t\varrho_\varepsilon\,\mathrm{d}x = \frac{1}{2\alpha^2}\frac{d}{dt}\int_{\mathbb{R}^{2d}} \omega_\alpha^L(y)[\varrho_\varepsilon*\omega^S(t,x) - \varrho_\varepsilon*\omega^S(t,x-y)]^2\,\mathrm{d}x\,\mathrm{d}y. \tag{4.3.10}$$

First, by recalling (4.1.3),

$$2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}(t,x)\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x = 2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^L(t,x)\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x + 2\int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^S(t,x)\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x.$$

As regards the first term on the right-hand side

$$
\begin{aligned}
2\int_{\mathbb{R}^d} \Phi^L_{\alpha,\varepsilon}(t,x)\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x &= -\frac{2}{\alpha^2}\int_{\mathbb{R}^d}[\omega^L_\alpha * \omega^S * \omega^S * \varrho_\varepsilon](t,x)\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x \\
&= -\frac{2}{\alpha^2}\int_{\mathbb{R}^d}[\omega^L_\alpha * \varrho_\varepsilon * \omega^S](t,x)\partial_t[\varrho_\varepsilon * \omega^S](t,x)\,\mathrm{d}x \\
&= -\frac{1}{\alpha^2}\int_{\mathbb{R}^{2d}}\omega^L_\alpha(y)[\varrho_\varepsilon * \omega^S](t,x-y)\partial_t[\varrho_\varepsilon * \omega^S](t,x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad -\frac{1}{\alpha^2}\int_{\mathbb{R}^{2d}}\omega^L_\alpha(y)[\varrho_\varepsilon * \omega^S](t,x)\partial_t[\varrho_\varepsilon * \omega^S](t,x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= -\frac{1}{\alpha^2}\frac{d}{dt}\int_{\mathbb{R}^{2d}}\omega^L_\alpha(y)[[\varrho_\varepsilon * \omega^S](t,x)\cdot[\varrho_\varepsilon * \omega^S](t,x-y)]\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

The second term on the right-hand side can be handled similarly and gives

$$
\begin{aligned}
2\int_{\mathbb{R}^d} \Phi^S_{\alpha,\varepsilon}(t,x)\,\partial_t\varrho_\varepsilon(t,x)\,\mathrm{d}x &= \frac{2}{\alpha^2}\int_{\mathbb{R}^d}[\omega^S * \varrho_\varepsilon](t,x)\partial_t[\varrho_\varepsilon * \omega^S](t,x)\,\mathrm{d}x \\
&= \frac{1}{\alpha^2}\frac{d}{dt}\int_{\mathbb{R}^d}[\varrho_\varepsilon * \omega^S]^2(t,x)\,\mathrm{d}x \\
&= \frac{1}{2\alpha^2}\frac{d}{dt}\int_{\mathbb{R}^{2d}}\omega^L_\alpha(y)\big[[\varrho_\varepsilon * \omega^S]^2(t,x) + [\varrho_\varepsilon * \omega^S]^2(t,x-y)\big]\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

By summing up the two previous identities we get (4.3.10), which, inserted in (4.3.9), concludes that

$$
\begin{aligned}
\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}^{2d}}|\xi|^2 f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi + \frac{\varepsilon^2}{2\alpha^2}\frac{d}{dt}\int_{\mathbb{R}^{2d}}\omega^L_\alpha(y)[\varrho_\varepsilon * \omega^S(t,x) - \varrho_\varepsilon * \omega^S(t,x-y)]^2\,\mathrm{d}x\,\mathrm{d}y \\
= \int_{\mathbb{R}^{2d}}|\xi|^2[\varrho_\varepsilon M(\xi) - f_\varepsilon]\,\mathrm{d}x\,\mathrm{d}\xi.
\end{aligned}
$$

□

Proposition 4.3.2 (Entropy relation). *The following estimate holds:*

$$
\frac{d}{dt}\int_{\mathbb{R}^{2d}} f_\varepsilon \log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^{2d}}[\varrho_\varepsilon M(\xi) - f_\varepsilon]\log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi. \tag{4.3.11}
$$

Proof. By multiplying (4.1.2) by $(1+\log f_\varepsilon)$ we obtain

$$
\varepsilon^2\frac{d}{dt}(f_\varepsilon\log f_\varepsilon) + \varepsilon\xi\cdot\nabla_x f_\varepsilon(1+\log f_\varepsilon) + \varepsilon\mathcal{F}_\varepsilon\nabla_\xi f_\varepsilon(1+\log f_\varepsilon) = [\varrho_\varepsilon M(\xi) - f_\varepsilon](1+\log f_\varepsilon)
$$

By taking the integrals with respect to x and ξ, the second and third terms in the above equation vanish and we obtain

$$
\varepsilon^2\frac{d}{dt}\int_{\mathbb{R}^{2d}} f_\varepsilon\log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi = \int_{\mathbb{R}^{2d}}[\varrho_\varepsilon M(\xi) - f_\varepsilon]\log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi
$$

as announced. □

With these two estimates, we can finally prove Theorem 4.2.

Proof of Theorem 4.2. From Propositions 4.3.1 and 4.3.2, we get the following result:

$$\frac{d}{dt}\mathcal{G}(t) = \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^{2d}} |\xi|^2[\varrho_\varepsilon M(\xi) - f_\varepsilon]\,\mathrm{d}x\,\mathrm{d}\xi + 2D\int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon]\log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi\right]$$
$$= \frac{1}{\varepsilon^2}\left[\int_{\mathbb{R}^{2d}} \varrho_\varepsilon M(\xi)|\xi|^2\,\mathrm{d}x\,\mathrm{d}\xi - \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi + 2D\int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon]\log f_\varepsilon\,\mathrm{d}x\,\mathrm{d}\xi\right]. \tag{4.3.12}$$

Using (4.1.7), we know that $\log(\varrho_\varepsilon M(\xi)) = \log\varrho_\varepsilon + C - \frac{|\xi|^2}{2D}$ for some constant C. Inserting this expression of $|\xi|^2$ in the first two terms on the righthand side of (4.3.12), we obtain

$$\frac{2D}{\varepsilon^2}\int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon]\,[\log\varrho_\varepsilon + C - \log(\varrho_\varepsilon M(\xi))]\,\mathrm{d}x\,\mathrm{d}\xi$$
$$= \frac{2D}{\varepsilon^2}\int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon]\,[-\log(\varrho_\varepsilon M(\xi))]\,\mathrm{d}x\,\mathrm{d}\xi.$$

Added to the third term on the righthand side of (4.3.12), we obtain the announced result. ☐

In order to prove Theorem 4.1, a major difficulty is to estimate the flux J_ε defined by (4.1.10). We start by establishing a useful inequality, recalling the notation (4.3.6).

Lemma 4.3.3 (Pointwise estimates on J_ε)**.** *For every $0 < r \leq 1$ and $(s,x) \in (0,T)\times\mathbb{R}^d$, we have*

$$|J_\varepsilon(s,x)| \leq r\varepsilon\|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi} + C\frac{1}{r^d}\exp\left(\frac{2C_M}{r^2}\right)\varrho_\varepsilon(s,x)^{1/2}\|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2}.$$

Proof. For $r > 0$, we decompose $J_\varepsilon(s,x) = J_\varepsilon^{(1)}(s,x) + J_\varepsilon^{(2)}(s,x)$, with

$$J_\varepsilon^{(1)} = \frac{1}{\varepsilon}\int_{\left\{\left|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\right| \geq \frac{|\xi|}{r}\right\}} \xi(f_\varepsilon - \varrho_\varepsilon M(\xi))\,\mathrm{d}\xi,$$

$$J_\varepsilon^{(2)} = \frac{1}{\varepsilon}\int_{\left\{\left|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\right| \leq \frac{|\xi|}{r}\right\}} \xi(f_\varepsilon - \varrho_\varepsilon M(\xi))\,\mathrm{d}\xi.$$

For $J_\varepsilon^{(1)}$, we write

$$|J_\varepsilon^{(1)}(s,x)| \leq \frac{r}{\varepsilon}\int_{\left\{\left|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\right| \geq \frac{|\xi|}{r}\right\}} \left|\log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right)\right|\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right|\mathrm{d}\xi \leq r\varepsilon\|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}.$$

For $J_\varepsilon^{(2)}$, we use the Cauchy-Schwarz inequality and, with $B(\xi) := \frac{|\xi|}{r(\exp(\frac{|\xi|}{r})-1)}$,

$$|J_\varepsilon^{(2)}(s,x)| \le \left(\int_{\mathbb{R}^d} |\xi|^2 \frac{\varrho_\varepsilon M}{B(\xi)}\, \mathrm{d}\xi\right)^{1/2} \left(\frac{1}{\varepsilon^2}\int_{\left\{|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})| \le \frac{|\xi|}{r}\right\}} \varrho_\varepsilon M \left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right|^2 B(\xi)\, \mathrm{d}\xi\right)^{1/2}.$$

Because $M(\xi)$ is a Gaussian and ϱ_ε depends only on (t,x), we obtain

$$|J_\varepsilon^{(2)}(s,x)| \le \varrho_\varepsilon(s,x)^{1/2} \left(\int_{\mathbb{R}^d} |\xi|^2 \frac{M(\xi)}{B(\xi)}\, \mathrm{d}\xi\right)^{1/2} (I_1 + I_2)^{1/2}.$$

Here we have split the second integral according to the sign of $\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})$. When it is negative, we may write, since $B(\xi) \le 1$,

$$I_1 := \frac{1}{\varepsilon^2}\int_{\left\{\frac{f_\varepsilon}{\varrho_\varepsilon M} \le 1\right\}} \varrho_\varepsilon M \left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right|^2 B(\xi)\, \mathrm{d}\xi \le$$
$$\le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \varrho_\varepsilon M \left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right| \left|\log \frac{f_\varepsilon}{\varrho_\varepsilon M}\right| \mathrm{d}\xi = \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}.$$

The second term is defined as

$$I_2 := \frac{1}{\varepsilon^2}\int_{\left\{0 \le \log(\frac{f_\varepsilon}{\varrho_\varepsilon M}) \le \frac{|\xi|}{r}\right\}} \varrho_\varepsilon M \left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right|^2 B(\xi)\, \mathrm{d}\xi.$$

Since log is a concave function, for $A > 1$ and $y \in [1, A]$, we have $y - 1 \le \log(y)\frac{A-1}{\log(A)}$. We choose $A = A(\xi) := \exp(\frac{|\xi|}{r})$ and $y = \frac{f_\varepsilon}{\varrho_\varepsilon M}$ so that $y \in [1, A]$ means exactly $0 \le \log(\frac{f_\varepsilon}{\varrho_\varepsilon M}) \le \frac{|\xi|}{r}$. Then, I_2 can be estimated as follows

$$I_2 \le \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \varrho_\varepsilon M \left|\frac{f_\varepsilon}{\varrho_\varepsilon M} - 1\right| \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \frac{r(\exp(\frac{|\xi|}{r}) - 1)}{|\xi|} B(\xi)\, \mathrm{d}\xi = \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}.$$

Therefore, for some constant C_M, defined through $M(\xi)$, we have

$$|J_\varepsilon^{(2)}(s,x)| \le$$
$$\le C \varrho_\varepsilon(s,x)^{1/2} \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2} \left(\int_{\mathbb{R}^d} r|\xi| \exp\left(\frac{-|\xi|^2}{C_M}\right)\left(\exp\left(\frac{|\xi|}{r}\right) - 1\right) \mathrm{d}\xi\right)^{1/2}.$$

It remains to treat the integral factor that we denote by I_3 and for r smaller than 1,

$$I_3 = \int_{\mathbb{R}^d} r|\xi| \exp\left(\frac{-|\xi|^2}{C_M}\right)\left(\exp\left(\frac{|\xi|}{r}\right) - 1\right) \mathrm{d}\xi \le \frac{C}{r^d}\exp\left(\frac{2C_M}{r^2}\right)$$

where C does not depend on r. This can be seen by splitting the integral in the zones $\{|\xi| \le \frac{2C_M}{r}\}$ and $\{|\xi| \ge \frac{2C_M}{r}\}$. Finally, we obtain

$$|J_\varepsilon| \le r\varepsilon \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi} + C\frac{1}{r^d}\exp\left(\frac{2C_M}{r^2}\right) \varrho_\varepsilon(s,x)^{1/2} \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2}.$$

□

From this lemma, we deduce the following L^1 bounds on J_ε

Proposition 4.3.4 (Estimate on J_ε in L^1)**.** *With the decomposition of Lemma 4.3.3, $J_\varepsilon(s,x) = J_\varepsilon^{(1)}(s,x) + J_\varepsilon^{(2)}(s,x)$, we have*

- $|J_\varepsilon^{(1)}(s,x)| \le \varepsilon \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}$,
- $|J_\varepsilon^{(2)}(s,x)| \le C \varrho_\varepsilon(s,x)^{1/2} \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2}$,
- $\|J_\varepsilon^{(2)}(s,\cdot) \log_+^{1/2} |J_\varepsilon^{(2)}(s,\cdot)|\|_{L^1_x} \le C \left[\|\varrho_\varepsilon(s,\cdot) \log_+ \varrho_\varepsilon(s,\cdot)\|_{L^1_x} + \|\mathcal{D}_\varepsilon(s,\cdot,\cdot)\|_{L^1_{x,\xi}}\right]$,
- $\|J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)\|_{L^1_{t,x}} \le C(\|\mathcal{D}_\varepsilon(s,\cdot,\cdot)\|_{L^1_{x,\xi}}, \|\varrho_\varepsilon(s,\cdot) \log_+ \varrho_\varepsilon(s,\cdot)\|_{L^1_{t,x}})$.

The first two estimates are similar to [119, Proposition 7.1] for the Vlasov-Poisson-Fokker-Planck system. Here, we have additionally included the last two controls and we give a different proof.

Proof. The first two estimates are a direct consequence of Lemma 4.3.3. The third estimate follows from the inequality, for $u \ge 1$, $v \ge 0$ and $uv \ge 1$,

$$(uv)^{1/2} \log^{1/2}(uv) \le u \log u + \sqrt{2} v.$$

The last result is given for the sake of completeness and its technical proof is postponed to Appendix 4.7. This concludes the proof of Proposition 4.3.4. □

With these estimates, we can now prove the main result of this section.

Proof of Theorem 4.1. Estimate (A) follows by mass conservation. The next bounds are deduced from the energy equality (4.3.4)-(4.3.5) which we write as

$$\int_{\mathbb{R}^{2d}} \left[2D \log(f_\varepsilon(t)) + |\xi|^2\right] f_\varepsilon(t) \, \mathrm{d}x \, \mathrm{d}\xi + 2D \int_0^t \|\mathcal{D}_\varepsilon(s,\cdot,\cdot)\|_{L^1_{x,\xi}} \, \mathrm{d}s \le \mathcal{G}(0), \quad (4.3.13)$$

where we ignore the nonnegative interaction term as it does not help in this computation. It is standard, see Appendix 4.6, to conclude from this inequality that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} \left[2D |\log(f_\varepsilon(t))| + \frac{1}{2}|\xi|^2\right] f_\varepsilon(t) \, \mathrm{d}x \, \mathrm{d}\xi + D \, \|\mathcal{D}_\varepsilon\|_{L^1_{t,x,\xi}} \\ \le \mathcal{G}(0) + C\left(\|\varrho_\varepsilon\|_{L^\infty_t L^1_x}, \|x f^0\|_{L^1_{x,\xi}}\right).\end{aligned} \quad (4.3.14)$$

The estimates (B) and (D) follow immediately. Then, estimate (E) follows from estimate (D) and the Csiszár-Kullback Inequality, see Lemma 4.6.1.

Estimate (C) is also very standard and we reproduce the proof from [159, Lemma 2.1]. We consider the convex function $\psi(\varrho) = \varrho \log(\varrho)$ and apply the Jensen inequality. We obtain

$$\begin{aligned}\varrho_\varepsilon \log(\varrho_\varepsilon) = \psi(\varrho_\varepsilon) = \psi\left(\int_{\mathbb{R}^d} \frac{f_\varepsilon}{M}\, M \,\mathrm{d}\xi\right) \leq \int_{\mathbb{R}^d} \psi\left(\frac{f_\varepsilon}{M}\right) M \,\mathrm{d}\xi = \\ = \int_{\mathbb{R}^d} \frac{f_\varepsilon}{M}\left[\log f_\varepsilon - \log M(\xi)\right] M \,\mathrm{d}\xi = \int_{\mathbb{R}^d} f_\varepsilon \left[\log f_\varepsilon + \frac{|\xi|^2}{2D}\right] \mathrm{d}\xi + C \int_{\mathbb{R}^d} f_\varepsilon \,\mathrm{d}\xi.\end{aligned}$$

The conclusion follows by taking the absolute values of both sides and integrating with respect to x.

Finally, estimate (F) is a direct consequence of Proposition 4.3.4, whereas (G) follows from (4.6.3). Concerning the weak compactness of $\{\varrho_\varepsilon\}$, it follows from estimates (C) and (G). Then, the weak local compactness of $\{J_\varepsilon\}$ is a direct consequence of Proposition 4.3.4 and the Dunford-Pettis theorem. Indeed, with the notations of Lemma 4.3.3, J_ε^1 converges strongly to 0 in $L^1((0,T)\times\mathbb{R}^d)$. For J_ε^2 we first have the weak local compactness in $L^1((0,T)\times\mathbb{R}^d)$ thanks to the third estimate of Proposition 4.3.4, bound (C) and the Dunford Pettis theorem. To prove the global weak compactness we only need to prove it for $J_\varepsilon^{(2)}$. We recall that, from Lemma 4.3.3, we have

$$|J_\varepsilon^{(2)}(s,x)| \leq C\varrho_\varepsilon(s,x)^{1/2} \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2}.$$

Therefore we can estimate with the Cauchy-Schwarz inequality

$$\|J_\varepsilon^{(2)}\,|x|^{1/2}\|_{L^1_{t,x}} \leq C\,\|\varrho_\varepsilon\,|x|\|_{L^1_{t,x}}^{1/2}\,\|\mathcal{D}_\varepsilon\|_{L^1_{t,x,\xi}}^{1/2}$$

which yields global weak compactness in $L^1((0,T)\times\mathbb{R}^d)$ with the Dunford-Pettis theorem. This ends the proof. □

4.4 The limit $\varepsilon \to 0$

We now perform the analysis allowing us to prove Theorem 4.1.1. We take $\alpha = \varepsilon$ where the parameter α defines the long range potential (4.1.5). Note, however, that different scaling between α and ε could possibly be considered.

Recalling the mass balance equation (4.1.10) and the ξ-moment equation (4.1.11), our aim is to take the limit $\varepsilon \to 0$ in these equations, and establish the relations

$$\partial_t \varrho(t,x) + \operatorname{div} J(t,x) = 0, \tag{4.4.1}$$

$$J(t,x) = -D\nabla \varrho(t,x) - \varrho \nabla \Phi(t,x), \qquad \Phi = -\delta \Delta[\omega^S * \omega^S * \varrho], \tag{4.4.2}$$

which are equivalent to (4.1.12).

A significant contribution comes from Theorem 4.1. The entropy bound for ϱ_ε, see (C), and the L^1 bound on J_ε, see Proposition 4.3.4, we immediately conclude that

• after extractions, ϱ_ε and $J_\varepsilon(t,x)$ admit weak limits in $L^1((0,T)\times\mathbb{R}^d)$, ϱ and J, see also Theorem 4.1,

• the equation (4.4.1) holds in the distributional sense.

The latter estimate on J_ε also tells us that $\varepsilon^2 \partial_t J_\varepsilon(t,x)$ converges to 0 in the distributional sense. Therefore, establishing the equation (4.4.2) from equation (4.1.11), is reduced to proving the two local weak limits in $L^1((0,T)\times\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \xi \otimes \xi \, f_\varepsilon(t,x,\xi)\, \mathrm{d}\xi \to D\varrho(t,x)\, \mathrm{I}, \qquad \varrho_\varepsilon \nabla \Phi_\varepsilon \to \varrho \nabla \Phi(t,x).$$

These follow directly from the following three lemmas

Lemma 4.4.1. *We have*

$$\int_{(0,T)\times\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \xi \otimes \xi (f_\varepsilon - \varrho_\varepsilon M)\, \mathrm{d}\xi \right| \mathrm{d}x\, \mathrm{d}t \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Lemma 4.4.2. *The sequence $\{\varrho_\varepsilon\}$ is compact in $L^p(0,T;L^1(\mathbb{R}^d))$ for all $1 \le p < \infty$.*

Lemma 4.4.3. *The potential $\Phi_\varepsilon(t,x)$ satisfies, uniformly in $\varepsilon \in (0,1)$,*

$$\|\Phi_\varepsilon\|_\infty \le C, \quad \|\nabla \Phi_\varepsilon\|_\infty \le C. \tag{4.4.3}$$

Moreover, we have for every $1 \le p < \infty$ the strong convergence in $L^p(0,T;L^\infty(\mathbb{R}^d))$,

$$\Phi_\varepsilon(t,x) \longrightarrow \Phi(t,x), \quad \nabla\Phi_\varepsilon(t,x) \longrightarrow \nabla\Phi(t,x), \quad \Phi(t,x) := -\delta\Delta[\omega^S * \omega^S * \varrho(t,x)]. \tag{4.4.4}$$

The end of the proof of Theorem 4.1.1 is thus to establish these results.

Proof of Lemma 4.4.3. Recalling the expressions of both long-range and short-range potentials and that $\alpha = \varepsilon$, we see that

$$\Phi_\varepsilon(t,x) = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \omega_\varepsilon^L(z)\left[\omega^S * \omega^S * \varrho_\varepsilon(t,x-z) - \omega^S * \omega^S * \varrho_\varepsilon(t,x)\right] \mathrm{d}z.$$

Let now set $y = \frac{z}{\varepsilon}$, so that from (4.1.6) we deduce that

$$\Phi_\varepsilon(t,x) = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \omega^L(y)\left[\omega^S * \omega^S * \varrho_\varepsilon(t,x-\varepsilon y) - \omega^S * \omega^S * \varrho_\varepsilon(t,x)\right] \mathrm{d}y.$$

Because the convolution terms are smooth (say $W^{3,\infty}$), we may use the Taylor expansion and obtain

$$\begin{aligned}\Phi_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}^d} &\nabla_x[\omega^S * \omega^S * \varrho_\varepsilon(t,x)]\cdot y\;\omega^L(y)\,\mathrm{d}y \\ &- \int_{\mathbb{R}^d} D_x^2[\omega^S * \omega^S * \varrho_\varepsilon(t,x)]y\cdot y\,\omega^L(y)\,\mathrm{d}y + O(\varepsilon)\end{aligned}$$

where the term $O(\varepsilon)$ converges to 0 in L^∞ since it is controlled by

$$C\varepsilon\int_{\mathbb{R}^d} |y|^3\omega^L(y)\|D_x^3\omega^S * \omega^S * \varrho_\varepsilon(t,\cdot)\|_\infty\,\mathrm{d}y,$$

and we recall the uniform bound (A). Moreover, recalling (4.1.6), we see that the first term in the right-hand side vanishes and the Hessian matrix reduces to the Laplacian, so that

$$\Phi_\varepsilon(t,x) = -\delta\Delta_x\left[\omega^S * \omega^S * \varrho_\varepsilon(t,x)\right] + O(\varepsilon) \tag{4.4.5}$$

from which we directly conclude from (A)

$$||\Phi_\varepsilon||_\infty \le C \quad \text{uniformly in } \varepsilon \in (0,1).$$

As far as $\nabla\Phi_\varepsilon$ is concerned, the properties of convolution with respect to derivatives gives

$$\nabla\Phi_\varepsilon(t,x) = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^d} \omega_\varepsilon^L(z)\left[\nabla\omega^S * \omega^S * \varrho_\varepsilon(t,x-z) - \nabla\omega^S * \omega^S * \varrho_\varepsilon(t,x)\right] \mathrm{d}z,$$

so that the $L^\infty((0,T)\times\mathbb{R}^d)$ bounded on $\nabla\Phi_\varepsilon$ follows from the previous argument assuming now that $\omega^S \in W^{4,\infty}$.

It remains to show that $\Phi_\varepsilon \to \Phi$ strongly in $L^p(0,T;L^\infty(\mathbb{R}^d))$, the convergence of $\nabla\Phi_\varepsilon$ uses the same arguments. The convergence follows from (4.4.5) since we have

$$\Phi_\varepsilon(t,x) - \Phi(t,x) = -\delta\left[\Delta\omega^S * \omega^S * (\varrho_\varepsilon - \varrho)(t,x)\right] + O(\varepsilon),$$

so that, thanks to the above control of the term $O(\varepsilon)$ and properties of the convolution,

$$\|\Phi_\varepsilon - \Phi\|_{L^p_t L^\infty_x} \le C\,\|\varrho_\varepsilon - \varrho\|_{L^p_t L^1_x} + C\varepsilon. \tag{4.4.6}$$

Using Lemma 4.4.2, we obtain the result. □

Proof of Lemma 4.4.2. This result is a consequence of the averaging lemma in kinetic theory. More precisely, we use Lemma 2.4.2. However, we cannot apply the averaging lemma directly on $\{f_\varepsilon\}$ because $\{f_\varepsilon\}$ is not bounded in $L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ and we follow the argument in [119] which follows idea of renormalized solutions [111]. We fix $\nu > 0$ and we consider the functions $\beta_\nu(f) = \frac{f}{1+\nu f}$ with derivative $\beta_\nu'(f) = \frac{1}{(1+\nu f)^2}$. Now we multiply (4.1.2) by $\beta_\nu'(f)$ and obtain

$$\varepsilon\partial_t\beta_\nu(f_\varepsilon) + \xi\cdot\nabla_x\beta_\nu(f_\varepsilon) = \frac{(\varrho_\varepsilon M - f)\beta_\nu'(f)}{\varepsilon} - \nabla_\xi\cdot(F_\varepsilon\beta_\nu(f_\varepsilon)).$$

We verify assumptions of Lemma 2.4.2. From (A) we see that $h^\varepsilon = \beta_\nu(f_\varepsilon)$ is bounded in both $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ and $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ so that it is bounded in $L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ by interpolation. The $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ bound on $h_0^\varepsilon = \frac{(\varrho_\varepsilon M - f)\beta_\nu'(f_\varepsilon)}{\varepsilon}$ is deduced from (E) and the $L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ bound on $\beta_\nu'(f_\varepsilon)$. Finally, since $\mathcal{F}_\varepsilon$ is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$ and $\beta_\nu(f_\varepsilon)$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ we see that $h_1^\varepsilon = -\mathcal{F}_\varepsilon\beta_\nu(f_\varepsilon)$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$.

The assumptions of Lemma 2.4.2 are satisfied and we obtain

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_\varepsilon)(t, x+y, \xi) - \beta_\nu(f_\varepsilon)(t, x, \xi))\, \psi(\xi)\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \to 0,$$

when $y \to 0$, uniformly in ε. As this is true for all $\nu > 0$, Lemma 2.4.3 implies

$$\left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x+y, \xi) - f_\varepsilon(t, x, \xi))\, \psi(\xi)\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \to 0, \tag{4.4.7}$$

when $y \to 0$, uniformly in ε.

The final step is to remove the weight ψ in the convergence (4.4.7) using uniform bound on $\{f_\varepsilon\, |\xi|^2\}$. To this end, consider a sequence of functions $\{\psi_n(\xi)\}_n$ in $\mathcal{D}(\mathbb{R}^d)$ such that $\psi_n(\xi) = 1$ for $|\xi| \le n$ and $\psi_n(\xi) = 0$ for $|\xi| \ge n+1$. Then,

$$\left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x, \xi)(1 - \psi_n(\xi))\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \le \left\| \int_{|\xi| \ge n} f_\varepsilon(t, x, \xi)\, \frac{|\xi|^2}{n^2}\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \le \frac{\|f_\varepsilon |\xi|^2\|_{L^1_{t,x,\xi}}}{n^2}$$

and similarly for the term with $f_\varepsilon(t, x+y, \xi)$. Hence, we may choose first n large enough and then for such n apply (4.4.7) to deduce

$$\|\varrho_\varepsilon(x+y) - \varrho_\varepsilon(x)\|_{L^1_{t,x}} = \left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x+y, \xi) - f_\varepsilon(t, x, \xi))\, \mathrm{d}\xi \right\|_{L^1_{t,x}} \to 0, \tag{4.4.8}$$

when $|y| \to 0$, uniformly in $\varepsilon > 0$. This yields compactness in space.

From Lemma 2.5.2 we know that $\{\varrho_\varepsilon\}$ is also compact in time (we use here estimate on $\{J_\varepsilon\}$ in $L^1((0,T) \times \mathbb{R}^d)$), and as a result

$$\begin{aligned}
\int_0^{T-h} \int_{\mathbb{R}^d} & |\varrho_\varepsilon(t+h, x+k) - \varrho_\varepsilon(t, x)|\, \mathrm{d}x\, \mathrm{d}t \\
&\le \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x+k) - \varrho_\varepsilon(t+h, x)|\, \mathrm{d}t\, \mathrm{d}x \\
&\quad + \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t, x)|\, \mathrm{d}t\, \mathrm{d}x \le \theta(h, k),
\end{aligned}$$

where $\theta(h, k) \to 0$ whenever $|h|, |k| \to 0$ uniformly in ε. This provides the equicontinuity of $\{\varrho_\varepsilon\}$ in $L^1_{t,x}$ which provides us with local compactness in x.
From (G) in Theorem 4.1 we know that

$$\sup_{0<\varepsilon<1} \int_{(0,T)\times\mathbb{R}^d} |x \varrho_\varepsilon(t, x)|\, \mathrm{d}t\, \mathrm{d}x < \infty,$$

and we obtain the strong convergence of the density in $L^1((0,T)\times\mathbb{R}^d)$ by Fréchet-Kolmogorov theorem, see also [248]. Using Estimate (A) we obtain by interpolation and [248, Theorem 1] the strong convergence in $L^p(0,T;L^1(\mathbb{R}^d))$ for every $1\le p<\infty$ and this concludes the proof of Lemma 4.4.2. $\square$

Proof of Lemma 4.4.1. We adapt the proof of Lemma 4.3.3. We write

$$\begin{aligned}
R_\varepsilon := \left|\int_{\mathbb{R}^d} \xi\otimes\xi(f_\varepsilon-\varrho_\varepsilon M)\,\mathrm{d}\xi\right| &\le \int_{\mathbb{R}^d} |\xi|^2|f_\varepsilon-\varrho_\varepsilon M|\,\mathrm{d}\xi\\
&\le \int_{\left\{\left|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\right|\ge\frac{|\xi|^2}{r}\right\}} |\xi|^2\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M}-1\right|\mathrm{d}\xi\\
&\quad+\int_{\left\{\left|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\right|\le\frac{|\xi|^2}{r}\right\}} |\xi|^2\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M}-1\right|\mathrm{d}\xi = I_1+I_2,
\end{aligned}$$

where r is chosen later. For the first term, we just write

$$I_1\le r\int_{\mathbb{R}^d}\log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right)\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M}-1\right|\mathrm{d}\xi\le r\varepsilon^2\left\|\mathcal{D}_\varepsilon\right\|_{L^1_\xi}.$$

The term I_2 is decomposed in two parts: where $f_\varepsilon\ge\varrho_\varepsilon M$ and $f_\varepsilon<\varrho_\varepsilon M$. The resulting integrals are called I_2^A and I_2^B. We only discuss I_2^A as I_2^B can be treated similarly as it was discussed in Lemma 4.3.3. We use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
I_2^A\le&\left(\int_{\left\{0\le\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\le\frac{|\xi|^2}{r}\right\}}|\xi|^4\frac{\varrho_\varepsilon M}{B(\xi)}\,\mathrm{d}\xi\right)^{1/2}\cdot\\
&\cdot\left(\int_{\left\{0\le\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\le\frac{|\xi|^2}{r}\right\}}\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M}-1\right|^2B(\xi)\,\mathrm{d}\xi\right)^{1/2} =: I_2^{A,1}\cdot I_2^{A,2},
\end{aligned}$$

where, as before, $B(\xi)=\frac{\log(A)}{A-1}=\frac{|\xi|^2}{r(\exp(\frac{|\xi|^2}{r})-1)}$, with $A=A(\xi):=\exp(\frac{|\xi|^2}{r})$. As in the proof of Lemma 4.3.3, we have the inequality $\log(y)\ge(y-1)\frac{\log(A)}{A-1}$ which yields with $y=\frac{f_\varepsilon}{\varrho_\varepsilon M}$

$$I_2^{A,2}\le\left(\int_{\left\{0\le\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})\le\frac{|\xi|^2}{r}\right\}}\varrho_\varepsilon M\left|\frac{f_\varepsilon}{\varrho_\varepsilon M}-1\right|\log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right)\mathrm{d}\xi\right)^{1/2}\le\varepsilon\left\|D_\varepsilon\right\|_{L^1_\xi}^{1/2}.$$

Now we choose r such that $M(\xi)\exp(\frac{|\xi|^2}{r})=C\exp(-a|\xi|^2)$ for some $a>0$. Then, we have

$$\int_{\mathbb{R}^d}|\xi|^4\frac{M(\xi)}{B(\xi)}\,\mathrm{d}\xi\le r\int_{\mathbb{R}^d}|\xi|^2\,M(\xi)\,\exp\left(\frac{|\xi|^2}{r}\right)\mathrm{d}\xi\le Cr\int_{\mathbb{R}^d}|\xi|^2\exp(-a|\xi|^2)\,\mathrm{d}\xi=:C^2.$$

It follows that $I_2^{A,1} \leq C\varrho_\varepsilon^{1/2}$ so that we obtain

$$R_\varepsilon \leq r\varepsilon^2 \|\mathcal{D}_\varepsilon\|_{L^1_\xi} + C\varepsilon\varrho_\varepsilon^{1/2}\|D_\varepsilon\|_{L^1_\xi}^{1/2}$$

and, using the Cauchy-Schwarz inequality, the proof of Lemma 4.4.1 is concluded.

□

This also concludes the proof of Theorem 4.1.1.

4.5 Conclusion and open problems

We proved that macroscopic densities $\{\varrho_\varepsilon\}$ formed from solutions of the Vlasov-Cahn-Hilliard equation (4.1.2) converge to the solutions of non-local degenerate Cahn-Hilliard (4.1.12). It is an open question whether one can obtain a local version of this equation by sending short-range interaction kernel ω^S to the Dirac mass δ_0. One expects in the limit the local degenerate Cahn-Hilliard equation:

$$\partial_t \varrho - D\Delta\varrho - \operatorname{div}(\varrho\nabla\Phi) = 0 \tag{4.5.1}$$

where $\Phi = -\delta\Delta\varrho$. One can try to perform this limit either on equation (4.1.12) or directly on (4.1.2), by sending $\omega_\alpha^L \overset{*}{\rightharpoonup} \delta_0$, $\omega^S \overset{*}{\rightharpoonup} \delta_0$ together, see Figure 4.1. Passing from (4.1.2) to (4.5.1), the main difficulty is the lack of entropy which gives integrability of second-order derivatives in the nondegenerate Cahn-Hilliard. On the other hand, when one tries to pass to the limit from (4.1.12) to (4.5.1), the entropy (that is, multiplying equation by $\log\varrho$) is available but it yields estimates only on

$$\Delta(\varrho * \omega^S) \text{ in } L^2((0,T)\times\mathbb{R}^d), \qquad \nabla\sqrt{\varrho} \text{ in } L^2((0,T)\times\mathbb{R}^d).$$

The minimal required information allowing to pass to the limit seems to be the strong compactness of $\{\nabla\varrho\}$ in $L^2((0,T)\times\mathbb{R}^d)$.

Moreover, it is also open to prove whether we can add the "usual" double-well Cahn-Hilliard interaction potential in the system. In fact, as far as this modification is concerned, it is not even clear if there exists a solution to the Vlasov-Cahn-Hilliard equation when the potential Φ is a function of the density ϱ.

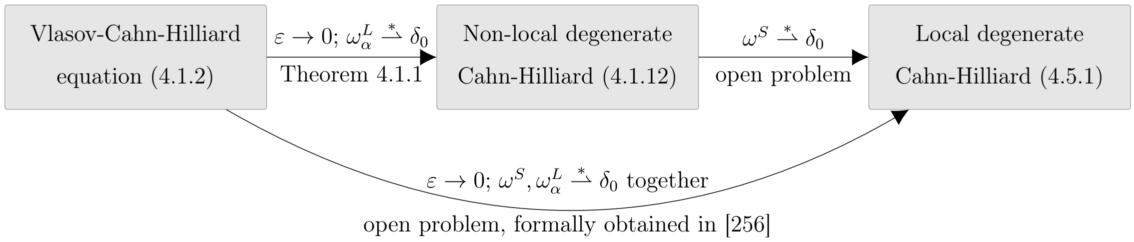

Figure 4.1: Relation between three types of the degenerate Cahn-Hilliard equations.

4.6 Appendix A: Csiszar-Kullback inequality and lower bound on the energy

We recall two lemmas which have been used in the proof of Theorem 4.1. The first one is a variant of the Csiszar-Kullback inequality.

Lemma 4.6.1. *Let* $f, g \geq 0$ *with* $\|f\|_1 = \|g\|_1$. *Then,*

$$\|f - g\|_1^2 \leq \|f\|_1 \int_{\mathbb{R}^d} (f - g)\,(\log f - \log g).$$

The second lemma is used to control $f \log_-(f)$ from $f \log f$, which immediately establishes the Inequality (4.6.1).

Lemma 4.6.2. *Let* $\log_-(f) := \max\{-\log(f), 0\}$. *Then*

$$\begin{aligned} \int_{\mathbb{R}^{2d}} 2D \log_-(f_\varepsilon(t)) f_\varepsilon(t) \,\mathrm{d}x \,\mathrm{d}\xi \leq C\left(\|\varrho_\varepsilon\|_{L^1_{t,x}}, \|x f^0\|_{L^1_{x,\xi}}\right) + \\ + \int_{\mathbb{R}^{2d}} \frac{|\xi|^2}{4} f_\varepsilon(t) \,\mathrm{d}\xi \,\mathrm{d}x + D \int_0^t ||\mathcal{D}_\varepsilon(s, \cdot, \cdot)||_{L^1_{x,\xi}} \,\mathrm{d}s. \end{aligned} \tag{4.6.1}$$

Proof of Lemma 4.6.1. Let $\|f\|_1 = \|g\|_1 = 1$. Usual the Csiszar-Kullback inequality gives us

$$\|f - g\|_1^2 \leq 2 \int_{\mathbb{R}^d} f \log\left(\frac{f}{g}\right).$$

By symmetry of the (LHS) we have

$$2\|f - g\|_1^2 \leq 2 \int_{\mathbb{R}^d} f \log\left(\frac{f}{g}\right) + 2 \int_{\mathbb{R}^d} g \log\left(\frac{g}{f}\right) = 2 \int_{\mathbb{R}^d} (f - g)(\log f - \log(g)).$$

The general case follows by rescaling. $\square$

Proof of Lemma 4.6.2. We proceed as in [119, Proposition 5.1].

We divide the domain in two parts:

$$\Omega_1 := \left\{ f_\varepsilon > \exp\left(-\frac{|x|}{4} - \frac{|\xi|^2}{8D} \right) \right\}, \qquad \Omega_2 := \left\{ f_\varepsilon \leq \exp\left(-\frac{|x|}{4} - \frac{|\xi|^2}{8D} \right) \right\},$$

On Ω_1, $\log_-(f_\varepsilon)$ is bounded so that we have

$$f_\varepsilon \log_-(f_\varepsilon) \leq \left(\frac{|x|}{4} + \frac{|\xi|^2}{8D} \right) f_\varepsilon,$$

while on Ω_2, $f_\varepsilon \leq 1$ so that $\sqrt{f_\varepsilon}\log_-(f_\varepsilon)$ is bounded by some constant C. Hence,

$$f_\varepsilon \log_-(f_\varepsilon) \leq C\sqrt{f_\varepsilon} \leq C\, \exp\left(-\frac{|x|}{8} - \frac{|\xi|^2}{16D} \right).$$

It follows that

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \log_-(f_\varepsilon(t)) f_\varepsilon(t) \,\mathrm{d}x \,\mathrm{d}\xi \leq \\ &\qquad \leq \int_{\mathbb{R}^{2d}} C \exp\left(-\frac{|x|}{8} - \frac{|\xi|^2}{16D} \right) + \left(\frac{|x|}{4} + \frac{|\xi|^2}{8D} \right) f_\varepsilon(t) \,\mathrm{d}\xi \,\mathrm{d}x. \end{aligned} \tag{4.6.2}$$

Now, we only need to bound the term $\int_{\mathbb{R}^{2d}} \frac{|x|}{4} f_\varepsilon(t) \,\mathrm{d}\xi \,\mathrm{d}x$. For this, we first observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x| f_\varepsilon(t) \,\mathrm{d}\xi = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} f_\varepsilon(t) \frac{x}{|x|} \xi \,\mathrm{d}\xi = \frac{x}{|x|} J_\varepsilon \leq 2 \|\mathcal{D}_\varepsilon(s,\cdot,\cdot)\|_{L^1_\xi} + C \varrho_\varepsilon,$$

where we have used Proposition 4.3.4 and Young's inequality (with $\varepsilon \leq 1$). Therefore, for all $t \geq 0$

$$\int_{\mathbb{R}^{2d}} |x| f_\varepsilon(t) \,\mathrm{d}\xi \,\mathrm{d}x \leq \int_{\mathbb{R}^{2d}} |x| f^0 \,\mathrm{d}\xi \,\mathrm{d}x + C \,\|\varrho_\varepsilon\|_{L^1_{t,x}} + 2 \int_0^t \|\mathcal{D}_\varepsilon(s,\cdot,\cdot)\|_{L^1_{x,\xi}} \,\mathrm{d}s. \tag{4.6.3}$$

Finally, equation (4.6.2) simplifies to give the desired result (4.6.1).

□

4.7 Appendix B: Estimate on $J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)$

From Lemma 4.3.3 we recall that for $0 < r \leq 1$

$$|J_\varepsilon(s,x)| \leq r\varepsilon \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi} + C \frac{1}{r^d} \exp\left(\frac{2C_M}{r^2} \right) \varrho_\varepsilon(s,x)^{1/2} \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}^{1/2}.$$

We can make further simplifications: applying a simple rescaling of r, ignoring ε, estimating $\frac{1}{r^d} \leq \exp(\frac{1}{r^d})$ and changing $r = \frac{1}{\alpha}$ we can assume

$$|J_\varepsilon(s,x)| \leq \frac{C}{\alpha}\|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi} + C\exp\left(\alpha^2\right)\varrho_\varepsilon(s,x)^{1/2}\|\mathcal{D}_\varepsilon(s,x,\cdot)\|^{1/2}_{L^1_\xi}. \tag{4.7.1}$$

To choose the best α in the inequality above, we let $u = \varrho_\varepsilon$, $v = \|\mathcal{D}_\varepsilon(s,x,\cdot)\|_{L^1_\xi}$ so that we can estimate

$$|J_\varepsilon(s,x)| \leq C\,v \min_{1<\alpha<\infty}\left[\frac{1}{\alpha} + \exp\left(\alpha^2\right)\sqrt{\frac{u}{v}}\right]. \tag{4.7.2}$$

Lemma 4.7.1. *Let $v \geq e$, $u \geq 0$, $v > e^2\,u$. The minimum in (4.7.2) is attained for $\alpha > 1$ which is the unique solution of*

$$2\,\alpha^3\,\exp(\alpha^2) = \sqrt{\frac{v}{u}}.$$

For such $\alpha > 1$ we have

$$v\left[\frac{1}{\alpha} + \exp\left(\alpha^2\right)\sqrt{\frac{u}{v}}\right] = v\left[\frac{1}{\alpha} + \frac{1}{2\alpha^3}\right] \leq \frac{2v}{\alpha}.$$

Then,

$$\frac{2v}{\alpha} \leq \begin{cases} \frac{2\sqrt{2}\,v}{\log_+^{1/2}\log_+^{1/2} v} & \text{if } v \geq u\log_+^{1/2}\, v, \\ 2\,u\log_+^{1/2} v & \text{if } v < u\log_+^{1/2} v. \end{cases} \tag{4.7.3}$$

Proof. The first statement is a consequence of simple calculus and we only have to prove that the minimum is attained for $\alpha > 1$. This follows from

$$\sqrt{\frac{v}{u}} = 2\alpha^3\exp(\alpha^2) \leq \exp(2\alpha^2) \implies \frac{1}{2}\log\left(\frac{v}{u}\right) \leq \alpha^2. \tag{4.7.4}$$

As $v > e^2\,u$, we deduce $\alpha > 1$.

We proceed to the estimates on $\frac{v}{2\alpha}$. Suppose that $v \geq u\log_+^{1/2} v$. Then, we have

$$\log v \geq \log u + \log\log_+^{1/2} v \implies \log_+^{1/2}\left(\frac{v}{u}\right) \geq \log_+^{1/2}\log_+^{1/2} v$$

(we use here $\frac{v}{u} > e^2$ and $v > e$ to write $\log_+$ instead of log). In view of (4.7.4), this gives lower bound on α which implies

$$\frac{2v}{\alpha} \leq \frac{2\sqrt{2}v}{\log_+^{1/2}\log_+^{1/2} v}.$$

We are left with the case $v < u\log_+^{1/2} v$. In this case we estimate directly using $\alpha > 1$:

$$\frac{2v}{\alpha} \leq 2v \leq 2\,u\log_+^{1/2} v.$$

$\square$

We proceed to estimating $J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e)$ in $L^1_{t,x}$. Let us observe that we can always restrict the set of integration to the points (t,x) where $\|\mathcal{D}_\varepsilon\|_{L^1_\xi}$ is arbitrarily large. Indeed, given $M \geq e$, we estimate

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e) \leq &\int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e)\mathbb{1}_{J_\varepsilon \leq M} \\ &+ \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e)\,\mathbb{1}_{\|\mathcal{D}_\varepsilon\|_{L^1_\xi} \leq e^2 \varrho_\varepsilon} \\ &+ \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2}\log^{1/2}\max(J_\varepsilon, e)\,\mathbb{1}_{\|\mathcal{D}_\varepsilon\|_{L^1_\xi} > e^2 \varrho_\varepsilon}\,\mathbb{1}_{J_\varepsilon > M}.\end{aligned}$$

The first integral is bounded by $\|J_\varepsilon\|_{L^1_{t,x}} \log^{1/2}\log^{1/2} M$. For the second integral, we note that (4.7.1) implies that $J_\varepsilon \leq C\,\varrho_\varepsilon$ so this integral is finite because we can use Young's inequality and $\log x \leq x$ to get

$$\varrho_\varepsilon \log^{1/2}\log^{1/2}\max(\varrho_\varepsilon, e) \leq \varrho_\varepsilon + \frac{1}{2}\varrho_\varepsilon \log\max(\varrho_\varepsilon, e).$$

In the third integral, by estimate (4.7.1) with $\alpha = 2$, we have $\|\mathcal{D}_\varepsilon\|_{L^1_\xi} \geq \frac{M}{C}$ for some constant C. It follows that $\|\mathcal{D}_\varepsilon\|_{L^1_\xi}$ can be assumed to be arbitrarily large by taking sufficiently large M. This allows us to apply Lemma 4.7.1.

Splitting the domain of integration for two subsets as in Lemma 4.7.1, it is sufficient to prove that the following functions

$$\begin{aligned}P^1_\varepsilon &:= \frac{\|\mathcal{D}_\varepsilon\|_{L^1_\xi}}{\log_+^{1/2}\log_+^{1/2}\|\mathcal{D}_\varepsilon\|_{L^1_\xi}} \log_+^{1/2}\log_+^{1/2}\left(\frac{\|\mathcal{D}_\varepsilon\|_{L^1_\xi}}{\log_+^{1/2}\log_+^{1/2}\|\mathcal{D}_\varepsilon\|_{L^1_\xi}}\right), \\ P^2_\varepsilon &:= \varrho_\varepsilon\, \log_+^{1/2}\|\mathcal{D}_\varepsilon\|_{L^1_\xi}\, \log_+^{1/2}\log_+^{1/2}\left(\varrho_\varepsilon\, \log_+^{1/2}\|\mathcal{D}_\varepsilon\|_{L^1_\xi}\right).\end{aligned}$$

are bounded in $L^1_{t,x}$ (here, we use that $\log_+^{1/2}\log_+^{1/2} v = \log^{1/2}\log^{1/2}\max(v, e)$).

For P^1_ε (this is the limiting case!), we restrict to the values of $\|\mathcal{D}_\varepsilon\|_{L^1_\xi}$ so large that

$$\log_+^{1/2}\log_+^{1/2}\|\mathcal{D}_\varepsilon\|_{L^1_\xi} > 1.$$

Then,

$$\log_+^{1/2} \log_+^{1/2} \left(\frac{\|\mathcal{D}_\varepsilon\|_{L^1_\xi}}{\log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi}} \right) \leq \log_+^{1/2} \log_+^{1/2} \left(\|\mathcal{D}_\varepsilon\|_{L^1_\xi} \right)$$

so that $P_\varepsilon^1 \leq \|\mathcal{D}_\varepsilon\|_{L^1_\xi}$.

For P_ε^2, we apply $\log x \leq x$, $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and $2\,x\,y \leq x^2 + y^2$ to get

$$\begin{aligned} P_\varepsilon^2 \leq \varrho_\varepsilon \, \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi} \, \log_+^{1/2} \left(\varrho_\varepsilon \, \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi} \right) \leq& \\ \leq \varrho_\varepsilon \, \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi} \, \log_+^{1/2} \varrho_\varepsilon + \varrho_\varepsilon \, \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi} \, \log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_\xi}& \\ \leq \varrho_\varepsilon \log_+ \varrho_\varepsilon + \varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L^1_\xi} + \varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L^1_\xi}& \end{aligned}$$

so it is sufficient to prove that $\varrho_\varepsilon \, \log_+ \|\mathcal{D}_\varepsilon\|_{L^1_\xi}$ is bounded in $L^1_{t,x}$. This follows from Fenchel-Young's inequality

$$\varrho_\varepsilon \, \log_+ \|\mathcal{D}_\varepsilon\|_{L^1_\xi} \leq \varrho_\varepsilon \log \varrho_\varepsilon + \varrho_\varepsilon + \|\mathcal{D}_\varepsilon\|_{L^1_\xi}.$$

Chapter 5

Degenerate Cahn-Hilliard: from nonlocal to local

The results in this chapter have been submitted for publication as the following preprints:

- C. Elbar, J. Skrzeczkowski. *Degenerate Cahn-Hilliard equation: From nonlocal to local.* Available at arXiv:2208.08955, cited as [123].
- J.A. Carrillo, C. Elbar, J. Skrzeczkowski. *Degenerate Cahn-Hilliard systems: from nonlocal to local.* In preparation, cited as [69]

The first paper [123] is concerned with a single equation as (5.1.3)–(5.1.4). The second paper [69] extends the result to the system (5.5.1a)–(5.5.1b) arising in biomathematics.

5.1 Motivation and the main result

Several recent papers [90, 91, 92, 215] have addressed the problem of deriving rigorously the Cahn-Hilliard equation from the nonlocal equation, also called aggregation equation [67]. In these works, only the case of non-degenerate mobilities is treated, which avoids the delicate question of defining the limit of low-order products that one encounters for nonlocal degenerate mobility that we present now. The degenerate model is written

$$\partial_t u = \operatorname{div}(u\nabla\mu), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d,$$
$$\mu = B[u] + F'(u), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d,$$

equipped with an initial datum $u^0 \geq 0$. Here, $\mathbb{T}^d$ is the d-dimensional flat torus, B is the nonlocal operator $B = B_\varepsilon$ defined with

$$B_\varepsilon[u_\varepsilon](x) = \frac{1}{\varepsilon^2}(u_\varepsilon(x) - \omega_\varepsilon * u_\varepsilon(x)) = \frac{1}{\varepsilon^2}\int_{\mathbb{T}^d} \omega_\varepsilon(y)(u_\varepsilon(x) - u_\varepsilon(x-y))\,\mathrm{d}y \tag{5.1.1}$$

for ε small enough and ω_ε is the usual radial mollification kernel $\omega_\varepsilon(x) = \frac{1}{\varepsilon^d}\omega(\frac{x}{\varepsilon})$ with ω compactly supported in the unit ball of $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \omega(y)\,\mathrm{d}y = 1, \qquad \int_{\mathbb{R}^d} y\,\omega(y)\,\mathrm{d}y = 0, \qquad \int_{\mathbb{R}^d} y_i y_j \omega\,\mathrm{d}y = \delta_{i,j}\frac{2D}{d}, \tag{5.1.2}$$

for some constant $D > 0$. Our target is to prove that as $\varepsilon \to 0$, the constructed solutions of

$$\partial_t u_\varepsilon = \operatorname{div}(u_\varepsilon\nabla\mu_\varepsilon), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d, \tag{5.1.3}$$
$$\mu_\varepsilon = B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d, \tag{5.1.4}$$

tend to the weak solution of the degenerate Cahn-Hilliard equation

$$\partial_t u = \operatorname{div}(u\nabla\mu), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d, \tag{5.1.5}$$
$$\mu = -D\Delta u + F'(u), \quad \text{in} \quad (0,+\infty)\times\mathbb{T}^d. \tag{5.1.6}$$

Our motivation for this work is fourfold.

- Firstly, the interest for the nonlocal Cahn-Hilliard equation is an old problem that can be traced back to Giacomin and Lebowitz [151, 152]. These seminal works establish the derivation of the degenerate nonlocal Cahn-Hilliard equation departing from interacting particle systems. However, they left open the question of deriving the local degenerate Cahn-Hilliard equation from the nonlocal one. This is the challenge we overcome here.

- Secondly, a revival of interest for this problem appeared in the last years with several papers [90, 91, 92, 215] deriving the local from the nonlocal Cahn-Hilliard equation in the non-degenerate case. Here, we study the Cahn-Hilliard equation with degenerate mobility which is mathematically more difficult and better motivated by physics. Indeed, the degenerate version is obtained as a limit of Vlasov equation (Chapter 4), high-friction limit (Chapter 6) and as an aforementioned limit of stochastic particle system [151, 152].

- Third, the nonlocal Cahn-Hilliard equation can be seen as a porous medium equation with a smooth advection term that is well understood, conversely to the local degenerate Cahn-Hilliard equation.

- Finally, the nonlocal Cahn-Hilliard equation (5.1.3)–(5.1.4) is in fact an example of an aggregation-diffusion equation with a nonlocal term corresponding to the aggregation effect [67]. Thus, in this paper, we show that if the nonlocal effect is appropriately scaled, one approaches Cahn-Hilliard equation. This limit was formally stated for instance in [32, 99, 133] and our work provides a rigorous mathematical argument for this approximation.

To formulate the main result, we first make the following assumptions on the potential F.

Assumption 5.1.1 (potential F)**.** For the interaction potential we assume that there exists $k \geq 2$ and a decomposition $F = F_1 + F_2$ such that

(A) F_1, F_2 are of class C^2,

(B) $F_1 = 0$ or F_1 is a convex function which has k-growth in the sense that for some nonnegative constants C_1, ..., C_8 we have

$$C_1|u|^k - C_2 \leq F_1(u) \leq C_3|u|^k + C_4.$$

$$C_5|u|^{(k-2)} - C_6 \leq F_1''(u) \leq C_7|u|^{(k-2)} + C_8,$$

(C) F_2 has bounded second derivative *i.e.* $\|F_2''\|_\infty < \infty$ and $F_2(u) \geq -C_9 - C_{10}\, u^2$ where C_{10} is sufficiently small: more precisely $4\, C_{10} < C_p$ with C_p being the constant in Lemma 5.8.1.

Example 5.1.2. The following potentials satisfy Assumption 5.1.1.

(1) power-type potential $F(u) = |u|^\gamma$, $\gamma > 2$ used in the context of tumour growth models [89, 118, 122, 229],

(2) double-well potential $F(u) = u^2\,(u-1)^2$ which is an approximation of logarithmic double-well potential often used in Cahn-Hilliard equation, see [218, Chapter 1],

(3) any $F \in C^2$ such that for some interval $I \subset \mathbb{R}$ we have $F''(u) > a > 0$ for $u \in \mathbb{R} \setminus I$ and

$$\begin{aligned} C\,|u|^k - C \leq F(u) \leq C\,|u|^k + C \qquad &\text{for all } u \in \mathbb{R} \setminus I, \\ C\,|u|^{k-2} - C \leq F''(u) \leq C\,|u|^{k-2} + C \qquad &\text{for all } u \in \mathbb{R} \setminus I, \end{aligned}$$

see Lemma 5.7.3 for details.

Note that (3) is a more general version of (2).

Notation 5.1.3 (exponents s and k)**.** In what follows we write

$$k = \begin{cases} 2 & \text{if } F_1 = 0, \\ k & \text{if } F_1 \neq 0. \end{cases}$$

We also define $s = \frac{2k}{k-1}$ and s' its conjugate exponent.

Now, we define weak solutions of the nonlocal and local degenerate Cahn-Hilliard equation.

Definition 5.1.4. *We say that* u^ε *is a weak solution of* (5.1.3)-(5.1.4) *if*

$$\begin{aligned} u^\varepsilon &\in L^\infty(0,T; L^k(\mathbb{T}^d)), \qquad & \partial_t u^\varepsilon &\in L^2(0,T; W^{-1,s'}(\mathbb{T}^d)) \\ \nabla u^\varepsilon &\in L^2((0,T) \times \mathbb{T}^d), \qquad & \sqrt{F_1''(u^\varepsilon)}\nabla u^\varepsilon &\in L^2((0,T) \times \mathbb{T}^d), \end{aligned}$$

$u(0,x) = u_0(x)$ *a.e. in* $\mathbb{T}^d$ *and for all* $\varphi \in L^2(0,T; W^{1,\infty}(\mathbb{T}^d))$

$$\begin{aligned} \int_0^T \langle \partial_t u^\varepsilon, \varphi \rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))} = &- \int_0^T \int_{\mathbb{T}^d} u^\varepsilon \nabla B_\varepsilon[u_\varepsilon] \cdot \nabla \varphi \\ &- \int_0^T \int_{\mathbb{T}^d} u^\varepsilon F''(u^\varepsilon) \nabla u^\varepsilon \cdot \nabla \varphi. \end{aligned} \tag{5.1.7}$$

Definition 5.1.5. *We say that u is a weak solution of* (5.1.5)-(5.1.6) *if*

$$u \in L^\infty(0,T;L^k(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d)), \quad \partial_t u \in L^2(0,T;W^{-1,s'}(\mathbb{T}^d)),$$
$$\sqrt{F_1''(u)}\nabla u \in L^2((0,T)\times\mathbb{T}^d),$$

$u(0,x) = u_0(x)$ *a.e. in* $\mathbb{T}^d$ *and if for all* $\varphi \in L^2(0,T;W^{2,\infty}(\mathbb{T}^d))$ *we have*

$$\int_0^T \langle \partial_t u, \varphi\rangle_{(W^{-1,s'}(\mathbb{T}^d),W^{1,s}(\mathbb{T}^d))} = -D\int_0^T\int_{\mathbb{T}^d} \Delta u\,\nabla u\cdot\nabla\varphi - D\int_0^T\int_{\mathbb{T}^d} u\,\Delta u\,\Delta\varphi$$
$$-\int_0^T\int_{\mathbb{T}^d} u\,F''(u)\,\nabla u\cdot\nabla\varphi.$$

Remark 5.1.6 (initial condition)**.** In Definitions 5.1.4 and 5.1.5 we can evaluate pointwise value $u(0,x)$ because by [239, Lemma 7.1], we know $u \in C(0,T;W^{-1,s'}(\mathbb{T}^d))$.

With these assumptions we can construct solutions to (5.1.3)-(5.1.4).

Theorem 5.1.7 (Existence of solutions for the nonlocal system)**.** *Let ε_0 be given by*

$$\varepsilon_0 := \min\left(\varepsilon_0^A, \varepsilon_0^B\left(\frac{1}{1+\|F_2''\|_\infty}\right)\right) \tag{5.1.8}$$

where ε_0^A and ε_0^B are given in Lemma 5.8.1 and 5.8.3 respectively. Let $\varepsilon < \varepsilon_0$. Let $u^0 \geq 0$ be an initial datum with finite energy and entropy $E_\varepsilon(u^0), \Phi(u^0) < \infty$ defined in (5.2.1)-(5.2.2)*. There exists a global weak solution u^ε of* (5.1.3)-(5.1.4) *in the sense defined by Definition 5.1.4. It satisfies the dissipation of energy and entropy* (5.2.3)-(5.2.4) *with $u = u^\varepsilon$, $\mu = \mu_\varepsilon$. Moreover, $u^\varepsilon \geq 0$.*

Our main result reads as follows.

Theorem 5.1.8 (Convergence of nonlocal to local Cahn-Hilliard equation on the torus)**.** *Let $u^0 \geq 0$ be an initial datum with finite energy and entropy $E(u^0), \Phi(u^0) < \infty$ defined in* (5.2.5) *and* (5.2.2)*. Let $\{u_\varepsilon\}$ be a sequence of solutions of the degenerate nonlocal Cahn-Hilliard equation* (5.1.3)-(5.1.4) *from Theorem 5.1.7. Then, up to a subsequence,*

$$u_\varepsilon \to u \text{ in } L^2(0,T;H^1(\mathbb{T}^d))$$

where u is a weak solution of the degenerate Cahn-Hilliard equation (5.1.5)-(5.1.6) *as in Definition 5.1.5.*

Remark 5.1.9. Note that by Lemma 5.8.2 condition $E(u^0) < \infty$ implies that $E_\varepsilon(u^0) < \infty$.

5.2 Important components of the proof

There are three main ingredients of the proof.

- Compactness for the system 5.1.3–5.1.4 is obtained from the energy E_ε and entropy Φ

$$E_\varepsilon[u] := \int_{\mathbb{T}^d} F(u)\,\mathrm{d}x + \frac{1}{4\varepsilon^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\varepsilon(y)|u(x)-u(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y, \tag{5.2.1}$$

$$\Phi[u] := \int_{\mathbb{T}^d} u(\log(u)-1)+1\,\mathrm{d}x. \tag{5.2.2}$$

Their dissipation is formally controlled by the identities

$$E_\varepsilon[u](t) + \int_0^t \int_{\mathbb{T}^d} u\,|\nabla \mu_\varepsilon|^2 \le E_\varepsilon[u^0], \tag{5.2.3}$$

$$\Phi[u](t) + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\varepsilon(y)\,|\nabla u(x) - \nabla u(x-y)|^2 + \int_{\mathbb{T}^d} F''(u)|\nabla u|^2 \le \Phi[u^0]. \tag{5.2.4}$$

According to the result of Bourgain-Brézis-Mironescu [41] which was improved later by Ponce [233], uniform bounds from (5.2.3), (5.2.4) together with Lions-Aubin lemma, yields strong convergence of $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ to $u, \nabla u$ in the space $L^2((0,T)\times\mathbb{T}^d)$. We note that in the limit $\varepsilon\to 0$, the energy $E_\varepsilon[u_\varepsilon]$ satisfy (see [41, Theorem 4] and [233, Theorem 1.2])

$$E[u] = \int_{\mathbb{T}^d} F(u)\,\mathrm{d}x + \frac{C(d)}{2}\int_{\mathbb{T}^d} |\nabla u(x)|^2\,\mathrm{d}x \le \liminf_{\varepsilon\to 0} E_\varepsilon[u_\varepsilon] \tag{5.2.5}$$

for some constant $C(d)$ depending only on the dimension d. Similarly, for the nonlocal term in the dissipation of the entropy we have

$$C(d)\sum_{i,j=1}^d \int_0^t\int_{\mathbb{T}^d} |\partial_{x_i}\partial_{x_j} u|^2 \le \liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)\,|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2$$

so in the limit $\varepsilon\to 0$ we gain one more derivative. We also point out that one can prove rigorously that (5.1.5)–(5.1.6) is a gradient flow of (5.2.5) [196, 214].

- In passing to the limit, we exploit the appropriate definition of weak solutions to (5.1.5)–(5.1.6). Indeed, first we prove convergence to the formulation

$$\int_0^T \langle \partial_t u, \varphi\rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))} = D\int_0^T\int_{\mathbb{T}^d} (\nabla u\otimes\nabla u) : D^2\varphi + $$
$$+ \frac{D}{2}\int_0^T\int_{\mathbb{T}^d} |\nabla u|^2\Delta\varphi + D\int_0^T\int_{\mathbb{T}^d} u\nabla u\cdot\nabla\Delta\varphi - \int_0^T\int_{\mathbb{T}^d} uF''(u)\nabla u\cdot\nabla\varphi.$$

Formally, it is obtained by integrating by parts twice using the formula

$$\nabla u \Delta u = \operatorname{div}(\nabla u \otimes \nabla u) - \frac{1}{2}\nabla |\nabla u|^2. \tag{5.2.6}$$

Its main advantage is that it exploits at most first-order derivatives so that we do not need any estimates on the second-order derivatives. This is important as they are not available for nonlocal degenerate Cahn-Hilliard. More precisely, the main difficulty is non-degeneracy of (5.1.3)–(5.1.4), that is we loose estimates on $\nabla \mu_\varepsilon$ whenever u_ε is approaching the zone $\{u_\varepsilon = 0\}$. For the non-degenerate equation studied in [90, 91, 92, 215],

$$\partial_t u_\varepsilon = \operatorname{div} \nabla \mu_\varepsilon, \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{5.2.7}$$

$$\mu_\varepsilon = B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{5.2.8}$$

one obtains immediately an estimate on $\nabla \mu_\varepsilon$ (by multiplying by μ_ε) and then one can identify its limit. Nevertheless, we point out that in [90, 91, 92, 215] the difficulty is rather the regularity of the potential and the kernel which we do not address in our work, assuming that F and ω are sufficiently smooth.

- For the nonlocal Laplacian operator given by B_ε defined in (5.1.1), we find an operator S_ε given in (5.4.4) which resembles gradient operator. It satisfies the integration by parts formula (S3) in Lemma 5.4.4 as well as the product rule (S2) in Lemma 5.4.4 with an error that vanishes when $\varepsilon \to 0$. This is necessary to perform usual calculus operations before sending $\varepsilon \to 0$, that is when we do not have Laplace operator in the equation.

5.3 Weak solutions to the nonlocal problem

The existence of weak solutions for the local Cahn-Hilliard equation with degenerate mobility usually follows the method from [124]. The idea is to apply a Galerkin scheme with a non-degenerate regularized mobility, *i.e.* , calling $m(n)$ the mobility, then one considers an approximation $m_\varepsilon(n) \geq \varepsilon$. Finally, using standard compactness methods one can prove the existence of weak solutions for the initial system. The uniqueness of the weak solutions is still an open question.

In the case of the nonlocal Cahn-Hilliard equation, we have to rely on a fixed point method. We first consider a nondegenerate mobility, and the fixed point argument is put on the nonlocality. Then, we pass to the limit to obtain the nonlocal Cahn-Hilliard equation with degenerate mobility.

Approximating solutions

Following the scheme above, we first consider a nondegenerate mobility and prove the existence of the following system

$$\partial_t u_\delta = \operatorname{div}(T_\delta(u_\delta)\nabla \mu_\delta) \qquad (0,+\infty)\times\mathbb{T}^d, \tag{5.3.1}$$

$$\mu_\delta = B_\varepsilon[u_\delta] + F'(u_\delta) \qquad (0,+\infty)\times\mathbb{T}^d, \tag{5.3.2}$$

where $\delta > 0$ is a small parameter such that $2\delta < \frac{1}{\delta} - 1$, $\delta < \frac{1}{4}$ and

$$T_\delta(u) = \begin{cases} \delta & \text{for } u \le \delta, \\ \text{smooth monotone interpolation} & \text{for } u \in [\delta, 2\delta], \\ u & \text{for } u \in [2\delta, \frac{1}{\delta} - 1], \\ \text{smooth monotone interpolation} & \text{for } u \in [\frac{1}{\delta} - 1, \frac{1}{\delta}], \\ \frac{1}{\delta} & \text{for } u \ge \frac{1}{\delta}. \end{cases} \tag{5.3.3}$$

The estimates for the sequence $\{u_\delta\}$ will be obtained from the dissipation of energy and entropy. The definition of energy E_ε remains the same as in (5.2.3). However, the definition of entropy has to be adapted to take into account the fact that we don't know if the solution remains nonnegative. To this end, we define a function ϕ_δ by an explicit formula

$$\phi_\delta(x) = \int_1^x \int_1^y \frac{1}{T_\delta(z)} \,\mathrm{d}z\,\mathrm{d}y. \tag{5.3.4}$$

Lemma 5.3.1. *Let ϕ_δ be defined with* (5.3.4) *and $\phi(x) = x(\log(x) - 1) + 1$. Then,*

(P1) $\phi_\delta''(x) = \frac{1}{T_\delta(x)}$ and $\phi_\delta(1) = \phi_\delta'(1) = 0$,

(P2) $\phi_\delta(x) \to \phi(x)$ for $x \ge 0$ as $\delta \to 0$,

(P3) $\phi_\delta(x) \ge 0$ for all $x \in \mathbb{R}$,

(P4) $\phi_\delta(x) \le \phi(x) + \frac{\delta}{2(\delta-1)}x^2 + 3$ for $x \ge 0$,

(P5) $\phi_\delta(x) \to \infty$ *when* $\delta \to 0$ *for all* $x < 0$.

The purely computational proof is presented in Section 5.7.

Theorem 5.3.2. *Let* $\delta > 0$, ε_0 *be as in* (5.1.8) *and* $F \in C^4$. *For* $\varepsilon < \varepsilon_0$ *there exists classical solution* (5.3.1)–(5.3.2). *Moreover, they satisfy the mass, energy, and entropy conservation: for all* $t > 0$

$$\int_{\mathbb{T}^d} u_\delta(t,\cdot)\,\mathrm{d}x = \int_{\mathbb{T}^d} u^0\,\mathrm{d}x, \tag{5.3.5}$$

$$E_\varepsilon[u_\delta](t) + \int_0^t \int_{\mathbb{T}^d} T_\delta(u_\delta)\,|\nabla \mu_\delta|^2 = E_\varepsilon[u^0], \tag{5.3.6}$$

$$\begin{aligned}\Phi_\delta[u_\delta](t) + \frac{1}{2\varepsilon^2}\int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\varepsilon(y)\,|\nabla u_\delta(x) - \nabla u_\delta(x-y)|^2 + \\ + \int_0^t \int_{\mathbb{T}^d} F''(u_\delta)|\nabla u_\delta|^2 = \Phi_\delta[u^0].\end{aligned} \tag{5.3.7}$$

Theorem 5.3.3. *Let* ε_0 *be as in* (5.1.8). *Let* F *satisfy Assumption 5.1.1 with an additional constraint* $2\,C_{10} < C_p$. *Then, the following sequences are bounded uniformly in* $\delta \in (0,1)$ *and* $\varepsilon \in (0,\varepsilon_0)$

(A1) $\{u_\delta\}$ *in* $L^\infty(0,T; L^k(\mathbb{T}^d))$,

(A2) $\{u_\delta\}$ *in* $L^k(0,T; L^{k\frac{d}{d-2}}(\mathbb{T}^d))$,

(A3) $\{\sqrt{T_\delta(u)}\,\nabla \mu_\delta\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$,

(A4) $\{\nabla u_\delta\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$,

(A5) $\{\partial_t u_\delta\}$ *in* $L^2(0,T; W^{-1,s'}(\mathbb{T}^d))$,

(A6) $\{\partial_t \nabla u_\delta\}$ *in* $L^2(0,T; W^{-2,s'}(\mathbb{T}^d))$,

(A7) $\{\sqrt{F_1''(u_\delta)}\nabla u_\delta\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$,

(A8) $\{\Phi_\delta[u_\delta]\}$ *in* $L^\infty(0,T)$,

where k *and* s *have been defined in Notation 5.1.3.*

To prove Theorem 5.3.2, we need to assume that $F \in C^4$ which allows us to use known results about classical solutions to uniformly parabolic equations.

Proof of Theorem 5.3.2. As $\delta > 0$ is fixed in this result, we write u instead of u_δ. Given w we consider an auxiliary equation

$$\partial_t u = \operatorname{div}\left(T_\delta(u)\nabla u\left(\frac{1}{\varepsilon^2} + F''(u)\right)\right) - \operatorname{div}\left(T_\delta(u)\frac{w * \nabla\omega_\varepsilon}{\varepsilon^2}\right). \tag{5.3.8}$$

Let $\alpha, \sigma, M, \kappa$ be parameters to be specified later. We want to apply Schauder fixed point theorem to the map

$$\begin{aligned} P &: X \to X \\ P &: w \mapsto u \quad \text{solution of (5.3.8)}, \end{aligned}$$

where X is defined as the set

$$X = \{w \in C^{\alpha,\alpha/2}([0,T]\times\mathbb{T}^d), \|w\|_{\infty,\sigma} \leq M\}$$

with the norm

$$\|w\|_X := \|w\|_{\infty,\sigma} + \kappa\,\|w\|_{\alpha,\alpha/2}$$

and the norm $\|\cdot\|_{\alpha,\alpha/2}$ is the usual Hölder seminorm in space-time. We also define

$$\|w\|_{\infty,\sigma} := \sup_{[0,T]\times\mathbb{T}^d} |u(t,x)|e^{-\sigma t}. \tag{5.3.9}$$

Note that the new norm is equivalent to the usual supremum norm so all topological properties do not change. We need to prove that P is continuous, P maps in fact X to X, and that $P(X)$ is relatively compact in X. First, we prove that $P(w) = u$ is the unique classical solution of equation (5.3.8) so that P is well defined and find Hölder estimates which will be useful to prove the continuity of the operator as well as its relative compactness.

Step 1: P is well defined and Hölder estimates. Equation (5.3.8) is equivalent to saying that u solves parabolic equation

$$\partial_t u = \operatorname{div} A(t,x,u,\nabla u) + B(t,x,u,\nabla u), \qquad (u)_{\mathbb{T}^d} = (u_0)_{\mathbb{T}^d}$$

with

$$A(t,x,z,p) = T_\delta(z)\,p\left(\frac{1}{\varepsilon^2} + F''(z)\right),$$

$$B(t,x,z,p) = -T_\delta'(z)\,p\cdot\frac{w * \nabla\omega_\varepsilon}{\varepsilon^2} - T_\delta(z)\,\frac{w * \Delta\omega_\varepsilon}{\varepsilon^2},$$

and we recall that $w \in X$ is Hölder continuous. The function A satisfies the strong parabolicity condition for sufficiently small $\varepsilon > 0$, i.e.

$$A(t, x, z, p) \cdot p \geq \delta\, p^2 \, \frac{1}{2\, \varepsilon_0^2}$$

for all $\varepsilon < \varepsilon_0$ (this uses Assumptions (B), (C) and (5.1.8)). Since the derivatives A_p, A_z, A_t, A_x and function B are Hölder continuous as functions of (t, x, z, p), [192, Theorems 12.10, 12.14] asserts that there exists a unique classical solution to (5.3.8) such that

$$\|u\|_{C^{1+\alpha, 1+\alpha/2}} \leq C(\delta, \varepsilon_0, \|w\|_{C^{\alpha,\alpha/2}}).$$

With this estimate, (5.3.8) can be considered as a linear equation so that the linear theory for parabolic equations [192, Theorem 5.14] implies

$$\|u\|_{C^{2+\alpha, 1+\alpha/2}} \leq C(\delta, \varepsilon_0, \|w\|_{C^{\alpha,\alpha/2}}). \tag{5.3.10}$$

Therefore u is a classical solution of (5.3.8) and it admits the Hölder bound (5.3.10).
<u>*Step 2: The operator P is continuous.*</u> We consider a sequence $\{w_n\}$ in X such that $\|w_n - w\|_X \to 0$. Then $u_n = P(w_n)$ is compact in $C^{2,1}$ from estimate (5.3.10) and Arzela-Ascoli. We choose subsequence such that $u_{n_k} \to u$ in $C^{2,1}$. These functions satisfy

$$\partial_t u_{n_k} = \operatorname{div}\left(T_\delta(u_{n_k}) \nabla u_{n_k} \left(\frac{1}{\varepsilon^2} + F''(u_{n_k})\right)\right) - \operatorname{div}\left(T_\delta(u_{n_k}) \frac{w_{n_k} * \nabla \omega_\varepsilon}{\varepsilon^2}\right). \tag{5.3.11}$$

Passing to the limit in (5.3.11) and using uniqueness of solutions to (5.3.8) from [192], we obtain that for every subsequence of $\{u_n\}$ we can extract a subsequence which converges to a unique limit $u = P(w)$. By a standard subsequence argument, this means that the whole sequence $\{u_n\}$ converges to $u = P(w)$. Therefore P is continuous.
<u>*Step 3: P maps X to X.*</u> We write the equation (5.3.8) in the form

$$\begin{aligned}\partial_t u = {} & T_\delta'(u) |\nabla u|^2 \left(\frac{1}{\varepsilon^2} + F''(u)\right) + T_\delta(u) \Delta u \left(\frac{1}{\varepsilon^2} + F''(u)\right) + T_\delta(u) |\nabla u|^2 F^{(3)}(u) \\ & - T_\delta'(u)\, \nabla u \cdot \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} - T_\delta(u) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2}.\end{aligned}$$

We substitute $u = v\, e^{\sigma t}$ and we compute PDE satisfied by v:

$$\partial_t v\, e^{\sigma t} + v\, \sigma\, e^{\sigma t} = T_\delta'(u)|\nabla v|^2 \left(\frac{1}{\varepsilon^2} + F''(u)\right) e^{2\sigma t} + T_\delta(u)\Delta v \left(\frac{1}{\varepsilon^2} + F''(u)\right) e^{\sigma t}$$
$$+ T_\delta(u)|\nabla v|^2 F^{(3)}(u) e^{2\sigma t} - T_\delta'(u)\, \nabla v \cdot \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} e^{\sigma t} - T_\delta(u) \frac{w * \Delta\omega_\varepsilon}{\varepsilon^2}.$$

Now, we multiply by v and evaluate the equation at the point (t_*, x_*) where v^2 attains its maximum. Therefore, all the terms with ∇v and $|\nabla v|^2$ vanish (as $|\nabla v|^2 v = \nabla v \cdot \nabla v^2/2$).

$$\frac{1}{2}\partial_t v^2 e^{\sigma t_*} + v^2 \sigma e^{\sigma t_*} = T_\delta(u)\, v\, \Delta v \left(\frac{1}{\varepsilon^2} + F''(u)\right) e^{\sigma t_*} - v\, T_\delta(u) \frac{w * \Delta\omega_\varepsilon}{\varepsilon^2}.$$

Using $v\,\Delta v = -|\nabla v|^2 + \Delta v^2 \leq 0$ and $\partial_t v^2 \geq 0$ we obtain

$$v^2\, \sigma\, e^{\sigma t_*} \leq -v\, T_\delta(u) \frac{w * \Delta\omega_\varepsilon}{\varepsilon^2},$$

so that

$$v^2(t_*, x_*)\, \sigma\, e^{\sigma t_*} \leq |v(t_*, x_*)| \frac{\|\Delta\omega_\varepsilon\|_1 \|w(t_*, \cdot)\|_\infty}{\delta\varepsilon^2}.$$

where we used the definition of T_δ. As v^2 attains maximum at (t^*, x^*), $|v(t_*, x_*)|$ also attains maximum at (t^*, x^*). Therefore, taking into account the initial condition

$$\|v\|_\infty \leq \max\left(\frac{\|\Delta\omega_\varepsilon\|_1 \|w(t_*, \cdot)\|_\infty}{\delta\varepsilon^2\sigma} e^{-\sigma t_*}, \|u_0\|_\infty\right) \leq \max\left(\frac{\|\Delta\omega_\varepsilon\|_1 \|w e^{-\sigma t}\|_\infty}{\delta\varepsilon^2\sigma}, \|u_0\|_\infty\right).$$

Choosing $\sigma = 2\|\Delta\omega_\varepsilon\|_1/(\delta\varepsilon^2)$, we obtain estimate

$$\|v\|_\infty \leq \max\left(\frac{1}{2}\, \|w e^{-\sigma t}\|_\infty, \|u_0\|_\infty\right).$$

By definition of the norm

$$\|Pw\|_{\infty,\sigma} \leq \max\left(\frac{1}{2}\, \|w\|_{\infty,\sigma}, \|u_0\|_\infty\right). \tag{5.3.12}$$

Moreover, the parabolic version of de Giorgi-Nash-Moser theory, see [183, Chap. V, Theorem 1.1], implies that there exists $\alpha = \alpha(\|w\|_{\infty,\sigma})$ such that the solution of (5.3.8) satisfy

$$\|u\|_{C^{\alpha,\alpha/2}} \leq f(\|w\|_{\infty,\sigma}).$$

Without loss of generality we may assume that $f(\|w\|_{\infty,\sigma})$ does not decrease and $\alpha(\|w\|_{\infty,\sigma})$ does not increase when $\|w\|_{\infty,\sigma}$ increases.

We proceed to choosing values of parameters M, α, κ and concluding the proof. We choose

$$M = 3\left\|u_0\right\|_{L^\infty}, \qquad \alpha = \alpha(M), \qquad \kappa = \frac{\left\|u_0\right\|_{L^\infty}}{2f(M)}.$$

Since w is in X and f is nondecreasing we obtain

$$\left\|Pw\right\|_X \leq \frac{1}{2}\|w\|_{\infty,\sigma}+\|u_0\|_\infty+\kappa f(\|w\|_{\infty,\sigma}) \leq \frac{M}{2}+\|u_0\|_\infty+\kappa f(M) \leq 3\left\|u_0\right\|_{L^\infty} = M.$$

This means that P maps X to X.

Step 4: $P(X)$ is relatively compact in X. The relative compactness of $P(X)$ follows from (5.3.10).

The proof is concluded. □

Proof of Theorem 5.3.3. To prove (A1) and (A3) we want to apply (5.3.6) and Assumption 5.1.1 on the potential. The energy identity reads:

$$\int_{\mathbb{T}^d} F(u_\delta)\,\mathrm{d}x+\frac{1}{4\varepsilon^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)|u_\delta(x)-u_\delta(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y+\int_0^t\int_{\mathbb{T}^d} T_\delta(u_\delta)\,|\nabla\mu_\delta|^2 = E_\varepsilon[u^0],$$

Applying Lemma 5.8.1, we deduce

$$\int_{\mathbb{T}^d} F(u_\delta)\,\mathrm{d}x + C_p\int_{\mathbb{T}^d}|u-(u)_{\mathbb{T}^d}|^2 + \int_0^t\int_{\mathbb{T}^d} T_\delta(u_\delta)\,|\nabla\mu_\delta|^2 \leq E_\varepsilon[u^0]$$

Splitting $F = F_1 + F_2$ and applying (C) in Assumption 5.1.1 we obtain

$$\int_{\mathbb{T}^d} F_1(u_\delta)\,\mathrm{d}x+C_p\int_{\mathbb{T}^d}|u_\delta-(u_\delta)_{\mathbb{T}^d}|^2+\int_0^t\int_{\mathbb{T}^d} T_\delta(u_\delta)\,|\nabla\mu_\delta|^2 \leq E_\varepsilon[u^0]+C_9\,|\mathbb{T}^d|+C_{10}\int_{\mathbb{T}^d}|u_\delta|^2$$

Note that by conservation of mass, $(u_\delta)_{\mathbb{T}^d} = (u^0)_{\mathbb{T}^d}$. Therefore, applying the simple inequality $|a+b|^2 \leq 2|a|^2+2|b|^2$ and $C_p > 2\,C_{10}$, we obtain an $L^\infty(0,T;L^2(\mathbb{T}^d))$ estimate on $\{u_\delta\}$ which can be improved to $L^\infty(0,T;L^k(\mathbb{T}^d))$ if $F_1 \neq 0$ cf. (B) in Assumption 5.1.1. Then, (A1) and so, (A3) is easily implied by the energy as all possibly negative terms are bounded.

Now, to prove (A4) we want to use the entropy equality (5.3.7):

$$\Phi_\delta[u_\delta](t)+\frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)\,|\nabla u_\delta(x)-\nabla u_\delta(x-y)|^2+\int_0^t\int_{\mathbb{T}^d} F''(u_\delta)|\nabla u_\delta|^2 = \Phi_\delta[u^0].$$

To exploit it, for γ to be chosen later, $\varepsilon \in (0, \widetilde{\varepsilon}_0(\gamma))$ we have by Lemma 5.8.3

$$\Phi_\delta[u_\delta](t) + \frac{1}{\gamma}\int_0^t \int_{\mathbb{T}^d} |\nabla u_\delta|^2 + \int_0^t \int_{\mathbb{T}^d} F_1''(u_\delta)|\nabla u_\delta|^2 \leq$$
$$\leq \Phi_\delta[u^0] + C(\gamma)\int_0^t \int_{\mathbb{T}^d} \|u_\delta\|^2_{L^2(\mathbb{T}^d)} + \|F_2''\|_\infty \int_0^t \int_{\mathbb{T}^d} |\nabla u_\delta|^2.$$

We choose $\gamma = \frac{1}{1+\|F_2''\|_\infty}$ which yields estimates (A4), (A7) and (A8) (here, we also exploit (P4) in Lemma 5.3.1 to control $\Phi_\delta[u^0]$). Now, to see (A5) we take a smooth test function φ and write thanks to the Hölder inequality

$$\begin{aligned}
\left|\int_0^T \int_{\mathbb{T}^d} \partial_t u_\delta\, \varphi \, \mathrm{d}x\, \mathrm{d}t\right| &= \left|\int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta)^{1/2} T_\delta(u_\delta)^{1/2} \nabla \mu_\delta \cdot \nabla\varphi \, \mathrm{d}x\, \mathrm{d}t\right| \\
&\leq \|T_\delta(u_\delta)^{1/2}\|_{L^\infty(0,T;L^{2k}(\mathbb{T}^d))} \|T_\delta(u_\delta)^{1/2}\nabla\mu_\delta\|_{L^2((0,T)\times\mathbb{T}^d)} \|\nabla\varphi\|_{L^2(0,T;L^s(\mathbb{T}^d))} \\
&\leq C\|\nabla\varphi\|_{L^2(0,T;L^s(\mathbb{T}^d))}.
\end{aligned}$$

In the last line we used estimates (A1), (A3) and the definition of T_δ. This concludes the proof for estimates (A5) and then (A6) easily follows.

Finally, we prove (A2). We note from (A7) that $\{\nabla u_\delta^{k/2}\}$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$ and from (A1) that $\{u_\delta^{k/2}\}$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$. Therefore, by Sobolev embedding, we obtain that $\{u_\delta^{k/2}\}$ is bounded in $L^2(0,T;L^{\frac{2d}{d-2}}(\mathbb{T}^d))$ so that $\{u_\delta\}$ is bounded in $L^k(0,T;L^{k\frac{d}{d-2}}(\mathbb{T}^d))$. □

Proof of existence result

Proof of Theorem 5.1.7.

Step 1: Approximation of the potential. For F as in Assumption 5.1.1, we consider its mollification $F_\delta = F * \eta_\delta$ where $\{\eta_\delta\}$ is the usual mollifier. We note that F_δ is C^4 and that F, F_δ satisfy Assumption 5.1.1 with comparable constants $C_1, ..., C_{10}$, see Lemma 5.7.2. The most important is constant C_{10} because there is a constraint on it in terms of C_p. More precisely, F satisfies Assumption 5.1.1 with $C_{10} < C_p/4$ so that from Lemma 5.7.2 we have that F_δ satisfies it with $2C_{10} < C_p/2$. This allows to apply Theorem 5.3.3 to otain uniform estimates. Moreover $F_\delta = F_{\delta,1} + F_{\delta,2}$ with $F^{(p)}_{\delta,(1,2)} \xrightarrow[\delta\to 0]{pointwise} F^{(p)}_{(1,2)}$ where $p = 0, 1, 2$ is the order of derivative.

Step 2: Compactness. Using Theorem 5.3.2, we can obtain u_δ such that for all $\varphi \in L^2(0,T;W^{1,\infty}(\mathbb{T}^d))$

$$\begin{aligned}\int_0^T \langle \partial_t u_\delta, \varphi\rangle_{(W^{1,s}(\mathbb{T}^d))',W^{1,s}(\mathbb{T}^d)} + \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta)\nabla B_\varepsilon[u_\delta]\cdot\nabla\varphi + \\ + \int_0^T\int_{\mathbb{T}^d} u_\delta F_\delta''(u_\delta)\nabla u_\delta\cdot\nabla\varphi = 0.\end{aligned} \tag{5.3.13}$$

The plan is to send $\delta \to 0$ in (5.3.13). By Theorem 5.3.3 and standard compactness results we can extract a subsequence (not relabelled) such that

(B1) $u_\delta \to u$ a.e. and in $L^2((0,T)\times\mathbb{T}^d), L^k((0,T)\times\mathbb{T}^d)$,

(B2) $\nabla u_\delta \rightharpoonup \nabla u$ in $L^2((0,T)\times\mathbb{T}^d)$,

(B3) $\partial_t u_\delta \rightharpoonup \partial_t u$ in $L^2(0,T;W^{-1,s'}(\mathbb{T}^d))$,

(B4) $\sqrt{F''_{1,\delta}(u_\delta)}\nabla u_\delta \rightharpoonup \xi$ in $L^2((0,T)\times\mathbb{T}^d)$ for some $\xi \in L^2((0,T)\times\mathbb{T}^d)$.

Only (B1) needs some justification. From (A1), (A4), (A5) and Aubin-Lions lemma, we obtain the strong convergence $u_\delta \to u$ a.e. and in $L^2((0,T)\times\mathbb{T}^d)$. To see the second strong convergence, we interpolate between spaces $L^\infty(0,T;L^k(\mathbb{T}^d))$ and $L^k(0,T;L^{k\frac{d}{d-2}}(\mathbb{T}^d))$ to prove that $\{u_\delta\}$ is bounded in $L^{k+\kappa}(0,T;L^{k+\kappa}(\mathbb{T}^d))$ for some $\kappa > 0$ because $k\frac{d}{d-2} > k$. Now, interpolating between $L^{k+\kappa}(0,T;L^{k+\kappa}(\mathbb{T}^d))$ and $L^2((0,T)\times\mathbb{T}^d)$ we obtain strong convergence in $L^k((0,T)\times\mathbb{T}^d)$.

Step 3: Nonnegativity of u. The plan is to obtain a contradiction with the uniform estimate of the entropy. For $\alpha > 0$, we define the sets

$$V_{\alpha,\delta} = \{(t,x) \in (0,T)\times\mathbb{T}^d : \, u_\delta(t,x) \le -\alpha\},$$

$$V_{\alpha,0} = \{(t,x) \in (0,T)\times\mathbb{T}^d : \, u(t,x) \le -\alpha\}.$$

By nonnegativity of ϕ_δ (see (5.3.4) as well as the properties below) and (A8) in Theorem 5.3.3, there is a constant $C(T)$ such that

$$\int_{V_{\alpha,\delta}} \phi_\delta(u_\delta)\,\mathrm{d}x\,\mathrm{d}t \le \int_{(0,T)\times\mathbb{T}^d} \phi_\delta(u_\delta)\,\mathrm{d}x\,\mathrm{d}t \le C(T).$$

For $u_\delta \le -\alpha$, we have $0 \le \phi_\delta(-\alpha) \le \phi_\delta(u_\delta)$ because $\phi_\delta'(x) \le 0$ for $x \le 0$, see (5.3.4). Therefore,

$$0 \le \phi_\delta(-\alpha)\int_{V_{\alpha,\delta}} 1\,\mathrm{d}x\,\mathrm{d}t = \int_{V_{\alpha,\delta}} \phi_\delta(x)\,\mathrm{d}x\,\mathrm{d}t \le C(T).$$

Sending $\delta \to 0$, exploiting (P5) in Lemma 5.3.1 and using the strong convergence of $u_\delta \to u$ we discover

$$\int_{V_{\alpha,0}} 1 \, \mathrm{d}x \, \mathrm{d}t = \lim_{\delta \to 0} \int_{V_{\alpha,\delta}} 1 \, \mathrm{d}x \, \mathrm{d}t = 0$$

(we use here the fact from measure theory asserting that on the measure space (X, μ) if $f_n, f : X \to \mathbb{R}$ and $f_n \to f$ in $L^1(X, \mu)$ then for $\alpha \in \mathbb{R}$ we have $\int_{f_n < \alpha} \mathrm{d}\mu \to \int_{f<\alpha} \mathrm{d}\mu$ as $n \to \infty$). This means that $V_{\alpha,0}$ is a null set for each $\alpha > 0$, concluding the proof of the nonnegativity.

Step 4: Identification $\xi = \sqrt{F_1''(u)}\nabla u$. We want to use (B4) so we have to identify ξ. For that purpose, we use the convergence a.e. of u_δ in (B1) and the pointwise convergence $F''_{\delta,1} \to F_1''$ to deduce that $F_{\delta,1}(u_\delta) \to F_1(u)$ a.e. Next, using Assumption (B) for $F_{\delta,1}$ and estimate (A1)

$$\left| \sqrt{F''_{\delta,1}(u_\delta)} \right|^2 \leq C_3 |u_\delta|^{k-2} + C_4.$$

As (RHS) is uniformly integrable by strong convergence (B1), we deduce that $\left| \sqrt{F''_{\delta,1}(u_\delta)} \right|^2$ is uniformly integrable so that the Vitali convergence theorem implies

$$\sqrt{F''_{\delta,1}(u_\delta)} \to \sqrt{F_1''(u)} \quad \text{in } L^2((0,T) \times \mathbb{T}^d)$$

Using weak convergence of gradient (B2), we finally obtain $\xi = \sqrt{F_1''(u)}\nabla u$.

Step 5: Passing to the limit in the first two terms of (5.3.13). Using (B3) it is easy to pass to the limit in the first term of (5.3.13). Now we focus on the second term. Note that

$$\nabla B_\varepsilon[u_\delta](x) = \frac{1}{\varepsilon^2}(\nabla u_\delta - \omega_\varepsilon * \nabla u_\delta).$$

The two terms of ∇B_ε are treated in the same way. We focus only on the harder term ∇u_δ which does not have regularizing properties of the convolution. For this term it is sufficient to prove that $T_\delta(u_\delta)\nabla u_\delta \rightharpoonup u \nabla u$ weakly in $L^2(0,T; L^1(\mathbb{T}^d))$. We first note that by definition of T_δ, the strong convergence (B1) and the nonnegativity of u, we obtain $T_\delta(u_\delta) \to u$ strongly in $L^2((0,T) \times \mathbb{T}^d)$. Hence, the result follows from weak convergence of the gradient (B2).

Step 6: Passing to the limit in the third term of (5.3.13). For the third term we write

$F_\delta'' = F_{\delta,1}'' + F_{\delta,2}''$ as discussed in Step 1. Then we decompose

$$\int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_\delta''(u_\delta) \nabla u_\delta \cdot \nabla \varphi = \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_{\delta,1}''(u_\delta) \nabla u_\delta \cdot \nabla \varphi + \\ + \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_{\delta,2}''(u_\delta) \nabla u_\delta \cdot \nabla \varphi = I_1 + I_2.$$

For I_1 we write

$$I_1 = \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)} \sqrt{F_{\delta,1}''(u_\delta)} \nabla u_\delta \cdot \nabla \varphi.$$

It remains to prove that $T_\delta(u_\delta)\sqrt{F_{\delta,1}''(u_\delta)}$ converges strongly in $L^2((0,T)\times\mathbb{T}^d)$. Note that since $u_\delta \to u \geq 0$ we have $T_\delta(u_\delta)\sqrt{F_{\delta,1}''(u_\delta)} \to u\sqrt{F_1''(u)}$ a.e. Moreover,

$$\left(T_\delta(u_\delta)\sqrt{F_{\delta,1}''(u_\delta)}\right)^2 \leq C_3 \left|u_\delta\right|^k + C_4$$

As the (RHS) is uniformly integrable by strong convergence, we deduce that (LHS) is uniformly integrable. Hence, the Vitali convergence theorem implies Assumption (B) and Estimate (A1) show that

$$T_\delta(u_\delta)\sqrt{F_{\delta,1}''(u_\delta)} \to u\sqrt{F_1''(u)} \text{ in } L^2((0,T)\times\mathbb{T}^d)$$

so that $I_1 \to \int_0^T \int_{\mathbb{T}^d} uF_1''(u)\nabla u \cdot \nabla \varphi$. For I_2, as $\nabla u_\delta \rightharpoonup \nabla u$ converges weakly in $L^2((0,T)\times\mathbb{T}^d)$, it is sufficient to prove the strong convergence of $T_\delta(u_\delta)F_{\delta,2}''(u_\delta)$ in $L^2((0,T)\times\mathbb{T}^d)$. Thanks to Assumption (C) on $F_{\delta,2}''$, this term is uniformly bounded so that trivially $|T_\delta(u_\delta)F_{\delta,2}''(u_\delta)| \leq \|F_2''\|_\infty |T_\delta(u_\delta)|$. Therefore, Vitali convergence theorem implies $T_\delta(u_\delta)F_{\delta,2}''(u_\delta)$ in $L^2((0,T)\times\mathbb{T}^d)$ and so

$$I_2 \to \int_0^T \int_{\mathbb{T}^d} uF_2''(u)\nabla u \cdot \nabla \varphi.$$

Step 7: Energy and entropy estimates. We pass to the limit $\delta \to 0$ in (5.3.6)-(5.3.7). With the above convergences and properties of the weak limit, we obtain the result. This ends the proof of Theorem 5.1.7. □

Now that weak solutions of the nonlocal Cahn-Hilliard equation have been constructed for a given initial datum, it remains to prove the convergence of the nonlocal system to the local one. This is the purpose of the next section.

5.4 Limit $\varepsilon \to 0$

Weak solutions of the local Cahn-Hilliard equation are understood in the sense of Definition 5.1.5. In order to prove the convergence of the nonlocal system to these solutions, we first collect the necessary estimates uniform in ε. Then we pass to the limit $\varepsilon \to 0$ to conclude the proof of Theorem 5.1.8.

Uniform estimates in $\varepsilon > 0$

We recall that in the previous section we had obtained the energy and entropy inequalities as well as estimates uniform in ε.

Lemma 5.4.1 (Mass, energy, entropy)**.** *The following identities hold true:*

$$\int_{\mathbb{T}^d} u_\varepsilon(t,\cdot)\,\mathrm{d}x = \int_{\mathbb{T}^d} u^0\,\mathrm{d}x, \tag{5.4.1}$$

$$E[u_\varepsilon(t,\cdot)] + \int_0^t \int_{\mathbb{T}^d} u_\varepsilon\,|\nabla \mu_\varepsilon|^2 \le E[u^0], \tag{5.4.2}$$

$$\begin{aligned}\Phi[u_\varepsilon(t,\cdot)] + \frac{1}{2\varepsilon^2}\int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\varepsilon(y)\,|\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y +& \\ + \int_0^t \int_{\mathbb{T}^d} F''(u_\varepsilon)|\nabla u_\varepsilon|^2 \le \Phi[u^0].&\end{aligned} \tag{5.4.3}$$

Lemma 5.4.2 (Uniform estimates)**.** *The following sequences are bounded:*

(A) $\{u_\varepsilon\}$ *in* $L^\infty(0,T;L^k(\mathbb{T}^d))$,

(B) $\{u_\varepsilon\}$ *in* $L^k(0,T;L^{k\frac{d}{d-2}}(\mathbb{T}^d))$,

(C) $\{\nabla u_\varepsilon\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$,

(D) $\{\sqrt{u_\varepsilon}\,\nabla\mu_\varepsilon\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$,

(E) $\{\partial_t u_\varepsilon\}$ *in* $L^2(0,T;W^{-1,s'}(\mathbb{T}^d))$,

(F) $\{\partial_t \nabla u_\varepsilon\}$ *in* $L^2(0,T;W^{-2,s'}(\mathbb{T}^d))$,

(G) $\{\sqrt{F_1''(u_\varepsilon)}\nabla u_\varepsilon\}$ *in* $L^2((0,T)\times\mathbb{T}^d)$.

Our last ingredient for the proof of Theorem 5.1.8 is about the compactness of $\{u_\varepsilon\}$ and its gradient.

Lemma 5.4.3 (Compactness). *Sequences $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ are strongly compact in $L^2((0,T)\times\mathbb{T}^d)$.*

Proof. The compactness of $\{u_\varepsilon\}$ follows from the Lions-Aubin lemma applied to estimates (A),(C) and (E). Then, for the compactness of $\{\nabla u_\varepsilon\}$, we recall the estimate provided by the entropy on the quantity:

$$\frac{1}{4\varepsilon^2}\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)\,|\nabla u_\varepsilon(x)-\nabla u_\varepsilon(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\le C.$$

By Theorem 2.6.3, it implies compactness in the spatial variable (2.6.3) for $\{\nabla u_\varepsilon\}$. Then, using the uniform bound on $\{\partial_t\nabla u_\varepsilon\}$ in $L^2(0,T;W^{-2,s'}(\mathbb{T}^d))$ (see also Remark 2.5.3 (A)) and Theorem 2.5.5, we obtain compactness of $\{\nabla u_\varepsilon\}$ in $L^2((0,T)\times\mathbb{T}^d)$. □

Nonlocal calculus

We want to pass to the limit $\varepsilon\to 0$ in Equations (5.1.5)-(5.1.6) and obtain weak solutions of the local Cahn-Hilliard equation. We have at most bounds on the gradient of u_ε and the limit equation has four derivatives. That means we need to mimic at the epsilon level integration by parts for nonlocal operators. For that purpose, we define the operator

$$S_\varepsilon[\varphi](x,y):=\frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2}\varepsilon}(\varphi(x-y)-\varphi(x)) \tag{5.4.4}$$

which has the following properties:

Lemma 5.4.4. *The operator S_ε satisfies:*

(S1) S_ε is a linear operator that commutes with derivatives with respect to x,

(S2) for all functions $f,g:\mathbb{T}^d\to\mathbb{R}$ we have

$$S_\varepsilon[fg](x,y)-S_\varepsilon[f](x,y)g(x)-S_\varepsilon[g](x,y)f(x)=\\ =\frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2}\varepsilon}[(f(x-y)-f(x))(g(x-y)-g(x))].$$

(S3) for all $u,\varphi\in L^2(\mathbb{T}^d)$

$$\langle B_\varepsilon[u](\cdot),\varphi(\cdot)\rangle_{L^2(\mathbb{T}^d)}=\langle S_\varepsilon[u](\cdot,\cdot),S_\varepsilon[\varphi](\cdot,\cdot)\rangle_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}.$$

(S4) if $\{u_\varepsilon\}$ is strongly compact in $L^2(0,T;H^1(\mathbb{T}^d))$ and $\varphi \in L^\infty((0,T)\times\mathbb{T}^d)$ we have
$$\int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2\,\varphi(t,x) \to D\int_0^T\int_{\mathbb{T}^d} |\nabla u(t,x)|^2\,\varphi(t,x)$$
where $D = \frac{1}{2}\int_{B_1}\omega(y)|y|^2\,\mathrm{d}y$.

Proof. The first one is trivial. For the second one, we just observe

$$\begin{aligned}&(f(x-y)-f(x))(g(x-y)-g(x)) = \\ &-(f(x-y)-f(x))\,g(x) - g(x-y)\,f(x) + f(x-y)\,g(x-y) = \\ &-(f(x-y)-f(x))\,g(x) - (g(x-y)-g(x))\,f(x) + (f(x-y)\,g(x-y) - f(x)g(x)).\end{aligned}$$

For the third one, we compute
$$\langle B_\varepsilon[u](\cdot),\varphi(\cdot)\rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\omega_\varepsilon(y)}{\varepsilon^2}(u(x)-u(x-y))\,\varphi(x)\,\mathrm{d}y\,\mathrm{d}x.$$
Changing variables $x' = x-y$, $y' = -y$ and using symmetry of the kernel
$$\langle B_\varepsilon[u](\cdot),\varphi(\cdot)\rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\omega_\varepsilon(y)}{\varepsilon^2}(u(x'-y')-u(x'))\,\varphi(x'-y')\,\mathrm{d}y'\,\mathrm{d}x'.$$
Therefore,
$$\begin{aligned}2\,\langle B_\varepsilon[u](\cdot),\varphi(\cdot)\rangle_{L^2(\mathbb{T}^d)} &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\omega_\varepsilon(y)}{\varepsilon^2}(u(x)-u(x-y))\,(\varphi(x)-\varphi(x-y))\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\sqrt{\omega_\varepsilon(y)}}{\varepsilon}(u(x)-u(x-y))\,\frac{\sqrt{\omega_\varepsilon(y)}}{\varepsilon}(\varphi(x)-\varphi(x-y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$
Finally, to prove (S4) we use the definition of ω_ε and change variables with respect to y to obtain:
$$\int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}(S_\varepsilon[u_\varepsilon])^2\,\varphi(t,x) = \int_{B_1}\omega(y)\int_0^T\int_{\mathbb{T}^d}\varphi(t,x)\frac{|u_\varepsilon(t,x)-u_\varepsilon(t,x-\varepsilon y)|^2}{2\varepsilon^2}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y.$$
For fixed y,
$$\int_0^T\int_{\mathbb{T}^d}\varphi(t,x)\frac{|u_\varepsilon(t,x)-u_\varepsilon(t,x-\varepsilon y)|^2}{\varepsilon^2}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_{\mathbb{T}^d}\varphi(t,x)|\nabla u(x)|^2|y|^2\,\mathrm{d}x\,\mathrm{d}t,$$
$$\left|\int_0^T\int_{\mathbb{T}^d}\varphi(t,x)\frac{|u_\varepsilon(t,x)-u_\varepsilon(t,x-\varepsilon y)|^2}{\varepsilon^2}\,\mathrm{d}x\,\mathrm{d}t\right| \le \|\varphi\|_\infty\,\sup_\varepsilon\|Du_\varepsilon\|_2^2\,|y|^2$$
due to Lemma 5.7.1. As the majorant is integrable, the dominated convergence theorem concludes the proof. $\square$

Since B_ε has a similar behavior as the Laplace operator, one can expect that S_ε acts like a gradient (in $L^2(\mathbb{T}^d)$). Nevertheless, note that $S_\varepsilon[\varphi](x,y)$ is a scalar. From now on, we write ∇S_ε for the gradient of S_ε with respect to the variable x *i.e.*

$$\nabla S_\varepsilon[\varphi](x,y) := \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2}\varepsilon}(\nabla\varphi(x-y) - \nabla\varphi(x)).$$

Proof of the main result

Proof of Theorem 5.1.8. We only have to explain how to pass to the limit in the term $\int_0^T \int_{\mathbb{T}^d} \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon)\varphi \,\mathrm{d}x\,\mathrm{d}t$ where $\varphi \in C^3([0,T]\times\mathbb{T}^d)$. Integrating by parts, we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon)\,\varphi \,\mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_{\mathbb{T}^d} u_\varepsilon \nabla\mu_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \\ &= \int_0^T \int_{\mathbb{T}^d} B_\varepsilon[u_\varepsilon]\nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} B_\varepsilon[u_\varepsilon] u_\varepsilon \Delta\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_{\mathbb{T}^d} u_\varepsilon F''(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t =: I_1 + I_2 + I_3.\end{aligned} \tag{5.4.5}$$

<u>*Step 1: Compactness*</u>. Using Lemma 5.4.2 and Lemma 5.4.3 we can choose a subsequence of $\{u^\varepsilon\}$ such that

(D1) $\partial_t u^\varepsilon \rightharpoonup \partial_t u$ weakly in $L^2(0,T;W^{-1,s'}(\mathbb{T}^d))$,

(D2) $u^\varepsilon \to u$ strongly in $L^2((0,T)\times\mathbb{T}^d)$,

(D3) $\nabla u^\varepsilon \to \nabla u$ strongly in $L^2((0,T)\times\mathbb{T}^d)$,

(D4) $\sqrt{F_1''(u_\varepsilon)}\nabla u_\varepsilon \rightharpoonup \xi$ weakly in $L^2((0,T)\times\mathbb{T}^d)$.

<u>*Step 1: Convergence of I_1*</u>. Using (S3) in Lemma 5.4.4 we write term I_1 as

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] S_\varepsilon(\nabla u_\varepsilon \cdot \nabla\varphi)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] S_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon]\nabla u_\varepsilon \cdot S_\varepsilon[\nabla\varphi]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + R_\varepsilon^{(1)} = J_1^{(1)} + J_2^{(1)} + R_\varepsilon^{(1)},\end{aligned}$$

where $R_\varepsilon^{(1)}$ is defined as

$$R_\varepsilon^{(1)} = \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon]\left(S_\varepsilon(\nabla u_\varepsilon\cdot\nabla\varphi) - S_\varepsilon(\nabla u_\varepsilon)\cdot\nabla\varphi - \nabla u_\varepsilon\cdot S_\varepsilon[\nabla\varphi]\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

For $J_1^{(1)}$ we use identity

$$S_\varepsilon[u_\varepsilon]\, S_\varepsilon(\nabla u_\varepsilon) = S_\varepsilon[u_\varepsilon]\, \nabla S_\varepsilon(u_\varepsilon) = \frac{1}{2}\nabla \left|S_\varepsilon[u_\varepsilon]\right|^2,$$

so after integration by parts we obtain

$$J_1^{(1)} = -\frac{1}{2}\int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2 \Delta\varphi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \to -\frac{D}{2}\int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{5.4.6}$$

due to (S4) in Lemma 5.4.4. For $J_2^{(1)}$ we change variables to have

$$\begin{aligned} J_2^{(1)} &= \frac{1}{2}\int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\varepsilon(y)\, \frac{u_\varepsilon(x-y)-u_\varepsilon(x)}{\varepsilon}\nabla u_\varepsilon(x)\cdot \frac{\nabla\varphi(x-y)-\nabla\varphi(x)}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \\ &= \frac{1}{2}\int_{\mathbb{T}^d} \omega(y) \int_0^T \int_{\mathbb{T}^d} \frac{u_\varepsilon(x-\varepsilon y)-u_\varepsilon(x)}{\varepsilon}\nabla u_\varepsilon(x)\cdot \frac{\nabla\varphi(x-\varepsilon y)-\nabla\varphi(x)}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y. \end{aligned}$$

We are first concerned with the inner integral. With Lemma 5.7.1 we have that for fixed $y \in \mathbb{T}^d$

$$\frac{u_\varepsilon(x-\varepsilon y)-u_\varepsilon(x)}{\varepsilon} \to -\nabla u(x)\cdot y \text{ in } L^2((0,T)\times\mathbb{T}^d).$$

Moreover, a Taylor expansion implies that

$$\frac{\nabla\varphi(x-\varepsilon y)-\nabla\varphi(x)}{\varepsilon} \to -D^2\varphi(x)y \text{ in } L^\infty((0,T)\times\mathbb{T}^d;\mathbb{R}^d).$$

Combining this with a strong convergence $\nabla u_\varepsilon \to \nabla u$ in $L^2((0,T)\times\mathbb{T}^d)$, we deduce

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \frac{u_\varepsilon(x-\varepsilon y)-u_\varepsilon(x)}{\varepsilon}\nabla u_\varepsilon(x)\cdot \frac{\nabla\varphi(x-\varepsilon y)-\nabla\varphi(x)}{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \to \\ \to \int_0^T \int_{\mathbb{T}^d} \nabla u(x)\cdot y\, \nabla u(x)\cdot (D^2\varphi(x)y)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Finally, we apply the dominated convergence theorem to the integral with respect to y with the dominating function $\|D^2\varphi\|_\infty \sup_\varepsilon \|\nabla u_\varepsilon\|_2^2 |y|^2$. We obtain

$$\begin{aligned} J_2^{(1)} \to \frac{1}{2}\int_{\mathbb{T}^d} \omega(y)|y|^2\,\mathrm{d}y \int_0^T \int_{\mathbb{T}^d} \nabla u(x)\cdot D^2\varphi(x)\nabla u(x)\,\mathrm{d}x\,\mathrm{d}t = \\ = D\int_0^T \int_{\mathbb{T}^d} (\nabla u(x)\otimes\nabla u(x)) : D^2\varphi(x)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{5.4.7}$$

where we also used the symmetry of $D^2\varphi$ and properties of ω defined in (5.1.2). It remains to deal with the error term. Using (S2) in Lemma 5.4.4 we can write

$$R_\varepsilon^{(1)} = \int_0^T \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon]\frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2}\varepsilon}[(\nabla u_\varepsilon(x-y)-\nabla u_\varepsilon(x))\cdot(\nabla\varphi(x-y)-\nabla\varphi(x))]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

We want to prove that $R_\varepsilon^{(1)}$ converges to 0. By Cauchy-Schwarz inequality (in time and space) as well as bounds on $S_\varepsilon[u_\varepsilon]$ it remains to prove that

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)|^2 |\nabla\varphi(x-y) - \nabla\varphi(x)|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \to 0. \quad (5.4.8)$$

Using Taylor's expansion we can estimate this integral with

$$\varepsilon \left\| D^2 \varphi \right\|_{L^\infty} \left(\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)|^2 \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}t \right)$$

which converges to zero by the bound from the entropy (5.4.3) so that (5.4.8) follows. We conclude that

$$I_1 \to -\frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi \, \mathrm{d}x \, \mathrm{d}t + D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2\varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Step 2: Convergence of I_2. We observe that the only differences between I_1 and I_2 are u_ε and $\Delta\varphi$ in place of ∇u_ε and $\nabla\varphi$ respectively. As we have the same (in fact, better) estimates for these quantities, the proof is the same and we conclude

$$I_2 \to D \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi \, \mathrm{d}x \, \mathrm{d}t + D \int_0^T \int_{\mathbb{T}^d} u \, \nabla u \cdot \nabla\Delta\varphi.$$

Step 3: Convergence of I_3. For I_3 the proof is similar to the reasoning in Steps 1, 3 and 6 of the proof of Theorem 5.1.7because we have to use the same estimates. Roughly speaking, one proves that $u^\varepsilon \to u$ strongly in $L^k((0,T) \times \mathbb{T}^d)$ by interpolation so that one can identify $\xi = \sqrt{F_1''(u)} \nabla u$. Next, convergence in $L^k((0,T) \times \mathbb{T}^d)$ allows also to prove strong convergence $u_\varepsilon \sqrt{F_1''(u_\varepsilon)} \to u \sqrt{F_1''(u)}$ in $L^2((0,T) \times \mathbb{T}^d)$ thanks to growth condition (B) while the convergence $u_\varepsilon \sqrt{F_2''(u_\varepsilon)} \to u \sqrt{F_2''(u)}$ in $L^2((0,T) \times \mathbb{T}^d)$ is trivial because $F_2'' \in L^\infty$. This shows that

$$I_3 \to - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Conclusion of Steps 1-3. In the limit $\varepsilon \to 0$ we obtain

$$\int_0^T \langle \partial_t u, \varphi \rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))} = D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2\varphi +$$
$$+ \frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi + D \int_0^T \int_{\mathbb{T}^d} u \, \nabla u \cdot \nabla\Delta\varphi - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla\varphi.$$

Step 4: Regularity of u and better weak formulation. Now we prove the regularity of the limit function u. This allows us to perform integration by parts on the different terms using the formula (5.2.6) and recover the Definition 5.1.5. In fact, in the limit $\varepsilon \to 0$, from the entropy we obtain (see [41, Theorem 4] and [233, Theorem 1.2])

$$\sum_{i,j=1}^{d} \int_0^t \int_{\mathbb{T}^d} |\partial_{x_i}\partial_{x_j} u|^2 \leq \liminf_{\varepsilon\to 0} \frac{1}{4\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y)\, |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2$$

so in the limit $\varepsilon \to 0$ we gain one more derivative. Then, since

$$D \int_0^T \int_{\mathbb{T}^d} u \nabla u \cdot \nabla \Delta\varphi = -D \int_0^T \int_{\mathbb{T}^d} \Delta\varphi\, |\nabla u|^2 - D \int_0^T \int_{\mathbb{T}^d} u\, \Delta u\, \Delta\varphi$$

and using formula (5.2.6), we compute

$$\begin{aligned} I_1 + I_2 &= D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2\varphi - \frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi - D \int_0^T \int_{\mathbb{T}^d} u\, \Delta u\, \Delta\varphi \\ &= -D \int_0^T \int_{\mathbb{T}^d} \Delta u\, \nabla u \cdot \nabla\varphi - D \int_0^T \int_{\mathbb{T}^d} u\, \Delta u\, \Delta\varphi. \end{aligned}$$

This ends the proof of Theorem 5.1.8. □

5.5 A similar result for the aggregation-diffusion system

There are many PDEs in mathematical biology for which one can study the similar question of convergence of the nonlocal problem to the local one. The most crucial tools are again control of the energy and the dissipation of the entropy. To illustrate it, we consider the aggregation-diffusion system, studied in [70, 133], which is used to model cell-cell adhesion:

$$\frac{\partial\rho}{\partial t} = \nabla \cdot \left(\rho \nabla \left(\kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma\rho - \beta\eta\right)\right), \qquad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{5.5.1a}$$

$$\frac{\partial\eta}{\partial t} = \nabla \cdot \left(\eta \nabla \left(\alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta\rho - \eta\right)\right), \qquad \text{in} \quad (0, +\infty) \times \mathbb{T}^d \tag{5.5.1b}$$

where B_ε is defined in (5.1.1).

From the modelling point of view, we consider two populations of the cells. Parameters $\kappa > 0$ and $\gamma > 0$ represent the relative self-adhesion strength of ρ with respect to η; while $\alpha > 0$ and $\beta \in \mathbb{R}$ give the relative strength of the cross-attraction forces. The local version of (5.5.1a)–(5.5.1b) (that is, after sending $\varepsilon \to 0$) is useful for analysis of its steady states [133] but the nonlocal version of (5.5.1a)–(5.5.1b) has much better regularity properties. Therefore, it is important to understand connection between the local and the nonlocal models.

Hence, our target is to prove that as $\varepsilon \to 0$, solutions to (5.5.1a)–(5.5.1b) tend to the weak solution of the local system:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \left(-\kappa \Delta \rho - \alpha \Delta \eta - \gamma \rho - \beta \eta\right)\right), \tag{5.5.2a}$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot \left(\eta \nabla \left(-\alpha \Delta \rho - \Delta \eta - \beta \rho - \eta\right)\right). \tag{5.5.2b}$$

which is the first result of nonlocal-to-local convergence for degenerate systems.

The nonlocal system is associated with the following formal energy/entropy structure

$$\begin{aligned} E_\varepsilon[\rho,\eta] :=& \frac{1}{4}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(\kappa|\rho(x)-\rho(x-y)|^2 + |\eta(x)-\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{\alpha}{2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(\rho(x)-\rho(x-y))(\eta(x)-\eta(x-y))\,\mathrm{d}x\,\mathrm{d}y \\ &- \int_{\mathbb{T}^d} \frac{\gamma}{2}\rho^2 + \frac{1}{2}\eta^2 + \beta\rho\eta\,\mathrm{d}x, \end{aligned} \tag{5.5.3}$$

$$\Phi[\rho,\eta] := \int_{\mathbb{T}^d} \rho(\log(\rho)-1) + \eta(\log(\eta)-1)\,\mathrm{d}x. \tag{5.5.4}$$

Their dissipation is formally controlled by the identities

$$E_\varepsilon[\rho,\eta](t) + \int_0^t\int_{\mathbb{T}^d} \rho\,|\nabla \mu_{\rho,\varepsilon}|^2 + \int_0^t\int_{\mathbb{T}^d} \eta\,|\nabla \mu_{\eta,\varepsilon}|^2 \le E_\varepsilon[\rho_0,\eta_0], \tag{5.5.5}$$

$$\Phi[\rho,\eta](t) + \mathcal{D}\Phi[\rho,\eta](t) \le \Phi[\rho_0,\eta_0]. \tag{5.5.6}$$

where $\mathcal{D}\Phi[\rho,\eta](t)$ is the dissipation of the entropy defined as

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(\kappa|\nabla\rho(x)-\nabla\rho(x-y)|^2 + |\nabla\eta(x)-\nabla\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &+ \alpha\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(\nabla\rho(x)-\nabla\rho(x-y))\cdot(\nabla\eta(x)-\nabla\eta(x-y))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &- \int_0^t\int_{\mathbb{T}^d} \gamma|\nabla\rho|^2 + |\nabla\eta|^2 + 2\beta\nabla\rho\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}s \end{aligned}$$

and chemical potentials $\mu_{\rho,\varepsilon}$, $\mu_{\eta,\varepsilon}$ are defined as:

$$\mu_{\rho,\varepsilon} = \kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma\rho - \beta\eta \qquad\qquad \mu_{\eta,\varepsilon} = \alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta\rho - \eta.$$

The main mathematical difficulty is that E_ε and $\mathcal{D}\Phi[\rho,\eta]$ does not have to be positive. Here, we show that one can control their sign (up to a constant) if $\kappa > 0$ and $\kappa > \alpha^2$. Indeed, we have the following

Proposition 5.5.1. *Suppose that $\kappa > 0$, $\kappa > \alpha^2$. Then, there exists $\varepsilon_0 > 0$ depending on κ, α, β, γ with the following property: for all $\varepsilon \in (0, \varepsilon_0)$ and ρ, η such that $\|\rho\|_{L^1(\mathbb{T}^d)} = \|\eta\|_{L^1(\mathbb{T}^d)} = 1$, up to a constant, the energy defined by* (5.5.3) *and the dissipation of the entropy defined in* (5.5.6) *are nonnegative and provide the estimates on the quantities*

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(|\rho(x)-\rho(x-y)|^2 + |\eta(x)-\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y \le C + E_\varepsilon[\rho,\eta],$$

$$\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2}(|\nabla\rho(x)-\nabla\rho(x-y)|^2 + |\nabla\eta(x)-\nabla\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \le$$
$$\le C + \mathcal{D}\Phi[\rho,\eta](t),$$

where C depends on κ, α, β and γ.

The main tool to establish nonnegativity (up to a constant) are non-local Poincare inequalities (see Lemma 5.8.3) with parameter which allows to handle the negative terms.

Proof of Proposition 5.5.1. We first focus on the energy. We can estimate

$$-\int_{\mathbb{T}^d} \frac{\gamma}{2}\rho^2 + \frac{1}{2}\eta^2 + \beta\rho\,\eta \ge -\frac{\gamma+|\beta|}{2}\int_{\mathbb{T}^d}\rho^2 - \frac{1+|\beta|}{2}\int_{\mathbb{T}^d}\eta^2.$$

Then, we use (5.8.1) in Lemma 5.8.3 with $\delta := \delta/\max\left(\frac{\gamma+|\beta|}{2}, \frac{1+|\beta|}{2}\right)$ and $\delta > 0$ to be chosen later (this also determines $\varepsilon_0 = \varepsilon_0^C(\delta)$) so that we obtain for $\varepsilon \in (0,\varepsilon_0)$

$$-\int_{\mathbb{T}^d} \frac{\gamma}{2}\rho^2 + \frac{1}{2}\eta^2 + \beta\rho\,\eta \ge -\frac{\delta}{4}\int_{\mathbb{T}^d\times\mathbb{T}^d}\omega_\varepsilon(y)\frac{|\rho(x)-\rho(x-y)|^2}{\varepsilon^2}\,\mathrm{d}x\,\mathrm{d}y$$
$$-\frac{\delta}{4}\int_{\mathbb{T}^d\times\mathbb{T}^d}\omega_\varepsilon(y)\frac{|\eta(x)-\eta(x-y)|^2}{\varepsilon^2}\,\mathrm{d}x\,\mathrm{d}y - C(\delta).$$

Therefore, using (5.5.3), we can bound the energy as follows

$$E_\varepsilon[\rho,\eta] \geq \frac{1}{4\varepsilon^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)((\kappa-\delta)\,|\rho(x)-\rho(x-y)|^2+(1-\delta)\,|\eta(x)-\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y$$
$$+\frac{\alpha}{2\varepsilon^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)(\rho(x)-\rho(x-y))(\eta(x)-\eta(x-y))\,\mathrm{d}x\,\mathrm{d}y - C(\delta).$$

Now, by continuity, we choose δ so small so that $\kappa-\delta>0$ and $(\kappa-\delta)(1-\delta)-\alpha^2>0$, i.e. so that the matrix $\begin{pmatrix}\kappa-\delta & \alpha\\ \alpha & 1-\delta\end{pmatrix}$ is positively defined. It follows that the assosciated quadratic form is bounded from below, that is there exists constant C (in fact, this constant is the smallest eigenvalue of the matrix) such that

$$E_\varepsilon[\rho,\eta] \geq \frac{C}{\varepsilon^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)(|\rho(x)-\rho(x-y)|^2+\,|\eta(x)-\eta(x-y)|^2)\,\mathrm{d}x\,\mathrm{d}y - C(\delta).$$

The proof for the dissipation of the entropy is the same: this time we use (5.8.2) in place of (5.8.1). □

Having Proposition 5.5.1, we obtain compactness for the solutions of the nonlocal system (5.5.1a)–(5.5.1b) and we can justify rigorously the limit for $\kappa>0$, $\kappa>\alpha^2$. Of course, one can ask what happens for the remaining set of parameters. However, in this case the situation is difficult as even the existence theory for (5.5.1a)–(5.5.1b) is tricky because (5.5.1a)–(5.5.1b) is not a strongly parabolic system.

5.6 Open problem concerning bounded domains

One can ask if the same results hold when $\mathbb{T}^d$ is replaced with some general bounded domain Ω. More precisely, we focus on the system

$$\partial_t u_\varepsilon = \operatorname{div}(u_\varepsilon\nabla\mu_\varepsilon), \quad \text{in} \quad (0,+\infty)\times\Omega, \tag{5.6.1}$$

$$\mu_\varepsilon = B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in} \quad (0,+\infty)\times\Omega. \tag{5.6.2}$$

Defining $\vec{n}$ the outward normal vector to $\partial\Omega$ we impose the Neumann boundary condition

$$u_\varepsilon\frac{\partial\mu_\varepsilon}{\partial\vec{n}} = 0 \quad \text{on } \partial\Omega. \tag{5.6.3}$$

The operator B_ε is now defined as

$$\begin{aligned} B_\varepsilon[u_\varepsilon](x) = \frac{1}{\varepsilon^2}\left(\int_\Omega \omega_\varepsilon(x-y)\,\mathrm{d}y\, u_\varepsilon(x) - \omega_\varepsilon * u_\varepsilon(x)\right) = \\ = \frac{1}{\varepsilon^2}\int_\Omega \omega_\varepsilon(x-y)(u_\varepsilon(x)-u_\varepsilon(y))\,\mathrm{d}y. \end{aligned} \tag{5.6.4}$$

Notice that in the case $\Omega = \mathbb{T}^d$, this definition is the same than (5.1.1) up to a change of variable in the integral. However, since u_ε is not a priori defined outside Ω we need to put the argument $(x-y)$ on ω_ε.

In the limit, we expect to obtain solutions to

$$\partial_t u = \mathrm{div}(u\nabla\mu), \quad \text{in} \quad (0,+\infty)\times\Omega, \tag{5.6.5}$$

$$\mu = -D\Delta u + F'(u), \quad \text{in} \quad (0,+\infty)\times\Omega \tag{5.6.6}$$

$$\frac{\partial u}{\partial \vec{n}} = u\,\frac{\partial \mu}{\partial \vec{n}} = 0, \quad \text{on} \quad \partial\Omega. \tag{5.6.7}$$

However, there are two difficult problems related to the equation posed on a bounded domain.

- **Lack of the entropy estimate.** In the case of bounded domain, we cannot use entropy estimate as in (5.2.4). This is because the nonlocal operator is defined as (5.6.4) rather than (5.1.1). As a consequence, we cannot symmetrize the expression with gradients and obtain the term
$$\frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\varepsilon(y)\,|\nabla u(x)-\nabla u(x-y)|^2$$
in the dissipation of the entropy.

- **Recovery of the Neumann boundary conditions.** The question is whether we can prove that in the limit $\frac{\partial u}{\partial \vec{n}} = 0$ on $\partial\Omega$. This is possible for the equation with constant mobility. More precisely, in [91], Authors were discussing the problem of nonlocal to local convergence for the Cahn-Hilliard equation with constant mobility. The constant mobility allows to obtain uniform bound on $\|B_\varepsilon(u_\varepsilon)\|_2$ which allows to conclude that $\frac{\partial u}{\partial \vec{n}} = 0$ on $\partial\Omega$. This is an extremely interesting phenomenon as this new boundary condition appears only in the limit. In our case, the estimate $\|B_\varepsilon(u_\varepsilon)\|_2$ seems unavailable.

A possible approach to overcome this problem is to apply Serfaty-Sandier approach on the convergence of gradient flows [242, 246].

5.7 Appendix A: Results from classical analysis

Difference quotients

Lemma 5.7.1. *Let $\{u^\varepsilon\}$ be a sequence strongly compact in $L^2(0,T;H^1(\mathbb{T}^d))$. Then, for fixed $y \in \mathbb{T}^d$,*

$$\frac{u_\varepsilon(t,x-\varepsilon y) - u_\varepsilon(t,x)}{\varepsilon} \to -\nabla u(t,x)\cdot y \textit{ strongly in } L^2((0,T)\times\mathbb{T}^d).$$

Proof. We write

$$\begin{aligned}\frac{u_\varepsilon(t,x-\varepsilon y) - u_\varepsilon(t,x)}{\varepsilon} &= -y\cdot\int_0^1 \nabla u_\varepsilon(t,x-\varepsilon\theta y)\,\mathrm{d}\theta \\ &= -y\cdot\int_0^1 (\nabla u_\varepsilon(t,x-\varepsilon\theta y) - \nabla u_\varepsilon(t,x))\,\mathrm{d}\theta - y\cdot\nabla u_\varepsilon(t,x).\end{aligned}$$

By assumption $y\cdot\nabla u_\varepsilon \to y\cdot\nabla u$ strongly in $L^2((0,T)\times\mathbb{T}^d)$ so we only have to prove that the first term on the (RHS) converges to 0. By Fubini's theorem and Cauchy-Schwarz inequality

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d}\Big|\int_0^1 (\nabla u_\varepsilon(t,x-\varepsilon\theta y) - \nabla u_\varepsilon(t,x))\,\mathrm{d}\theta\Big|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\le C\int_0^1\int_0^T\int_{\mathbb{T}^d}|\nabla u_\varepsilon(t,x-\varepsilon\theta y) - \nabla u_\varepsilon(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\theta \\ &\qquad= C\int_0^1 \|\tau_{\varepsilon\theta y}\nabla u_\varepsilon - \nabla u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{T}^d)}\,\mathrm{d}\theta,\end{aligned}$$

where τ is the translation operator. The last term converges to 0 when $\varepsilon\to 0$ by the Fréchet Kolmogorov theorem. $\square$

Growth estimates on mollified nonlinearity

Lemma 5.7.2. *Let F satisfies Assumption 5.1.1 with constants C_1, ..., C_{10}. Then, $F_\delta = F * \eta_\delta$ with $0 \leq \delta \leq 1$ satisfies Assumption 5.1.1 with constants*

$$\widetilde{C_1} = 2^{1-k} C_1, \qquad \widetilde{C_2} = C_1 + C_2, \qquad \widetilde{C_3} = 2^{k-1}\, C_3,$$
$$\widetilde{C_4} = \widetilde{C_3} + C_4, \qquad \widetilde{C_5} = \min(2^{3-k}, 1)\, C_5, \qquad \widetilde{C_6} = C_5 + C_6,$$
$$\widetilde{C_7} = \max(2^{k-3}, 1)\, C_7, \qquad \widetilde{C_8} = \widetilde{C_7} + C_8, \qquad \widetilde{C_9} = C_9 + 2\, C_{10},$$
$$\widetilde{C_{10}} = 2\, C_{10}.$$

Proof. We decompose $F_{\delta,1} = F_1 * \eta_\delta$ and $F_{\delta,2} = F_2 * \eta_\delta$. Suppose that $F_1(u) \leq C_3 |u|^k + C_4$. Then,

$$F_{\delta,1}(u) = \int_{\mathbb{R}} F_1(u-s)\, \eta_\delta(s)\, \mathrm{d}s \leq C_3 \int_{\mathbb{R}} |u-s|^k \eta_\delta(s)\, \mathrm{d}s + C_4 \leq 2^{k-1} C_3 |u|^k + 2^{k-1} C_3 + C_4$$

where we used inequality valid for $p \geq 0$

$$|u-s|^p \leq \max(1, 2^{p-1})\, (|u|^p + |s|^p). \tag{5.7.1}$$

It follows that $\widetilde{C_3} = 2^{k-1} C_3$ and $\widetilde{C_4} = 2^{k-1} C_3 + C_4$. In a similar way, we compute constants $\widetilde{C_7}$, $\widetilde{C_8}$. For $\widetilde{C_1}, \widetilde{C_2}$, $\widetilde{C_5}, \widetilde{C_6}$ the reasoning is the same but we have to use a lower bound of the form

$$|u-s|^p \geq \min(1, 2^{1-p})\, |u|^p - |s|^p.$$

so that, for example, if $F_1 \geq C_1 |u|^k - C_2$ we have

$$F_{\delta,1}(u) = \int_{\mathbb{R}} F_1(u-s)\, \eta_\delta(s)\, \mathrm{d}s \geq C_1 \int_{\mathbb{R}} |u-s|^k \eta_\delta(s)\, \mathrm{d}s - C_2 \geq 2^{1-k}\, C_1\, |u|^p - C_1 - C_2.$$

For the constants $\widetilde{C_9}, \widetilde{C_{10}}$ we argue using (5.7.1) once again

$$F_{\delta,2}(u) \geq -C_9 - C_{10} \int_{\mathbb{R}} |u-s|^2 \eta_\delta(s)\, \mathrm{d}s \geq -C_9 - 2\, C_{10} - 2\, C_{10}\, |u|^2.$$

□

Potentials satisfying Assumption 5.1.1

Lemma 5.7.3. *Let F be as in (3) in Example 5.1.2. Then, F satisfies Assumption 5.1.1.*

Proof. On $\mathbb{R} \setminus I$ we define $F_1(u) = F(u)$. By [266, Theorem 3.2], there exists a C^2 extension of F_1 to $\mathbb{R}$ denoted by F_1 which preserves convexity, i.e. $F_1''(u) > b > 0$ for some $b > 0$. Moreover, F_1 has k-growth on $\mathbb{R}$ (in fact, the behaviour of F_1 on I can be included in constants C_2, C_4, C_6 and C_8 in Assumption 5.1.1). We finally define

$$F_2 = \begin{cases} F(u) - F_1(u) & \text{on } I, \\ 0 & \text{on } \mathbb{R} \setminus I. \end{cases}$$

Function F_2 is C^2 because at the endpoints of interval I we have $F'' = F_1''$ as F_1 is C^2 extension of F. Finally, F_2 satisfies condition (C) in Assumption 5.1.1 with $F_2(u) \geq -\|F_2\|_\infty$. $\square$

Proof of Lemma 5.3.1

Proof. First, we note the formula which will be useful

$$\phi(x) = \int_1^x \int_1^y \frac{1}{z} \, \mathrm{d}z \, \mathrm{d}y.$$

Now, we proceed to the proof. First, (1) follows from the definition. Next, (2) follows from writing

$$\phi_\delta(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{T_\delta(z)} \mathrm{sgn}(y-1) \, \mathrm{sgn}(x-1) \, \mathbb{1}_{y \in [1,x]} \, \mathbb{1}_{z \in [1,y]} \, \mathrm{d}z \, \mathrm{d}y, \tag{5.7.2}$$

and dominated convergence (for fixed $x > 0$). Then, (3) follows from $T_\delta \geq 0$ and the observation that $x \geq 1$, $x < 1$ implies $y \geq 1$, $y < 1$ respectively.

To see (4), we distinguish three cases.

- When $x \geq \frac{1}{\delta} - 1$, we split the integrals and use the estimate $T_\delta(x) \geq \frac{1}{\delta} - 1$ so that

$$\phi_\delta(x) \leq \int_1^{\frac{1}{\delta}-1} \int_1^y \frac{1}{z} \, \mathrm{d}z \, \mathrm{d}y + \int_{\frac{1}{\delta}-1}^x \int_1^y \frac{1}{\frac{1}{\delta} - 1} \, \mathrm{d}z \, \mathrm{d}y \leq$$
$$\leq \phi\left(\frac{1}{\delta} - 1\right) + \frac{\delta}{2(\delta - 1)} x^2 \leq \phi(x) + \frac{\delta}{\delta - 1}(x-1)^2,$$

because $\phi(x)$ is non-decreasing for $x \geq 1$.

- When, $x \in \left(2\delta, \frac{1}{\delta} - 1\right)$ we have $\phi_\delta = \phi$ because on this set $T_\delta(z) = z$.

- When $x \in [0, 2\delta]$ we have a lower bound $T_\delta(x) \geq \delta$ so that

$$\phi_\delta(x) \leq \int_x^{2\delta} \int_y^1 \frac{1}{\delta} + \int_{2\delta}^1 \int_y^1 \frac{1}{z} \, \mathrm{d}z \leq 2 + \phi(2\delta) \leq 3$$

  as $\phi(2\delta) \leq \phi(0) = 1$ because $\phi(x)$ is decreasing for $x \in (0, 1)$.

Finally, to see (5), let $x < 0$. Then,

$$\phi_\delta(x) \geq \int_x^0 \int_y^0 \frac{1}{\delta} \, \mathrm{d}z \, \mathrm{d}y = \frac{1}{\delta} \int_x^0 -y \, \mathrm{d}y = \frac{x^2}{2\delta}.$$

□

5.8 Appendix B: Nonlocal Poincaré inequalities

Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a smooth function, supported in the unit ball such that $\int_{\mathbb{R}^d} \omega(x) \, \mathrm{d}x = 1$. Consider $\omega_\varepsilon = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$.

Lemma 5.8.1. *There exists C_p and ε_0^A such that*

$$\int_{\mathbb{T}^d} |f - (f)_{\mathbb{T}^d}|^2 \leq \frac{1}{4\, C_p} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f(t,x) - f(t,y)|^2}{\varepsilon^2} \omega_\varepsilon(|x - y|) \, \mathrm{d}x \, \mathrm{d}y$$

for every $f \in L^2(\mathbb{T}^d)$ and $\varepsilon \leq \varepsilon_0^A$.

For the proof, we refer to Ponce [233, Theorem 1.1] with kernel given by (2.6.4). We also have an opposite inequality from [41, Theorem 1]:

Lemma 5.8.2. *For all $f \in H^1(\mathbb{T}^d)$*

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f(x) - f(y)|^2}{\varepsilon^2} \omega_\varepsilon(x - y) \, \mathrm{d}x \, \mathrm{d}y \leq C(\mathbb{T}^d) \, \|f\|_{H^1(\mathbb{T}^d)}^2.$$

Finally, we formulate a variant of Lemma 5.8.1 which does not require an average on the left-hand side.

Lemma 5.8.3. *For each $\gamma \in (0, 1)$ there exists $\varepsilon_0^B(\gamma)$ and constant $C(\gamma)$ such that for all $\varepsilon \in (0, \varepsilon_0^B)$ and all $f \in H^1(\mathbb{T}^d)$ we have*

$$\|f\|_{H^1(\mathbb{T}^d)}^2 \leq \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla f(x) - \nabla f(y)|^2}{\varepsilon^2} \omega_\varepsilon(|x - y|) \, \mathrm{d}x \, \mathrm{d}y + C(\gamma) \|f\|_{L^2(\mathbb{T}^d)}^2.$$

Similarly, for each $\delta \in (0,1)$ there exists $\varepsilon_0^C(\delta)$ and constant $C(\delta)$ such that for all $\varepsilon \in (0, \varepsilon_0^C)$ and all $f \in H^1(\mathbb{T}^d)$ we have:

$$\|f\|_2^2 \leq \delta \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) \frac{|f(x) - f(x-y)|^2}{\varepsilon^2} \, \mathrm{d}x \, \mathrm{d}y + C(\delta) \|f\|_1^2, \tag{5.8.1}$$

$$\|\nabla f\|_2^2 \leq \delta \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) \frac{|\nabla f(x) - \nabla f(x-y)|^2}{\varepsilon^2} \, \mathrm{d}x \, \mathrm{d}y + C(\delta) \|f\|_1^2. \tag{5.8.2}$$

Proof. Aiming at a contradiction, suppose that there exists γ with the following property: there exists sequence $\{\varepsilon_n\}$ with $0 < \varepsilon_n < \frac{1}{n}$ and sequence $\{f_n\}$ such that

$$\|f_n\|_{H^1(\mathbb{T}^d)}^2 > \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla f_n(x) - \nabla f_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) \, \mathrm{d}x \, \mathrm{d}y + n \, \|f_n\|_{L^2(\mathbb{T}^d)}^2.$$

As $\|f_n\|_{H^1(\mathbb{T}^d)} > 0$, we may define $g_n := \frac{f_n}{\|f_n\|_{H^1(\mathbb{T}^d)}}$. Note that $\|g_n\|_{H^1(\mathbb{T}^d)} = 1$ and

$$1 > \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla g_n(x) - \nabla g_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) \, \mathrm{d}x \, \mathrm{d}y + n \, \|g_n\|_{L^2(\mathbb{T}^d)}^2.$$

The first term gives compactness of the gradients (because $\{g_n\}$ is bounded in $H^1(\mathbb{T}^d)$ and Proposition 2.6.1) so that, together with Rellich-Kondrachov, there exists function g such that $g_n \to g$ in $H^1(\mathbb{T}^d)$ (after passing to a subsequence). But then $g = 0$ because $n \, \|g_n\|_{L^2(\mathbb{T}^d)} < 1$. This is however contradiction with $\|g\|_{H^1(\mathbb{T}^d)} = \lim_{n \to \infty} \|g_n\|_{H^1(\mathbb{T}^d)} = 1$.

Inequalities (5.8.1), (5.8.2) are proved in a similar manner. □

Chapter 6

High friction limit for the Euler-Korteweg equation

- C. Elbar, P. Gwiazda, J. Skrzeczkowski, A. Świerczewska–Gwiazda, *From nonlocal Euler-Korteweg to local Cahn-Hilliard*. In preparation, cited as [120].

6.1 Introduction

We consider the nonlocal Euler-Korteweg system re-scaled in time *i.e.* $t \to \frac{t}{\varepsilon}$ and with high friction coefficient $\frac{1}{\varepsilon}$

$$\partial_t \rho + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u}) = 0, \tag{6.1.1}$$

$$\partial_t(\rho \mathbf{u}) + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\varepsilon^2} \rho \mathbf{u} - \frac{1}{\varepsilon} \rho \nabla (F'(\rho) + B_\eta[\rho]), \tag{6.1.2}$$

considered on $(0, +\infty) \times \mathbb{T}^d$. This equation models the long-time asymptotics of the motion of a compressible fluid with density ρ, velocity $\mathbf{u}$ which is in fact a liquid-vapor mixture. The fluid experiences a high friction (due to the term $-\frac{1}{\varepsilon^2}\rho\mathbf{u}$) and additional capillary effects in the transition zone between liquid and vapour (due to the term $-\frac{1}{\varepsilon}\rho\nabla(F'(\rho) + B_\eta[\rho])$ as proposed by Korteweg [182]).

Concerning the notation, $\mathbb{T}^d$ is the d-dimensional flat torus, $\varepsilon > 0$, B_η is the nonlocal operator approximating $-\Delta$ operator, defined by

$$B_\eta[\rho](x) = \frac{1}{\eta^2}(\rho(x) - \omega_\eta * \rho(x)) = \frac{1}{\eta^2} \int_{\mathbb{T}^d} \omega_\eta(y)(\rho(x) - \rho(x-y)) \, \mathrm{d}y$$

for $\eta > 0$ small enough and ω_η is the usual radial mollification kernel $\omega_\eta(x) = \frac{1}{\eta^d}\omega(\frac{x}{\eta})$ with ω compactly supported in the unit ball of $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \omega(y)\,\mathrm{d}y = 1, \quad \int_{\mathbb{R}^d} y\,\omega(y)\,\mathrm{d}y = 0, \quad \int_{\mathbb{R}^d} y_i y_j \omega\,\mathrm{d}y = \delta_{i,j}\frac{2D}{d} < \infty. \tag{6.1.3}$$

We also define

$$\mu = F'(\rho) + B_\eta[\rho].$$

When ε is very small, the friction is so big, that we mostly observe a phase separation phenomenon between the liquid and the vapor. More rigorously, when $\varepsilon, \eta \to 0$ in some scaling to be determined, we prove that the constructed solution of (6.1.1)-(6.1.2) converge to solutions of the local Cahn-Hilliard equation

$$\partial_t \rho = \operatorname{div}(\rho \nabla \mu), \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{6.1.4}$$

$$\mu = -D\Delta\rho + F'(\rho), \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{6.1.5}$$

which describes the dynamics of phase separation.

Our proof relies on the relative entropy method, which uses similar arguments to the weak-strong uniqueness method. It usually requires the existence of classical solutions of the limit system, which is in this case the local Cahn-Hilliard equation. As the existence of the latter is still an open question (on arbitrary intervals of time), we introduce an intermediate step and consider the nonlocal Cahn-Hilliard equation by introducing the parameter η. Since we know from Chapter 5 that the nonlocal Cahn-Hilliard equation converges to the local Cahn-Hilliard equation when $\eta \to 0$, it remains to prove that the nonlocal Euler-Korteweg system tends to the nonlocal Cahn-Hilliard equation when $\varepsilon \to 0$. Then, sending ε and η to 0 with the appropriate scaling, we prove the result.

The main motivation for our work is the paper of Lattanzio and Tzavaras [187], who prove the convergence of the local Euler Korteweg system to the local Cahn-Hilliard equation. They assume the existence of dissipative (that is, satisfying energy inequality) weak solutions of the first system and classical solutions of the second

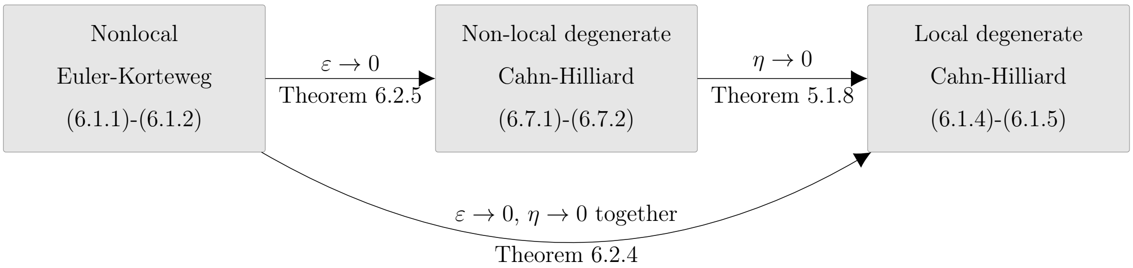

Figure 6.1: Relation between the three equations considered in this chapter.

one. The first assumption is a drawback as dissipative weak solutions existing on arbitrary intervals of times are not known to exist for most models in fluids dynamics. One can try to construct the solutions via convex integration method but these solutions will have a jump in the energy at the initial time so they will not be dissipative. The second assumption of [187] is also difficult to be satisfied as so far, there is no theory of classical solutions to the local Cahn-Hilliard equation with degenerate mobility on arbitrary interval of time. Similarly, there is no maximum principle which is necessary in [187] to deduce that the classical solution is strictly positive using positivity of the initial condition.

We propose to overcome the first problem by the concept of dissipative measure-valued solutions, introduced by DiPerna [110] in the context of hyperbolic conservation laws in one dimension and by DiPerna and Majda [114] for the incompressible Euler equations. Roughly speaking, they are defined as the weak limit of classical solutions of appropriate approximating problems. As weak compactness is not sufficient to pass to the limit in nonlinear terms, the definition of measure-valued solution includes the Young measure $\nu_{t,x}$ and the concentration measure m to represent weak limits as in (2.2.2).

While measure-valued solutions are weaker than the usual weak solutions (because they include potential concentration terms as in Proposition 2.2.1), they are dissipative and they are known to exist. Moreover, their importance comes from the fact that they enjoy the property called weak-strong uniqueness: they coincide with the strong solution whenever the latter exists. The dissipativity is important both for the weak-strong uniqueness and application of the relative entropy method: the weak-strong uniqueness does not hold for weak or measure-valued solutions without any condition on the energy as demonstrated by solutions arising by the convex integration method [96, 252].

Since the weak-strong uniqueness property was observed by Brenier, De Lellis and Székelyhidi in [46], dissipative measure-valued solutions were studied for several systems including compressible fluid models [165], isentropic Euler system [146], polyconvex elastodynamics [101], Euler-Poisson system [68], general hyperbolic conservation laws [162]. Moreover, for many equations describing compressible fluids, the measure-valued formulation has been significantly simplified [1, 27, 135]: it boils down to the usual distributional identity modulo the so-called Reynolds stress tensor.

Concerning the problem of the existence of classical solutions, we propose to introduce a nonlocality in the equation and introduce an intermediate step in the convergence analysis as outlined in Figure 6.1. The advantage is that the nonlocal Cahn-Hilliard equation is in fact a porous medium equation. In particular, it satisfies the maximum principle and so, if the initial condition is positive, the solution remains positive and one can prove existence and uniqueness of classical solution, see Section 6.7. Furthermore, we know that the nonlocal Cahn-Hilliard equation converges to the local one (see Chapter 5) so that at the end, the nonlocality can be removed.

To prove the convergence, we use the relative entropy method. The method is based on introducing a functional called relative entropy (or energy), which measures the dissipation between two solutions of the system. Essentially, the same method is used to prove the aforementioned weak-strong uniqueness when the relative entropy measures the distance between weak (measure-valued) and strong solution. This strategy has been applied for several singular limits [10, 71, 72, 80, 172, 186, 187] and we also refer to the excellent review on weak-strong uniqueness [261].

Our proof via the relative entropy method is based on an important assumption that the initial datum is well-prepared. In our case, this means that the initial velocity $\mathbf{u}_0$ vanishes as the parameter $\varepsilon \to 0$ cf. (6.2.1) and (6.2.2) so that the initial kinetic energy is very small. Such an assumption is necessary to guarantee that the relative entropy $\Theta(0)$ at time $t = 0$ converges to 0 as $\varepsilon \to 0$ so that $\Theta(t) \to 0$, cf. (6.8.5), which implies the main result. Let us however remark that one can also study similar problems via compactness methods and this approach is also effective for ill-prepared initial data. Nevertheless, its applicability is restricted to some special cases like one spatial dimension (which allows to use div-curl lemma in the time-space setting) [202] or presence of viscosity terms yielding compactness [136].

6.2 Rigorous formulation of the main result

We make the following assumptions on the potential F.

Assumption 6.2.1 (potential F)**.** For the interaction potential we assume that there exists $k \geq 2$ and constant C such that F can be written as $F = F_1 + F_2$ where

1. $F_1 \in C^4(\mathbb{R})$ is a convex, nonnegative function having k-growth

$$\frac{1}{C}|u|^k - C \leq F_1(u) \leq C|u|^k + C,$$

$$\frac{1}{C}|u|^{(k-2)} - C \leq F_1''(u) \leq C|u|^{(k-2)} + C$$

   and satisfying $|uF_1'(u)| \leq C(F_1(u) + 1)$, $|uF_1^{(3)}(u)| \leq C(F_1''(u) + 1)$,

2. $F_2 \in C^4(\mathbb{R})$ is such that $F_2, F_2', F_2'', sF_2^{(3)}(s) \in L^\infty(\mathbb{R})$ are bounded on the whole line. Moreover, $\|F_2''\|_\infty < C_p$ where C_p is a constant in Lemma 5.8.1.

We also define $s := \frac{2k}{k-1}$.

Example 6.2.2. The following potentials satisfy Assumption 6.2.1.

(1) power-type potential $F(u) = |u|^\gamma$, $\gamma > 2$ used in the context of tumour growth models [89, 118, 122, 229],

(2) double-well potential $F(u) = u^2\,(u-1)^2$ which is an approximation of logarithmic double-well potential often used in Cahn-Hilliard equation, see [218, Chapter 1].

Now, we define weak solutions of the local degenerate Cahn-Hilliard equation.

Definition 6.2.3. *We say that ρ is a weak solution of* (6.1.4)-(6.1.5) *if*

$$\rho \in L^\infty(0,T;L^k(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d)), \quad \partial_t \rho \in L^2(0,T;W^{-1,s'}(\mathbb{T}^d)),$$
$$\sqrt{F_1''(\rho)}\nabla\rho \in L^2((0,T)\times\mathbb{T}^d),$$

$\rho(0,x) = \rho_0(x)$ a.e. in $\mathbb{T}^d$ and if for all $\varphi \in L^2(0,T;W^{2,\infty}(\mathbb{T}^d))$ we have

$$\int_0^T \langle \partial_t \rho, \varphi\rangle_{(W^{-1,s'}(\mathbb{T}^d),W^{1,s}(\mathbb{T}^d))} = -D\int_0^T\int_{\mathbb{T}^d} \Delta\rho\,\nabla\rho\cdot\nabla\varphi - D\int_0^T\int_{\mathbb{T}^d}\rho\,\Delta\rho\,\Delta\varphi - \int_0^T\int_{\mathbb{T}^d}\rho\,F''(\rho)\,\nabla\rho\cdot\nabla\varphi.$$

Our main theorem is as follows.

Theorem 6.2.4. *Let ρ_0 be an initial density satisfying*

$$\rho_0 \geq \nu > 0, \qquad \rho_0 \in C^3(\mathbb{T}^d)$$

for some $\nu > 0$. Let $\boldsymbol{u}_{0,\varepsilon}$ be an initial velocity satisfying

$$\|\boldsymbol{u}_{0,\varepsilon}\|_{L^2(\mathbb{T}^d)} \to 0 \text{ as } \varepsilon \to 0. \tag{6.2.1}$$

Let $(\rho_{\eta,\varepsilon}, \overline{\sqrt{\rho_{\eta,\varepsilon}}\boldsymbol{u}_{\eta,\varepsilon}}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$ be an admissible dissipative measure-valued solution of (6.1.1)–(6.1.2) *with the initial condition $(\rho_0, \boldsymbol{u}_{0,\varepsilon})$ and parameters ε, η as defined in Definitions 6.5.1, 6.5.4 and 6.5.5. Then, for each sequence $\eta_k \to 0$, there exists a subsequence $\{\eta_k\}$ (not relabelled) and a sequence $\{\varepsilon_k\}$ depending on $\{\eta_k\}$ such that $\varepsilon_k \to 0$ and $\rho_{\eta_k,\varepsilon_k} \to \rho$ in $L^2((0,T)\times\mathbb{T}^d)$, where ρ is a weak solution of* (6.1.4)–(6.1.5) *with initial condition ρ_0 as defined in Definition 6.2.3.*

Let us briefly comment that the measure-valued solution has in fact four components. While the first component $\rho_{\eta,\varepsilon}$ is the most important since it converges to the Cahn-Hilliard equation, we can also characterize what happens with the other ones, see

Theorem 6.9.2. Roughly speaking, the second converges to 0 in $L^\infty(0,T;L^2(\mathbb{T}^d))$ which represents that in the high-friction limit, the kinetic energy converges to 0. The parametrized measure $\nu^{\eta,\varepsilon}$ converges in the second Wasserstein metric $\mathcal{W}_2$ to the Dirac mass $\delta_{\rho(t,x)} \otimes \delta_{\mathbf{0}}$:

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathcal{W}_2(\nu^{\eta_k,\varepsilon_k}, \delta_{\rho(t,x)} \otimes \delta_{\mathbf{0}})\right]^2 \mathrm{d}x\, \mathrm{d}t \to 0 \text{ as } \varepsilon_k, \eta_k \to 0$$

while the concentration measure m_{η_k,ε_k} converges to 0 in the total variation norm. The estimate in the Wasserstein metric is in the spirit of [137].

Theorem 6.2.4 is valid only for a subsequence as the convergence from non-local Cahn-Hilliard to the local one is based on the compactness arguments (and there is no uniqueness for the limit equation). On the other hand, the passage from nonlocal Euler-Korteweg equation to the nonlocal Cahn-Hilliard equation is based on the relative entropy method and so the convergence is satisfied for any sequence. We state this result below.

Theorem 6.2.5. *Let $\eta \in (0, \eta_0)$ where $\eta_0 = \varepsilon_0^A$ is a number defined in Lemma 5.8.1. Let ρ_0 be an initial density satisfying*

$$\rho_0 \geq \nu > 0, \qquad \rho_0 \in C^3(\mathbb{T}^d)$$

for some $\nu > 0$. Let $\boldsymbol{u}_{0,\varepsilon}$ be an initial velocity satisfying

$$\|\boldsymbol{u}_{0,\varepsilon}\|_{L^2(\mathbb{T}^d)} \to 0 \text{ as } \varepsilon \to 0. \tag{6.2.2}$$

Let $(\rho_{\eta,\varepsilon}, \overline{\sqrt{\rho_{\eta,\varepsilon}}\boldsymbol{u}_{\eta,\varepsilon}}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$ be an admissible dissipative measure-valued solution of (6.1.1)–(6.1.2) *with initial condition $(\rho_0, \boldsymbol{u}_{0,\varepsilon})$ and parameters ε, η as defined in Definitions 6.5.1, 6.5.4, 6.5.5 and let ρ_η be the solution of non-local Cahn-Hilliard* (6.7.1)-(6.7.2) *with the same initial condition ρ_0. Then, $\rho_{\eta,\varepsilon} \to \rho_\eta$ in $L^\infty(0,T;L^2(\mathbb{T}^d))$.*

Similarly as for Theorem 6.2.4, we can prove convergence of the other components of the measure-valued solution $\overline{\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}}$, $\nu^{\eta,\varepsilon}$, $m_{\eta,\varepsilon}$, see Theorem 6.9.1.

6.3 The Euler-Korteweg equation

The compressible Euler–Korteweg equation models the motion of liquid-vapor mixtures with possible phase transitions. It combines the classical Euler equation with Korteweg tensor introduced in [182]. The equation reads

$$\begin{aligned}&\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0,\\ &\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla(p(\rho)) = -\zeta\rho\mathbf{u} + \rho\nabla(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2).\end{aligned} \tag{6.3.1}$$

Here, ρ is the density of the fluid, $\mathbf{u}$ is its velocity, $K(\rho)$ corresponds to the capillary coefficient, ζ is the friction coefficient and p is the pressure function. In a liquid-vapor system, the tensor K takes into account that the liquid and vapour are separated by a thin layer of finite thickness and describes the capillary effects in this transition zone. There are numerous mathematical results concerning well-(and ill-)posedness of solutions to (6.3.1), see [14, 19, 31, 47, 48, 115]. For the physical background of (6.3.1) (in particular, the form of the Korteweg tensor) we refer to [117,170,177] but it is a fairly complicated matter.

The viscous version of (6.3.1), that is the Navier-Stokes-Korteweg system, was also studied in the mathematical literature [15, 149]. In particular, several papers are concerned with the case of the nonlocal equation, where $-\Delta\rho$ is approximated by the nonlocal operator B_η In [237], the author proves the short time well-posedness while in [74], the global well-posedness as well as the convergence of the nonlocal Navier-Stokes-Korteweg to the local one is established. We also refer to [73] for a variant of this system.

6.4 High-friction limit

The high-friction limit (also referred to in the literature as the relaxation limit) is a part of a long research programme of establishing a connection between nonlinear hyperbolic systems and degenerate diffusion equations. One of the first results in this direction [202] states that the solutions to the compressible Euler equations in one dimension

$$\begin{aligned} &\partial_t\rho + \partial_x(\rho\, u) = 0, \\ &\varepsilon^2\partial_t(\rho\, u) + \partial_x(\varepsilon^2\,\rho\, u^2 + p(\rho)) = -u \end{aligned} \tag{6.4.1}$$

converge, as $\varepsilon \to 0$, to the porous media equation

$$\partial_t\rho = \partial_x\left(\rho\,\partial_x p(\rho)\right)$$

where $p(\rho)$ is the pressure function of the form $p(\rho) = \rho^\gamma$. To connect (6.4.1) with our system (6.1.1)–(6.1.2), it is sufficient to rescale $\widetilde{u} = \varepsilon\, u$ so that we have

$$\begin{aligned} &\partial_t\rho + \frac{1}{\varepsilon}\partial_x(\rho\,\widetilde{u}) = 0, \\ &\partial_t(\rho\,\widetilde{u}) + \frac{1}{\varepsilon}\,\partial_x(\,\rho\,\widetilde{u}^2 + p(\rho)) = -\frac{\widetilde{u}}{\varepsilon^2}. \end{aligned} \tag{6.4.2}$$

Intuitively, it is easy to understand from (6.4.2) that the flow of the fluid with big damping or friction (caused by the term $-\frac{\widetilde{u}}{\varepsilon^2}$) and very small kinetic energy (caused by the initial condition) ressembles a flow through a porous media. Several other limit passages have been studied between porous medium equation and hyperbolic equations [17, 201, 258]. The revival of interest in this type of problems appeared recently with an observation that one can study these problems by the relative entropy method [80, 153, 186, 187].

In our case, we consider (6.3.1) with $K(\rho) = 1$, large friction coefficient $\zeta = \frac{1}{\varepsilon}$, we approximate the Laplace operator $-\Delta$ by the nonlocal operator B_η with η small enough, and we perform a rescaling in time $t \to \frac{t}{\varepsilon}$. Then, we let both $\varepsilon, \eta \to 0$ and in the limit we obtain the Cahn-Hilliard equation. Again, it is intuitive that due to the very large damping and small kinetic energy, we observe mostly a phase separation process. The latter is described by the Cahn-Hilliard equation so that it is not surprising that it is the limiting PDE.

6.5 Measure-valued solutions to the nonlocal Euler-Korteweg equation

Let us motivate the definition of a measure-valued solution by their construction. We will consider a sequence of approximating solutions $\{(\rho_\delta, \mathbf{u}_\delta)\}$ satisfying the estimates (uniform in δ)

$$\{\rho_\delta\} \text{ in } L^\infty(0,T;L^2(\mathbb{T}^d)), \qquad \{F(\rho_\delta)\} \text{ in } L^\infty(0,T;L^1(\mathbb{T}^d)),$$

$$\{\sqrt{\rho_\delta}\mathbf{u}_\delta\} \text{ in } L^\infty(0,T;L^2(\mathbb{T}^d)).$$

As we do not have estimates on $\{\mathbf{u}_\delta\}$ itself, we will consider in fact the sequence $\{(\rho_\delta, \sqrt{\rho_\delta}\,\mathbf{u}_\delta)\}$. Up to a subsequence, we have as $\delta \to 0$

$$\rho_\delta \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty(0,T;L^2(\mathbb{T}^d)) \qquad \sqrt{\rho_\delta}\mathbf{u}_\delta \overset{*}{\rightharpoonup} \overline{\sqrt{\rho}\mathbf{u}} \text{ in } L^\infty(0,T;L^2(\mathbb{T}^d)), \tag{6.5.1}$$

where $\overline{\sqrt{\rho}\mathbf{u}}$ is a *definition* of a weak limit of $\sqrt{\rho_\delta}\mathbf{u}_\delta$. Let $\{\nu_{t,x}\}$ be the Young measure generated by this sequence as in Theorem 2.1.1. We will use dummy variables $(\lambda_1, \lambda') \in \mathbb{R}^+ \times \mathbb{R}^d$ when integrating with respect to $\nu_{t,x}$:

$$\langle F(\lambda_1,\lambda'), \nu_{t,x}\rangle := \int_{\mathbb{R}^+\times\mathbb{R}^d} F(\lambda_1,\lambda')\,\mathrm{d}\nu_{x,t}(\lambda_1,\lambda'), \tag{6.5.2}$$

with λ_1 representing ρ variable and λ' as representing $\overline{\sqrt{\rho}\mathbf{u}}$ variable. In terms on Young measures we can write weak convergence (6.5.1) as

$$\rho = \langle \lambda_1, \nu\rangle, \qquad \overline{\sqrt{\rho}\mathbf{u}} = \langle \lambda', \nu\rangle, \tag{6.5.3}$$

as there is no concentration measure because of integrability in $L^2((0,T)\times\mathbb{T}^d)$. Using notation (2.2.2) we can represent weak limits (as $\delta \to 0$) of all the terms that should appear in the weak formulation and the energy

$$\overline{\rho^2} = \langle \lambda_1^2, \nu\rangle + m^{\rho^2}, \tag{6.5.4}$$

$$\overline{\rho\mathbf{u}} = \langle \sqrt{\lambda_1}\lambda', \nu\rangle, \tag{6.5.5}$$

$$\overline{\rho\mathbf{u}\otimes\mathbf{u}} = \langle \lambda'\otimes\lambda', \nu\rangle + m^{\rho\mathbf{u}\otimes\mathbf{u}}, \tag{6.5.6}$$

$$\overline{\rho|\mathbf{u}|^2} = \langle |\lambda'|^2, \nu\rangle + m^{\rho|\mathbf{u}|^2}, \tag{6.5.7}$$

$$\overline{F(\rho)} = \langle F(\lambda_1), \nu\rangle + m^{F(\rho)} \tag{6.5.8}$$

$$\overline{\rho\, F'(\rho)} = \langle \lambda_1 F'(\lambda_1), \nu \rangle + m^{\rho F'(\rho)}, \tag{6.5.9}$$

$$\overline{p(\rho)} = \overline{\rho\, F'(\rho)} - \overline{F(\rho)} + \frac{1}{2\eta^2}\, \overline{\rho^2}, \tag{6.5.10}$$

where $p(\rho) := \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$.

Moreover, we will identify weak limits of several nonlinearities which will be used in this work. By linearity of weak limits, we have the following identities:

$$\overline{\int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y} = \overline{\rho^2} + \overline{\rho^2} * \omega_\eta - 2\,\rho\, \omega_\eta * \rho \tag{6.5.11}$$

Similarly, for all bounded functions $P : (0, T) \times [0, +\infty) \to \mathbb{R}^+$ and vector fields $\mathbf{U} : (0, T) \times \mathbb{T}^d \to \mathbb{R}^d$ we have

$$|\rho - \mathrm{P}|^2 = \overline{\rho^2} + \mathrm{P}^2 - 2\rho\, \mathrm{P} \tag{6.5.12}$$

$$\overline{\rho|\mathbf{u} - \mathbf{U}|^2} = \overline{\rho\, |\mathbf{u}|^2} + \rho\, |\mathbf{U}|^2 - 2\, \overline{\rho \mathbf{u}} \cdot U, \tag{6.5.13}$$

$$\overline{\rho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} = \langle (\lambda' - \sqrt{\lambda_1}\mathbf{U}) \otimes (\lambda' - \sqrt{\lambda_1}\mathbf{U}), \nu_{t,x} \rangle + m^{\rho\mathbf{u}\otimes\mathbf{u}}, \tag{6.5.14}$$

$$\begin{aligned} &\overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} = \\ &\quad = \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y} + \int_{\mathbb{T}^d} \omega_\eta(y)|P(x) - P(x-y)|^2 \,\mathrm{d}y \\ &\quad\; - 2\int_{\mathbb{T}^d} \omega_\eta(y)(P(x) - P(x-y))(\rho(x) - \rho(x-y) \,\mathrm{d}y, \end{aligned} \tag{6.5.15}$$

$$\begin{aligned} \overline{F(\rho|\mathrm{P})} &:= \overline{F(\rho)} - F(\mathrm{P}) - F'(\mathrm{P})(\rho - \mathrm{P}), \\ \overline{p(\rho|\mathrm{P})} &:= \overline{p(\rho)} - p(\mathrm{P}) - p'(\mathrm{P})(\rho - \mathrm{P}) \end{aligned} \tag{6.5.16}$$

where nonlinearities are defined as

$$\begin{aligned} F(\rho|\mathrm{P}) &= F(\rho) - F(\mathrm{P}) - F'(\mathrm{P})(\rho - \mathrm{P}), \\ p(\rho|\mathrm{P}) &= p(\rho) - p(\mathrm{P}) - p'(\mathrm{P})(\rho - \mathrm{P}). \end{aligned} \tag{6.5.17}$$

Now, we define measure-valued solutions by *inverting* this discussion.

Definition 6.5.1 (Measure-valued solution)**.** *We say that* $(\rho, \overline{\sqrt{\rho}\boldsymbol{u}}, \nu, m)$ *where*

$$\nu = \{\nu_{t,x}\} \in L^\infty_{w^*}((0,T)\times\mathbb{T}^d; \mathcal{P}([0,+\infty)\times\mathbb{R}^d)$$

$$\rho = \langle \lambda_1, \nu\rangle = \int_{\mathbb{R}^+\times\mathbb{R}^d} \lambda_1 \,\mathrm{d}\nu_{x,t}(\lambda_1,\lambda') \in L^\infty(0,T;L^2(\mathbb{T}^d)),$$

$$\overline{\sqrt{\rho}\boldsymbol{u}} = \langle \lambda', \nu\rangle = \int_{\mathbb{R}^+\times\mathbb{R}^d} \lambda' \,\mathrm{d}\nu_{x,t}(\lambda_1,\lambda') \in L^\infty(0,T;L^2(\mathbb{T}^d)),$$

$$m = \left(m^{\rho^2}, m^{\rho\boldsymbol{u}\otimes\boldsymbol{u}}, m^{\rho|\boldsymbol{u}|^2}, m^{F(\rho)}, m^{\rho F'(\rho)}\right)$$

with

$$m^{\rho^2}, m^{\rho|\boldsymbol{u}|^2}, m^{F(\rho)} \in L^\infty((0,T);\mathcal{M}^+(\mathbb{T}^d)), \qquad m^{\rho F'(\rho)} \in L^\infty((0,T);\mathcal{M}(\mathbb{T}^d))$$
$$m^{\rho\boldsymbol{u}\otimes\boldsymbol{u}} \in L^\infty((0,T);\mathcal{M}(\mathbb{T}^d)^{d\times d})$$

and

$$|m^{\varrho\boldsymbol{u}\otimes\boldsymbol{u}}| \le m^{\varrho|\boldsymbol{u}|^2} \tag{6.5.18}$$

$$|m^{\rho\, F'(\rho)}| \le C_F\, m^{F(\rho)} + C_F\, m^{\rho^2}, C_F \text{ defined in } (6.10.2) \tag{6.5.19}$$

is a measure-valued solution of (6.1.1)-(6.1.2) *with initial data* $(\rho_0, \boldsymbol{u}_0)$ *if for every* $\psi \in C^1_c([0,T)\times\mathbb{T}^d;\mathbb{R})$, $\phi \in C^1_c([0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ *it holds that*

$$\int_0^T \int_{\mathbb{T}^d} \partial_t\psi\,\rho + \frac{1}{\varepsilon}\nabla\psi\cdot\overline{\rho\boldsymbol{u}}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d}\psi(x,0)\rho_0\,\mathrm{d}x = 0, \tag{6.5.20}$$

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \partial_t\phi\cdot\overline{\rho\boldsymbol{u}} + \frac{1}{\varepsilon}\nabla\phi : \overline{\rho\boldsymbol{u}\otimes\boldsymbol{u}} - \frac{1}{\varepsilon^2}\phi\cdot\overline{\rho\boldsymbol{u}} + \frac{1}{\varepsilon}\operatorname{div}\phi\,\overline{p(\rho)}\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\int_{\mathbb{T}^d}\frac{1}{\varepsilon\eta^2}\phi\cdot\rho\nabla\omega_\eta * \rho\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d}\phi(x,0)\cdot\rho_0u_0\,\mathrm{d}x = 0.\end{aligned} \tag{6.5.21}$$

where $p(\rho) = \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$ *and all the terms are defined in* (6.5.3)–(6.5.10).

Definition 6.5.2 (nonlinear functions)**.** *Let $(\rho, \overline{\sqrt{\rho}\boldsymbol{u}}, \nu, m)$ be a measure-valued solution. For all bounded $P : (0,T) \times [0,+\infty) \to \mathbb{R}^+$, $\boldsymbol{U} : (0,T) \times \mathbb{T}^d \to \mathbb{R}^d$, we define nonlinear quantities*

$$\overline{\int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y}, \qquad \overline{|\rho - \mathrm{P}|^2}, \qquad \overline{\rho|\boldsymbol{u} - \boldsymbol{U}|^2}, \qquad \overline{F(\rho|\mathrm{P})}, \qquad \overline{p(\rho|\mathrm{P})},$$

$$\overline{\rho(\boldsymbol{u} - \boldsymbol{U}) \otimes (\boldsymbol{u} - \boldsymbol{U})}, \qquad \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y}$$

by formulas (6.5.11)–(6.5.16)*.*

Definition 6.5.3 (energy)**.** *Given a measure-valued solution $(\rho, \overline{\sqrt{\rho}\boldsymbol{u}}, \nu, m)$ for a.e. $t \in (0,T)$ we define the energy as*

$$E_{mvs}(t) := \int_{\mathbb{T}^d} \frac{1}{2}\overline{\rho|\boldsymbol{u}|^2} + \overline{F(\rho)} \,\mathrm{d}x + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x,$$

where the nonlocal term is defined by (6.5.11)*. We also define*

$$E_0 := \int_{\mathbb{T}^d} \frac{1}{2}\rho_0|u_0|^2(x) + F(\rho_0) \,\mathrm{d}x + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y)|\rho_0(x) - \rho_0(x-y)|^2 \,\mathrm{d}x \,\mathrm{d}y.$$

This energy is well-defined because, by Proposition 2.2.1, a concentration measure $m \in L^\infty(0,T;\mathcal{M}(\mathbb{T}^d))$ admit disintegration $\mathrm{d}m(t,x) = m(t,\mathrm{d}x)\,\mathrm{d}t$ where $m(t,\cdot)$ is a well-defined measure on $\mathbb{T}^d$ for a.e. $t \in (0,T)$.

Definition 6.5.4 (Dissipativite measure-valued solution)**.** *We say that a measure-valued solution $(\rho, \sqrt{\rho}\boldsymbol{u}, \nu, m)$ is dissipative if*

$$E_{mvs}(t) + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho|\boldsymbol{u}|^2} \,\mathrm{d}x \,\mathrm{d}t \leq E_0 \tag{6.5.22}$$

for almost every $t \in (0,T)$.

Definition 6.5.5 (Admissible measure-valued solution)**.** *A measure-valued solution $(\rho, \overline{\sqrt{\rho}\boldsymbol{u}}, \nu, m)$ with initial condition ρ_0 is admissible if it satisfies nonlocal Poincare inequality: for a.e. $t \in (0,T)$ and all bounded $\mathrm{P} : \Omega_T \to [0,+\infty)$ such that $(\mathrm{P})_{\mathbb{T}^d} = (\rho_0)_{\mathbb{T}^d}$ we have*

$$\int_{\mathbb{T}^d} \overline{|\rho - P|^2} \,\mathrm{d}x \leq \frac{1}{4\,C_p\,\eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x. \tag{6.5.23}$$

where the constant C_p is given by Lemma 5.8.1.

Let us remark that in Lemma 6.8.2, we will prove that any measure-valued solution satisfies

$$\int_{\mathbb{T}^d} |\rho - P|^2 \,\mathrm{d}x \leq \frac{1}{4\, C_p\, \eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x.$$

which is a weaker version of (6.5.23). Nevertheless, (6.5.23) will be necessary to estimate several terms appearing in the application of the relative entropy method in Section 6.8. Let us also point out that similar Poincare-type inequalities are usually assumed for measure-valued solutions to several different PDEs, see for instance [134, eq. (2.23)].

We conclude with a simple observation concerning the energy.

Lemma 6.5.6. *The energy E_{mvs} defined by* (6.5.22) *is nonnegative.*

Proof. The lemma seems to be trivial from the point of view of our discussion about weak limits at the beginning of this section. However, the measure-valued solution is defined by Definition 6.5.1 so that we can argue only using Definitions 6.5.1 and 6.5.2. Clearly, $\overline{\frac{1}{2}\rho|\mathbf{u}|^2}$ and $\overline{F(\rho)}$ are nonnegative so that we only have to study the nonlocal term. By (6.5.11),

$$\int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y} = 2 \int_{\mathbb{T}^d} \overline{\rho^2} - 2 \int_{\mathbb{T}^d} \rho\, \omega_\eta * \rho.$$

By Cauchy-Schwarz and Young convolution inequalities:

$$2 \int_{\mathbb{T}^d} \rho\, \omega_\eta * \rho \,\mathrm{d}x \leq 2 \int_{\mathbb{T}^d} \rho^2 \,\mathrm{d}x.$$

Using Jensen's inequality (measure $\nu_{t,x}$ is the probability measure with respect to both coordinates)

$$\int_{\mathbb{T}^d} \rho^2 \,\mathrm{d}x = \int_{\mathbb{T}^d} \langle \lambda_1, \nu_{t,x} \rangle^2 \,\mathrm{d}x \leq \int_{\mathbb{T}^d} \langle \lambda_1^2, \nu_{t,x} \rangle \,\mathrm{d}x \leq \int_{\mathbb{T}^d} \overline{\rho^2} \,\mathrm{d}x \tag{6.5.24}$$

so that the nonlocal term is nonnegative. □

6.6 Existence of measure-valued solutions

The approximating system

To construct a measure-valued solution we use a method as outlined in [199, Section 5.5], see also [68, 161]. This is a fairly standard procedure based on regularizing density by a positive parameter

$$\rho_{0,\delta} = \rho_0 + \delta, \quad \rho_0 \in C^1(\mathbb{T}^d),\, \rho_0 > 0, \quad \mathbf{u}_{0,\delta}(x) = \mathbf{u}_0(x) \in W^{3,2}(\mathbb{T}^d)^d, \tag{6.6.1}$$

which makes the density ρ_δ globally bounded from below. We will only discuss the main steps and for the full presentation, we refer to [199, Section 5.5].

We work in $W^{3,2}(\mathbb{T}^d)^d$ (but for dimensions d higher than 3, we need to work even in $W^{1+d,2}(\mathbb{T}^d)$) because of the embedding $W^{3,2}(\mathbb{T}^d) \subset C^1(\mathbb{T}^d)$ which will be important for certain estimates. We use notation $(\!(\cdot,\cdot)\!)$ for the standard scalar product in $W^{3,2}(\mathbb{T}^d)^d$. By [199, Appendix, Theorem 4.11], we take $\{\boldsymbol{\omega_i}\}$ to be an orthonormal basis of $W^{3,2}(\mathbb{T}^d)^d$ which are $C^\infty(\mathbb{T}^d)^d$ functions. Finally, we define Π^N to be the projection operator into span$\{\boldsymbol{\omega_1}, ..., \boldsymbol{\omega_N}\}$ which satisfies $\left\|\Pi^N \mathbf{u}\right\|_{W^{3,2}} \leq \|\mathbf{u}\|_{W^{3,2}}$ and $\left\|\Pi^N \mathbf{u}\right\|_{L^2} \leq \|\mathbf{u}\|_{L^2}$.

We will find solution $(\rho_\delta, \mathbf{u}_\delta)$ such that

$$\begin{aligned}
&\rho_\delta \in L^\infty((0,T)\times\mathbb{T}^d) \cap L^2(0,T;W^{1,2}(\mathbb{T}^d)), \quad \frac{\partial \rho_\delta}{\partial t} \in L^2((0,T)\times\mathbb{T}^d) \\
&\mathbf{u}_\delta \in L^\infty(0,T;W^{3,2}(\mathbb{T}^d)), \quad \frac{\partial \mathbf{u}_\delta}{\partial t} \in L^2((0,T)\times\mathbb{T}^d),
\end{aligned} \tag{6.6.2}$$

to the following problem: for all $\psi \in C_c^1([0,T)\times\mathbb{T}^d;\mathbb{R})$, $\phi \in C_c^1([0,T)\times\mathbb{T}^d;\mathbb{R}^d)$ it holds that

$$\int_0^T \int_{\mathbb{T}^d} \partial_t\psi\rho_\delta + \frac{1}{\varepsilon}\nabla\psi\cdot\rho_\delta\mathbf{u}_\delta \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} \psi(x,0)\rho_{0,\delta}\,\mathrm{d}x = 0, \tag{6.6.3}$$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \partial_t\phi\cdot\rho_\delta\mathbf{u}_\delta + \frac{1}{\varepsilon}\nabla\phi : \rho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta - \frac{1}{\varepsilon^2}\phi\cdot\rho_\delta\mathbf{u}_\delta + \frac{1}{\varepsilon}\operatorname{div}\phi\, p(\rho_\delta)\,\mathrm{d}x\,\mathrm{d}t + \\
&\quad \int_0^T \int_{\mathbb{T}^d} \frac{1}{\varepsilon\eta^2}\phi\cdot\rho_\delta\nabla\omega_\eta * \rho_\delta\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d}\phi(x,0)\cdot\rho_{0,\delta}\mathbf{u}_{0,\delta}\,\mathrm{d}x = \delta\int_0^T (\!(\mathbf{u}_\delta,\phi)\!)\,\mathrm{d}t.
\end{aligned} \tag{6.6.4}$$

To find the solution to (6.6.3)–(6.6.4), we use the method of Galerkin approximations. We look for $\mathbf{u}^N$ of the form

$$\mathbf{u}^N = \sum_{j=1}^N c_j^N(t)\,\boldsymbol{\omega_j}$$

solving

$$\frac{\partial \rho^N}{\partial t} + \frac{1}{\varepsilon}\operatorname{div}(\rho^N \mathbf{u}^N) = 0, \tag{6.6.5}$$

$$\begin{aligned}\int_{\mathbb{T}^d}\left(\rho^N\partial_t\mathbf{u}^N + \frac{1}{\varepsilon}\rho^N\mathbf{u}^N\nabla\mathbf{u}^N + \frac{1}{\varepsilon^2}\rho^N\mathbf{u}^N + \frac{1}{\varepsilon}\nabla p(\rho^N)\right)\cdot \boldsymbol{\omega_i}\,\mathrm{d}x + \\ -\int_{\mathbb{T}^d}\frac{1}{\varepsilon\,\eta^2}\rho^N\nabla\omega_\eta * \rho^N\cdot\boldsymbol{\omega_i}\,\mathrm{d}x + \delta(\!(\mathbf{u}^N,\boldsymbol{\omega_i})\!) = 0,\end{aligned} \tag{6.6.6}$$

for $i = 1, ..., N$ with intial conditions $\rho^N(0) = \rho_{0,\delta}$, $\mathbf{u}^N(0) = \Pi^N\mathbf{u}_{0,\delta}$.

The proof of existence to (6.6.5)–(6.6.6) follows 3 steps: using a fixed point argument to prove the existence on a small interval, deriving a priori estimates on this interval, extending the procedure on the whole interval. The crucial point is the lower bound on ρ^N in terms of δ. This is obtained by the method of characteristics. Indeed,

$$\begin{aligned}\rho^N(t,x) \geq \operatorname{ess\,inf}_{x\in\mathbb{T}^d}\rho_{0,\delta}\,\exp\left(-\frac{1}{\varepsilon}\int_0^T\|\mathrm{div}u^N\|_\infty\,\mathrm{d}t\right) \geq \\ \geq \delta\,\exp\left(-\frac{1}{\varepsilon}\int_0^T\|u^N\|_{W^{3,2}}\,\mathrm{d}t\right)\end{aligned} \tag{6.6.7}$$

by the well-known formula for the continuity equation. On the other hand, thanks to the regularizing term, $\|u^N\|_{L^2(0,T;W^{3,2}(\mathbb{T}^d))} \leq \frac{C}{\delta}$. This gives uniform lower (and also upper) bound on ρ^N and allows to look at (6.6.6) as a system of ODEs. We refer to [199, Section 5.5] and omit the details. We obtain the following lemma:

Lemma 6.6.1. *For fixed N, there exists a solution to* (6.6.5)–(6.6.6) *such that $\rho^N \in C^1([0,T]\times\mathbb{T}^d)$, $\boldsymbol{u}^N \in C^1([0,T];W^{3,2}(\mathbb{T}^d)^d)$. Moreover, we have the energy estimate: for all times $\tau \in [0,T]$*

$$\begin{aligned}&\int_{\mathbb{T}^d}\frac{1}{2}\rho^N|\boldsymbol{u}^N|^2 + F(\rho^N)\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|\rho^N(x)-\rho^N(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y\\ &+\delta\int_0^\tau\|u^N\|^2_{W^{3,2}}\,\mathrm{d}t + \frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{T}^d}\rho^N|\boldsymbol{u}^N|^2\,\mathrm{d}x \leq\\ &\leq \int_{\mathbb{T}^d}\frac{1}{2}\rho_{0,\delta}|\boldsymbol{u}_0|^2 + F(\rho_{0,\delta})\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|\rho_{0,\delta}(x)-\rho_{0,\delta}(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{6.6.8}$$

as well as the following estimates

$$\rho^N(t,x) \geq C\left(\frac{1}{\delta}\right), \tag{6.6.9}$$

$$\left\|\rho^N\right\|_{L^\infty((0,T)\times\Omega)} + \int_0^\tau \left\|\partial_t \rho^N\right\|^2_{L^2(\mathbb{T}^d)} + \int_0^\tau \left\|\nabla\rho^N\right\|^2_{L^2(\mathbb{T}^d)} \leq C\left(\frac{1}{\delta}\right), \tag{6.6.10}$$

$$\int_0^T \left\|\partial_t \boldsymbol{u}^N\right\|^2_{L^2(\mathbb{T}^d)} + \delta\left\|\boldsymbol{u}^N\right\|_{L^\infty((0,T);W^{3,2}(\mathbb{T}^d)^d)} \leq C\left(\frac{1}{\delta}\right), \tag{6.6.11}$$

where $C\left(\frac{1}{\delta}\right)$ is a constant depending on $\frac{1}{\delta}$ and other fixed parameters (like ε).

Proof. The energy estimate follows by testing (6.6.6) by $\mathbf{u}^N$ (in the Galerkin sense: we multiply (6.6.6) by c_i^N and sum for $i = 1, ..., N$). Estimate (6.6.9) follows from the characteristics as explained in (6.6.7). Similarly, we obtain the upper bound. Concerning the estimates on derivatives of ρ^N, they follow by differentiating the formula from the method of characteristics and using the bound $\|u^N\|_{L^2(0,T;W^{3,2}(\mathbb{T}^d))} \leq \frac{C}{\delta}$. Finally, (6.6.11) is a consequence of testing (6.6.6) by $\partial_t \mathbf{u}^N$. □

Using the estimates in Lemma 6.6.1, up to a subsequence, we can pass to the limit $N \to \infty$

$$\begin{aligned} \rho^N &\to \rho_\delta \quad \text{strongly in } L^2((0,T)\times\mathbb{T}^d), \\ \mathbf{u}^N &\to \mathbf{u}_\delta \quad \text{strongly in } L^2((0,T)\times\mathbb{T}^d)^d \end{aligned}$$

(the convergence is true even in better spaces). We also have an energy inequality:

$$\begin{aligned} &\int_{\mathbb{T}^d} \frac{1}{2}\rho_\delta|\mathbf{u}_\delta|^2 + F(\rho_\delta)\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\eta(y)|\rho_\delta(x) - \rho_\delta(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &+ \delta\int_0^\tau \|u_\delta\|^2_{W^{3,2}}\,\mathrm{d}t + \frac{1}{\varepsilon^2}\int_0^\tau\int_{\mathbb{T}^d} \rho_\delta|\mathbf{u}_\delta|^2\,\mathrm{d}x \leq \\ &\leq \int_{\mathbb{T}^d} \frac{1}{2}\rho_{0,\delta}|\mathbf{u}_0|^2 + F(\rho_{0,\delta})\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \omega_\eta(y)|\rho_{0,\delta}(x) - \rho_{0,\delta}(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{6.6.12}$$

This concludes the proof of existence of $(\rho_\delta, \mathbf{u}_\delta)$ satisfying (6.6.3)–(6.6.4).

Existence of dissipative admissible measure-valued solutions

It remains to pass to the limit $\delta \to 0$ in (6.6.3)–(6.6.4). First we gather some uniform bounds in δ, being a simple consequence of (6.6.9) and (6.6.12), in the following lemma:

Lemma 6.6.2. *Let* $(\rho^\delta, \boldsymbol{u}^\delta)$ *be weak solutions of* (6.6.3)–(6.6.4) *as constructed above. Then, there exists a constant* $C > 0$ *independent of* δ *such that*

$$\rho_\delta \geq 0 \quad a.e. \text{ in } (0,T)\times\mathbb{T}^d,$$

$$\|\sqrt{\rho_\delta}\boldsymbol{u}_\delta\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C \quad \|F(\rho_\delta)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \leq C, \quad \|\rho_\delta\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C,$$

$$\delta\,\|\boldsymbol{u}_\delta\|^2_{L^2(0,T;W^{3,2}(\mathbb{T}^d))} \leq C$$

$$\|\partial_t\rho_\delta\|_{L^2(0,T;(W^{1,4}(\mathbb{T}^d))')} \leq C.$$

In fact, the proof of existence of dissipative measure-valued solution follows now the method described at the beginning of Section 6.5. By Lemma 6.6.2, we have sufficient estimates to have convergence (6.5.1) which allows us to define the Young measure $\{\nu_{t,x}\}$ as in (6.5.2)–(6.5.3). Then, the representations formulas for weak limits of nonlinearities (6.5.4)–(6.5.7) are a consequence of Lemma 2.2.1 and the estimate on $\left\|\sqrt{\rho_\delta}\mathbf{u}_\delta\right\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}$ which guarantees that all of the considered quantites are at least in $L^\infty(0,T;L^1(\mathbb{T}^d))$. Note that $m^{\rho\mathbf{u}} = 0$ because we have a uniform bound $\|\rho_\delta\mathbf{u}_\delta\|_{L^\infty(0,T;L^{\frac{4}{3}}(\mathbb{T}^d))} \leq C$. Next, (6.5.8) follows from the estimate on $\|F(\rho_\delta)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))}$. Here, the measure $m^{F(\rho)}$ is nonnegative because $F = F_1 + F_2$ where $F_1 \geq 0$ while F_2 is bounded so that the only concentration effect can arise from F_1. Similarly, by Assumption 6.2.1, $\|\rho_\delta\,F'(\rho_\delta)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \leq C$ so that (6.5.9) follows. Finally, (6.5.10) is a consequence of linearity and uniqueness of weak limits. This allows to pass to the limit $\delta \to 0$ in almost all of the terms in formulation (6.6.3)–(6.6.4).

Concerning the regularizing term on the (RHS) of (6.6.4), we observe that

$$\left|\delta\int_0^T ((\mathbf{u}_\delta,\phi))\,\mathrm{d}t\right| \leq \delta\,\|\mathbf{u}_\delta\|_{L^2((0,T);W^{3,2}(\mathbb{T}^d))}\,\|\phi\|_{L^2((0,T);W^{3,2}(\mathbb{T}^d))} \leq$$
$$\leq C\sqrt{\delta}\,\|\phi\|_{L^2((0,T);W^{3,2}(\mathbb{T}^d))} \to 0.$$

When it comes to the nonlocal terms, we observe that we can identify its weak limits because the convolution upgrades a weak convergence to the strong one. More precisely, if $\rho_\delta \xrightharpoonup{*} \rho$ in $L^\infty(0,T);L^2(\mathbb{T}^d))$, then $\rho_\delta * \omega_\eta \to \rho * \omega_\eta$ in $L^p(0,T;L^p(\mathbb{T}^d))$ strongly, for all $1 \leq p < \infty$. This follows by Lions-Aubin lemma and a standard subsequence argument as the sequence $\{\rho_\delta * \omega_\eta\}_\delta$ has uniformly bounded derivatives in the spatial derivaties while its time derivative is bounded in some negative Sobolev space by the estimate on $\{\partial_t \rho_\delta\}$ in Lemma 6.6.2.

Concerning (6.5.18), we notice that it is a consequence of the inequality

$$|\lambda' \otimes \lambda'| = \left(\sum_{i,j=1}^d \left(\lambda_i' \lambda_j'\right)^2 \right)^{1/2} = \sum_{i=1}^d |\lambda_i'|^2 = |\lambda'|^2$$

and Lemma 2.2.2. Similarly, (6.5.19) follows by virtue of Proposition 2.2.2 and inequality (6.10.2).

Next, the constructed measure-valued solution is dissipative in the sense of Definition 6.5.4 because we can pass to the limit in (6.6.12) using indetified weak limits (rigorously, one multiplies (6.6.12) with a nonnegative test function of time, passes to the limit and then perform a standard localization argument, see the proof below).

Finally, the constructed solution is admissible in the sense of Definition 6.5.5. Indeed, by Lemma 5.8.1 we have for all bounded and nonnegative $\varphi : [0,T] \to [0,\infty)$

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - \mathrm{P}) - \delta|^2 \leq$$
$$\leq \frac{1}{4\, C_p\, \eta^2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - \mathrm{P})(x) - (\rho_\delta - \mathrm{P})(x-y)|^2 \omega_\eta(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t$$

because $(\rho_\delta - \mathrm{P})_{\mathbb{T}^d} = \delta$. The (LHS) can be written as

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - \mathrm{P}) - \delta|^2 = \int_0^T \int_{\mathbb{T}^d} \varphi(t) \left((\rho_\delta - \mathrm{P})^2 + \delta^2 - 2\, \delta\, (\rho_\delta - \mathrm{P}) \right).$$

As $\rho_\delta - \mathrm{P}$ is bounded $L^\infty(0,T; L^2(\mathbb{T}^d))$, the last two terms vanish in the limit $\delta \to 0$. Finally, the term $(\rho_\delta - \mathrm{P})^2$ has weak limit $\overline{\rho^2} + \mathrm{P}^2 - 2\, \mathrm{P}\, \rho$ which is exactly $\overline{(\rho - \mathrm{P})^2}$, cf. (6.5.12). Similarly, we consider the term on the (RHS) so that we obtain

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) \overline{|\rho - \mathrm{P}|^2} \leq$$
$$\leq \frac{1}{4\, C_p\, \eta^2} \int_0^T \varphi(t) \overline{\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \, \mathrm{d}y \, \mathrm{d}x} \, \mathrm{d}t.$$

As this inequality holds for all φ, we conclude the proof of admissibility.

6.7 Classical solutions to the nonlocal Cahn-Hilliard equation

To prove the convergence of the measure-valued solution of the nonlocal Euler-Korteweg to a solution of the Cahn-Hilliard equation, we use arguments similar to weak-strong uniqueness. Therefore, we study below the classical solutions of the nonlocal Cahn-Hilliard equation. More precisely, we consider the equation

$$\partial_t \rho = \operatorname{div}(\rho \nabla \mu), \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{6.7.1}$$

$$\mu = B_\eta[\rho] + F'(\rho), \quad \text{in} \quad (0, +\infty) \times \mathbb{T}^d, \tag{6.7.2}$$

The initial condition is a smooth positive function, more precisely we consider for some $\alpha, \nu > 0$

$$\rho(0, x) = \rho_0(x), \quad \rho_0 \in C^{2+\alpha}(\mathbb{T}^d), \quad \rho_0(x) \geq \nu \quad \forall x \in \mathbb{T}^d. \tag{6.7.3}$$

We also suppose that $F \in C^4$ which is required by the parabolic regularity theory exploited in Lemma 6.7.2. Equations (6.7.1)-(6.7.2) can be rewritten as

$$\partial_t \rho - \Delta(\phi(\rho)) + \operatorname{div}(\rho\, b(\rho)) = 0 \tag{6.7.4}$$

with

$$\phi(\rho) = \frac{\rho^2}{2\eta^2} + \int_0^\rho s F''(s)\, \mathrm{d}s, \quad b(\rho) = \frac{\nabla \omega_\eta * \rho}{\eta^2}.$$

Theorem 6.7.1. *Equation* (6.7.4) *with initial condition* u_0 *satisfying* (6.7.3) *admits a classical unique solution.*

To prove this theorem we first consider an approximate problem and we define T_δ a smooth function such that

$$T_\delta(0) = \frac{\delta}{2}, \quad T_\delta(\rho) = \rho \text{ if } u \geq \delta, \quad T_\delta \text{ is increasing.}$$

The plan is to approximate (6.7.1) with

$$\partial_t \rho = \operatorname{div}(T_\delta(\rho) \nabla \mu). \tag{6.7.5}$$

We also define

$$\phi_\delta(\rho) := \int_0^\rho \frac{T_\delta(s)}{\eta^2}\,\mathrm{d}s + \int_0^\rho T_\delta(s)F''(s)\,\mathrm{d}s = \int_0^\rho T_\delta(s)\left(\frac{1}{\eta^2}+F''(s)\right)\mathrm{d}s \qquad (6.7.6)$$

so that (6.7.5) can be rewritten as a porous media equation

$$\partial_t\rho - \Delta(\phi_\delta(\rho)) + \mathrm{div}(\rho\, b(\rho)) = 0 \quad \rho(0,x) = \rho_0(x). \qquad (6.7.7)$$

From the properties of F we note that $\phi_\delta \geq 0$ and $\phi_\delta' \geq 0$.

Lemma 6.7.2 (existence)**.** *There exists a classical solution to* (6.7.7)*. Moreover, the solution obeys the maximum principle*

$$\underline{\rho}(t) := \nu\exp\left(-\int_0^t \|\mathrm{div}\, b(\rho)\|_{L^\infty}(s)\,\mathrm{d}s\right) \leq \rho(t,x) \leq \nu\exp\left(\int_0^t \|\mathrm{div}\, b(\rho)\|_{L^\infty}(s)\,\mathrm{d}s\right).$$

Proof. The existence follows from Theorem 5.3.2 in Chapter 5. To prove the maximum principle, we denote $w = \rho - \underline{\rho}$ so that

$$\partial_t w - \Delta(\phi_\delta(\rho)) + \mathrm{div}(w\, b(\rho)) + \underline{\rho}(\mathrm{div}(b(\rho)) - \|\mathrm{div}(b(\rho))\|_{L^\infty}) = 0,$$

$$w(0,x) = \rho_0(x) - \underline{\rho} \geq 0.$$

We multiply this equation by $\mathrm{sgn}^-(w) := \begin{cases} -1 \text{ if } w < 0 \\ 0 \text{ if } w \geq 0. \end{cases}$. We obtain, with $w^- = \min\{w,0\}$, $|w^-| = -\min\{w,0\}$.

$$\partial_t|w^-| + \Delta(\phi_\delta(\rho))\,\mathrm{sgn}^-(w) + \mathrm{div}(|w^-|\, b(\rho)) \leq 0.$$

Therefore integrating in space and using the inequality

$$\int_{\mathbb{T}^d} \Delta\phi_\delta(\rho)\,\mathrm{sgn}^-(w) \geq 0,$$

we obtain

$$\partial_t \int_{\mathbb{T}^d} |w^-| \leq 0.$$

Using the initial condition we conclude $|w^-| = 0$.

Since the solutions to (6.7.5) satisfy uniform lower bound, we obtain $T_\delta(\rho) = \rho$ for sufficiently small δ and thus classical solutions of Theorem 6.7.1. □

Lemma 6.7.3 (uniqueness)**.** *Classical nonnegative solutions to* (6.7.4) *are unique.*

Proof. We want to adapt usual L^1 contraction principle [259, Proposition 3.5] to the case with additional continuity equation term. Let ρ_1, ρ_2 be solutions to (6.7.7) and let $w = \rho_1 - \rho_2$. Equation for w reads

$$\partial_t w - \Delta(\phi(\rho_1) - \phi(\rho_2)) + \mathrm{div}(\rho_1\, b(\rho_1) - \rho_2\, b(\rho_2)) = 0.$$

We multiply this equation by $p_\varepsilon(\phi(\rho_1) - \phi(\rho_2))$ where p_ε approximates $p(u) = \mathbb{1}_{u>0}$ and $p_\varepsilon' \geq 0$. Then,

$$\int_{\mathbb{T}^d} \Delta(\phi(\rho_1) - \phi(\rho_2))\, p_\varepsilon(\phi(\rho_1) - \phi(\rho_2))\,\mathrm{d}x = -\int_{\mathbb{T}^d} p_\varepsilon'\, |\nabla(\phi(\rho_1) - \phi(\rho_2))|^2\,\mathrm{d}x \leq 0.$$

Concerning the other terms we notice that after sending $\varepsilon \to 0$ we arrive at the term $p(\phi(\rho_1) - \phi(\rho_2)) = p(\rho_1 - \rho_2)$ by monotonicity of ϕ. Therefore,

$$\int_{\mathbb{T}^d} \partial_t w\, p(\rho_1 - \rho_2)\,\mathrm{d}x = \partial_t \int_{\mathbb{T}^d} |w|_+\,\mathrm{d}x.$$

Now, concerning divergence term, we split it for two parts:

$$\mathrm{div}(\rho_1\, b(\rho_1) - \rho_2\, b(\rho_2)) = [\rho_1\, \mathrm{div} b(\rho_1) - \rho_2\, \mathrm{div} b(\rho_2)] + [\nabla\rho_1\, b(\rho_1) - \nabla\rho_2\, b(\rho_2)] = A + B.$$

The term A can be estimated in $L^1(\mathbb{T}^d)$ with

$$\begin{aligned}
\|A\|_1 &\leq \|\rho_1\, \mathrm{div} b(\rho_1) - \rho_2\, \mathrm{div} b(\rho_1)\|_1 + \|\rho_2\, \mathrm{div} b(\rho_1) - \rho_2\, \mathrm{div} b(\rho_2)\|_1 \\
&\leq \|\rho_1 - \rho_2\|_1\, \|\mathrm{div} b(\rho_1)\|_\infty + \frac{1}{\eta^2}\|\rho_2\|_1\, \|D^2\omega_\eta\|_\infty \|\rho_1 - \rho_2\|_1 \\
&\leq \frac{\|D^2\omega_\eta\|_\infty}{\eta^2}(\|\rho_1\|_1 + \|\rho_2\|_1)\, \|\rho_1 - \rho_2\|_1
\end{aligned}$$

where we used Young's convolutional inequality. Therefore,

$$\int_{\mathbb{T}^d} p\, A\,\mathrm{d}x \leq \|p\, A\|_1 \leq \frac{\|D^2\omega_\eta\|_\infty}{\eta^2}(\|\rho_1\|_1 + \|\rho_2\|_1)\, \|\rho_1 - \rho_2\|_1.$$

where we denoted for simplicity $p = p(\rho_1 - \rho_2)$. Concerning term B we write similarly

$$B = (\nabla\rho_1\, b(\rho_1) - \nabla\rho_2\, b(\rho_1)) + (\nabla\rho_2\, b(\rho_1) - \nabla\rho_2\, b(\rho_2)) =: B_1 + B_2.$$

As above, we easily obtain

$$\|B_2\|_1 \leq \frac{\|\nabla\omega_\eta\|_\infty}{\eta^2}\, \|\nabla\rho_2\|_1\, \|\rho_1 - \rho_2\|_1, \qquad \int_{\mathbb{T}^d} p\, B_2\,\mathrm{d}x \leq \frac{\|\nabla\omega_\eta\|_\infty}{\eta^2}\, \|\nabla\rho_2\|_1\, \|\rho_1 - \rho_2\|_1.$$

The term B_1 is more tricky. Keeping in mind that everything is multiplied by the term $p(\rho_1 - \rho_2)$ we have

$$\int_{\mathbb{T}^d} (\nabla\rho_1 - \nabla\rho_2)\ p(\rho_1 - \rho_2)\, b(\rho_1)\, \mathrm{d}x = \int_{\mathbb{T}^d} \nabla|\rho_1 - \rho_2|_+\, b(\rho_1)\, \mathrm{d}x =$$
$$= - \int_{\mathbb{T}^d} |\rho_1 - \rho_2|_+ \mathrm{div} b(\rho_1)\, \mathrm{d}x \leq \|\rho_1 - \rho_2\|_1 \, \|\rho_1\|_1 \, \frac{\|D^2\omega_\eta\|_\infty}{\eta^2}.$$

We conclude that for some constant C depending on L^1 norms of ρ_1, ρ_2 and $\nabla\rho_2$ we have

$$\partial_t \int_{\mathbb{T}^d} |\rho_1 - \rho_2|_+ \, \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\rho_1 - \rho_2| \, \mathrm{d}x.$$

Replacing ρ_1 and ρ_2 we obtain

$$\partial_t \int_{\mathbb{T}^d} |\rho_1 - \rho_2| \, \mathrm{d}x \leq C \int_{\mathbb{T}^d} |\rho_1 - \rho_2| \, \mathrm{d}x.$$

so that we conclude $\rho_1 = \rho_2$. □

6.8 Convergence of nonlocal Euler-Korteweg to nonlocal Cahn-Hilliard

To prove the convergence of nonlocal Euler-Korteweg equation to the nonlocal Cahn-Hilliard equation, we first rewrite the latter as a nonlocal Euler-Korteweg equation with an error term:

$$\partial_t \mathrm{P} + \frac{1}{\varepsilon}\, \mathrm{div}(\mathrm{P}\mathbf{U}) = 0, \tag{6.8.1}$$
$$\partial_t(\mathrm{P}\mathbf{U}) + \frac{1}{\varepsilon}\, \mathrm{div}\,(\mathrm{P}\mathbf{U} \otimes \mathbf{U}) = -\frac{1}{\varepsilon^2}\mathrm{P}\mathbf{U} - \frac{1}{\varepsilon}\mathrm{P}\nabla(F'(\mathrm{P}) + B_\eta[\mathrm{P}]) + e(\mathrm{P}, \mathbf{U}). \tag{6.8.2}$$

Here, velocity $\mathbf{U}$ is given by

$$\mathbf{U} = -\varepsilon\nabla(F'(\mathrm{P}) - B_\eta(\mathrm{P})) \tag{6.8.3}$$

and the error term is given by

$$\begin{aligned} e(\mathrm{P},\mathbf{U}) &= \partial_t(\mathrm{P}\mathbf{U}) + \frac{1}{\varepsilon}\, \mathrm{div}\,(\mathrm{P}\mathbf{U} \otimes \mathbf{U}) \\ &= \varepsilon\, \mathrm{div}(\mathrm{P}\nabla(F'(\mathrm{P} + B_\eta[\mathrm{P}])) \otimes \nabla(F'(\mathrm{P} + B_\eta[\mathrm{P}]))) - \varepsilon\partial_t(\mathrm{P}\nabla(F'(\mathrm{P} + B_\eta[\mathrm{P}]))). \end{aligned}$$

Finally, given strong solution $(P, \mathbf{U})$ and measure-valued solution represented by $(\rho, \overline{\sqrt{\rho}\mathbf{u}}, \nu, m)$ we define the relative entropy as

$$\begin{aligned} \Theta(t) = \int_{\mathbb{T}^d} \frac{1}{2}\overline{\rho|\mathbf{u} - \mathbf{U}|^2} + \overline{F(\rho|\mathrm{P})} \,\mathrm{d}x + \\ + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x. \end{aligned} \tag{6.8.4}$$

where nonlinearity $F(\rho|\mathrm{P})$ is defined in (6.5.17) and measure-valued terms are defined by (6.5.13), (6.5.15) and (6.5.16). The main result reads:

Theorem 6.8.1. *Let $(\rho, \overline{\sqrt{\rho}\boldsymbol{u}}, \nu, m)$ be an admissible dissipative measure valued solution of* (6.1.1)–(6.1.2) *and let* $(\mathrm{P}, \boldsymbol{U})$ *be classical solutions of* (6.8.1)–(6.8.2)*. Then, for a constant independent of ε and η we have*

$$\Theta(t) \leq \left(\Theta(0) + \varepsilon^4\, C(\|\mathrm{P}\|_{C^{2,1}}) \left\|\frac{1}{\mathrm{P}}\right\|_\infty^2\right) e^{TC(\|\mathrm{P}\|_{C^{2,1}})/\eta^{d+3}}. \tag{6.8.5}$$

Lemma 6.8.2. *Let $\eta \in (0, \eta_0)$. Then, the relative entropy defined by* (6.8.4) *is nonnegative: there exists a $\kappa \in (0,1)$ such that*

$$\int_{\mathbb{T}^d} \overline{\rho|\boldsymbol{u} - \boldsymbol{U}|^2} \,\mathrm{d}x + \frac{\kappa}{4\eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x \leq \Theta(t) \tag{6.8.6}$$

where both terms on the (LHS) are nonnegative. Moreover, for the constant C_p (defined in Lemma 5.8.1) we have an estimate

$$\|\rho - \mathrm{P}\|_{L^2(\mathbb{T}^d)}^2 \leq \frac{1}{4\, C_p\, \eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathrm{P})(x) - (\rho - \mathrm{P})(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x. \tag{6.8.7}$$

Proof of Theorem 6.8.1. We study the three terms appearing in (6.8.4) separately. First, for (6.5.13) we write by Fubini theorem

$$\begin{aligned} \overline{\rho|\mathbf{u} - \mathbf{U}|^2} = \left\langle |\lambda'|^2 + \lambda_1 |\mathbf{U}|^2 - 2\sqrt{\lambda_1}\, \lambda' \mathbf{U}, \nu_{t,x} \right\rangle + m^{\rho|\mathbf{u}|^2} = \\ = \left\langle |\lambda' - \sqrt{\lambda_1}\mathbf{U}|^2, \nu_{t,x} \right\rangle + m^{\rho|\mathbf{u}|^2}, \end{aligned}$$

so that, after integration in space, it is positive (for a.e. $t \in (0,T)$). Now, we study the nonlocal term. We claim that (after integration)

$$\begin{aligned} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y} \,\mathrm{d}x \geq \\ \geq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x-y)|^2 \,\mathrm{d}y \,\mathrm{d}x. \end{aligned} \tag{6.8.8}$$

Indeed, by definition (6.5.11), the (LHS) equals $2\int_{\mathbb{T}^d}\overline{\rho^2} - \int_{\mathbb{T}^d} 2\,\rho\,\omega_\eta * \rho$. By (6.5.24), we know that $\int_{\mathbb{T}^d}\overline{\rho^2} \geq \int_{\mathbb{T}^d}\rho^2$. To conclude the proof of (6.8.8), it is sufficient to observe

$$2\int_{\mathbb{T}^d}\rho^2 - \int_{\mathbb{T}^d} 2\,\rho\,\omega_\eta * \rho = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta|\rho(x)-\rho(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x.$$

Now, combining (6.5.15) and (6.8.8), we obtain

$$\begin{aligned}\int_{\mathbb{T}^d}\overline{\int_{\mathbb{T}^d}\omega_\eta(y)|(\rho-\mathrm{P})(x)-(\rho-\mathrm{P})(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x} \geq \\ \geq \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|(\rho-\mathrm{P})(x)-(\rho-\mathrm{P})(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Using Lemma 5.8.1, we conclude the proof of (6.8.7) and nonnegativity of the nonlocal term.

It remains to study the term $\overline{F(\rho|\mathrm{P})}$. The concentration measure $m^{F(\rho)}$ is nonnegative and will neglected in the estimate below. We split $F = F_1 + F_2$ (where F_1, F_2 are defined in Assumption 6.2.1) in (6.5.16) so that from (6.5.8) and (6.5.16)

$$\begin{aligned}\overline{F(\rho|\mathrm{P})} &= \langle F_1(\lambda_1) - F_1(\mathrm{P}) - F_1'(\mathrm{P})(\lambda_1-\mathrm{P}), \nu_{t,x}\rangle \\ &\quad + \langle F_2(\lambda_1) - F_2(\mathrm{P}) - F_2'(\mathrm{P})(\lambda_1-\mathrm{P}), \nu_{t,x}\rangle + m^{F(\rho)}\end{aligned} \tag{6.8.9}$$

The first term is nonnegative by convexity of F_1. The second can be estimated from below (by Taylor's expansion) with $-\|F_2''\|_\infty\,\langle(\lambda_1-\mathrm{P})^2,\nu_{t,x}\rangle$. Now, recall that $\|F_2''\|_\infty < C_p$ (cf. Assumption 6.2.1). In particular, there exists $\kappa\in(0,1)$ such that $\|F_2''\|_\infty < (1-\kappa)\,C_p$. Using admissibility (Definition 6.5.5) and the fact that the concentration measure m^{ρ^2} is nonnegative we have

$$\begin{aligned}&-\|F_2''\|_\infty\int_{\mathbb{T}^d}\big\langle(\lambda_1-\mathrm{P})^2,\nu_{t,x}\big\rangle\,\mathrm{d}x \\ &\quad\geq -\|F_2''\|_\infty\int_{\mathbb{T}^d}\overline{(\rho-\mathrm{P})^2}\,\mathrm{d}x \geq -(1-\kappa)\,C_p\int_{\mathbb{T}^d}\overline{(\rho-P)^2}\,\mathrm{d}x \\ &\quad\geq -\frac{1-\kappa}{4\eta^2}\int_{\mathbb{T}^d}\overline{\int_{\mathbb{T}^d}\omega_\eta(y)|(\rho-\mathrm{P})(x)-(\rho-\mathrm{P})(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x}.\end{aligned} \tag{6.8.10}$$

Therefore, we can compensate a possibly negative term with the positive nonlocal term appearing in (6.8.4). □

Proof of Theorem 6.8.1. We split the reasoning for several steps.

Step 1: Energy identities. First, we recall that the dissipative measure valued solutions satisfy

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{1}{2}\overline{\rho|\mathbf{u}|^2} + \overline{F(\rho)}\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\overline{\int_{\mathbb{T}^d}\omega_\eta(y)|\rho(x)-\rho(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x} \\ +\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\overline{\rho|\mathbf{u}|^2}\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathbb{T}^d}\frac{1}{2}\rho_0|u_0|^2(x) + F(\rho_0)\,\mathrm{d}x \\ +\frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|\rho_0(x)-\rho_0(x-y)|^2\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{6.8.11}$$

where the quantities on the (LHS) of (6.8.11) are evaluated at time t. Similarly, the classical solutions $(\mathrm{P},\mathbf{U})$ satisfy

$$\begin{aligned}&\int_{\mathbb{T}^d}\frac{1}{2}\mathrm{P}|\mathbf{U}|^2 + F(\mathrm{P})\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|\mathrm{P}(x)-\mathrm{P}(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{\mathbb{T}^d}\frac{1}{2}\mathrm{P}_0|u_0|^2(x) + F(\mathrm{P}_0)\,\mathrm{d}x + \frac{1}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\omega_\eta(y)|\mathrm{P}_0(x)-\mathrm{P}_0(x-y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad -\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\mathrm{P}|\mathbf{U}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot e(\mathrm{P},\mathbf{U})\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{6.8.12}$$

Identity (6.8.12) can be obtained from testing (6.8.1)–(6.8.2) by $\mathbf{U}$ and performing several integration by parts.

Step 2: Estimate for the mixed terms $F'(\mathrm{P})(\rho-\mathrm{P})$, $B_\eta[\mathrm{P}]$ and $\rho\,|\boldsymbol{U}|^2$. We consider weak solutions of the mass equation satisfied by the differences between the measure valued solutions and classical solutions:

$$\int_0^T\int_{\mathbb{T}^d}\partial_t\psi(\rho-\mathrm{P}) + \frac{1}{\varepsilon}\nabla\psi\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d}\psi(x,0)(\rho_0-\mathrm{P}_0)\,\mathrm{d}x = 0. \tag{6.8.13}$$

We set

$$\theta_\delta(t) := \begin{cases} 1 & \text{for } 0\le\tau<t, \\ \frac{t-\tau}{\delta}+1 & \text{for } t\le\tau<t+\delta, \\ 0 & \text{for } \tau\ge t+\delta. \end{cases}$$

Note that $\theta'(t)$ is an approximation of the dirac mass $-\delta_t$. We consider test function in (6.8.13) defined as $\psi = \theta_\delta(t)\left(F'(\mathrm{P}) + B_\eta[\mathrm{P}] - \frac{1}{2}|\mathbf{U}|^2\right)$ so that after letting $\delta\to 0$

we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)(\rho-\mathrm{P})\Big|_{\tau=0}^{t}\,\mathrm{d}x=\\
&=+\int_0^t\int_{\mathbb{T}^d}\partial_\tau\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)(\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\nabla\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{6.8.14}
$$

Step 3: Estimate for the mixed term $\overline{\rho\mathbf{u}}\,\boldsymbol{U}$. We consider weak solutions of the momentum equation satisfied by the differences between the measure valued solutions and classical solutions:

$$
\begin{aligned}
&\int_0^\infty\int_{\mathbb{T}^d}\partial_t\phi\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})+\frac{1}{\varepsilon}\nabla\phi:(\overline{\rho\mathbf{u}\otimes\mathbf{u}}-\mathrm{P}\mathbf{U}\otimes\mathbf{U})-\frac{1}{\varepsilon^2}\phi\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\\
&\int_0^\infty\int_{\mathbb{T}^d}\frac{1}{\varepsilon}\operatorname{div}\phi(\overline{p(\rho)}-p(\mathrm{P}))+\frac{1}{\varepsilon\eta^2}\phi\cdot(\rho\nabla\omega_\eta*\rho-\mathrm{P}\cdot\nabla\omega_\eta*\mathrm{P})-\phi\cdot e(\mathrm{P},\mathbf{U})\\
&\qquad\qquad+\int_{\mathbb{T}^d}\phi(x,0)\cdot(\rho_0\mathbf{u}_0-\mathrm{P}_0\mathbf{U}_0)\,\mathrm{d}x=0.
\end{aligned}
$$

We consider the test function $\phi=\theta_\delta(t)\mathbf{U}$ so that after letting $\delta\to 0$ we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\Big|_{\tau=0}^{t}\,\mathrm{d}x=\int_0^t\int_{\mathbb{T}^d}\partial_\tau\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau\\
&+\int_0^t\int_{\mathbb{T}^d}\frac{1}{\varepsilon}\nabla\mathbf{U}:(\overline{\rho\mathbf{u}\otimes\mathbf{u}}-\mathrm{P}\mathbf{U}\otimes\mathbf{U})+\frac{1}{\varepsilon}\operatorname{div}(\mathbf{U})(\overline{p(\rho)}-p(\mathrm{P}))\,\mathrm{d}x\,\mathrm{d}\tau\\
&-\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{\varepsilon\eta^2}\int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot(\rho\nabla\omega_\eta*\rho-\mathrm{P}\cdot\nabla\omega_\eta*\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau\\
&-\int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot e(\mathrm{P},\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}
\tag{6.8.15}
$$

Step 4: First estimate on the relative entropy. Let us observe that when we subtract (6.8.12), (6.8.14) and (6.8.15) from (6.8.11) we obtain an estimate for $\Theta(t)-\Theta(0)$. To see this, let us write explicitly the (LHS) after the subtraction (we omit integral with respect to x for simplicity and we consider only terms at time $\tau=t$; of course, for $\tau=0$, they will be analogous):

$$
\begin{aligned}
&\frac{1}{2}\overline{\rho|\mathbf{u}|^2}+\overline{F(\rho)}+\frac{1}{4\eta^2}\overline{\int_{\mathbb{T}^d}\omega_\eta(y)|\rho(x)-\rho(x-y)|^2\,\mathrm{d}y}-\frac{1}{2}\mathrm{P}|\mathbf{U}|^2-F(\mathrm{P})\\
&-\frac{1}{4\eta^2}\int_{\mathbb{T}^d}\omega_\eta(y)|\mathrm{P}(x)-\mathrm{P}(x-y)|^2\,\mathrm{d}y-\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)(\rho-\mathrm{P})-\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U}).
\end{aligned}
$$

We claim that this expression equals $\Theta(t)$. Indeed, the terms containing both density and velocity sum up to the term $\overline{\rho|\mathbf{u}-\mathbf{U}|^2}$ as in (6.5.13). Similarly, terms with the potential F and its derivative F' can be combined to (6.5.16). Finally, for the nonlocal term, the claim is the consequence of two identities:

$$B_\eta[\mathrm{P}]\,\rho = \frac{1}{2\eta^2}\int_{\mathbb{T}^d}\omega_\eta(y)(\mathrm{P}(x)-\mathrm{P}(x-y))\,(\rho(x)-\rho(x-y))\,\mathrm{d}y$$

and the similar one for $B_\eta[\mathrm{P}]\,\mathrm{P}$ we can easily see that this expression equals $\Theta(t)$. Subtracting all the terms on the (RHS) of (6.8.12), (6.8.14),(6.8.15) from (RHS) of (6.8.11) we obtain

$$\begin{aligned}\Theta(t)-\Theta(0) \le &-\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\overline{\rho|\mathbf{u}|^2}-\mathrm{P}|\mathbf{U}|^2-\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\int_0^t\int_{\mathbb{T}^d}\partial_\tau\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)(\rho-\mathrm{P})+\partial_\tau\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\nabla\left(F'(\mathrm{P})+B_\eta[\mathrm{P}]-\frac{1}{2}|\mathbf{U}|^2\right)\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\nabla\mathbf{U}:(\overline{\rho\mathbf{u}\otimes\mathbf{u}}-\mathrm{P}\mathbf{U}\otimes\mathbf{U})+\mathrm{div}(\mathbf{U})(\overline{p(\rho)}-p(\mathrm{P}))\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot(\rho\nabla\omega_\eta*\rho-\mathrm{P}\cdot\nabla\omega_\eta*\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{6.8.16}$$

Step 5: Terms with $\partial_\tau U$ in (6.8.16). To estimate the right-hand side of (6.8.16) we first try to eliminate time derivative from (6.8.16). To this end, we compute $\partial_t\mathbf{U}$ from the equations (6.8.1)-(6.8.2) to obtain that $\mathbf{U}$ satisfies

$$\partial_t\mathbf{U}+\frac{1}{\varepsilon}(\mathbf{U}\cdot\nabla)\,\mathbf{U} = -\frac{1}{\varepsilon^2}\mathbf{U}-\frac{1}{\varepsilon}\nabla(F'(\mathrm{P})+B_\eta[\mathrm{P}])+\frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}. \tag{6.8.17}$$

We take the scalar product of this equation with $\overline{\rho\,\mathbf{u}}-\rho\,\mathbf{U}$ which yields

$$\begin{aligned}&\partial_t\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})+\frac{1}{2}\partial_t|\mathbf{U}|^2(\mathrm{P}-\rho)+\frac{1}{\varepsilon}\nabla\mathbf{U}:(\overline{\rho\mathbf{u}}\otimes\mathbf{U}-\rho\mathbf{U}\otimes\mathbf{U})\\ &=\frac{1}{\varepsilon^2}(\rho|\mathbf{U}|^2-\mathbf{U}\cdot\overline{\rho\mathbf{u}})-\frac{1}{\varepsilon}\nabla(F'(\mathrm{P})+B_\eta[\mathrm{P}])\cdot(\overline{\rho\mathbf{u}}-\rho\mathbf{U})+\frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}\cdot(\overline{\rho\mathbf{u}}-\rho\mathbf{U}).\end{aligned}$$

where we used identities

$$\frac{1}{\varepsilon}(\mathbf{U}\cdot\nabla)\mathbf{U}\cdot(\overline{\rho\,\mathbf{u}}-\rho\mathbf{U}) = \frac{1}{\varepsilon}\nabla\mathbf{U}:(\overline{\rho\mathbf{u}}\otimes\mathbf{U}-\rho\mathbf{U}\otimes\mathbf{U}),$$

$$\partial_t\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\rho\mathbf{U}) = \partial_t\mathbf{U}\cdot(\overline{\rho\mathbf{u}}-\mathrm{P}\mathbf{U})+\frac{1}{2}\partial_t|\mathbf{U}|^2(\mathrm{P}-\rho).$$

Finally, using matrix identity $x\,A\,y = A : x \otimes y$ where $x, y \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$ we easily deduce the formula

$$\nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \mathbf{U} - \rho\mathbf{U}\otimes\mathbf{U}) = \nabla\mathbf{U} : (\overline{\rho\mathbf{u}\otimes\mathbf{u}} - \mathrm{P}\mathbf{U}\otimes\mathbf{U})$$
$$- \nabla\mathbf{U} : \overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})} - \nabla\left(\frac{1}{2}|\mathbf{U}|^2\right)(\overline{\rho\mathbf{u}} - \mathrm{P}\mathbf{U}),$$

where

$$\overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})} := \overline{\rho\mathbf{u}\otimes\mathbf{u}} - \overline{\rho\mathbf{u}}\otimes\mathbf{U} - \mathbf{U}\otimes\overline{\rho\mathbf{u}} + \rho\mathbf{U}\otimes\mathbf{U}.$$

We obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t|\mathbf{U}|^2\,(\mathrm{P}-\rho) + \partial_t(\mathbf{U})\cdot(\overline{\rho\mathbf{u}} - \mathrm{P}\mathbf{U}) - \frac{1}{\varepsilon}\nabla\left(\frac{1}{2}|\mathbf{U}|^2\right)\cdot(\overline{\rho\mathbf{u}} - \mathrm{P}\mathbf{U}) + \\
&+ \frac{1}{\varepsilon}\nabla\mathbf{U} : (\overline{\rho\mathbf{u}\otimes\mathbf{u}} - \mathrm{P}\mathbf{U}\otimes\mathbf{U}) = \frac{1}{\varepsilon}\nabla\mathbf{U} : \overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})} + \\
&= \frac{1}{\varepsilon^2}(\rho|\mathbf{U}|^2 - \mathbf{U}\cdot\overline{\rho\mathbf{u}}) - \frac{1}{\varepsilon}\nabla(F'(\mathrm{P}) + B_\eta[\mathrm{P}])\cdot(\overline{\rho\mathbf{u}} - \rho\mathbf{U}) + \frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}\cdot(\overline{\rho\mathbf{u}} - \rho\mathbf{U}).
\end{aligned} \tag{6.8.18}$$

Note that this gives us an estimate on four terms appearing on the (RHS) of (6.8.16).

Step 6: Terms with F' and B_η in (6.8.16). We now consider the expression

$$-\int_0^t\int_{\mathbb{T}^d}\partial_\tau\left(F'(\mathrm{P}) + B_\eta[\mathrm{P}]\right)(\rho - \mathrm{P}) + \frac{1}{\varepsilon}\nabla\left(F'(\mathrm{P}) + B_\eta[\mathrm{P}]\right)\cdot(\overline{\rho\mathbf{u}} - \mathrm{P}\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \int_0^t\int_{\mathbb{T}^d}\frac{1}{\varepsilon}\nabla(F'(\mathrm{P}) + B_\eta[\mathrm{P}])\cdot(\overline{\rho\mathbf{u}} - \rho\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau.$$

The first integral comes from (6.8.16) while the second from (6.8.18) plugged into (6.8.16). We can simplify this to get

$$-\int_0^t\int_{\mathbb{T}^d}\partial_\tau\left(F'(\mathrm{P}) + B_\eta[\mathrm{P}]\right)(\rho - \mathrm{P}) + \frac{1}{\varepsilon}\nabla\left(F'(\mathrm{P}) + B_\eta[\mathrm{P}]\right)\cdot\mathbf{U}(\rho - \mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.8.19}$$

We split the term with $B_\eta[\mathrm{P}] = \frac{\mathrm{P}}{\eta^2} - \frac{\mathrm{P}*\omega_\eta}{\eta^2}$ for the local and non-local parts. Now, concerning the terms with potential F, we use (6.8.1) to deduce

$$\partial_t F'(\mathrm{P}) = F''(\mathrm{P})\,\partial_t\mathrm{P} = -\frac{1}{\varepsilon}F''(\mathrm{P})\,\nabla\mathrm{P}\cdot\mathbf{U} - \frac{1}{\varepsilon}F''(\mathrm{P})\,\mathrm{P}\,\mathrm{div}\mathbf{U} =$$
$$= -\frac{1}{\varepsilon}\nabla F'(\mathrm{P})\cdot\mathbf{U} - \frac{1}{\varepsilon}F''(\mathrm{P})\,\mathrm{P}\,\mathrm{div}\mathbf{U}.$$

Similarly,

$$\frac{1}{\eta^2}\partial_t \mathrm{P} = -\frac{1}{\varepsilon\,\eta^2}\nabla \mathrm{P}\cdot\mathbf{U} - \frac{1}{\varepsilon\,\eta^2}\mathrm{P}\,\mathrm{div}\mathbf{U}.$$

Therefore, the local parts of (6.8.19) sum up to

$$-\frac{1}{\varepsilon}\left(F''(\mathrm{P})\,\mathrm{P} + \frac{1}{\eta^2}\,\mathrm{P}\right)\,\mathrm{div}\mathbf{U}\,(\rho-\mathrm{P}) = -\frac{1}{\varepsilon}p'(\mathrm{P})\,\mathrm{div}\mathbf{U}\,(\rho-\mathrm{P})$$

which together with $-\frac{1}{\varepsilon}\,\mathrm{div}(\mathbf{U})(\overline{p(\rho)} - p(\mathrm{P}))\,\mathrm{div}\mathbf{U}$ from (6.8.16) gives $\overline{p(\rho|\mathrm{P})}\,\mathrm{div}\mathbf{U}$, where

$$\overline{p(\rho|\mathrm{P})} := \overline{p(\rho)} - p(\mathrm{P}) - p'(\mathrm{P})\,(\rho-\mathrm{P}).$$

Now, we consider the nonlocal parts in (6.8.19) and the last nonlocal term coming from (6.8.16). which equals

$$\begin{aligned}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\partial_\tau\,(\mathrm{P}*\omega_\eta)\,(\rho-\mathrm{P}) + \frac{1}{\varepsilon\,\eta^2}\nabla\,(\mathrm{P}*\omega_\eta)\cdot\mathbf{U}(\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau \\ -\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\mathbf{U}\cdot(\rho\nabla\omega_\eta*\rho - \mathrm{P}\cdot\nabla\omega_\eta*\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned} \tag{6.8.20}$$

Using (6.8.1) and properties of the convolution we can rewrite the first term in (6.8.20):

$$\begin{aligned}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\partial_\tau\,(\mathrm{P}*\omega_\eta)\,(\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau &= -\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\mathrm{div}(\mathrm{P}\mathbf{U})\,(\omega_\eta*(\rho-\mathrm{P}))\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}\mathrm{P}\mathbf{U}\cdot(\nabla\omega_\eta*(\rho-\mathrm{P}))\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned}$$

so that (6.8.20) boils down to

$$-\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}(\rho-\mathrm{P})\mathbf{U}\cdot\nabla\omega_\eta*(\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau.$$

Step 7: Final estimate on the relative entropy. Using the steps above and (6.8.16) we obtain

$$\begin{aligned}\Theta(t)-\Theta(0) \leq {}& -\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\overline{\rho|\mathbf{u}-\mathbf{U}|^2}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_{\mathbb{T}^d}\frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}(\overline{\rho\mathbf{u}}-\rho\mathbf{U})\,\mathrm{d}x\,\mathrm{d}\tau \\ & -\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\nabla\mathbf{U} : \overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})}\,\mathrm{d}x\,\mathrm{d}\tau - \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\mathrm{div}(\mathbf{U})\,\overline{p(\rho|\mathrm{P})}\,\mathrm{d}x\,\mathrm{d}\tau \\ & -\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}(\rho-\mathrm{P})\mathbf{U}\cdot\nabla\omega_\eta*(\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau =: A+B+C+D+E.\end{aligned} \tag{6.8.21}$$

By definition of $\mathbf{U}$ we notice that

$$\|\mathbf{U}\|_\infty, \|\nabla\mathbf{U}\|_\infty, |e(\mathrm{P},\mathbf{U})| \leq \varepsilon\, C(\|\mathrm{P}\|_{C^{2,1}}),$$

where $C(\|\mathrm{P}\|_{C^{2,1}})$ is a numerical constant which depends on $\|\mathrm{P}\|_{C^{2,1}}$ and blows up when $\eta \to 0$ since we don't have estimates in C^2 of the solutions of the local Cahn-Hilliard equation. Now, we estimate the terms appearing on the (RHS) of (6.8.21).

<u>*Term E.*</u> For the nonlocal term E we use boundedness of $\mathbf{U}$ to have

$$\left|\frac{1}{\varepsilon}\frac{1}{\eta^2}\int_0^t\int_{\mathbb{T}^d}(\rho-\mathrm{P})\mathbf{U}\cdot\nabla\omega_\eta * (\rho-\mathrm{P})\,\mathrm{d}x\,\mathrm{d}\tau\right| \leq$$
$$\leq \frac{C\,\|\mathbf{U}\|_\infty}{\eta^2}\,\|\rho-\mathrm{P}\|_2\,\|\nabla\omega_\eta * (\rho-\mathrm{P})\,\|_2 \leq \frac{C\|\mathbf{U}\|_\infty}{\eta^{d+3}}\|\rho-\mathrm{P}\|_2^2.$$

Using (6.8.7) for $\eta \in (0,\eta_0)$ we obtain

$$E \leq \frac{C(\|\mathrm{P}\|_{C^{2,1}})}{4\eta^{d+3}}\int_0^t \Theta(\tau)\,\mathrm{d}\tau.$$

<u>*Term B.*</u> Using (6.5.3) and (6.5.5) we can write

$$B = -\int_0^t\int_{\mathbb{T}^d}\left\langle \frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}\sqrt{\lambda_1}(\lambda' - \sqrt{\lambda_1}\mathbf{U}), \nu_{t,x}\right\rangle \mathrm{d}x\,\mathrm{d}\tau$$

Using Cauchy-Schwartz with a parameter

$$B \leq \int_0^t\int_{\mathbb{T}^d}\left\langle \frac{\varepsilon^2}{2}\left|\frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}\right|^2\lambda_1 + \frac{1}{2\varepsilon^2}|\lambda' - \sqrt{\lambda_1}\mathbf{U}|^2, \nu_{t,x}\right\rangle \mathrm{d}x\,\mathrm{d}\tau$$

Now, $\left|\frac{e(\mathrm{P},\mathbf{U})}{\mathrm{P}}\right| \leq \varepsilon\, C(\|\mathrm{P}\|_{C^{2,1}})\,\left\|\frac{1}{\mathrm{P}}\right\|_\infty$. Moreover, expanding the square in $|\lambda' - \sqrt{\lambda_1}\mathbf{U}|^2$ and using (6.5.5), (6.5.7) we recognize that

$$\int_0^t\int_{\mathbb{T}^d}\left\langle|\lambda' - \sqrt{\lambda_1}\mathbf{U}|^2, \nu_{t,x}\right\rangle \leq \int_0^t\int_{\mathbb{T}^d}\overline{\rho|\mathbf{u}-\mathbf{U}|^2}.$$

Therefore, we have the estimate

$$B \leq \varepsilon^4\, C(\|\mathrm{P}\|_{C^{2,1}})\,\left\|\frac{1}{\mathrm{P}}\right\|_\infty^2 + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^d}\overline{\rho|\mathbf{u}-\mathbf{U}|^2}.$$

<u>*Term C.*</u> We have

$$\left|\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\nabla\mathbf{U} : \overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})}\,\mathrm{d}x\,\mathrm{d}\tau\right| \leq$$
$$\leq \frac{C\,\|\nabla U\|_\infty}{\varepsilon}\int_0^t\int_{\mathbb{T}^d}\left|\overline{\rho(\mathbf{u}-\mathbf{U})\otimes(\mathbf{u}-\mathbf{U})}\right|\mathrm{d}x\,\mathrm{d}\tau$$

Estimating directly under the integral in (6.5.14)

$$\left|\langle(\lambda' - \sqrt{\lambda_1}\mathbf{U}) \otimes (\lambda' - \sqrt{\lambda_1}\mathbf{U}), \nu_{t,x}\rangle\right| \leq \langle|\lambda' - \sqrt{\lambda_1}\mathbf{U}|^2, \nu_{t,x}\rangle$$

and using (6.5.18) we arrive at

$$C \leq C(\|P\|_{C^{2,1}}) \int_0^t \int_{\mathbb{T}^d} \overline{\rho|\mathbf{u} - \mathbf{U}|^2} \, \mathrm{d}x \, \mathrm{d}\tau$$

<u>*Term D.*</u> Using (6.5.16) and (6.5.10), we can write

$$|\overline{p(\rho|\mathrm{P})}| \leq \langle p(\lambda_1) - p(\mathrm{P}) - p'(\mathrm{P})(\lambda_1 - \mathrm{P}), \nu_{t,x}\rangle + |m^{\rho\, F'(\rho)}| + m^{F(\rho)} + \frac{1}{\eta^2}\, m^{\rho^2}.$$

The first part can be estimated using (6.10.1):

$$\langle p(\lambda_1|\mathrm{P}), \nu_{t,x}\rangle \leq C_{F,R}\langle F(\lambda_1|\mathrm{P}), \nu_{t,x}\rangle + \left(C_{F,R} + \frac{1}{\eta^2}\right)\langle(\lambda_1 - P)^2, \nu_{t,x}\rangle \tag{6.8.22}$$

The concentration measures part can be estimated using (6.5.19):

$$|m^{\rho\, F'(\rho)}| + m^{F(\rho)} + \frac{1}{\eta^2}\, m^{\rho^2} \leq (C_F + 1)m^{F(\rho)} + \left(C_F + \frac{1}{\eta^2}\right) m^{\rho^2}. \tag{6.8.23}$$

Summing up (6.8.22) and (6.8.23) we obtain

$$|\overline{p(\rho|\mathrm{P})}| \leq C\, \overline{F(\rho|\mathrm{P})} + C\left(1 + \frac{1}{\eta^2}\right)\overline{|\rho - \mathrm{P}|^2}.$$

The last term can be estimated by the nonlocal term appearing in the definition of Θ due to the admissibility condition (6.5.23). As $\overline{F(\rho|\mathrm{P})}$ also appears in the definition of Θ we obtain

$$D \leq \left|\frac{1}{\varepsilon}\int_0^t \int_{\mathbb{T}^d} \mathrm{div}(\mathbf{U})\overline{p(\rho|\mathrm{P})} \, \mathrm{d}x \, \mathrm{d}\tau\right| \leq C(\|\mathrm{P}\|_{C^{2,1}})\left(1 + \frac{1}{\eta^2}\right)\int_0^t \Theta(\tau)\, \mathrm{d}\tau.$$

We conclude that for $\eta < 1$:

$$\Theta(t) \leq \Theta(0) + \frac{C(\|\mathrm{P}\|_{C^{2,1}})}{4\eta^{d+3}}\int_0^t \Theta(\tau)\, \mathrm{d}\tau + \varepsilon^4\, C(\|\mathrm{P}\|_{C^{2,1}})\left\|\frac{1}{\mathrm{P}}\right\|_\infty^2$$

Using Gronwall's lemma, we obtain (6.8.5). □

Proof of Theorem 6.2.5. The proof is a direct consequence of (6.8.5). Indeed, we consider the relative entropy Θ as in (6.8.4) with $\rho = \rho_{\eta,\varepsilon}$, $\mathbf{u} = \mathbf{u}_{\eta,\varepsilon}$, $\mathrm{P} = \rho_\eta$ and

$\mathbf{U} = -\varepsilon\nabla(F'(\rho_\eta) - B_\eta(\rho_\eta))$. As $\eta \in (0, \eta_0)$ is fixed, P (which depends on η!) is a $C^{2,1}$ function bounded away from 0 (Theorem 6.7.1, Lemma 6.7.2). Furthermore,

$$\Theta(0) \le C\,(\varepsilon^2 + \|\mathbf{u}_{0,\varepsilon}\|^2_{L^2(\mathbb{T}^d)}) \to 0 \tag{6.8.24}$$

(here, we use that the initial density ρ_0 belongs to C^3 so that $\|\mathbf{U}(0,x)\|_{L^\infty(\mathbb{T}^d)} \le C\,\varepsilon$, cf. (6.8.3)). Therefore, we get that $\Theta(t) \to 0$ as $\varepsilon \to 0$. By (6.8.6) and (6.8.7), we obtain convergence in $L^2(\mathbb{T}^d)$, even uniformly in time. □

Proof of Theorem 6.2.4. We write ρ_η (note that it does not depend on ε, cf. (6.8.1) and (6.8.3)) for for solutions to (6.8.1)–(6.8.2) and we note that they depend on η. From [123] we know that there exists a subsequence $\eta_k \to 0$ such that

$$\|\rho_{\eta_k} - \rho\|_{L^2((0,T)\times\mathbb{T}^d)} \to 0,$$

where ρ is a weak solution to the local Cahn-Hilliard equation. Now, let $\rho_{\eta_k,\varepsilon_k}$ be a measure-valued solution of non-local Euler-Korteweg equation. Using (6.8.24), (6.8.6) and (6.8.7), we have

$$\begin{aligned}\|\rho_{\eta_k} - \rho_{\eta_k,\varepsilon_k}\|_{L^2((0,T)\times\mathbb{T}^d)} &\le \\ &\le C\left(\varepsilon_k^2 + \|\mathbf{u}_{0,\varepsilon_k}\|^2_{L^2(\mathbb{T}^d)} + \varepsilon_k^4\,\|\rho_{\eta_k}\|^2_{C^{2,1}}\left\|\frac{1}{\rho_{\eta_k}}\right\|^2_\infty\right) e^{CT\|\rho_{\eta_k}\|_{C^{2,1}}/\eta^{d+3}}.\end{aligned}$$

Of course, the quantity $\|\rho_{\eta_k}\|^2_{C^{2,1}}\left\|\frac{1}{\rho_{\eta_k}}\right\|^2_\infty e^{CT\|\rho_{\eta_k}\|_{C^{2,1}}/\eta^{d+3}}$ is blowing up as $\eta_k \to 0$ (because we loose parabolicity), nevertheless we can choose ε_k so small to obtain convergence to 0. The conclusion follows by triangle inequality. □

6.9 Convergence result for the parametrized measure $\nu^{\eta,\varepsilon}$ and the concentration measures $m_{\eta,\varepsilon}$

Theorems 6.2.4 and 6.2.5 answer the question of what happens with the function $\rho_{\eta,\varepsilon}$ when $\eta, \varepsilon \to 0$. However, the measure-valued solution $(\rho_{\eta,\varepsilon}, \overline{\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$ is in fact a collection of four components. Below, we address the question of convergence of the other components: $\overline{\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}}$, $\nu^{\eta,\varepsilon}$, $m_{\eta,\varepsilon}$. We provide a detailed proof only for the situation in Theorem 6.2.5. Adaptation to the case analyzed in Theorem 6.2.4

is straightforward.

We first recall some basic notions from measure theory. We consider the set $\mathbb{R}^+ \times \mathbb{R}^d$ and we write (λ_1, λ') for a given element of this set where $\lambda_1 \in \mathbb{R}^+$ and $\lambda' \in \mathbb{R}^d$ as in Section 6.5. For two probability measures μ, ν on $\mathbb{R}^+ \times \mathbb{R}^d$ with a finite second moment, that is,

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} \left(|\lambda_1|^2 + |\lambda'|^2\right) \mathrm{d}\mu(\lambda_1, \lambda') < \infty$$

the Wasserstein distance $\mathcal{W}_2(\mu, \nu)$ is defined as

$$\mathcal{W}_2(\mu,\nu)^2 = \inf_{\pi \in \Pi(\mu,\nu)} \int_{(\mathbb{R}^+ \times \mathbb{R}^d)^2} \left[\left|\lambda_1 - \widetilde{\lambda_1}\right|^2 + \left|\lambda' - \widetilde{\lambda'}\right|^2 \right] \mathrm{d}\pi \left(\lambda_1, \lambda', \widetilde{\lambda_1}, \widetilde{\lambda'}\right), \quad (6.9.1)$$

where the set $\Pi(\mu, \nu)$ is the set of couplings between μ, ν; that is, the set of measures π on the product $(\mathbb{R}^+ \times \mathbb{R}^d)^2$ such that

$$\pi(A \times (\mathbb{R}^+ \times \mathbb{R}^d)) = \mu(A), \qquad \pi((\mathbb{R}^+ \times \mathbb{R}^d) \times B) = \nu(B).$$

Furthermore, for a measure μ on some space X, the total variation of μ is defined as

$$\|\mu\|_{TV} = |\mu|(X),$$

where $|\mu|(A) = \mu^+(A) - \mu^-(A)$ and μ^+, μ^- are positive and negative parts of μ, respectively. Note that if μ is a nonnegative measure, $\|\mu\|_{TV} = \mu(X)$. For more on spaces of measures and related norms, we refer to [116, Chapter 1].

Theorem 6.9.1. *Under the notation of Theorem 6.2.5, the function* $\overline{\sqrt{\rho_{\eta,\varepsilon}}\boldsymbol{u}_{\eta,\varepsilon}}$ *converges to 0 in* $L^\infty(0,T; L^2(\mathbb{T}^d))$*:*

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} |\overline{\sqrt{\rho_{\eta,\varepsilon}}\boldsymbol{u}_{\eta,\varepsilon}}|^2 \, \mathrm{d}x \to 0 \ \textit{as}\ \varepsilon \to 0. \quad (6.9.2)$$

Moreover, the parametrized measure $\nu^{\eta,\varepsilon} \in L^\infty_{weak}((0,T) \times \mathbb{T}^d; \mathcal{P}([0,+\infty) \times \mathbb{R}^d))$ *converges to* $\delta_{\rho_\eta(t,x)} \otimes \delta_{\mathbf{0}}$ *in the following sense*

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} \left[\mathcal{W}_2(\nu^{\eta,\varepsilon}, \delta_{\rho_\eta(t,x)} \otimes \delta_{\mathbf{0}})\right]^2 \mathrm{d}x \to 0 \ \textit{as}\ \varepsilon \to 0. \quad (6.9.3)$$

Furthermore, the concentration measures vector $m_{\eta,\varepsilon}$ *converges to 0 in the total variation norm, uniformly in time:*

$$\operatorname{ess\,sup}_{t\in(0,T)} \|m_{\eta,\varepsilon}(t,\cdot)\|_{TV} \to 0 \ \textit{as}\ \varepsilon \to 0. \quad (6.9.4)$$

Proof. From the proof of Theorem 6.2.5, we know that $\sup_{t\in(0,T)}\Theta(t) \to 0$ where $\Theta(t)$ is defined as in (6.8.4) with $\rho := \rho_{\eta,\varepsilon}$, $\mathrm{P} := \rho_\eta$, $\mathbf{u} := \mathbf{u}_{\eta,\varepsilon}$ and

$$\mathbf{U} := -\varepsilon\nabla(F'(\rho_\eta) - B_\eta(\rho_\eta)). \tag{6.9.5}$$

Due to Lemma 6.8.2 this yields

$$\begin{aligned}\sup_{t\in(0,T)} \int_{\mathbb{T}^d} \frac{1}{2}\overline{\rho_{\eta,\varepsilon}|\mathbf{u}_{\eta,\varepsilon} - \mathbf{U}|^2}\,\mathrm{d}x + \\ + \frac{\kappa}{4\eta^2}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho_{\eta,\varepsilon} - \rho_\eta)(x) - (\rho_{\eta,\varepsilon} - \rho_\eta)(x-y)|^2}\,\mathrm{d}y\,\mathrm{d}x \to 0\end{aligned} \tag{6.9.6}$$

and these two quantities are nonnegative. First, by admissibility (Definition 6.5.5) and (6.5.12), we have

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d}\int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda_1 - \rho_\eta(t,x)|^2\,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}x + m_{\eta,\varepsilon}^{\rho^2}(t,\mathbb{T}^d) \to 0. \tag{6.9.7}$$

In particular,

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda_1 - \widetilde{\lambda_1}(t,x)|^2\,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}\delta_{\rho_\eta(t,x)}(\widetilde{\lambda_1})\,\mathrm{d}x \to 0. \tag{6.9.8}$$

Second, due to (6.5.13), we can expand the term $\int_{\mathbb{T}^d}\frac{1}{2}\overline{\rho_{\eta,\varepsilon}|\mathbf{u}_{\eta,\varepsilon} - \mathbf{U}|^2}$ into three integrals:

$$\frac{1}{2}\int_{\mathbb{T}^d}\overline{\rho_{\eta,\varepsilon}\,|\mathbf{u}_{\eta,\varepsilon}|^2}\,\mathrm{d}x - \int_{\mathbb{T}^d}\overline{\rho_{\eta,\varepsilon}\,\mathbf{u}_{\eta,\varepsilon}}\cdot\mathbf{U}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d}\rho_{\eta,\varepsilon}\,|\mathbf{U}|^2\,\mathrm{d}x. \tag{6.9.9}$$

We claim that the second and third term converge to 0. For the third term, we can deduce it from (6.9.5), conservation of mass $\int_{\mathbb{T}^d}\rho_{\eta,\varepsilon}\,\mathrm{d}x = \int_{\mathbb{T}^d}\rho_0\,\mathrm{d}x$ and nonnegativity of $\rho_{\eta,\varepsilon}$. Concerning the second term, by the dissipativity (Definition 6.5.4) and nonnegativity of the energy (Lemma 6.5.6), we have the uniform estimate

$$\begin{aligned}&\left|\int_{\mathbb{T}^d}\overline{\rho_{\eta,\varepsilon}\,\mathbf{u}_{\eta,\varepsilon}}\,\mathrm{d}x\right| = \left|\int_{\mathbb{T}^d}\left\langle\sqrt{\lambda_1}\,\lambda',\nu^{\eta,\varepsilon}\right\rangle\mathrm{d}x\right| \le \frac{1}{2}\int_{\mathbb{T}^d}\langle\lambda_1,\nu^{\eta,\varepsilon}\rangle\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d}\langle|\lambda'|^2,\nu^{\eta,\varepsilon}\rangle\,\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{T}^d}\rho_{\eta,\varepsilon}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d}\overline{|\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}|^2}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{T}^d}\rho_0\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^d}\overline{|\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}|^2}\,\mathrm{d}x \le C\end{aligned}$$

As $|\mathbf{U}| \le C\varepsilon$, we conclude that $\operatorname{ess\,sup}_{t\in(0,T)}\left|\int_{\mathbb{T}^d}\overline{\rho_{\eta,\varepsilon}\,\mathbf{u}_{\eta,\varepsilon}}\cdot\mathbf{U}\,\mathrm{d}x\right| \to 0$ as $\varepsilon \to 0$ so that (6.9.9) implies

$$\operatorname{ess\,sup}_{t\in(0,T)}\frac{1}{2}\int_{\mathbb{T}^d}\overline{\rho_{\eta,\varepsilon}\,|\mathbf{u}_{\eta,\varepsilon}|^2}\,\mathrm{d}x \to 0.$$

Again, we can write it as

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda'|^2 \,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}x + m_{\eta,\varepsilon}^{\rho|\mathbf{u}|^2}(t,\mathbb{T}^d) \to 0 \tag{6.9.10}$$

which implies

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda' - \widetilde{\lambda}'|^2 \,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}\delta_{\mathbf{0}}(\widetilde{\lambda}')\,\mathrm{d}x \to 0. \tag{6.9.11}$$

Now, as the product measure $\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda') \otimes \delta_{\rho_\eta}(\widetilde{\lambda_1}) \otimes \delta_{\mathbf{0}}(\widetilde{\lambda}')$ is an admissible coupling between $\nu_{t,x}^{\eta,\varepsilon}$ and $\delta_{\rho_\eta} \otimes \delta_{\mathbf{0}}$ we can estimate the infimum in (6.9.1) by

$$\begin{aligned}\left[\mathcal{W}_2\left(\nu_{t,x}^{\eta,\varepsilon}, \delta_{\rho_\eta} \otimes \delta_{\mathbf{0}}\right)\right]^2 \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda_1 - \widetilde{\lambda_1}(t,x)|^2 \,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}\delta_{\rho_\eta(t,x)}(\widetilde{\lambda_1}) + \\ + \int_{\mathbb{R}^d} \int_{\mathbb{R}^+\times\mathbb{R}^d} |\lambda' - \widetilde{\lambda}'|^2 \,\mathrm{d}\nu_{t,x}^{\eta,\varepsilon}(\lambda_1,\lambda')\,\mathrm{d}\delta_{\mathbf{0}}(\widetilde{\lambda}')\end{aligned}$$

so that integrating over $\mathbb{T}^d$ and taking $\operatorname{ess\,sup}_{t\in(0,T)}$ we conclude the proof of (6.9.3) due to (6.9.8) and (6.9.11). Furthermore, by Jensen's inequality

$$\int_{\mathbb{T}^d} |\overline{\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}}|^2 = \int_{\mathbb{T}^d} |\langle \lambda', \nu^{\eta,\varepsilon}\rangle|^2 \,\mathrm{d}x \leq \int_{\mathbb{T}^d} \langle |\lambda'|^2, \nu^{\eta,\varepsilon}\rangle.$$

Taking $\operatorname{ess\,sup}_{t\in(0,T)}$ and using (6.9.10), we arrive at (6.9.2).

Finally, we study the concentration measures. From (6.9.7) and (6.9.10) we know that

$$\operatorname{ess\,sup}_{t\in(0,T)} m_{\eta,\varepsilon}^{\rho^2}(t,\mathbb{T}^d), \quad \operatorname{ess\,sup}_{t\in(0,T)} m_{\eta,\varepsilon}^{\rho|\mathbf{u}|^2}(t,\mathbb{T}^d) \to 0 \text{ as } \varepsilon \to 0.$$

Using (6.5.18) we obtain the same for $|m_{\eta,\varepsilon}^{\rho\,\mathbf{u}\otimes\mathbf{u}}|$. It remains to study $m_{\eta,\varepsilon}^{F(\rho)}$ and $m_{\eta,\varepsilon}^{\rho\,F'(\rho)}$. In fact, since $m_{\eta,\varepsilon}^{F(\rho)}$ is nonnegative, if we prove that $\operatorname{ess\,sup}_{t\in(0,T)} m_{\eta,\varepsilon}^{F(\rho)}(t,\mathbb{T}^d)$ converges to 0 as $\varepsilon \to 0$, the same will be true for $\left|m_{\eta,\varepsilon}^{\rho\,F'(\rho)}\right|$ due to (6.5.19).

By $\sup_{t\in(0,T)} \Theta(t) \to 0$ and (6.9.6), we have that

$$\sup_{t\in(0,T)} \int_{\mathbb{T}^d} \overline{F(\rho_{\eta,\varepsilon}|\rho_\eta)} \to 0 \text{ as } \varepsilon \to 0.$$

We can write $\overline{F(\rho_{\eta,\varepsilon}|\rho_\eta)}$ as (cf. (6.8.9))

$$\begin{aligned}\overline{F(\rho_{\eta,\varepsilon}|\rho_\eta)} = \langle F_1(\lambda_1) - F_1(\rho_\eta) - F_1'(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon}\rangle + \\ + \langle F_2(\lambda_1) - F_2(\rho_\eta) - F_2'(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon}\rangle + m_{\eta,\varepsilon}^{F(\rho)}.\end{aligned} \tag{6.9.12}$$

The first term is nonnegative while the second converges to 0. Indeed, it can be bounded by $\|F_2''\|_\infty \langle (\lambda_1 - \rho_\eta)^2, \nu_{t,x}\rangle$ which can be estimated due to inequality (cf. (6.8.10)):

$$\|F_2''\|_\infty \int_{\mathbb{T}^d} \left\langle (\lambda_1 - \rho_\eta)^2, \nu_{t,x} \right\rangle \mathrm{d}x \leq$$
$$\leq \frac{1-\kappa}{4\eta^2} \int_{\mathbb{T}^d} \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |(\rho - \rho_\eta)(x) - (\rho - \rho_\eta)(x-y)|^2 \,\mathrm{d}y \,\mathrm{d}x}$$

for some $\kappa \in (0,1)$. Thanks to (6.9.6),

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} |\langle F_2(\lambda_1) - F_2(\rho_\eta) - F_2'(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon}\rangle| \,\mathrm{d}x \to 0 \text{ as } \varepsilon \to 0.$$

Due to (6.9.12), the proof of (6.9.4) is concluded. □

We can also formulate a similar result to Theorem 6.9.1 in the context of Theorem 6.2.4. The proof is the same as the one of Theorem 6.9.1.

Theorem 6.9.2. *Under the notation of Theorem 6.2.4, the function* $\overline{\sqrt{\rho_{\eta_k,\varepsilon_k}}\boldsymbol{u}_{\eta_k,\varepsilon_k}}$ *converges to 0 in* $L^\infty(0,T;L^2(\mathbb{T}^d))$*:*

$$\operatorname{ess\,sup}_{t\in(0,T)} \int_{\mathbb{T}^d} |\overline{\sqrt{\rho_{\eta_k,\varepsilon_k}}\boldsymbol{u}_{\eta_k,\varepsilon_k}}|^2 \,\mathrm{d}x \to 0 \ \text{as } \varepsilon_k, \eta_k \to 0.$$

Moreover, the parametrized measure $\nu^{\eta_k,\varepsilon_k} \in L^\infty_{weak}((0,T)\times\mathbb{T}^d; \mathcal{P}([0,+\infty)\times\mathbb{R}^d))$ *converges to* $\delta_{\rho(t,x)} \otimes \delta_{\mathbf{0}}$ *in the following sense:*

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathcal{W}_2(\nu^{\eta_k,\varepsilon_k}, \delta_{\rho(t,x)} \otimes \delta_{\mathbf{0}})\right]^2 \mathrm{d}x \,\mathrm{d}t \to 0 \ \text{as } \varepsilon_k, \eta_k \to 0.$$

Furthermore, the concentration measures vector m_{η_k,ε_k} *converges to 0 in the total variation norm, uniformly in time:*

$$\operatorname{ess\,sup}_{t\in(0,T)} \|m_{\eta_k,\varepsilon_k}(t,\cdot)\|_{TV} \to 0 \ \text{as } \varepsilon_k, \eta_k \to 0.$$

6.10 Appendix: Some inequalities

Lemma 6.10.1. *Let* $\nu > 0$ *and a final time* $T > 0$*. Let* $\underline{u}$ *be defined by* $\underline{u}(t) = \nu \exp\left(-\int_0^t \|\operatorname{div} b\|_{L^\infty}(s)\,\mathrm{d}s\right)$ *and* ϕ_δ *defined in* (6.7.6)*. Then*

$$\int_{\mathbb{T}^d} \Delta\phi_\delta(u) \operatorname{sgn}^-(u - \underline{u}) \geq 0.$$

Proof. We note f_τ a concave approximation as $\tau \to 0$ of the function $f : x \mapsto \min\{x, 0\}$. Then f'_τ approximates $f' : x \mapsto \mathrm{sgn}^-(x)$. We have

$$\int_{\mathbb{T}^d} \Delta\phi_\delta(u) f'_\tau(u - \underline{u}) = -\int_{\mathbb{T}^d} \phi'_\delta(u) f''_\tau(u - \underline{u})|\nabla u|^2.$$

Since $\phi'_\delta \geq 0$, $f''_\tau \leq 0$ by concavity and we conclude by sending $\tau \to 0$. □

Lemma 6.10.2. *Let F satisfy Assumption* (6.2.1)*, $p(\rho) = \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$ and $F(\rho|\mathrm{P})$, $p(\rho|\mathrm{P})$ be defined by* (6.5.17)*. Then there exists a constant $C_{F,R}$ such that $p(\rho|\mathrm{P})$ is bounded in terms of $F(\rho|\mathrm{P})$ and $|\rho - \mathrm{P}|^2$* i.e.

$$p(\rho|\mathrm{P}) \leq C_{F,R}\, F(\rho|\mathrm{P}) + \left(C_{F,R} + \frac{1}{\eta^2}\right)|\rho - \mathrm{P}|^2. \tag{6.10.1}$$

Similarly, there exists constant C_F such that

$$|\rho\, F'(\rho)| \leq C_F\, F(\rho) + C_F\, \rho^2 + C_F. \tag{6.10.2}$$

Proof. We write

$$p(\rho|\mathrm{P}) = (\rho - \mathrm{P})^2 \int_0^1 \int_0^\tau p''(s\rho + (1-s)\mathrm{P})\, \mathrm{d}s\, \mathrm{d}\tau,$$

$$F(\rho|\mathrm{P}) = (\rho - \mathrm{P})^2 \int_0^1 \int_0^\tau F''(s\rho + (1-s)\mathrm{P})\, \mathrm{d}s\, \mathrm{d}\tau.$$

We note $h(s) = s\rho + (1-s)\mathrm{P}$ to simplify the notations. By definition $p'(\rho) = \rho\,(F''(\rho) + \frac{1}{\eta^2})$. Therefore we obtain

$$\begin{aligned}
p(\rho|\mathrm{P}) =& (\rho - \mathrm{P})^2 \int_0^1 \int_0^\tau F_1''(h(s)) + F_2''(h(s)) + h(s)F_1^{(3)}(h(s)) + h(s)F_2^{(3)}(h(s))\, \mathrm{d}s\, \mathrm{d}\tau \\
&+ \frac{1}{\eta^2}|\rho - \mathrm{P}|^2 \\
=& F(\rho|\mathrm{P}) + (\rho - \mathrm{P})^2 \int_0^1 \int_0^\tau h(s)F_1^{(3)}(h(s)) + h(s)F_2^{(3)}(h(s))\, \mathrm{d}s\, \mathrm{d}\tau + \frac{1}{\eta^2}|\rho - \mathrm{P}|^2.
\end{aligned}$$

We note $I_1 = \int_0^1 \int_0^\tau h(s)F_1^{(3)}(h(s))\, \mathrm{d}s\, \mathrm{d}\tau$ and $I_2 = \int_0^1 \int_0^\tau h(s)F_2^{(3)}(h(s))\, \mathrm{d}s\, \mathrm{d}\tau$. By assumptions on $|uF_1^{(3)}|$ we obtain

$$I_1 \leq C + C\int_0^1 \int_0^\tau F_1''(h(s))\, \mathrm{d}s\, \mathrm{d}\tau \leq C + C\int_0^1 \int_0^\tau F_1''(h(s)) + F_2''(h(s))\, \mathrm{d}s\, \mathrm{d}\tau,$$

where the value of C changed in the last inequality, using the boundedness assumption on F_2''. For I_2 we simply use boundedness of $|uF_2^3(u)|$ so that

$$I_2 \leq C.$$

This concludes the proof of (6.10.1). Concerning (6.10.2), we have

$$\rho\, F'(\rho) = \rho\, F_1'(\rho) + \rho\, F_2'(\rho) \le C(1 + F_1(\rho)) + C\,\rho \le C\,(1 + F(\rho)) + C\left(\frac{1}{2} + \frac{\rho^2}{2}\right) + C,$$

where we used estimate on $\rho\, F_1'(\rho)$, boundedness of F_2', F_2 and inequality $2\rho \le 1+\rho^2$. The proof is concluded. $\square$

Part II

Rough behavior

Chapter 7

Non-standard growth spaces

We now briefly recall theory of Musielak - Orlicz spaces. They are natural generalization of L^p spaces. Recall that $f \in L^p(\Omega_T)$ if

$$\int_{\Omega_T} |f(t,x)|^p \, \mathrm{d}x \, \mathrm{d}t < \infty.$$

Musielak-Orlicz spaces appear when one replaces function $\xi \mapsto |\xi|^p$ with a general function, called N-function, $M(t,x,\xi)$. Of course, there are some conditions on M that one has to assume in order to be able to define assosciated Banach space with M and they will be given in Definition 7.1.2. Nevertheless, let us point out that the content of this chapter will be our basic toolbox in Chapters 8–10. For a detailed discussion of Musielak-Orlicz spaces, we refer the reader to the classical book [223] as well as to a modern presentation [77] aimed at applications in PDEs.

7.1 N-functions

In what follows, $\Omega \subset \mathbb{R}^d$ denotes a bounded domain and $T > 0$ is arbitrary. We set $\Omega_T := (0,T) \times \Omega$.

Definition 7.1.1 (Young function). *We say that $m : [0,\infty) \to [0,\infty)$ is a Young function if the following holds true:*

(Y1) $m(s) = 0 \iff s = 0$,

(Y2) m is convex,

(Y3) *m is superlinear, i.e.* $\lim_{s\to 0} \frac{m(s)}{s} = 0$ *and* $\lim_{s\to\infty} \frac{m(s)}{s} = \infty$.

Definition 7.1.2 (N-function)**.** *We say that* $M : \Omega_T \times \mathbb{R}^d \to \mathbb{R}$ *is N-function if the following holds true:*

(M1) $M(t,x,\xi) = M(t,x,-\xi)$ *for a.e.* $(t,x) \in \Omega_T$ *and all* $\xi \in \mathbb{R}^d$,

(M2) $M(t,x,\xi)$ *is a Carathéodory function, i.e. for a.e.* $(t,x) \in \Omega_T$*, the mapping* $\mathbb{R}^d \ni \xi \mapsto M(t,x,\xi)$ *is continuous and for all* $\xi \in \mathbb{R}^d$*, the mapping* $\Omega_T \ni (t,x) \mapsto M(t,x,\xi)$ *is measurable,*

(M3) *for a.e.* $(t,x) \in \Omega_T$*, the map* $\mathbb{R}^d \ni \xi \mapsto M(t,x,\xi)$ *is convex,*

(M4) *there exist two Young functions* m_1*,* m_2 *such that for almost all* $(t,x) \in \Omega_T$ *and all* $\xi \in \mathbb{R}^d$ *we have*

$$m_1(|\xi|) \leq M(t,x,\xi) \leq m_2(|\xi|).$$

Example 7.1.3. The motivation for introducing Definitions 7.1.1 and 7.1.2 is to generalize the role of the function $\xi \mapsto |\xi|^p$ in the definition of the L^p space. Examples of the Young functions include $|\xi|^p$, $|\xi|\, e^{|\xi|}$ and $|\xi| \log(1 + |\xi|)$ (this type of growth appears naturally in the kinetic theory). Examples of N-functions include $|\xi|^{p(t,x)}$ and $|\xi|^{p(t,x)} + a(t,x)|\xi|^{q(t,x)}$ where $p(t,x)$, $q(t,x)$ are strictly separated from 1 and $+\infty$ while a is a bounded, nonnegative function.

Definition 7.1.4 (Convex conjugate)**.** *Let m be a Young function. Then, we define its convex conjugate m^* as*

$$m^*(s) = \sup_{t\in[0,\infty)} (st - m(t)).$$

Similarly, if M is an N-function, we define its convex conjugate M^* as

$$M^*(t,x,\eta) = \sup_{\xi\in\mathbb{R}^d} (\xi \cdot \eta - M(t,x,\xi)).$$

Remark 7.1.5. The motivation for Definition 7.1.4 is that we want to generalize the natural duality between L^p and $L^{p'}$ space where p' is the usual Hölder conjugate of p, see Lemma 7.2.3. Indeed, if $m(s) = \frac{1}{p}|s|^p$, then $m^*(s) = \frac{1}{p'}|s|^{p'}$.

Remark 7.1.6. Condition (M4) is not necessary for the basic functional analytic properties of Musielak-Orlicz spaces (for instance, being a Banach space or basic convergence properties outlined in Section 7.3). We should think about this condition as the one guaranteeing good properties of the dual spaces. First, it is needed for M^* to be well-defined (in the critical case of $M(\xi) = |\xi|$, M^* is not well-defined). Of course, one can imagine a situation when for each fixed (t,x), the map $\xi \mapsto M(t,x,\xi)$ is superlinear so that M^* is well-defined, yet (M4) is not satisfied. Nevertheless, autonomous lower and upper bounds in terms of Young functions are necessary for the proof of Lemma 7.2.9 (see [77, Theorem 3.5.3]) which is essential in most applications in PDEs. Indeed, it provides the good representation of the preduals $L_M(\Omega_T) = (E_{M^*}(\Omega_T))^*$ and $L_{M^*}(\Omega_T) = (E_M(\Omega_T))^*$ which allows to apply weak* compactness in these spaces.

Lemma 7.1.7 (Properties of N-functions)**.** *Let m be a Young function and M be an N-function. Then:*

(N1) function $\frac{m(t)}{t}$ is nondecreasing,

(N2) m^* is a Young function,

(N3) M^* is an N-function,

(N4) $\lim_{|\xi|\to 0} \operatorname{ess\,sup}_{(t,x)\in\Omega_T} \frac{M(t,x,\xi)}{|\xi|} = 0$ and $\lim_{|\xi|\to\infty} \operatorname{ess\,inf}_{(t,x)\in\Omega_T} \frac{M(t,x,\xi)}{|\xi|} = \infty$,

(N5) if $f_n : \Omega_T \to \mathbb{R}^d$ is a sequence of functions and $\int_{\Omega_T} M(t,x,f_n(t,x))\,\mathrm{d}x\,\mathrm{d}t \le C$ independently of n, then $\{f_n\}$ is equi-integrable,

(N6) if $f_n : \Omega_T \to \mathbb{R}^d$ is a sequence of functions and $\int_{\Omega_T} M(t,x,f_n(t,x))\,\mathrm{d}x\,\mathrm{d}t \le C$ for some $C > 1$ then $\|f_n\|_{L_M} \le C$,

(N7) if $f_n : \Omega_T \to \mathbb{R}^d$ is a sequence of functions such that $f_n \to f$ a.e. in Ω_T and $\|f_n\|_\infty \le C$ independently of n, then

$$\int_{\Omega_T} M(t,x,f_n(t,x))\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} M(t,x,f(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $t \le s$. By convexity of m, we have

$$\frac{m(t)}{t} = \frac{1}{t} m\left(\frac{t}{s}s + \left(1 - \frac{t}{s}\right)0\right) \le \frac{1}{t}\frac{t}{s} m(s) = \frac{m(s)}{s},$$

which proves (N1).

To see property (N2), we observe directly from Definition 7.1.4 that $m^*(0) = 0$ as $m \geq 0$ and $m(0) = 0$. The convexity of m^* follows as it is a supremum of affine maps. Hence, it remains to check (Y3) in Definition 7.1.1. For any $\lambda > 0$

$$\liminf_{s\to\infty} \frac{m^*(s)}{s} \geq \frac{\lambda s - m(\lambda)}{s} \geq \lambda$$

which proves $\lim_{s\to\infty} \frac{m^*(s)}{s} = \infty$. Now, let $\delta > 0$ and $s \in (0, \delta)$ be arbitrary. Then,

$$\frac{m^*(s)}{s} = \sup_{t\in[0,\infty)} \left(t - \frac{m(t)}{s}\right) = \sup_{t\in[0,\infty)} t\left(1 - \frac{m(t)}{t}\frac{1}{s}\right) \leq \sup_{t\in[0,\infty)} t\left(1 - \frac{m(t)}{t}\frac{1}{\delta}\right)$$

However, for t such that $\frac{m(t)}{t} \geq \delta$, the maximized expression is negative. By property (N1) and (Y3) in Definition 7.1.1, we find t_δ, such that $\frac{m(t_\delta)}{t_\delta} = \delta$ and we get that

$$\frac{m^*(s)}{s} \leq \sup_{t\in[0,t_\delta]} t\left(1 - \frac{m(t)}{t}\frac{1}{\delta}\right) \leq t_\delta.$$

We claim that $t_\delta \to 0$ as $\delta \to 0$. For if not, $C_2 \geq t_\delta \geq C_1 > 0$ for some constants C_1 and C_2. But then

$$\delta = \frac{m(t_\delta)}{t_\delta} \geq \frac{m(C_1)}{C_2} > \frac{m(0)}{C_2} = 0,$$

since m is strictly increasing and $m(0) = 0$. This proves (N2). To see (N3), we observe that

$$m_1(|\xi|) \leq M(t,x,\xi) \leq m_2(\xi) \implies m_2^*(|\xi|) \leq M^*(t,x,\xi) \leq m_1^*(\xi).$$

Since m_1^* and m_2^* are Young functions, the conclusion follows. Property (N4) is a consequence of (M4) in Definition 7.1.2 and superlinearity of Young functions (Y3). To deduce (N5), we note that

$$\int_{\Omega_T} m_1(|f_n(t,x)|)\,\mathrm{d}x\,\mathrm{d}t \leq C$$

and it is well-known that such bound for superlinear function m_1 is equivalent to uniform integrability on bounded domains, see [11, Proposition 1.27]. Property (N6) follows by convexity:

$$\int_{\Omega_T} M\left(t,x,\frac{f_n(t,x)}{C}\right)\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{C}\int_{\Omega_T} M\left(t,x,f_n(t,x)\right)\mathrm{d}x\,\mathrm{d}t \leq 1.$$

Finally, as Young function are increasing, property (N7) follows by Dominated Convergence Theorem. □

Remark 7.1.8. In previous works on PDEs in Musielak - Orlicz spaces, N-functions were defined slightly differently using combination of conditions in Definition 7.1.1, Definition 7.1.2 and Lemma 7.1.7 (see, for instance, [53, 78, 79]). We believe that Definition 7.1.2 makes our work more accessible for readers not familiar with this setting.

7.2 Musielak - Orlicz spaces

Definition 7.2.1 (Musielak - Orlicz space $L_M(\Omega_T)$)**.** *Let M be an N - function. Then, the Musielak - Orlicz space $L_M(\Omega_T)$ is defined as*

$$L_M(\Omega_T) = \left\{\xi : \Omega_T \to \mathbb{R}^d : \exists \lambda > 0 \text{ such that } \int_{\Omega_T} M\left(t, x, \frac{\xi(t,x)}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t < \infty\right\}.$$

This is a Banach space equipped with the norm

$$\|\xi\|_{L_M} = \inf\left\{\lambda > 0 : \int_{\Omega_T} M\left(t, x, \frac{\xi(t,x)}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t \le 1\right\}. \tag{7.2.1}$$

If m is a Young function, we can similarly define the Musielak - Orlicz space $L_m(\Omega_T)$.

Remark 7.2.2. The idea to define the norm (7.2.1) in a variational way comes from the fact that the usual $L^p(\Omega_T)$ norm can be equivalently defined as

$$\inf\left\{\lambda > 0 : \int_{\Omega_T} \left|\frac{\xi(t,x)}{\lambda}\right|^p \mathrm{d}x\,\mathrm{d}t \le 1\right\}.$$

The advantage of this approach is that one does not take p-th root which allows to define, for instance, $L^{p(t,x)}$ spaces. They will be discussed in detail in Section 7.4.

The following form of the Young and the Hölder inequalities are true in Musielak-Orlicz spaces (see [263, Lemma 2.4]):

Lemma 7.2.3. *Let M be an N-function and M^* be its convex conjugate. Then, for all $\xi \in L_M(\Omega_T)$ and $\eta \in L_{M^*}(\Omega_T)$:*

(I1) $\int_{\Omega_T} \xi(t,x)\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} M\left(t, x, \xi(t,x)\right) \mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} M^*\left(t, x, \eta(t,x)\right) \mathrm{d}x\,\mathrm{d}t,$

(I2) $\int_{\Omega_T} \xi(t,x)\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t \leq 2\|\xi\|_{L_M}\|\eta\|_{L_{M^*}}$.

As convergence in norm in space $L_M(\Omega_T)$ seems to be too strong for applications in PDEs, we introduce the concept of modular convergence.

Definition 7.2.4 (Modular convergence in $L_M(\Omega_T)$)**.** *We say that sequence of functions $\{\xi_n\} \subset L_M(\Omega_T)$ converges to ξ modularly if there exists $\lambda > 0$ such that*

$$\int_{\Omega_T} M\left(t, x, \frac{\xi_n(t,x) - \xi(t,x)}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t \to 0.$$

We write $\xi_n \xrightarrow{M} \xi$. By convexity, if follows that if $\{\xi_n\} \subset L_M(\Omega_T)$ and $\xi_n \xrightarrow{M} \xi$ then $\xi \in L_M(\Omega_T)$.

Note that modularly converging sequences converge in $L^1(\Omega_T)$ and so, they have a subsequence converging a.e. As in the case of classical Lebesgue spaces, simple functions are dense in $L_M(\Omega_T)$ with respect to the modular convergence:

Lemma 7.2.5 (Density of simple functions)**.** *Let $\xi \in L_M(\Omega_T)$. Then, there is a sequence $\{\xi_n\}$ of simple functions such that $\xi_n \xrightarrow{M} \xi$.*

Due to Vitali Convergence Theorem (Theorem 7.3.2), we have the following characterization of modular convergence and its corollary.

Theorem 7.2.6. *Let $\{\xi_n\} \subset L_M(\Omega_T)$ and $\xi \in L_M(\Omega_T)$. Then, $\xi_n \xrightarrow{M} \xi$ if and only if the following hold:*

(V1) $\{\xi_n\}$ converges to ξ in measure,

(V2) $\left\{M\left(t, x, \frac{\xi_n}{\lambda}\right)\right\}$ is uniformly equi-integrable for some $\lambda > 0$.

Corollary 7.2.7. Let $\{\varphi_j\} \subset L_M(\Omega_T)$ and $\{\phi_j\} \subset L_{M^*}(\Omega_T)$. Suppose that $\varphi_j \xrightarrow{M} \varphi$ and $\phi_j \xrightarrow{M^*} \phi$. Then, $\varphi_j\,\phi_j \to \varphi\,\phi$ in $L^1(\Omega_T)$.

Proof. By Theorem 7.2.6, $\varphi_j \to \varphi$ and $\phi_j \to \phi$ in measure, and so $\varphi_j \cdot \phi_j \to \varphi \cdot \phi$ also in measure. To conclude, we have to prove uniform integrability of $\{\varphi_j \cdot \phi_j\}$. However, by Young's inequality, for any $Q \subset \Omega_T$:

$$\begin{aligned} &\int_Q \frac{\varphi_j(t,x) \cdot \phi_j(t,x)}{\lambda}\,\mathrm{d}x\,\mathrm{d}t \leq \\ &\qquad \leq \int_Q M\left(t, x, \frac{\varphi_j(t,x)}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t + \int_Q M^*\left(t, x, \frac{\phi_j(t,x)}{\lambda}\right) \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{7.2.2}$$

Again, Theorem 7.2.6 implies existence of $\lambda_1, \lambda_2 > 0$ such that both sequences $\left\{M\left(t,x,\frac{\varphi_j(t,x)}{\lambda_1}\right)\right\}$ and $\left\{M^*\left(t,x,\frac{\phi_j(t,x)}{\lambda_2}\right)\right\}$ are uniformly integrable. Taking $\lambda = \max(\lambda_1, \lambda_2)$ in (7.2.2), we conclude the proof. $\square$

Finally, we discuss some compactness results allowing to extract converging subsequences.

Definition 7.2.8 (Subspace $E_M(\Omega_T)$). *$E_M(\Omega_T)$ is a closure of bounded functions in the norm* (7.2.1)*.*

It is easy to see by approximation with simple functions that $E_M(\Omega_T)$ is separable. Therefore, [263, Theorem 2.6] and the Banach-Alaoglu-Bourbaki Theorem (cf. [49, Theorem 3.16 and Corollary 3.30]) yields:

Lemma 7.2.9. *We have the following duality characterization*

$$(E_M(\Omega_T))^* = L_{M^*}(\Omega_T).$$

In particular, if $\{\xi_n\}$ is a bounded sequence in $L_{M^*}(\Omega_T)$, it has a weakly-$*$ converging subsequence.

For Young functions, we also define Orlicz–Sobolev spaces and we recall their basic properties (cf. [5, Chapter 8]).

Definition 7.2.10 (Orlicz–Sobolev space). *Let $m : \mathbb{R} \to \mathbb{R}$ be a Young function. We define Orlicz–Sobolev spaces $W_0^1 L_m(\Omega_T)$ as*

$$W_0^1 L_m(\Omega_T) = \left\{\xi \in L^1(0,T; W_0^{1,1}(\Omega)) : \|\xi\|_{L_m}, \|\nabla\xi\|_{L_m} < \infty\right\}$$

and we equip it with the norm

$$\|\xi\|_{W^1 L_m} = \|\xi\|_{L_m} + \|\nabla\xi\|_{L_m}.$$

We also consider its subset $W_0^1 E_m(\Omega_T)$:

$$W_0^1 E_m(\Omega_T) = \left\{\xi \in W_0^1 L_m : \xi \in E_m(\Omega_T) \text{ and } \nabla\xi \in E_m(\Omega_T)\right\}$$

Lemma 7.2.11 (Properties of $W_0^1 E_m(\Omega_T)$ and $W_0^1 L_m(\Omega_T)$). *Spaces $W_0^1 E_m(\Omega_T)$ and $W_0^1 L_m(\Omega_T)$ have the following properties:*

(P1) $W_0^1 E_m(\Omega_T)$ *is separable,*

(P2) space $C_0^\infty((0,T)\times\Omega)$ *is dense in* $W_0^1 E_m(\Omega_T)$ *with respect to* $\|\cdot\|_{L_m}$ *norm,*

(P3) (Poincaré inequality, cf. [76, Corollary 4.1]) there are constants c_1 *and* c_2 *such that for all* $u \in W_0^1 L_m(\Omega_T)$,

$$\int_{\Omega_T} m(c_1|u|)\,\mathrm{d}x\,\mathrm{d}t \le c_2 \int_{\Omega_T} m(|\nabla u|)\,\mathrm{d}x\,\mathrm{d}t.$$

In particular, $\|\nabla u\|_{L_m}$ *is an equivalent norm on* $W_0^1 L_m(\Omega_T)$.

7.3 The Δ_2 condition and variable exponent spaces

In Chapter 9 and 10 we will make an additional assumption on the N-function, namely that M satisfies so-called Δ_2 condition, i.e.

$$M(t,x,2\xi) \le C\,M(t,x,\xi), \tag{7.3.1}$$

for some constant C. We collect the main consequences of (7.3.1) below. The most important one is that if (7.3.1) holds then the modular and strong convergences coincide.

Lemma 7.3.1. *Let* $f, f_n : \Omega_T \to \mathbb{R}^d$. *Then,*

(C1) The following are equivalent: $\|f\|_{L_M} < \infty \iff \int_{\Omega_T} M(t,x,c\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t < \infty$ *for some* $c > 0 \iff \int_{\Omega_T} M(t,x,c\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t < \infty$ *for all* $c > 0$,

(C2) $\|f_n - f\|_{L_M} \to 0 \iff$ *for some* $c > 0$ $\int_{\Omega_T} M(t,x,c\,(f_n(t,x) - f(t,x)))\,\mathrm{d}x\,\mathrm{d}t \to 0 \iff$ *for all* $c > 0$ $\int_{\Omega_T} M(t,x,c\,(f_n(t,x) - f(t,x)))\,\mathrm{d}x\,\mathrm{d}t \to 0$,

(C3) if $\|f\|_{L_M} < \infty$ *and any of the conditions in (C2) is satisfied then we have* $\int_{\Omega_T} M(t,x,f_n(t,x))\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} M(t,x,f(t,x))\,\mathrm{d}x\,\mathrm{d}t$,

(C4) if $f_n \to f$ *a.e. on* Ω_T, $\|f\|_{L_M} < \infty$ *and the sequence* $\{M(t,x,f_n(x))\}_{n\in\mathbb{N}}$ *is uniformly integrable then* $\|f_n - f\|_{L_M} \to 0$.

For the proof we need the following convergence result (see [36, Theorem 4.5.4]).

Theorem 7.3.2. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$). Let $\{f_n\} \subset L^1(X, \mathcal{F}, \mu)$ and f be an $\mathcal{F}$-measurable function. Then, $f_n \to f$ in $L^1(X, \mathcal{F}, \mu)$ if and only if $f_n \to f$ in measure and $\{f_n\}$ is uniformly integrable, i.e.*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \in \mathcal{F} \qquad \mu(A) < \delta \implies \sup_{n \in \mathbb{N}} \int_A |f_n| \, \mathrm{d}\mu < \varepsilon.$$

In fact, we will apply the following corollary.

Corollary 7.3.3. Let $(X, \mathcal{F}, \mu)$ be a finite measure space (i.e. $\mu(X) < \infty$). Let $\{f_n\} \subset L^1(X, \mathcal{F}, \mu)$ be a nonnegative sequence and f be an $\mathcal{F}$-measurable function. Suppose that

(J1) $f_n \to f$ in measure,

(J2) there exists a sequence of functions $\{g_n\}$ convergent in $L^1(X, \mathcal{F}, \mu)$ and function $h \in L^1(X, \mathcal{F}, \mu)$ such that

$$0 \le f_n \le g_n + h.$$

Then, $f_n \to f$ in $L^1(X, \mathcal{F}, \mu)$.

Proof. In view of Theorem 7.3.2, it is sufficient to prove that $\{f_n\}$ is uniformly integrable. To this end, for an arbitrary set A, we have

$$\int_A |f_n| \, \mathrm{d}\mu = \int_A f_n \, \mathrm{d}\mu \le \int_A g_n \, \mathrm{d}\mu + \int_A h \, \mathrm{d}\mu \le \int_A |g_n + h| \, \mathrm{d}\mu.$$

Let $\varepsilon > 0$. As $\{g_n\}$ is convergent in $L^1(X, \mathcal{F}, \mu)$, the same is true for $\{g_n + h\}$. It follows that $\{g_n + h\}$ is uniformly integrable. Therefore, there exists $\delta > 0$ such that if $\mu(A) < \delta$, we have $\int_A |g_n + h| \, \mathrm{d}\mu < \varepsilon$. It follows that

$$\int_A |f_n| \, \mathrm{d}\mu < \varepsilon.$$

$\square$

Proof of Lemma 7.3.1. The first equivalence in (C1) follows directly from definition of the norm so in fact it is sufficient to prove that if $\int_{\Omega_T} M(t, x, c\, f(t, x)) \, \mathrm{d}x \, \mathrm{d}t < \infty$

for some $c > 0$ then $\int_{\Omega_T} M(t,x,d\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t < \infty$ for all $d > 0$. First, if $d < c$, this follows by convexity and Jensen's inequality:

$$\begin{aligned}&\int_{\Omega_T} M(t,x,d|f(t,x)|)\,\mathrm{d}x\,\mathrm{d}t = \\ &\quad = \int_{\Omega_T} M\left(t,x,\frac{d}{c}\,cf(t,x)+0\right)\mathrm{d}x\,\mathrm{d}t \le \frac{d}{c}\int_{\Omega_T} M(t,x,c\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.3.2}$$

If $d > c$, we find $k \in \mathbb{N}$ such that $d \le 2^k\,c$ and apply (7.3.1) k times:

$$\begin{aligned}&\int_{\Omega_T} M(t,x,d\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t \le \\ &\le C^k \int_{\Omega_T} M\left(t,x,\frac{d}{2^k}\,f(t,x)\right)\mathrm{d}x\,\mathrm{d}t \le C^k\frac{d}{2^k c}\int_{\Omega_T} M(t,x,c\,f(t,x))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{7.3.3}$$

where we used the first part.

Concerning (C2), we first prove equivalence:

$$\|f_n - f\|_{L_M} \to 0 \iff \int_{\Omega_T} M(t,x,c\,(f_n - f))\,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ for all } c > 0.$$

To prove ($\Rightarrow$) we fix $c > 0$ and we note that there exists n_c such that for all $n \ge n_c$ we have $c\,\|f_n - f\|_{L_M} < 1$. By definition (7.2.1), there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}}$ convergent to 0 such that $c\,\|f_n - f\|_{L_M} + \delta_k < 1$ and

$$\int_{\Omega_T} M\left(t,x,\frac{c\,(f_n - f)}{c\,\|f_n - f\|_{L_M} + \delta_k}\right)\mathrm{d}x\,\mathrm{d}t \le 1.$$

Using convexity of M and equality $M(t,x,0) = 0$ we obtain

$$\begin{aligned}&\int_{\Omega_T} M\left(t,x,c\,(f_n - f)\right)\mathrm{d}x\,\mathrm{d}t \le \\ &\quad (\|f_n - f\|_{L_M} + \delta_k)\int_{\Omega_T} M\left(t,x,\frac{c\,(f_n - f)}{\|f_n - f\|_{L_M} + \delta_k}\right)\mathrm{d}x\,\mathrm{d}t \le c\,\|f_n - f\|_{L_M} + \delta_k.\end{aligned}$$

Letting $k \to \infty$ (so that $\delta_k \to 0$) and $n \to \infty$ we conclude the proof. For ($\Leftarrow$), we note that for each $c > 0$, there exists n_c such that for all $n \ge n_c$ we have $\int_{\Omega_T} M(t,x,c\,(f_n - f))\,\mathrm{d}x\,\mathrm{d}t \le 1$, i.e. $\|f_n - f\|_{L_M} \le \frac{1}{c}$. The conclusion follows by letting $c \to \infty$. We are left to prove equivalence

$$\begin{aligned}&\int_{\Omega_T} M(t,x,c\,(f_n - f))\,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ for all } c > 0 \iff \\ &\qquad\qquad \iff \int_{\Omega_T} M(t,x,c\,(f_n - f))\,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ for some } c > 0.\end{aligned}$$

This follows from (7.3.2) and (7.3.3).

To prove (C3) we assume that $\int_\Omega M(t,x,(f_n - f))\,\mathrm{d}x\,\mathrm{d}t \to 0$ and $\|f\|_{L_M} < \infty$ which implies $\int_{\Omega_T} M(t,x,f)\,\mathrm{d}x\,\mathrm{d}t < \infty$. First, we deduce that $f_n \to f$ at least in measure. Second, we can estimate by convexity

$$0 \le M(t,x,f_n) \le M\left(t,x,\frac{1}{2}2(f_n - f) + \frac{1}{2}2\,f\right) \le$$
$$\le \frac{1}{2}M(t,x,2(f_n - f)) + \frac{1}{2}M(t,x,2f) \le \frac{C}{2}M(t,x,f_n - f) + \frac{C}{2}M(t,x,f).$$

Corollary 7.3.3 implies that $M(t,x,f_n) \to M(t,x,f)$ in $L^1(\Omega_T)$ so in particular, $\int_{\Omega_T} M(t,x,f_n)\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} M(t,x,f)\,\mathrm{d}x\,\mathrm{d}t$.

Concerning (C4), in view of Vitali convergence theorem cf. Theorem 7.3.2, it is sufficient to prove that the sequence $\{M(t,x,f_n - f)\}_n$ is uniformly integrable. Using convexity and Δ_2 condition (7.3.1) we obtain

$$0 \le M(t,x,f_n - f) \le M\left(t,x,\frac{1}{2}2\,f_n + \frac{1}{2}2\,f\right) \le$$
$$\le \frac{1}{2}M(t,x,2\,f_n) + \frac{1}{2}M(t,x,2\,f) \le \frac{C}{2}M(t,x,f_n) + \frac{C}{2}M(t,x,f).$$

It follows that $M(t,x,f_n - f) \to 0$ in $L^1(\Omega)$ and the conclusion follows from (C2).

□

7.4 Variable exponent spaces

We conclude with a particular example of N-function that satisfy (7.3.1). The related space is called the variable exponent space. We consider $M(t,x,\xi) = |\xi|^{s(t,x)}$ where $s(t,x)$ satisfy $1 < s_- \le s(t,x) \le s_+$ for some s_-, s_+.

Definition 7.4.1. *Given a measurable function $s(t,x) : \Omega_T \to [1,\infty)$, we let*

$$L^{s(t,x)}(\Omega_T) = \left\{\xi : \Omega_T \to \mathbb{R}^d : \text{ there is } \lambda > 0 \text{ such that } \int_{\Omega_T} \left|\frac{\xi(t,x)}{\lambda}\right|^{s(t,x)} \mathrm{d}x\,\mathrm{d}t < \infty\right\}.$$

This definition is equivalent to

$$L^{s(t,x)}(\Omega_T) = \left\{ \xi : \Omega_T \to \mathbb{R}^d : \int_{\Omega_T} |\xi(t,x)|^{s(t,x)} \,\mathrm{d}x\,\mathrm{d}t < \infty \right\}.$$

This is the Banach space with the norm

$$\|\xi\|_{L^{s(t,x)}} = \inf\left\{ \lambda > 0 : \int_{\Omega_T} \left| \frac{\xi(t,x)}{\lambda} \right|^{s(t,x)} \mathrm{d}x\,\mathrm{d}t \le 1 \right\}. \tag{7.4.1}$$

Let us observe that due to Lemma 7.3.1

$$\xi_n \to \xi \text{ in } L^{s(t,x)}(\Omega_T) \iff \int_{\Omega_T} |\xi_n - \xi|^{s(t,x)} \,\mathrm{d}x\,\mathrm{d}t \to 0$$

and, as a consequence of Theorem 7.2.6, we have:

Theorem 7.4.2. *Suppose that $\phi_n \to \phi$ in $L^{s(t,x)}(\Omega_T)$ and $\psi_n \to \psi$ in $L^{s'(t,x)}(\Omega_T)$ where $\frac{1}{s(t,x)} + \frac{1}{s'(t,x)} = 1$. Then, $\phi_n\,\psi_n \to \phi\,\psi$ in $L^1(\Omega_T)$.*

Now, we consider the special case of exponent depending only on the time variable, that is $s(t,x) = q(t)$. In this case, many inequalities including Poincare's and Korn's inequality are valid. We formulate it using a these two examples but Reader can easily adapt it to several other inequalities.

Lemma 7.4.3. *For all $f : \Omega_T \to \mathbb{R}$ such that $f \in L^1(0,T; W_0^{1,1}(\Omega))$ and $\nabla f \in L^{q(t)}(\Omega_T)$ we have*

$$\|f\|_{L^{q(t)}} \le C\, \|\nabla f\|_{L^{q(t)}}.$$

for some constant C which is independent of f. Similarly, if $f \in L^1(0,T; W_0^{1,1}(\Omega))$ and $Df \in L^{q(t)}(\Omega_T)$ then $\nabla f \in L^{q(t)}(\Omega_T)$ and we have the following Korn's inequality

$$\|\nabla f\|_{L^{q(t)}} \le C\, \|Df\|_{L^{q(t)}}.$$

Proof. For a.e. $t \in (0,T)$, we have $f \in W_0^{1,1}(\Omega)$ and $\int_\Omega |\nabla f|^{q(t)} \,\mathrm{d}x < \infty$ due to (C1) in Lemma 7.3.1. We consider a cube Q of side $2\,\mathrm{diam}(\Omega)$ such that $\Omega \Subset Q$. Therefore, by usual Poincare inequality

$$\int_Q |\varphi|^{q(t)} \,\mathrm{d}x \le (2\,\mathrm{diam}(\Omega))^{q(t)} \int_Q |\nabla \varphi|^{q(t)} \,\mathrm{d}x \qquad \forall \varphi \in C_c^\infty(Q).$$

As $f(t,\cdot) \in W_0^{1,1}(\Omega)$, we may extend it with 0. Then, we consider a usual mollification sequence which after passing to the limits implies

$$\int_{\Omega} |f|^{q(t)} \, \mathrm{d}x \leq (2 \operatorname{diam}(\Omega))^{q(t)} \int_{\Omega} |\nabla f|^{q(t)} \, \mathrm{d}x = \int_{\Omega} |2 \operatorname{diam}(\Omega) \nabla f|^{q(t)} \, \mathrm{d}x.$$

We divide by $\lambda^{q(t)}$ and integrate in time to obtain

$$\int_{\Omega_T} \left|\frac{f}{\lambda}\right|^{q(t)} \mathrm{d}x \, \mathrm{d}t \leq \int_{\Omega_T} \left|\frac{2 \operatorname{diam}(\Omega) \nabla f}{\lambda}\right|^{q(t)} \mathrm{d}x \, \mathrm{d}t.$$

By definition of the norm (Definition 7.4.1), we choose a sequence

$$\lambda_n \to \|2 \operatorname{diam}(\Omega) \nabla f\|_{L^{q(t)}}, \qquad \int_{\Omega_T} \left|\frac{2 \operatorname{diam}(\Omega) \nabla f}{\lambda}\right|^{q(t)} \mathrm{d}x \, \mathrm{d}t \leq 1$$

so that

$$\int_{\Omega_T} \left|\frac{f}{\lambda_n}\right|^{q(t)} \mathrm{d}x \, \mathrm{d}t \leq 1 \implies \|f\|_{L^{q(t)}} \leq \lambda_n.$$

The conclusion follows by sending $n \to \infty$. For the proof of Korn's inequality, the strategy is the same: we use classical Korn's inequality [100, Chapter 7]:

$$\|\nabla f\|_{L^q(\Omega)} \leq C(q,\Omega) \, \|Df\|_{L^q(\Omega)}$$

for fixed value of t and we integrate in time. The small difficulty here is that the constant is not explicit with respect to the exponent q. However, one can quantify it in terms of the norm of Riesz transform and the norm of the maximal operator (see [100, Chapter 7]) which shows that the constant is continuous with respect to the exponent. Thus, the final constant will depend on q_- and q_+. □

Chapter 8

Parabolic equations with roughly changing growth

The results in this chapter have been published in:

- M. Bulíček, P. Gwiazda, J. Skrzeczkowski. *Parabolic equations in Musielak – Orlicz spaces with discontinuous in time N-function.* Journal of Differential Equations, 290, 17-56, 2021, cited as [56].

8.1 Introduction and the main results

In this chapter we consider parabolic PDEs where the parabolic operator changes discontinuously with respect to time. To motivate, consider the following equation

$$u_t = \begin{cases} \operatorname{div}\nabla u & \text{in } (0,1] \times \Omega, \\ \operatorname{div}\left(|\nabla u|^2 \nabla u\right) & \text{in } (1,2] \times \Omega, \end{cases} \tag{8.1.1}$$

which can be solved piecewisely (first on time interval $(0, 1]$ and then on $(1, 2]$) so one can develop well-posedness theory for (8.1.1). Moreover, ∇u is expected to belong to the space $L^{p(t)}(\Omega_T)$ where $p(t) = \begin{cases} 2 & t \in [0,1], \\ 4 & t \in (1,2], \end{cases}$ so that the natural functional space changes discontinuously in time.

A continuous generalization of (8.1.1) is the $p(t,x)$-Laplace equation (studied, for instance, in [16, 39, 191])

$$u_t = \mathrm{div}(\nabla u|\nabla u|^{p(t,x)-2}) + f \tag{8.1.2}$$

where $p(t,x)$ is discontinuous in time. Here, we expect $\nabla u \in L^{p(t,x)}(\Omega_T)$ so again, the space changes discontinuously with respect to t. The target of this chapter is to establish well-posedness (existence and uniqueness) of solutions without any regularity assumption with respect to time. This is the first result of this type as all the previous works assumed so-called log-Holder continuity of $p(t,x)$, cf. (1.2.3).

To generalize a bit, we write both equations (8.1.1) and (8.1.2) as

$$u_t(t,x) = \mathrm{div} A(t,x,\nabla u(t,x)) + f(t,x) \text{ in } (0,T)\times\Omega. \tag{8.1.3}$$

We equip (8.1.3) with the homogeneous Dirichlet boundary condition and the initial value $u_0(x)$. Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain, T denotes the length of time interval, $f:(0,T)\times\Omega\to\mathbb{R}$ is a measurable bounded function and A is a monotone operator with coercivity and *non-standard* growth controlled by a so - called N-function $M:(0,T)\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ (see Definition 7.1.2), i.e. for almost all $(t,x)\in(0,T)\times\Omega$ and all $\xi\in\mathbb{R}^d$, we have:

$$M(t,x,\xi) + M^*(t,x,A(t,x,\xi)) \le c\, A(t,x,\xi)\cdot\xi + h(t,x) \tag{8.1.4}$$

where M^* denotes the convex conjugate to M (see Definition 7.1.4) and $h\in L^1((0,T)\times\Omega)$. Many equations fall into this abstract setting (see, for instance, [78, Corollary 1.2]); for instance, (8.1.2) can be written in the form of (8.1.3) with

$$A(t,x,\xi) = \xi\,|\xi|^{p(t,x)-2}, \qquad M(t,x,\xi) = |\xi|^{p(t,x)},$$

where we need to assume that $1 < p_- < p(t,x) < p_+ < \infty$, see Example 8.2.6. Let us also comment that as we will see in Chapter 9, the abstract theory developed in this chapter can be applied to non-Newtonian and electrorheological fluids.

Originally, problem (8.1.3) was solved with $M(t,x,\xi) = |\xi|^p$ where $1 < p < \infty$. In this classical setting, (8.1.4) implies that A, understood as a map

$$L^p(0,T;W_0^{1,p}(\Omega)) \ni u \mapsto A(t,x,\nabla u) \in \left(L^p\left(0,T;W_0^{1,p}(\Omega)\right)\right)^*,$$

is a bounded continuous operator and standard approaches (Galerkin method and compactness in Sobolev-Bochner spaces) applies (see [50,185] and references therein) showing that the Sobolev space is an appropriate functional setting for problem (8.1.3). However, if the N-function M appearing in (8.1.4) has not a polynomial growth with respect to ξ and is (t,x)-dependent, one has to look for a solution u such that its gradient ∇u belongs to the Musielak - Orlicz space $L_M((0,T)\times\Omega)$, i.e. the space of measureable functions $\xi : (0,T)\times\Omega \to \mathbb{R}^d$ which satisfy

$$\int_{(0,T)\times\Omega} M\left(t,x,\frac{\xi(t,x)}{\lambda}\right) \mathrm{d}t\,\mathrm{d}x < \infty$$

for some $\lambda > 0$, see Definition 7.2.1.

A modern approach to such equations is based on looking for hypothesis on M implying that $C_0^\infty((0,T)\times\Omega)$ is a dense subset of $L_M((0,T)\times\Omega)$ (at least in the sense of modular convergence, see Definition 7.2.4) so that one can test (8.1.3) with the solution itself. It is a classical fact that for variable Lebesgue spaces (i.e. $M(t,x,\xi) = |\xi|^{p(t,x)}$) some continuity of p in (t,x) is in general necessary (see [84, Example 6.12]) and it is quite simple to understand why this is the case. Let $u \in L_M((0,T)\times\Omega)$ and consider its mollification in spatial variable $u_\varepsilon = u * \eta_\varepsilon$. Then, even to prove that $u_\varepsilon \in L_M((0,T)\times\Omega)$ we need some continuity of M in the spatial variable x. Indeed,

$$\begin{aligned}\int_0^T \int_\Omega M\left(t,x,\frac{1}{\lambda}\int_{B(0,\epsilon)} \eta_\epsilon(y)\, u(t,x-y)\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \leq \\ \leq \int_0^T \int_\Omega \int_{B(0,\epsilon)} \eta_\epsilon(y)\, M\left(t,x,\frac{u(t,x-y)}{\lambda}\right)\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{8.1.5}$$

and continuity is necessary for approximating

$$M\left(t,x,\frac{u(t,x-y)}{\lambda}\right) \approx M\left(t,x-y,\frac{u(t,x-y)}{\lambda}\right)$$

so that the (RHS) of (8.1.5) can be seen as a convolution of two integrable functions. Similar reasoning works for the mollification in the time variable t. There are many recent works developing approximation and well-posedness theory for (8.1.3) in this spirit [8, 78, 79, 251].

Motivated by (8.1.1), in this chapter we establish the existence of solutions to (8.1.3) in the Musielak - Orlicz space $L_M((0,T)\times\Omega)$ without any assumption on continuity of $M(t,x,\xi)$ with respect to t (see Theorem 8.2.8). Moreover, for isotropic N-functions of the form $M(t,x,|\xi|)$ we obtain the uniqueness in a given class. To illustrate the result, in the particular case of (8.1.2), for $p(t,x) := p(t)$, the only assumption we need on $p(t)$ is that $1 < p_- \leq p(t) \leq p_+ < \infty$ and p is measurable. Our result seems to be completely unexpected as the natural functional space for (8.1.2), that is $L^{p(t,x)}(\Omega_T)$, changes discontinuously in time but still we can prove the complete well-posedness theory for (8.1.2).

The main idea to obtain this result is that we do not try to approximate *every* function in $L_M((0,T)\times\Omega)$ but only the distributional solution to (8.1.3). If $u_\varepsilon = u*\eta_\varepsilon$ denotes mollification in spatial variable x and u solves (8.1.3), then

$$(u_\varepsilon)_t(t,x) = \mathrm{div} A_\varepsilon(t,x,\nabla u(t,x)) + f_\varepsilon(t,x) \text{ in } (0,T)\times\Omega. \tag{8.1.6}$$

Therefore, if u is mollified in space, it immiedately gains Sobolev derivative in time. As we will see, this makes u_ε an admissible test function for (8.1.3). Similar approaches have been used for renormalized solutions to the transport equation, see [112] and [95, Section 2.1].

The approximation is necessary to identify the limit of $A(t,x,\nabla u_n)$ where u_n is a suitable approximation of the desired solution. When $\nabla u_n \to \nabla u$ (in some weak sense), we do not know a priori that $A(t,x,\nabla u_n) \rightharpoonup A(t,x,\nabla u)$ (at least weakly, in some topology). A priori, we only know that $A(t,x,\nabla u_n) \rightharpoonup \alpha$ and we need to prove that $\alpha = A(t,x,\nabla u)$. To prove the latter, we use classical Minty's monotonicity trick [217] which (informally) amounts to proving

$$\limsup_{n\to\infty} \int_{\Omega_T} A(t,x,\nabla u_n)\cdot \nabla u_n\, \psi(x)\,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} \alpha\cdot\nabla u\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$

for all test functions $\psi \in C_c^\infty(\Omega)$. The information about $A(t,x,\nabla u_n)\cdot\nabla u_n$ and $\alpha\cdot\nabla u$ comes from testing equations for u_n and u by u_n and u respectively. The latter is rigorously correct if we first approximate u_n and u. Let us also remark that this identification argument can be performed locally in space: in particular, for the existence part the regularity of the boundary does not play any role in the argument.

Finally, let us note that there is another strategy to study PDEs with non-standard growth. This is based on assuming that the N-function M and its convex conjugate M^* satisfy Δ_2 condition (7.3.1). The first implies that $L_M(\Omega_T) = E_M(\Omega_T)$ while the second that $L_{M^*}(\Omega_T) = E_{M^*}(\Omega_T)$, cf. [77, Theorem 3.3.2]. Combined with Lemma 7.2.9, they imply that

$$(L_M(\Omega_T))^* = (E_M(\Omega_T))^* = L_{M^*}(\Omega_T) \qquad (L_{M^*}(\Omega_T))^* = (E_{M^*}(\Omega_T))^* = L_M(\Omega_T).$$

so that together they imply that $L_M(\Omega_T)$ is reflexive. For elliptic equations (where we have no time variable), one can use this to construct solutions via Galerkin method (with the basis being a dense subset of $W_0^1E_M$, the space of functions in $W_0^{1,1}(\Omega)$ such that $\nabla u\in E_M(\Omega)$). In fact, in this case it is sufficient to assume that only M or M^* satisfies Δ_2 [53]. However, in the parabolic case, even for equation (8.1.6), we cannot use Galerkin method because space $L^{p(t,x)}(\Omega_T)$ cannot be factorized into the Bochner form $L^{p(t)}(0,T;L^{q(x)}(\Omega))$ for some exponents $p(t)$, $q(x)$ so that the basis cannot be time independent. Nevertheless, in some special cases, for instance when $p(t,x) = p(x)$ (the exponent is time independent) one can still obtain well-posedness result without continuity assumptions basing only on Δ_2 condition, see [167].

8.2 Rigorous formulation of the main results

We start with the continuity assumption on N-function M. Its formulation for a general function M is fairly difficult but it simplifies a lot for isotropic N-function, that is when $M(t,x,\xi)$ depends only on t, x and $|\xi|$, see Remark 8.2.2.

Assumption 8.2.1 (Assumptions on M)**.** We assume that $M : \Omega_T \times \mathbb{R}^d \to \mathbb{R}$ is an N-function. Moreover, we assume that there is a function $\Theta : (0,T)\times[0,1]\times[0,\infty) \to [0,\infty)$, which is nondecreasing with respect to the second and the third variable, such that

$$\forall C > 1\ \forall \delta_0 > 0\ \exists R > 0 \text{ such that for a.e. } t \in (0,T)$$
$$\text{and all } \delta \leq \delta_0 \text{ there holds } \Theta(t,\delta,C\delta^{-1}) \leq R.$$

This function describes relation between $M_Q(t,\xi) = \operatorname{ess\,inf}_{x\in\Omega\cap 5Q} M(t,x,\xi)$ and $M(t,x,\xi)$, where $Q \subset \mathbb{R}^d$ is an arbitrary cube and $5Q$ is a cube with the same center as Q with five times longer edge. More precisely, we assume that there exists $\xi_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that for every cube $Q \subset \mathbb{R}^d$ with edge $\delta \in (0,\delta_0)$ and all $\xi \in \mathbb{R}^d$ with $|\xi| > \xi_0$ we have

$$\frac{M(t,x,\xi)}{M_Q^{**}(t,\xi)} \leq \Theta(t,\delta,|\xi|), \tag{8.2.1}$$

where $M_Q^{**} = (M_Q^*)^*$ is the second convex conjugate to M_Q, see Definition 7.1.4. □

Remark 8.2.2. In the particular case of an isotropic N-function $M(t,x,|\xi|)$, Assumption 8.2.1 boils down to existence of the function $\Theta : (0,T) \times [0,1] \times [0,\infty) \to [0,\infty)$ which is nondecreasing with respect to second and third variable such that

$$\limsup_{\delta\to 0^+} \Theta(t,\delta,C\delta^{-1}) \text{ is bounded uniformly in time } t \in (0,T) \tag{8.2.2}$$

and

$$\frac{M(t,x,r)}{M(t,y,r)} \leq \Theta(t,|x-y|,r).$$

See [78, Lemma A.4] for the proof.□

We remark that Assumption 8.2.1 mimics the one made in [78], namely

$$\frac{M(t,x,\xi)}{M_{Q,I}^{**}(\xi)} \leq \Theta(\delta,|\xi|),$$

where $M_{Q,I}(\xi) = \operatorname{ess\,inf}_{x\in\Omega\cap 3Q, t\in I\cap(0,T)} M(t,x,\xi)$, Q is a cube with edge of length δ, I is a subinterval of $\mathbb{R}$ with $|I| \leq \delta$ and function Θ satisfies:

$$\forall C > 0\ \forall \delta_0 > 0\ \exists R > 0 \text{ such that for a.e. } t \in (0,T)$$
$$\text{and all } \delta \leq \delta_0 \text{ there holds } \Theta(\delta,C\delta^{-d}) \leq R.$$

Several other equivalent conditions can be formulated, see condition (A1') in [168, Definition 4.1.1], (A1-n) in [168, Chapter 7.3] and [169].

The relaxed regularity in time allows for N-functions which are merely measurable in time. On the other hand, we need to control the quantity $\Theta(t, \delta, C\,\delta^{-1})$ rather than $\Theta(\delta, C\,\delta^{-d})$ which results in better exponents regimes for some well-known examples of N-functions, see Example 8.2.3. This improvement is based on the observation that in the approximation result one needs to approximate in the modular topology functions of the form

$$\nabla(T_k(u) + \varphi) \text{ where } \nabla u \in L_M(\Omega_T),\, \varphi \in C_c^\infty(\Omega_T)$$

The observation described above can be easily implemented in the previous works on this topic cf. [78, 79]. This will be also of great importance for our new results concerning double-phase functionals presented in Chapter 10.

Example 8.2.3. We list here N-functions satisfying Assumptions 8.2.1. For the proof, we refer to Appendix 8.8.

(E1) $M(t,x,\xi) = |\xi|^{p(t,x)}$ with $1 < p_- \leq p(t,x) \leq p_+ < \infty$ and $p(t,x) \in L^\infty(0,T; C_{\log}(\Omega))$. Here, $C_{\log}(\Omega)$ is the space of log-Hölder continuous functions on Ω, i.e. functions $v : \Omega \to \mathbb{R}$ such that

$$|v(x) - v(y)| \leq -\frac{C}{\log|x-y|}$$

for all $x, y \in \Omega$ and some constant C. Note that only very low regularity of $p(t,x)$ in time is required.

(E2) $M(t,x,\xi) = |\xi|^{p(t,x)} + a(t,x)\,|\xi|^{q(t,x)}$ where

- $1 < p_- \leq p(t,x) < p^+ < \infty$, $1 < q_- \leq q(t,x) < q^+ < \infty$,
- $p(t,x), q(t,x) \in L^\infty(0,T; C_{\log}(\Omega))$,
- $a(t,x) \in L^\infty(0,T; C^\alpha(\Omega))$ for some $\alpha \in (0,1)$ and $a \geq 0$,
- $q(t,x) - p(t,x) \leq \alpha$.

Here, $C^\alpha(\Omega)$ is the space of α-Hölder continuous functions on Ω. We stress that only very low regularity of $p(t,x)$ and $q(t,x)$ in time is required. We also observe that for $p_- < d$, our admissible regime of exponents is better than $q(t,x) - p(t,x) \leq \frac{\alpha p_-}{d}$ known from [78].

Assumption 8.2.4 (Assumptions on A)**.** We assume that $A : \Omega_T \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:

(A1) A is a Carathéodory's function, i.e. for a.e. $(t,x) \in \Omega_T$, map $\mathbb{R}^d \ni \xi \mapsto A(t,x,\xi)$ is continuous and for all $\xi \in \mathbb{R}^d$, map $\Omega_T \ni (t,x) \mapsto A(t,x,\xi)$ is measurable,

(A2) (coercivity and growth bound) there is a constant c and function $h \in L^\infty(\Omega_T)$ such that for all $\xi \in \mathbb{R}^d$ and a.e. $(t,x) \in \Omega_T$:

$$M(t,x,\xi) + M^*(t,x,A(t,x,\xi)) \leq c\, A(t,x,\xi) \cdot \xi + h(t,x),$$

(A3) (monotonicity) for all $\eta, \xi \in \mathbb{R}^d$ and a.e. $(t,x) \in \Omega_T$:

$$(A(t,x,\xi) - A(t,x,\eta)) \cdot (\xi - \eta) \geq 0,$$

(A4) for a.e. $(t,x) \in \Omega_T$ we have $A(t,x,0) = 0$.

Remark 8.2.5. In classical papers, condition (A4) could be deduced from coercivity and growth bounds. Here, (A2) implies only that

$$0 \leq M^*(t,x,A(t,x,0)) \leq h(t,x).$$

We believe that (A4) can be waived. Nevertheless, we make this assumption as it is natural and it simplifies many technical computations.

Example 8.2.6. We list here functions $\mathcal{A}$ corresponding to N-functions in Example 8.2.3 which satisfy Assumptions 8.2.4. For the proof, we refer to Appendix 8.7.

(F1) $A(t,x,\xi) = |\xi|^{p(t,x)-2}\xi$ leads to the equation with $p(t,x)$-Laplacian

$$u_t(t,x) = \operatorname{div}\left[|\nabla u(t,x)|^{p(t,x)-2}\, \nabla u(t,x)\right] + f(t,x)$$

and the governing N-function $M(t, x, \xi)$ is given by (E1) in Example 8.2.3. Such problems have been considered recently for instance in [6, 18] under assumption that $p(t, x)$ is log-Hölder continuous jointly in t and x. In our setting, we only need $p(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$.

(F2) $A(t, x, \xi) = |\xi|^{p(t,x)-2}\, \xi + a(t, x)\, |\xi|^{q(t,x)-2}\, \xi$ leads to the double phase problem

$$u_t(t, x) = \operatorname{div}\left[|\nabla u(t, x)|^{p(t,x)-2}\, \nabla u(t, x) + a(t, x)\, |\nabla u(t, x)|^{q(t,x)-2}\, \nabla u(t, x)\right].$$

Such problems were studied with variational methods [38,207] but mostly with constant or only x-dependent exponents. The case of $p(t, x)$ and $q(t, x)$ which are log-Hölder continuous jointly in t and x was studied in [78]. Our theory requires only $p(t, x), q(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$.

Lemma 8.2.7. *Let A satisfy Assumption 8.2.4. Then, for every $K > 0$, there exists a constant $C(K)$ depending on K such that $|A(t, x, \xi)| \le C(K)$ for a.e. $(t, x) \in \Omega_T$ and all $\xi \in \mathbb{R}^d$ fulfilling $|\xi| \le K$.*

Proof. Let $|\xi| \le K$. Assumption (A2) implies that

$$M^*(t, x, A(t, x, \xi)) \le c\, A(t, x, \xi) \cdot \xi + h(t, x). \tag{8.2.3}$$

Let m be a Young function such that $m(|\xi|) \le M^*(t, x, \xi)$ for a.e. $(t, x) \in \Omega_T$ as in point (M4) in Definition 7.1.2. If $|A(t, x, \xi)| \le 1$, the assertion follows by choosing $C(K) \ge 1$. Otherwise, (8.2.3) implies

$$\frac{m(|A(t, x, \xi)|)}{|A(t, x, \xi)|} \le c\, |\xi| + \|h\|_\infty \le c\, K + \|h\|_\infty.$$

Since map $s \mapsto \frac{m(s)}{s}$ is nondecreasing (property (N1) in Lemma 7.1.7) and m is superlinear (property (Y3) in Definition 7.1.1), the assertion follows. □

Next, we define a function space relevant for the problem (8.1.3) as follows:

$$V_T^M = \big\{u : \Omega_T \to \mathbb{R} \text{ such that } u \in L^1(0, T; W_0^{1,1}(\Omega)), \nabla u \in L_M(\Omega_T) \\ \text{and } u \in L^\infty(0, T; L^2(\Omega))\big\}.$$

The main results of this paper read:

Theorem 8.2.8 (Existence of solutions)**.** *Suppose that Assumptions 8.2.1 and 8.2.4 are satisfied. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$. Then, there exists $u \in V_T^M(\Omega)$ which is a weak solution to* (8.1.3)*. More precisely, there exists $u \in V_T^M(\Omega)$ such that $A(t,x,\nabla u) \in L_{M^*}(\Omega_T)$ and for all $\varphi \in C_0^\infty([0,T) \times \Omega)$, there holds:*

$$\begin{aligned}-\int_{\Omega_T} u(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega u_0(x)\varphi(0,x)\,\mathrm{d}x +& \\ + \int_{\Omega_T} A(t,x,\nabla u)\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x &= \int_{\Omega_T} f(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

In addition, u satisfies the global energy inequality, i.e. for all $t \in [0,T]$ there holds

$$\begin{aligned}\frac{1}{2}\int_\Omega \left[u^2(t,x) - u_0^2(x)\right]\mathrm{d}x \leq &-\int_0^t\int_\Omega A(s,x,\nabla u(s,x))\cdot\nabla u(s,x)\ \mathrm{d}x\,\mathrm{d}s \\ &+\int_0^t\int_\Omega f(s,x)\,u(s,x)\ \mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{8.2.4}$$

Theorem 8.2.9 (Uniqueness of solutions)**.** *Let all assumptions of Theorem 8.2.8 be satisfied. Moreover, suppose that the N-function M is isotropic, i.e. it is of the form $M(t,x,|\xi|)$. Then, weak solution to* (8.1.3) *is unique and it satisfies the energy equality, i.e. for all $t \in [0,T]$ there holds*

$$\begin{aligned}\frac{1}{2}\int_\Omega \left[u^2(t,x) - u_0^2(x)\right]\mathrm{d}x = &-\int_0^t\int_\Omega A(s,x,\nabla u(s,x))\cdot\nabla u(s,x)\ \mathrm{d}x\,\mathrm{d}s \\ &+\int_0^t\int_\Omega f(s,x)\,u(s,x)\ \mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{8.2.5}$$

8.3 Approximation in Musielak-Orlicz spaces

In this section we prove that if $u \in V_T^M(\Omega)$, then u can be approximated in the modular topology of the gradients. We formulate this result locally in Ω but we remark that the similar approach has already been used in [78, Theorem 3.1], where approximation was performed globally for Lipschitz domains Ω by using a decomposition on star-shaped sets, see [145, Lemma II.1.3].

First, we recall the definition of truncation and mollification operators:

Definition 8.3.1 (Truncation)**.** *Function*

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\frac{s}{|s|} & \text{otherwise,} \end{cases}$$

is called truncation at level k. We also denote by G_k its primitive function, i.e. we set

$$G_k(s) = \int_0^s T_k(\sigma)\,\mathrm{d}\sigma.$$

Definition 8.3.2 (Mollification with respect to the spatial variable)**.** *Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a standard regularizing kernel, i.e. η is a smooth nonnegative function compactly supported in a ball of radius one and fulfills $\int_{\mathbb{R}^d} \eta(x)\,\mathrm{d}x = 1$. Then, we set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)$ and for arbitrary $u : \Omega \to \mathbb{R}$ and $\Omega' \Subset \Omega$, we define $u^\varepsilon : \Omega' \to \mathbb{R}$ as*

$$u^\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)\,\mathrm{d}y.$$

Furthermore, if $u : \Omega_T \to \mathbb{R}$, then u^ε denotes mollification in space, i.e.

$$u^\varepsilon(t,x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(t,y)\,\mathrm{d}y.$$

Definition 8.3.3 (Mollification with respect to time)**.** *Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a standard regularizing kernel, i.e. ζ is a smooth nonnegative function compactly supported in a ball of radius one and fulfills $\int_{\mathbb{R}} \zeta(x)\,\mathrm{d}x = 1$. Then, we set $\zeta_\varepsilon(x) = \frac{1}{\varepsilon}\zeta\left(\frac{x}{\varepsilon}\right)$ and for arbitrary $u : \mathbb{R} \times \Omega \to \mathbb{R}$, we define $\mathcal{R}^\varepsilon u : \mathbb{R} \times \Omega \to \mathbb{R}$ as*

$$\mathcal{R}^\varepsilon u(t,x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s)u(s,x)\,\mathrm{d}s.$$

For properties of mollified functions, the reader may consult [130, Appendix C.4]. Finally, we formulate the approximative properties of the mollifications defined above, which is the most essential tool used in the paper.

Theorem 8.3.4. *Let $\Omega \subset \mathbb{R}^d$, $\psi : \Omega \to \mathbb{R}$ be compactly supported satisfying $0 \le \psi \le 1$ and $u \in V_T^M(\Omega)$. Suppose that Assumption 8.2.1 is satisfied. Then, there exists $\varepsilon_0 > 0$:*

(S1) $(T_k(u^\varepsilon)\psi)^\varepsilon \in L^1(0,T;C_0^\infty(\Omega))$ *for all* $\varepsilon \in (0,\varepsilon_0)$*,*

(S2) $T_k(u^\varepsilon)\psi \to T_k(u)\psi$ *a.e. in* Ω_T *and in* $L^1(0,T;L^1(\Omega))$ *as* $\varepsilon \to 0^+$,

(S3) $\nabla\left(T_k(u^\varepsilon)\psi\right)^\varepsilon \xrightarrow{M} \nabla\left(T_k(u)\psi\right)$ *as* $\varepsilon \to 0^+$, *where the modular convergence* $\xrightarrow{M}$ *is defined in Definition 7.2.4.*

The key estimate needed for the proof of Theorem 8.3.4 is formulated in the following lemma.

Lemma 8.3.5. *Suppose that Assumption 8.2.1 is satisfied,* $v : \Omega_T \to \mathbb{R}^d$ *and* $v \in L_M(\Omega_T)$ *with* $\int_{\Omega_T} M\left(t,x,v(t,x)\right)\mathrm{d}x\,\mathrm{d}t < \infty$. *Assume that* $v = \nabla u + \varphi$ *for some* $u \in V_T^M(\Omega)$ *and* $\varphi \in L^\infty(\Omega_T)$. *Then, there is a constant* C *such that for any compactly supported* $\psi : \Omega \to \mathbb{R}$ *with* $0 \le \psi \le 1$ *and for all* $k \in \mathbb{N}$,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{\Omega_T} M\left(t,x,\left(\mathbb{1}_{|u^\varepsilon|\le k}\, v^\varepsilon(t,x)\psi(x)\right)^\varepsilon\right)\mathrm{d}x\,\mathrm{d}t \le\\ \le \int_{\Omega_T} m_2\left(|v(t,x)|\psi(x)\right)\mathbb{1}_{|v(t,x)|\psi(x)\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x + C\int_{\Omega_T} M\left(t,x,v(t,x)\right)\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where ξ_0 *is a constant from Assumption 8.2.1 and* m_2 *is a Young function as in (M4) in Definition 7.1.2.*

Remark 8.3.6. Since $v \in L_M(\Omega_T)$, the condition $\int_{\Omega_T} M\left(t,x,v(t,x)\right)\mathrm{d}x\,\mathrm{d}t < \infty$ can be always satisfied by considering appropriate scaling if necessary.

Proof of Lemma 8.3.5. We denote $z_\varepsilon(t,x) = \left(\mathbb{1}_{|u^\varepsilon|\le k}\, v^\varepsilon(t,x)\psi(x)\right)^\varepsilon$ and write:

$$\begin{aligned}\int_{\Omega_T} M\left(t,x,\left(\mathbb{1}_{|u^\varepsilon|\le k}\, v^\varepsilon(t,x)\psi(x)\right)^\varepsilon\right)\mathrm{d}x\,\mathrm{d}t \le\\ \le \int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right)\mathbb{1}_{|z_\varepsilon(t,x)|\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x + \int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right)\,\mathbb{1}_{|z_\varepsilon(t,x)|>\xi_0}\,\mathrm{d}t\,\mathrm{d}x.\end{aligned} \tag{8.3.1}$$

For the first term, we use (M4) in Definition 7.1.2 to observe:

$$\int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right)\mathbb{1}_{|z_\varepsilon(t,x)|\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x \le \int_{\Omega_T} m_2\left(t,x,z_\varepsilon(t,x)\right)\mathbb{1}_{|z_\varepsilon(t,x)|\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x$$

and so, by (N7) in Lemma 7.1.7 we get

$$\limsup_{\varepsilon\to 0}\int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right)\mathbb{1}_{|z_\varepsilon(t,x)|\le\xi_0} \le \int_{\Omega_T} m_2\left(|v(t,x)|\psi(x)\right)\mathbb{1}_{|v(t,x)|\psi(x)\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x. \tag{8.3.2}$$

Hence, it is sufficient to focus on the second term in (8.3.1). Let $\{Q_j\}_{j=1}^{N_\varepsilon}$ be a family of closed cubes with edge ε such that $\mathrm{int}Q_j \cap \mathrm{int}Q_i = \emptyset$ for $i \neq j$ and $\Omega \subset \cup_{i=1}^{N_\varepsilon} Q_i$. Moreover, let $3Q_i$ and $5Q_i$ be the cubes with the same center as Q_i and edges 3ε and 5ε, respectively. Then,

$$\int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right) \mathbb{1}_{|z_\varepsilon(t,x)|>\xi_0} \,\mathrm{d}t\,\mathrm{d}x =$$
$$= \sum_{i=1}^{N_\varepsilon} \int_0^T \int_{Q_i\cap\Omega} \frac{M\left(t,x,z_\varepsilon(t,x)\right)}{M^{**}_{Q_i}(t,z_\varepsilon(t,x))} M^{**}_{Q_i}(t,z_\varepsilon(t,x)) \mathbb{1}_{|z_\varepsilon(t,x)|>\xi_0} \,\mathrm{d}x\,\mathrm{d}t,$$

where $M^{**}_{Q_i}$ is defined in Assumption 8.2.1. Note that we assume that $v = \nabla u + \varphi$ for some $u \in V_T^M(\Omega)$ and $\varphi \in L^\infty(\Omega_T)$. We note that

$$z_\varepsilon(t,x) = \left(\nabla T_k(u^\varepsilon(t,x))\psi(x)\right)^\varepsilon + \left(\mathbb{1}_{|u^\varepsilon|\leq k}\, \varphi^\varepsilon(t,x)\psi(x)\right)^\varepsilon := z^1_\varepsilon(t,x) + z^2_\varepsilon(t,x).$$

Clearly, using Young's convolutional inequality, we have $|z^2_\varepsilon(t,x)| \leq \|\varphi\|_\infty \|\psi\|_\infty$. Moreover,

$$z^1_\varepsilon(t,x) = -\left(T_k(u^\varepsilon)\ \mathrm{div}\,\psi\right) * \eta_\varepsilon(t,x) + \left(T_k(u^\varepsilon)\psi\right) * \nabla\eta_\varepsilon(t,x)$$

so applying Young's convolutional inequality we have:

$$|z^1_\varepsilon(t,x)| \leq k\,\|\,\mathrm{div}\,\psi\|_\infty + \frac{k\,\|\psi\|_\infty\,\|\nabla\eta_\varepsilon\|_1}{\varepsilon}.$$

We conclude that $|z_\varepsilon(t,x)| \leq \frac{C(k,\varphi,\eta)}{\varepsilon}$ for $\varepsilon < 1$ and therefore, using (8.2.1), we get that for $x \in Q_i \cap \Omega$ the following inequality

$$\frac{M\left(t,x,z_\varepsilon(t,x)\right)}{M^{**}_{Q_i}(t,z_\varepsilon(t,x))} \leq \Theta\left(t,\delta,\frac{C(k,\varphi,\eta)}{\varepsilon}\right) \leq C$$

holds true for sufficiently small ε. Consequently,

$$\int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right) \mathbb{1}_{|z_\varepsilon(t,x)|>\xi_0} \,\mathrm{d}t\,\mathrm{d}x \leq C \sum_{i=1}^{N_\varepsilon} \int_0^T \int_{Q_i\cap\Omega} M^{**}_{Q_i}(t,z_\varepsilon(t,x))\,\mathrm{d}x\,\mathrm{d}t. \tag{8.3.3}$$

To estimate the right hand side in the above inequality, we focus on each summand separately. Using Jensen's and Young's convolutional inequalities we deduce:

$$\begin{aligned}
&\int_0^T \int_{Q_i\cap\Omega} M_{Q_i}^{**}(t, z_\varepsilon(t,x))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^T \int_{Q_i\cap\Omega} M_{Q_i}^{**}\left(t, \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\Big(v^\varepsilon(t,x-y)\psi(x-y)\mathbb{1}_{|u^\varepsilon|\le k}(x-y)\Big)\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_0^T \int_{Q_i\cap\Omega}\int_{B(0,\varepsilon)} \eta_\varepsilon(y) M_{Q_i}^{**}\big(t, v^\varepsilon(t,x-y)\psi(x-y)\mathbb{1}_{|u^\varepsilon|\le k}(x-y)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_0^T \int_{\mathbb{R}^d}\int_{B(0,\varepsilon)} \eta_\varepsilon(y) M_{Q_i}^{**}\big(t, v^\varepsilon(t,x-y)\psi(x-y)\mathbb{1}_{3Q_i\cap\Omega}(x-y)\big)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_0^T \int_{\mathbb{R}^d} M_{Q_i}^{**}\big(t, v^\varepsilon(t,x)\psi(x)\mathbb{1}_{3Q_i\cap\Omega}(x)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= \int_0^T \int_{3Q_i\cap\Omega} M_{Q_i}^{**}\big(t, v^\varepsilon(t,x)\psi(x)\big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{8.3.4}$$

where we used the fact that $\|\eta_\varepsilon\|_{L^1} = 1$ and the fact that $M_{Q_i}^{**}(t,\xi) = 0 \iff \xi = 0$. Next, by convexity of $\xi \mapsto M_{Q_i}^{**}(t,\xi)$ and thanks to $0 \le \psi(x) \le 1$, we can simply estimate the last term as

$$\int_0^T \int_{3Q_i\cap\Omega} M_{Q_i}^{**}\big(t, v^\varepsilon(t,x)\psi(x)\big)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{3Q_i\cap\Omega\cap\mathrm{supp}(\psi)} M_{Q_i}^{**}\big(t, v^\varepsilon(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Then, repeating the procedure from (8.3.4), we deduce

$$\int_0^T \int_{3Q_i\cap\Omega\cap\mathrm{supp}(\psi)} M_{Q_i}^{**}\big(t, v^\varepsilon(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{5Q_i\cap\Omega\cap\mathrm{supp}(\psi)} M_{Q_i}^{**}\big(t, v(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, as $M_{Q_i}(t,\xi) = \text{ess inf}_{x\in\Omega\cap 5Q_i} M(t,x,\xi)$ and since $M_{Q_i}^{**}(t,\xi) \le M_{Q_i}(t,\xi)$, we can estimate each summand by the above inequality to get:

$$\int_0^T \int_{5Q_i\cap\Omega\cap\mathrm{supp}(\psi)} M_{Q_i}^{**}\big(t, v(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_{5Q_i\cap\Omega} M(t,x,v(t,x))\,\mathrm{d}x\,\mathrm{d}t.$$

Coming back to (8.3.3), we obtain

$$\int_{\Omega_T} M\left(t,x,z_\varepsilon(t,x)\right)\mathbb{1}_{|z_\varepsilon(t,x)|>\xi_0}\,\mathrm{d}t\,\mathrm{d}x \le \int_{\Omega_T} M\left(t,x,v(t,x)\right)\mathrm{d}t\,\mathrm{d}x \tag{8.3.5}$$

for some possibly different constant C which can be increased due to integration over repeating parts of overlaping cubes $\{5Q_i\}_{i=1}^{N_\varepsilon}$. Combining (8.3.2) with (8.3.5), we finish the proof. □

Proof of Theorem 8.3.4. First two properties follow from properties of mollification and continuity of the truncation. To show also the third property, we first compute:

$$\nabla \left(T_k(u^\varepsilon)\psi\right)^\varepsilon = \left(\mathbb{1}_{|u^\varepsilon|\le k}(\nabla u)^\varepsilon \psi\right)^\varepsilon + \left(T_k(u^\varepsilon)\nabla\psi\right)^\varepsilon.$$

Then, due to (N7) in Lemma 7.1.7, $(T_k(u^\varepsilon)\nabla\psi)^\varepsilon \xrightarrow{M} T_k(u)\nabla\psi$ and so, it is sufficient to focus only on the first term. Using Lemma 7.2.5, we find a sequence of simple functions $\{\varphi_n\}$ such that $\varphi_n \to \nabla u$ a.e. and $\varphi_n \xrightarrow{M} \nabla u$ as $n\to\infty$, i.e. there is $\widetilde{\lambda} > 0$ such that

$$\int_{\Omega_T} M\left(t,x,\frac{\nabla u(t,x) - \varphi_n(t,x)}{\widetilde{\lambda}}\right) \mathrm{d}x\,\mathrm{d}t \to 0.$$

Then, for some $\lambda_1,\lambda_2,\lambda_3$ to be chosen later, $\lambda = \lambda_1+\lambda_2+\lambda_3$ and some $n\in\mathbb{N}$ we write:

$$\begin{aligned}
\int_{\Omega_T} M&\left(t,x,\frac{\left(\mathbb{1}_{|u^\varepsilon|\le k}(\nabla u)^\varepsilon\psi\right)^\varepsilon - \mathbb{1}_{|u|\le k}\nabla u\psi}{\lambda}\right)\mathrm{d}x\,\mathrm{d}t \le \\
&\le \frac{\lambda_1}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{\left(\mathbb{1}_{|u^\varepsilon|\le k}(\nabla u)^\varepsilon\psi\right)^\varepsilon - \left(\mathbb{1}_{|u^\varepsilon|\le k}(\varphi_n)^\varepsilon\psi\right)^\varepsilon}{\lambda_1}\right)\mathrm{d}x\,\mathrm{d}t + \\
&+ \frac{\lambda_2}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{\left(\mathbb{1}_{|u^\varepsilon|\le k}(\varphi_n)^\varepsilon\psi\right)^\varepsilon - \mathbb{1}_{|u|\le k}\varphi_n\psi}{\lambda_2}\right)\mathrm{d}x\,\mathrm{d}t \\
&+ \frac{\lambda_3}{\lambda}\int_{\Omega_T} M\left(t,x,\mathbb{1}_{|u|\le k}\psi\frac{\varphi_n-\nabla u}{\lambda_3}\right)\mathrm{d}x\,\mathrm{d}t =: A^{n,\varepsilon} + B^{n,\varepsilon} + C^{n,\varepsilon}.
\end{aligned}$$

Using (N7) in Lemma 7.1.7, for any $n\in\mathbb{N}$ and $\lambda_2>0$, $\limsup_{\varepsilon\to 0} B^{n,\varepsilon} = 0$. Also, we note that

$$\begin{aligned}
\frac{\lambda_1}{\lambda}\int_{\Omega_T} M&\left(t,x,\frac{\left(\mathbb{1}_{|u^\varepsilon|\le k}(\nabla u)^\varepsilon\psi\right)^\varepsilon - \left(\mathbb{1}_{|u^\varepsilon|\le k}(\varphi_n)^\varepsilon\psi\right)^\varepsilon}{\lambda_1}\right)\mathrm{d}x\,\mathrm{d}t \le \\
&\le \frac{\lambda_1}{\lambda}\int_{\Omega_T} M\left(t,x,\frac{\left(\mathbb{1}_{|u^\varepsilon|\le k}(\nabla u-\varphi_n)^\varepsilon\psi\right)^\varepsilon}{\lambda_1}\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Therefore, if we choose $\lambda_1 = \lambda_3 = \widetilde{\lambda}$ and use Lemma 8.3.5, we obtain

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\left(A^{n,\varepsilon} + C^{n,\varepsilon}\right) \le &\int_{\Omega_T} M\left(t,x,\mathbb{1}_{|u|\le k}\psi\frac{\varphi_n-\nabla u}{\widetilde{\lambda}}\right)\mathrm{d}x\,\mathrm{d}t + \\
&+ \int_{\Omega_T} m_2\left(\left|\frac{\varphi_n-\nabla u}{\widetilde{\lambda}}\right|\psi(x)\right)\mathbb{1}_{\left|\frac{\varphi_n-\nabla u}{\widetilde{\lambda}}\right|\psi(x)\le\xi_0}\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Since $\varphi_n\to\nabla u$ a.e. in Ω_T and $\varphi_n \xrightarrow{M} \nabla u$, we conclude the proof. $\square$

8.4 Standard techniques for PDEs with non-standard growth

In this section we present two techniques usually used in analysis of the non-standard growth PDEs. The first one allows to approximate PDEs with operator A by a sequence of PDEs with operators A_n whose growth and coercivity is controlled by an isotropic Young function. The second one is a local version of monotonicity trick which allows

8.4.1 Regularization of the operator

In this section, we formulate well-posedness theory for parabolic equations in Musielak-Orlicz spaces with Young functions. This allows us to construct solution to our problem by a limiting procedure. The following result was proven by Elmahi and Meskine [125, Theorem 2] using Galerkin's approximation and mollification as in Section 8.3 (however here N-function is homogeneous and isotropic so the result can be established significantly easier).

Theorem 8.4.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with segment property. Let $m : \mathbb{R} \to \mathbb{R}$ be a Young function. Suppose that $a : \Omega_T \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:*

(R1) a is a Carathéodory's function, i.e. for a.e. $(t,x) \in \Omega_T$, map $\mathbb{R}^d \ni \xi \mapsto a(t,x,\xi)$ is continuous and for all $\xi \in \mathbb{R}^d$, map $\Omega_T \ni (t,x) \mapsto a(t,x,\xi)$ is measurable,

(R2) there are $c \in E_{m^*}(\Omega_T)$ with $c \geq 0$ and nonnegative constant β and γ such that

$$|a(t,x,\xi)| \leq \beta\Big(c(t,x) + (m^*)^{-1}(m(\gamma|\xi|))\Big),$$

(R3) there are $d \in L^1(\Omega_T)$ and nonnegative constants α and λ such that

$$a(t,x,\xi) \cdot \xi + d(t,x) \geq \alpha\, m\left(\frac{|\xi|}{\lambda}\right),$$

(R4) a is stronly monotone, i.e. for all $\eta, \xi \in \mathbb{R}^d$ and a.e. $(t,x) \in \Omega_T$:

$$(a(t,x,\xi) - a(t,x,\eta)) \cdot (\xi - \eta) > 0.$$

Then, the problem

$$u_t = \operatorname{div} a(t, x, \nabla u) + g$$

with $u(0,x) = u_0(x) \in L^\infty(\Omega_T)$, $u(t,x) = 0$ *for* $x \in \partial\Omega$ *and* $g \in L^\infty(\Omega_T)$ *has the unique weak solution* $u \in C((0,T); L^2(\Omega)) \cap W^1 L_m(\Omega_T)$ *(see Definition 7.2.10).*

Using Theorem 8.4.1, one can define a sequence approximating solutions to (8.1.3) as follows:

Lemma 8.4.2. *Suppose* A *satisfies Assumption 8.2.4,* M *is an* N*-function and* m *is a Young function such that* $M(t,x,\xi) \leq m(|\xi|)$*. For* $\theta \in (0,1]$*, consider regularized operator*

$$A_\theta(t,x,\xi) = A(t,x,\xi) + \theta \nabla_\xi m(|\xi|). \tag{8.4.1}$$

Then, there exists a weak solution to the problem

$$u_t^\theta = \operatorname{div} A_\theta(t,x,\nabla u^\theta) + g \tag{8.4.2}$$

with $u(0,x) = u_0(x) \in L^\infty(\Omega_T)$, $u(t,x) = 0$ *for* $x \in \partial\Omega$ *and* $g \in L^\infty(\Omega_T)$*. More precisely,*

$$u^\theta \in C((0,T); L^2(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega)).$$

Moreover, u^θ *satisfies the global energy equality:*

$$\begin{aligned}\int_\Omega \left[(u^\theta(t,x))^2 - (u_0(x))^2\right] \mathrm{d}x = -\int_0^t \int_\Omega A^\theta(s,x,\nabla u^\theta(s,x)) \cdot \nabla u^\theta(s,x) \, \mathrm{d}x \, \mathrm{d}s \\ + \int_0^t \int_\Omega g(s,x)\, u^\theta(s,x) \, \mathrm{d}x \, \mathrm{d}s.\end{aligned} \tag{8.4.3}$$

We also have bounds which are uniform in θ*:*

(C1) sequence $\{u^\theta\}$ *is uniformly bounded in* $L^\infty(0,T; L^2(\Omega))$*,*

(C2) sequence $\{\nabla u^\theta\}$ *is uniformly bounded in* $L_M(\Omega_T)$*,*

(C3) sequence $\{A(t,x,\nabla u^\theta)\}$ *is uniformly bounded in* $L_{M^*}(\Omega_T)$*,*

(C4) sequence $\{\theta m^*(\nabla_\xi m(|\nabla u^\theta|))\}$ *is uniformly bounded in* $L^1(\Omega_T)$*.*

Proof. First, we observe we observe from the definition of the convex conjugate that

$$\nabla_\xi m(|\xi|) \cdot \xi = m(|\xi|) + m^*(|\nabla_\xi m(|\xi|)|). \tag{8.4.4}$$

We also note that $\nabla_\xi m(|\xi|) = m'(|\xi|)\frac{\xi}{|\xi|}$ so that $\nabla_\xi m(|\xi|)\xi \geq 0$. Let us check that assumptions of Theorem 8.4.1 are satisfied with operator (8.4.1) controlled by N-function m. Assumption (R1) is fulfilled trivially. To verify (R2), we use (8.4.4), (A2) in Assumption 8.2.4 and the convexity, to obtain:

$$\begin{aligned} cA_\theta(t,x,\xi)\cdot\xi &\geq M(t,x,\xi) + M^*(t,x,A(t,x,\xi)) - h(t,x) + c\,\theta\,\nabla_\xi m(|\xi|)\cdot\xi \\ &\geq 0 + m^*(|A(t,x,\xi)|) - h(t,x) + c\,\theta\, m^*(|\nabla_\xi m(|\xi|)|) \\ &\geq 2\min(1,c)\left(\frac{1}{2}m^*(|A(t,x,\xi)|) + \frac{1}{2}\,m^*(\theta|\nabla_\xi m(|\xi|)|)\right) - |h(t,x)| \\ &\geq 2\min(1,c)\, m^*\left(\frac{1}{2}\,|A_\theta(t,x,\xi)|\right) - |h(t,x)|. \end{aligned} \tag{8.4.5}$$

On the other hand, by Young's inequality

$$cA_\theta(t,x,\xi)\cdot\xi \leq \min(1,c)\, m\left(\frac{c}{\min^2(1,c)}|\xi|\right) + \min(1,c)\, m^*\left(\frac{1}{2}\,|A_\theta(t,x,\xi)|\right). \tag{8.4.6}$$

Hence, we combine (8.4.5) and (8.4.6) to deduce

$$\min(1,c)\, m^*\left(\frac{1}{2}\,|A_\theta(t,x,\xi)|\right) \leq \min(1,c)\, m\left(\frac{c}{\min^2(1,c)}|\xi|\right) + |h(t,x)|.$$

Next, we abbreviate $c_1 = 1/\min(1,c)$ and $c_2 = \frac{c}{\min^2(1,c)}$. Furthermore, since m^* is increasing and convex, then $(m^*)^{-1}$ is increasing and concave. Moreover $(m^*)^{-1}(0) = 0$ so $(m^*)^{-1}$ is subadditive and therefore

$$\frac{1}{2}\,|A_\theta(t,x,\xi)| \leq (m^*)^{-1}\Big(m\,(|\xi|) + c_1|h(t,x)|\Big) \leq (m^*)^{-1}\big(m\,(|\xi|)\,\big) + (m^*)^{-1}\big(c_1|h(t,x)|\big),$$

which proves (R2) since $h \in L^\infty(\Omega_T)$. Then, repeating computation in (8.4.5) and applying (8.4.4) we deduce:

$$\begin{aligned} cA_\theta(t,x,\xi)\cdot\xi &\geq M(t,x,\xi) + M^*(t,x,A(t,x,\xi)) - h(t,x) + c\,\theta\,\nabla_\xi m(|\xi|)\cdot\xi \\ &\geq c\,\theta\, m(|\xi|) - h(t,x), \end{aligned} \tag{8.4.7}$$

which proves (R3). Finally, (R4) follows easily as the function m can be always assumed to be strictly convex (otherwise, one can add a strictly convex function to m). Therefore, Theorem 8.4.1 applies so we conclude that for each $\theta \in (0,1]$ there is a unique solution u^θ as desired. Moreover, energy equality (8.4.3) is valid.

Now, we intend to establish uniform estimates (C1)–(C4). Let m_1 be a Young function such that $m_1(|\xi|) \le M(t,x,\xi)$ as in point (M4) in Definition 7.1.2. We estimate by using the Hölder inequality:

$$\begin{aligned}\int_{\Omega_t} f(s,x)u^\theta(s,x)\,\mathrm{d}s\,\mathrm{d}x \le \|f\|_\infty \int_{\Omega_t} |u^\theta(s,x)|\,\mathrm{d}s\,\mathrm{d}x \le \\ \le \|f\|_\infty \int_{\Omega_t} \big(|u^\theta(s,x)|^2+1\big)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

Using energy equality (8.4.3) and noting that $A^\theta(s,x,\nabla u^\theta(s,x))\cdot\nabla u^\theta(s,x) \ge 0$ we deduce that for a.e. $t\in(0,T)$

$$\int_\Omega \big(u^\theta(t,x)\big)^2\,\mathrm{d}x \le \int_\Omega (u_0(x))^2\,\mathrm{d}x + \|f\|_\infty \int_0^t \int_\Omega \big(|u^\theta(s,x)|^2+1\big)\,\mathrm{d}x\,\mathrm{d}s.$$

Therefore, Grönwall's lemma implies that u^θ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Moreover, (A2) in Assumption 8.2.4 leads to the estimate:

$$\begin{aligned}\int_{\Omega_t} \big(M^*(s,x,A(s,x,\nabla u^\theta(s,x))) + M(s,x,\nabla u^\theta(s,x)) - h(s,x)\big)\,\mathrm{d}s\,\mathrm{d}x \le \\ \le c\int_{\Omega_t} A(s,x,\nabla u^\theta(s,x))\cdot\nabla u^\theta(s,x)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

As $\int_\Omega \big(u^\theta(t,x)\big)^2\,\mathrm{d}x$ and $\int_{\Omega_t} f(s,x)u^\theta(s,x)\,\mathrm{d}s\,\mathrm{d}x$ are uniformly bounded, we deduce from energy equality (8.4.3) that for a.e. $(t,x)\in\Omega_T$, the quantity

$$\begin{aligned}\int_{\Omega_t} M^*(s,x,A(s,x,\nabla u^\theta(s,x)))\,\mathrm{d}s\,\mathrm{d}x + \int_{\Omega_t} M(s,x,\nabla u^\theta(s,x))\,\mathrm{d}s\,\mathrm{d}x + \\ + \int_{\Omega_t} \theta\nabla_\xi m(|\nabla u^\theta(s,x)|)\cdot\nabla u^\theta(s,x)\,\mathrm{d}s\,\mathrm{d}x \le C(f,h,u_0),\end{aligned}$$

the constant $C(f,h,u_0)$ is independent of θ. Due to (N6) in Lemma 7.1.7, we have that $\{\nabla u^\theta\}$ is uniformly bounded in $L_M(\Omega_T)$ and $\{A(t,x,\nabla u^\theta)\}$ is uniformly bounded in $L_{M^*}(\Omega_T)$. Finally, using (8.4.4) we deduce that sequence $\{\theta m^*(\nabla_\xi m(|\nabla u^\theta|))\}$ is uniformly bounded in $L^1(\Omega_T)$. □

Thanks to the uniform bounds established in Lemma 8.4.2, we can now let $\theta \to 0$ in (8.4.2). The starting point for this limiting procedure is the observation that the approximative term vanishes in the limit, which is formulated in the next lemma.

Lemma 8.4.3. *Under notation and assumptions of Lemma 8.4.2, for any $\varphi : \Omega_T \mapsto \mathbb{R}^d$ such that $\varphi \in L^\infty(\Omega_T; \mathbb{R}^d)$, we have*

$$\lim_{\theta \to 0} \int_{\Omega_T} \theta \nabla_\xi m(|\nabla u^\theta|) \cdot \varphi \, \mathrm{d}t \, \mathrm{d}x = 0.$$

Proof. This was also proved in [78] but it was not formulated as a separate result so we provide the proof here. Consider $\Omega_T^R = \{(t,x) \in \Omega_T : |\nabla u^\theta| \le R\}$ and write

$$\int_{\Omega_T} \left|\theta \nabla_\xi m(|\nabla u^\theta|)\right| = \int_{\Omega_T^R} \left|\theta \nabla_\xi m(|\nabla u^\theta|)\right| + \int_{\Omega_T \setminus \Omega_T^R} \left|\theta \nabla_\xi m(|\nabla u^\theta|)\right|. \tag{8.4.8}$$

For any $R > 0$, the first term converges to 0 as $\theta \to 0$. Note that by convexity,

$$m^*(\theta \nabla_\xi m(|\nabla u^\theta|)) \le m^*(\nabla_\xi m(|\nabla u^\theta|))$$

so that due to (N5) in Lemma 7.1.7, sequence $\{\theta \nabla_\xi m(|\nabla u^\theta|)\}$ is uniformly integrable. Therefore, as $R \to \infty$, the second term in (8.4.8) tends to 0 and the conclusion follows. □

The next result deals with the time derivatives of u^θ and will be used to deduce the pointwise convergence.

Lemma 8.4.4. *Under notation and assumptions of Lemma 8.4.2, for every $\theta > 0$, we have $\partial_t u^\theta \in (W^1 E_m(\Omega_T))^*$ where m is defined in Lemma 8.4.2. Moreover, for all $\varphi \in W^1 E_m(\Omega_T)$ we have the following inequality:*

$$\left(\partial_t u^\theta, \varphi\right) \le C \|\varphi\|_{W^1 L_m}, \tag{8.4.9}$$

where the constant C is independent of θ.

Proof. First, let $\varphi \in C_0^\infty((0,T) \times \Omega)$. By the weak formulation of (8.4.2) we have

$$\begin{aligned} -\int_{\Omega_T} u^\theta(t,x) \partial_t \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x + \int_{\Omega_T} A(t,x,\nabla u^\theta) \cdot \nabla \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x + \\ + \int_{\Omega_T} \theta_n \nabla_\xi m(|\nabla u^\theta|) \cdot \nabla \varphi \, \mathrm{d}t \, \mathrm{d}x = \int_{\Omega_T} f(t,x) \varphi(t,x) \, \mathrm{d}t \, \mathrm{d}x. \end{aligned}$$

Thus, we can estimate the left hand side using Lemma 7.2.3 as follows:

$$\left|\int_{\Omega_T} u^\theta(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| \leq \left\|A(t,x,\nabla u^\theta)\right\|_{L_{m^*}}\left\|\nabla\varphi\right\|_{L_m} +$$
$$+\,\theta_n\left\|\nabla_\xi m(|\nabla u^\theta|)\right\|_{L_{m^*}}\left\|\nabla\varphi\right\|_{L_m} + |\Omega_T|\, m^*\left(\|f\|_\infty\right)\left\|\varphi\right\|_{L_m}.$$

Note that $M(t,x,\xi) \leq m(|\xi|)$ implies $m^*(|\xi|) \leq M^*(t,x,\xi)$ and so,

$$\left\|A(t,x,\nabla u^\theta)\right\|_{L_{m^*}} \leq \left\|A(t,x,\nabla u^\theta)\right\|_{L_{M^*}}.$$

Therefore, we can use uniform bounds provided by Lemma 8.4.2 and this (after application of the Poincaré inequality from Lemma 7.2.11) concludes the proof of (8.4.9) for $\varphi \in C_0^\infty((0,T)\times\Omega)$. The general case follows by the density (in norm!) of $C_0^\infty((0,T)\times\Omega)$ in $W_0^1E_m(\Omega_T)$ (cf. (P2) in Lemma 7.2.11). $\square$

Finally, note that uniform bounds in Lemma 8.4.2 guarantees the existence of subsequences (that we do not relabel) converging weakly-$*$ in appropriate spaces (cf. Lemma 7.2.9). We will also need stronger compactness provided by the following result.

Lemma 8.4.5. *Under notation and assumptions of Lemma 8.4.2, the sequence $\{u^\theta\}$ is relatively compact in $L^1(0,T;L^1(\Omega))$. In particular, it has a subsequence converging a.e. in Ω_T.*

Proof. We recall a version of Aubin-Lions Lemma (cf. [248]):

Aubin-Lions Lemma. Let X_0, X and X_1 be Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1. Suppose that sequence of functions $\{f_n\}$ is bounded in $L^q(0,T;X)$ and $L^1(0,T;X_0)$. Moreover, assume that sequence of distributional time derivatives $\{\partial_t f_n\}$ is bounded in $L^1(0,T;X_1)$. Then, $\{f_n\}$ is relatively compact in $L^p(0,T;X)$ for any $1 \leq p < q$.

We want to apply this result with $X_0 = W_0^{1,1}(\Omega)$, $X = L^1(\Omega)$ and $X_1 = W^{-2,r}(\Omega)$ for r such that $W_0^{2,r}(\Omega)$ is continuously embedded in $C^1(\Omega)$ ($r > d$ is sufficient, cf. [154, Corollary 7.11]).

- By Rellich-Kondrachov Theorem (or Arzela-Ascoli Theorem if $d = 1$), X_0 is compactly embedded in X.

- Let $f \in L^1(\Omega)$. Then, for $\varphi \in W_0^{2,r}(\Omega)$,

$$\left| \int_\Omega f\, \varphi \right| \leq \|f\|_{L^1} \|\varphi\|_{L^\infty} \leq C \|f\|_{L^1} \|\varphi\|_{W^{2,r}},$$

  for some constant C so that X is continuously embedded in X_1.

- Sequence $\{u^\theta\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$ and $\{\nabla u^\theta\}$ is uniformly bounded in $L_{M^*}(\Omega_T)$. In particular, $\{u^\theta\}$ is uniformly bounded in $L^1(0,T;W_0^{1,1}(\Omega))$ and $L^2(0,T;L^1(\Omega))$.

- Let $\varphi \in L^\infty(0,T;W_0^{2,r}(\Omega))$ with $\left\|\varphi\right\|_{L^\infty(0,T;W_0^{2,r}(\Omega))} \leq 1$ and the plan is to prove that $\left(\partial_t u^\theta, \varphi\right)$ is uniformly bounded in φ and $\theta \in (0,1]$. By the choice of r, there is a constant C such that $\left|\varphi\right| \leq C$ and $\left|\nabla\varphi\right| \leq C$. In particular, $\varphi \in W_0^1 E_m(\Omega_T)$ and $\left\|\varphi\right\|_{W^1 L_m} \leq C$ for some possibly different constant C. Using Lemma 8.4.4, we establish assertion. By duality, this shows that $\{\partial_t u^\theta\}$ is uniformly bounded in $L^1(0,T;W^{-2,r}(\Omega))$.

Aubin-Lions Lemma implies that $\left\{u^\theta\right\}$ is relatively compact in $L^1(0,T;L^1(\Omega))$. □

8.4.2 Local version of monotonicity method

The following procedure allows us to identify weak-$*$ limit of $A(t,x,\nabla u_n)$. We formulate here its local version and provide the proof that is almost identical to the global case presented in [78, Lemma A.5].

Lemma 8.4.6. *Let A satisfy Assumption 8.2.4 and M be an N-function. Assume that there are $\alpha \in L_{M^*}(\Omega_T;\mathbb{R}^d)$ and $\xi \in L_M(\Omega_T;\mathbb{R}^d)$ such that*

$$\int_{\Omega_T} (\alpha - A(t,x,\eta)) \cdot (\xi - \eta)\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x \geq 0 \tag{8.4.10}$$

for all $\eta \in L^\infty(\Omega_T;\mathbb{R}^d)$ and $\psi \in C_0^\infty(\Omega)$ with $0 \leq \psi \leq 1$. Then,

$$A(t,x,\xi) = \alpha(t,x) \text{ a.e. in } \Omega_T.$$

Proof. Consider subsets $\Omega_T^k = \{(t,x) \in \Omega_T : |\xi(t,x)| \le k\}$ and note that if $j < i$ then $\Omega_T^j \subset \Omega_T^i$. We use the assumption (8.4.10) with $\eta = \xi \mathbb{1}_{\Omega_T^i} + h\, z\, \mathbb{1}_{\Omega_T^j}$ where $h > 0$ and $z \in L^\infty(\Omega_T; \mathbb{R}^d)$ and we obtain

$$\int_{\Omega_T} \Big(\alpha - A(t,x,\xi\mathbb{1}_{\Omega_T^i} + h\, z\, \mathbb{1}_{\Omega_T^j})\Big) \cdot (\xi - \xi\mathbb{1}_{\Omega_T^i} - h\, z\, \mathbb{1}_{\Omega_T^j})\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x \ge 0.$$

Considering integral on Ω_T^i and $\Omega_T \setminus \Omega_T^i$ we deduce

$$\int_{\Omega_T \setminus \Omega_T^i} (\alpha - A(t,x,0)) \cdot \xi\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x + h \int_{\Omega_T^j} (A(t,x,\xi + h\, z) - \alpha) \cdot z\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x \ge 0.$$

Note that $A(s,x,0) = 0$ due to (A4) in Assumption 8.2.4. Therefore, by integrability, the first term tends to 0 as $i \to \infty$. Therefore,

$$\int_{\Omega_T^j} (A(t,x,\xi + h\, z) - \alpha) \cdot z\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x \ge 0.$$

Now, we want to let $h \to 0$. We have convergence $A(t,x,\xi + h\, z) \to A(t,x,\xi)$ due to (A1) in Assumption 8.2.4. Moreover, $\xi + h\, z$ is uniformly bounded on Ω_T^j. Therefore, (N7) in Lemma 7.1.7 implies:

$$\int_{\Omega_T^j} (A(t,x,\xi) - \alpha) \cdot z\, \psi(x) \,\mathrm{d}t \,\mathrm{d}x \ge 0.$$

Finally, choosing $z(t,x) = -\frac{A(t,x,\xi) - \alpha(t,x)}{|A(t,x,\xi) - \alpha(t,x)|} \mathbb{1}_{A(t,x,\xi) - \alpha(t,x) \neq 0}$, we deduce

$$A(t,x,\xi) = \alpha(t,x) \qquad \text{for a.e.}(t,x) \in \Omega_T^j \cap \operatorname{supp}\psi.$$

Since j and ψ are arbitrary, the assertion follows. □

8.5 Equation $u_t = \operatorname{div} \alpha + f$ for $\alpha \in L_{M^*}(\Omega_T)$ and $f \in L^\infty(\Omega_T)$

In this section we study the equation

$$u_t = \operatorname{div} \alpha + f$$

or more precisely, the following distributional identity required to be satisfied for all $\varphi \in C_0^\infty([0,T) \times \Omega)$:

$$\begin{aligned} -\int_{\Omega_T} u(t,x) \partial_t \varphi(t,x) \,\mathrm{d}t \,\mathrm{d}x - \int_{\Omega} u_0(x) \varphi(0,x) \,\mathrm{d}x + \\ + \int_{\Omega_T} \alpha(t,x) \cdot \nabla \varphi(t,x) \,\mathrm{d}t \,\mathrm{d}x = \int_{\Omega_T} f(t,x) \varphi(t,x) \,\mathrm{d}t \,\mathrm{d}x, \end{aligned} \tag{8.5.1}$$

which is obtained in Section 8.6 as the limit of (8.4.2). For $u : \Omega_T \to \mathbb{R}$ solving (8.5.1), we write $\widetilde{u}$ to denote its extension:

$$\widetilde{u}(t,x) = \begin{cases} 0 & \text{for } t > T, \\ u(t,x) & \text{for } t \in (0,T], \\ u_0(x) & \text{for } t \leq 0. \end{cases} \tag{8.5.2}$$

We also extend α and f to be zero for $t \in \mathbb{R} \setminus (0,T)$:

$$\overline{\alpha}(t,x) = \begin{cases} \alpha(t,x) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in \mathbb{R} \setminus (0,T), \end{cases} \qquad \overline{f}(t,x) = \begin{cases} f(t,x) & \text{for } t \in (0,T), \\ 0 & \text{for } t \in \mathbb{R} \setminus (0,T). \end{cases} \tag{8.5.3}$$

Our goal is to obtain some form of energy equality which will be crucial in developing the existence theory for (8.1.3). Classical approach (cf. [78]) was based on appropriate mollification in space and time which required some continuity assumptions on $M(t,x,\xi)$ both in t and x. Below, we show that mollification of the solution u only in space has already Sobolev regularity in space and time.

Lemma 8.5.1. *Suppose that $u \in V_T^M(\Omega)$, $\alpha \in L_{M^*}(\Omega_T)$ and $f \in L^\infty(\Omega_T)$. Consider extensions $\widetilde{u}$, $\overline{\alpha}$ and $\overline{f}$ defined in* (8.5.2) *and* (8.5.3)*. Then,*

$$\begin{aligned} -\int_\Omega \int_{-T}^T & \widetilde{u}(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \\ &= -\int_\Omega \int_{-T}^T \overline{\alpha}(t,x)\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega \int_{-T}^T \overline{f}(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x, \end{aligned} \tag{8.5.4}$$

for arbitrary $\varphi \in C_0^\infty((-T,T)\times\Omega)$. Moreover, $\widetilde{u}^\varepsilon \in W^{1,1}((-T,T)\times\Omega')$ where $\Omega' \Subset \Omega$.

Proof. To verify (8.5.4), let $\varphi \in C_0^\infty((-T,T)\times\Omega)$. We compute using (8.5.1):

$$\begin{aligned} -\int_\Omega \int_{-T}^T & \widetilde{u}(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \\ &= -\int_\Omega \int_{-T}^0 \widetilde{u}(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega \int_0^T \widetilde{u}(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \\ &= -\int_\Omega u_0(x)\varphi(0,x)\,\mathrm{d}x - \int_\Omega \int_0^T u(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \\ &= -\int_\Omega \int_{-T}^T \overline{\alpha}(t,x)\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_\Omega \int_{-T}^T \overline{f}(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

Mollifying (8.5.4) in space (by testing with mollified test function), we deduce $\partial_t u^\varepsilon \in L^1((-T,T)\times\Omega')$ proving the Sobolev regularity in time. Asserted regularity in space is obvious. $\square$

Remark 8.5.2. Extension procedure above can be applied to obtain that $u^\varepsilon \in W^{1,1}((-M,T)\times\Omega')$ for any $0 < M < T$. However, we only need Sobolev regularity on $(-\delta,T)\times\Omega'$ for some $\delta > 0$ which can be arbitrarily small.

Lemma 8.5.3 (Local energy equality)**.** *Suppose that $u \in V_T^M(\Omega)$ is a solution to (8.5.1) with $\alpha \in L_{M^*}(\Omega_T)$, $f \in L^\infty(\Omega_T)$ and Assumption 8.2.1 is satisfied. Then, for arbitrary $k \in \mathbb{N}$, for arbitrary $\psi \in C_0^\infty(\Omega)$ fulfilling $0 \le \psi \le 1$ and for a.e. $t \in (0,T)$, the following energy equality is satisfied:*

$$\begin{aligned}\int_\Omega \psi(x)\big[G_k(u(t,x)) - G_k(u_0(x))\big]\,\mathrm{d}x =& \\ = -\int_0^t\int_\Omega \alpha(s,x)\cdot\nabla\left[T_k(u(s,x))\,\psi(x)\right]\mathrm{d}x\,\mathrm{d}s& \qquad (8.5.5)\\ + \int_0^t\int_\Omega f(s,x)\,T_k(u(s,x))\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s,&\end{aligned}$$

where the function G_k and the function T_k are defined in Definition 8.3.1.

Proof. For $s_1, s_2 \in \mathbb{R}$ and $\tau > 0$ we define the approximation of $\mathbb{1}_{[s_1,s_2]}$:

$$\gamma^\tau_{s_1,s_2}(s) = \begin{cases} 0 & \text{for } s \le s_1 - \tau \text{ or } s \ge s_2 + \tau, \\ 1 & \text{for } s \in [s_1, s_2], \\ \text{affine} & \text{for } s \in [s_1 - \tau, s_1] \cup [s_2, s_2 + \tau]. \end{cases}$$

Let $\psi \in C_0^\infty(\Omega)$, $k \in \mathbb{N}$, $\varepsilon, \delta, \tau$ be small positive parameters and $\eta, \beta \in (0,T)$. Consider test function in (8.5.4):

$$\varphi^{\delta,\tau,\varepsilon}_{\eta,\beta}(t,x) = \left(\mathcal{R}^\delta\left(T_k(\widetilde{u}^\varepsilon(t,x))\,\psi(x)\,\gamma^\tau_{-\eta,\beta}(t)\right)\right)^\varepsilon \in C_0^\infty((-T,T)\times\Omega),$$

see Definitions 8.3.2 and 8.3.3 for mollification operators and Definition 8.3.1 for truncation T_k. Note that since $\psi \in C_0^\infty(\Omega)$, mollification in space is well-defined for sufficiently small $\varepsilon > 0$.

Now, we want to take limits in (8.5.4): first $\delta \to 0$, then $\tau \to 0$ and finally $\varepsilon \to 0$. We denote:

$$\begin{aligned}
A^{\delta,\tau,\varepsilon}_{\eta,\beta} &= -\int_\Omega \int_{-T}^{T} \widetilde{u}(t,x)\, \partial_t \varphi^{\delta,\tau,\varepsilon}_{\eta,\beta}(t,x)\, \mathrm{d}t\, \mathrm{d}x, \\
B^{\delta,\tau,\varepsilon}_{\eta,\beta} &= -\int_\Omega \int_{-T}^{T} \overline{\alpha}(t,x) \cdot \nabla \varphi^{\delta,\tau,\varepsilon}_{\eta,\beta}(t,x)\, \mathrm{d}t\, \mathrm{d}x, \\
C^{\delta,\tau,\varepsilon}_{\eta,\beta} &= \int_\Omega \int_{-T}^{T} \overline{f}(t,x)\, \varphi^{\delta,\tau,\varepsilon}_{\eta,\beta}\, \mathrm{d}t\, \mathrm{d}x.
\end{aligned}$$

and we study each term separately.

<u>Term $A^{\delta,\tau,\varepsilon}_{\eta,\beta}$.</u> Note that Sobolev derivatives and mollification commute so using Sobolev regularity in time from Lemma 8.5.1:

$$A^{\delta,\tau,\varepsilon}_{\eta,\beta} = \int_\Omega \int_{-T}^{T} \partial_t \widetilde{u}^\varepsilon(t,x) \Big(\mathcal{R}^\delta \Big(T_k(\widetilde{u}^\varepsilon(t,x))\psi(x)\gamma^\tau_{-\eta,\beta}(t)\Big)\Big)\, \mathrm{d}t\, \mathrm{d}x.$$

Using Dominated Convergence (we still have $\varepsilon > 0$),

$$\lim_{\tau\to 0}\lim_{\delta\to 0} A^{\delta,\tau,\varepsilon}_{\eta,\beta} = \int_\Omega \int_{-\eta}^{\beta} \partial_t \widetilde{u}^\varepsilon(t,x)\, T_k(\widetilde{u}^\varepsilon(t,x))\, \psi(x)\ \mathrm{d}t\, \mathrm{d}x =: A^\varepsilon_{\eta,\beta}.$$

As function $G(s) = \int_0^s T_k(\sigma)\, \mathrm{d}\sigma$ is C^1 with uniformly bounded derivative so standard chain rule for Sobolev maps [154, Theorem 7.8] together with Sobolev regularity in time from Lemma 8.5.1 shows that $G(\widetilde{u}^\varepsilon(t,x))\psi(x)$ is in $W^{1,1}((-T,T)\times\Omega)$, in particular it has Sobolev derivative in time. Moreover,

$$\partial_t G(\widetilde{u}^\varepsilon(t,x)) = T_k(\widetilde{u}^\varepsilon(t,x))\, \partial_t \widetilde{u}^\varepsilon(t,x)$$

Therefore, we can write:

$$A^\varepsilon_{\eta,\beta} = \int_\Omega \int_{-\eta}^{\beta} \partial_t G\left(\widetilde{u}^\varepsilon(t,x)\right)\ \mathrm{d}t\, \psi(x)\ \mathrm{d}x.$$

Now, using absolute continuity on lines for Sobolev maps [131, Theorem 4.21], fundamental theorem of calculus applies for a.e. $x \in \Omega$ and $\eta, \beta \in (0,T)$ so we obtain

$$A^\varepsilon_{\eta,\beta} = \int_\Omega \left[G_k\left(\widetilde{u}^\varepsilon(\beta,x)\right) - G_k\left(\widetilde{u}^\varepsilon(-\eta,x)\right)\right]\ \psi(x)\ \mathrm{d}x.$$

However, using definition of extension (8.5.2), this can be rewritten as

$$A^\varepsilon_{\eta,\beta} = \int_\Omega \left[G_k\left(\widetilde{u}^\varepsilon(\beta,x)\right) - G_k\left(u^\varepsilon_0(x)\right)\right]\ \psi(x)\ \mathrm{d}x.$$

Note that this step would not be achieved without extension for negative times as then, absolute continuity of Sobolev functions could be only applied for almost all times in $(0,T)$. Finally, using a.e. convergence of mollification and Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0} A^{\varepsilon}_{\eta,\beta} = \int_\Omega \left[G_k\left(\widetilde{u}(\beta,x)\right) - G_k\left(u_0(x)\right)\right] \psi(x)\,\mathrm{d}x$$

for almost all $\beta > 0$.

Term $B^{\delta,\tau,\varepsilon}_{\eta,\beta}$. First, we use commutating properties of mollification to write:

$$\begin{aligned} B^{\delta,\tau,\varepsilon}_{\eta,\beta} &= -\int_\Omega \int_{-T}^{T} \overline{\alpha}^{\varepsilon}(t,x)\cdot \nabla \mathcal{R}^{\delta}\Big(T_k(\widetilde{u}^{\varepsilon}(t,x))\psi(x)\gamma^{\tau}_{-\eta,\beta}(t)\Big)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_\Omega \int_{-T}^{T} \operatorname{div}\overline{\alpha}^{\varepsilon}(t,x)\psi(x)\; \mathcal{R}^{\delta}\Big(T_k(\widetilde{u}^{\varepsilon}(t,x))\gamma^{\tau}_{-\eta,\beta}(t)\Big)\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

Note that as $\delta\to 0$ and $\tau \to 0$, $\mathcal{R}^{\delta}\Big(T_k(\widetilde{u}^{\varepsilon}(t,x))\gamma^{\tau}_{-\eta,\beta}(t)\Big) \to T_k(\widetilde{u}^{\varepsilon}(t,x))\mathbb{1}_{[-\eta,\beta]}(t)$ a.e. in $(-T,T)\times\Omega'$ for $\Omega' \Subset \Omega$. As $\operatorname{div}\overline{\alpha}^{\varepsilon}(t,x)\psi(x) \in L^1(0,T;C_0^\infty(\Omega))$, we use Dominated Convergence Theorem to obtain

$$\lim_{\tau\to 0}\lim_{\delta\to 0} B^{\delta,\tau,\varepsilon}_{\eta,\beta} = \int_\Omega\int_{-\eta}^{\beta} \operatorname{div}\overline{\alpha}^{\varepsilon}(t,x)\psi(x)\; T_k(\widetilde{u}^{\varepsilon}(t,x))\,\mathrm{d}t\,\mathrm{d}x := B^{\varepsilon}_{\eta,\beta}.$$

Then, we write:

$$B^{\varepsilon}_{\eta,\beta} = -\int_\Omega\int_{-\eta}^{\beta} \overline{\alpha}(t,x)\cdot\nabla\left(\psi(x)T_k(\widetilde{u}^{\varepsilon}(t,x))\right)^{\varepsilon}\,\mathrm{d}t\,\mathrm{d}x.$$

Due to Theorem 8.3.4, $\nabla\left(\psi(x)T_k(\widetilde{u}^{\varepsilon}(t,x))\right)^{\varepsilon} \xrightarrow{M} \nabla\left(\psi(x)T_k(\widetilde{u}(t,x))\right)$ so using Corollary 7.2.7 we finally conclude that $B^{\eta,\tau,\varepsilon}_{\eta,\beta}$ converges to

$$-\int_\Omega\int_{-\eta}^{\beta} \overline{\alpha}(t,x)\cdot\nabla\left(\psi(x)T_k(\widetilde{u}(t,x))\right)\mathrm{d}t\,\mathrm{d}x =$$

$$= -\int_\Omega\int_0^{\beta} \alpha(t,x)\cdot\nabla\left(\psi(x)T_k(u(t,x))\right)\mathrm{d}t\,\mathrm{d}x.$$

Term $C^{\delta,\tau,\varepsilon}_{\eta,\beta}$. This is the easiest part. Note that $\varphi^{\delta,\tau,\varepsilon}_{\eta,\beta} \to T_k(\widetilde{u}(t,x))\psi(x)\mathbb{1}_{[-\eta,\beta]}(t)$ a.e. in $(-T,T)\times\Omega$ as $\delta\to 0$, $\tau\to 0$ and $\varepsilon\to 0$. Moreover, since $f\in L^\infty(\Omega_T)$ and $\left|\varphi^{\delta,\tau,\varepsilon}_{\eta,\beta}\right| \le k$, we use Dominated Convergence Theorem to deduce

$$C^{\delta,\tau,\varepsilon}_{\eta,\beta} \to \int_\Omega\int_{-\eta}^{\beta} \overline{f}(t,x)T_k(\widetilde{u}(t,x))\psi(x)\,\mathrm{d}t\,\mathrm{d}x = \int_\Omega\int_0^{\beta} f(t,x)T_k(u(t,x))\psi(x)\,\mathrm{d}t\,\mathrm{d}x.$$

Finally, we obtain (8.5.5) for $t=\beta$ concluding the proof. □

Remark 8.5.4. The same energy equality as (8.5.5) is satisfied by the solution to (8.4.2). Indeed, as the operator (8.4.1) is controlled by a Young function, Assumption 8.2.1 is satisfied. Therefore, for $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi(x) \leq 1$ and a.e. $t \in (0, T)$:

$$\begin{aligned}\int_\Omega \psi(x) \left[G_k(u^\theta(t,x)) - G_k(u_0(x))\right] \mathrm{d}x = \\ = -\int_0^t \int_\Omega A_\theta(s,x,\nabla u^\theta) \cdot \nabla \left[T_k(u^\theta(s,x))\, \psi(x)\right] \mathrm{d}x\, \mathrm{d}s \\ + \int_0^t \int_\Omega f(s,x)\, T_k(u^\theta(s,x))\, \psi(x)\, \mathrm{d}x\, \mathrm{d}s.\end{aligned} \tag{8.5.6}$$

Note that u^θ also satisfies the global energy equality (8.4.3), see Lemma 8.4.2.

8.6 Proof of existence result (Theorem 8.2.8)

Consider sequence of solutions $\{u^\theta\}_{\theta \in (0,1]}$ to the regularized problem (8.4.2). Using Lemma 8.4.5 as well as uniform bounds from Lemmata 8.4.2 and 7.2.9, we can extract a subsequence denoted with $u_n := u^{\theta_n}$ and $\theta_n \to 0$ such that:

- $u_n \to u$ in $L^1(0,T;L^1(\Omega))$ and a.e. in Ω_T,
- $u_n \overset{*}{\rightharpoonup} u$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$,
- $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ weakly-$*$ in $L_M(\Omega_T)$,
- $u_n \rightharpoonup u$ weakly in $L^1(0,T;W^{1,1}(\Omega))$,
- $A(\cdot,\cdot,\nabla u_n) \overset{*}{\rightharpoonup} \alpha$ weakly-$*$ in $L_{M^*}(\Omega_T)$,
- $u_n(t,x) \to u(t,x)$ in $L^2(\Omega)$ for a.e. $t \in (0,T)$ (see Lemma 8.6.1 after the proof of existence)

for some $u \in V_T^M(\Omega)$ and $\alpha \in L_{M^*}(\Omega_T)$.

For solutions to the regularized problem (8.4.2) we have the weak formulation.

Namely, for all $\varphi \in C_0^\infty([0,T)\times\Omega)$:

$$
\begin{aligned}
-\int_{\Omega_T} u_n(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega u_0(x)\varphi(0,x)\,\mathrm{d}x + \int_{\Omega_T} A(t,x,\nabla u_n)\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x \\
+\int_{\Omega_T}\theta_n\nabla_\xi m(|\nabla u_n|)\cdot\nabla\varphi\,\mathrm{d}t\,\mathrm{d}x = \int_{\Omega_T} f(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
\tag{8.6.1}
$$

Using Lemma 8.4.3, we can pass to the limit with $n\to\infty$ (or $\theta_n\to 0$) in (8.6.1) to obtain:

$$
\begin{aligned}
&-\int_{\Omega_T} u(t,x)\partial_t\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x - \int_\Omega u_0(x)\varphi(0,x)\,\mathrm{d}x = \\
&\qquad = -\int_{\Omega_T}\alpha\cdot\nabla\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x + \int_{\Omega_T} f(t,x)\varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
\tag{8.6.2}
$$

Thanks to (8.6.2), the theory from Section 8.5 can be applied and by using Lemma 8.5.3 we obtain that for $\psi\in C_0^\infty(\Omega)$ with $0\le\psi\le 1$ and a.e. $t\in(0,T)$:

$$
\begin{aligned}
\int_\Omega \psi(x)\left[G_k(u(t,x)) - G_k(u_0(x))\right]\mathrm{d}x = -\int_0^t\int_\Omega \alpha(s,x)\cdot\nabla\big(T_k(u(s,x))\big)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s \\
-\int_0^t\int_\Omega \alpha(s,x)\cdot\nabla\psi(x)\,T_k(u(s,x))\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega f(s,x)\,T_k(u(s,x))\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
\tag{8.6.3}
$$

Due to Remark 8.5.4, a similar energy equality holds for sequence $\{u_n\}_{n\in\mathbb{N}}$:

$$
\begin{aligned}
&\int_\Omega \psi(x)\left[G_k(u_n(t,x)) - G_k(u_0(x))\right]\mathrm{d}x = \\
&= -\int_0^t\int_\Omega A_{\theta_n}(s,x,\nabla u_n)\cdot\nabla\left[T_k(u_n)\,\psi(x)\right]\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega f(s,x)\,T_k(u_n)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}
\tag{8.6.4}
$$

We note that the term with operator $A_{\theta_n}(s,x,\nabla u_n)$ can be decomposed into four parts:

- $\int_0^t\int_\Omega A(s,x,\nabla u_n)\cdot\nabla\big(T_k(u_n)\big)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s$
- $\int_0^t\int_\Omega A(s,x,\nabla u_n)\cdot\nabla\psi(x)\,T_k(u_n)\,\mathrm{d}x\,\mathrm{d}s$ which, due to $A(s,x,\nabla u_n)\overset{*}{\rightharpoonup}\alpha$, $u_n\to u$ a.e. (so that $T_k(u_n)\to T_k(u)$ a.e.) and Dominated Convergence Theorem, converges to $\int_0^t\int_\Omega \alpha\cdot\nabla\psi(x)\,T_k(u)\,\mathrm{d}x\,\mathrm{d}s$,
- $\int_0^t\int_\Omega \theta_n\nabla_\xi m(|\nabla u_n|)\cdot\nabla\big(T_k(u_n)\big)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}s$, which is nonnegative,

- $\int_0^t \int_\Omega \theta_n \nabla_\xi m(|\nabla u_n|) \cdot \nabla\psi(x)\, T_k(u_n)\, \mathrm{d}x\, \mathrm{d}s$, converging to 0 due to Lemma 8.4.3.

Therefore, (8.6.4) implies:

$$\begin{aligned}\limsup_{n\to\infty} \int_0^t \int_\Omega A(s,x,\nabla u_n) \cdot \nabla\big(T_k(u_n(s,x))\big)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}s \leq \\ \leq - \int_0^t \int_\Omega \alpha \cdot \nabla\psi(x)\, T_k(u(s,x))\, \mathrm{d}x\, \mathrm{d}s - \int_\Omega \psi(x)\, [G_k(u(t,x)) - G_k(u_0(x))]\, \mathrm{d}x \\ + \int_0^t \int_\Omega f(s,x)\, T_k(u(s,x))\, \psi(x)\, \mathrm{d}x\, \mathrm{d}s,\end{aligned}$$

which combined with (8.6.3) yields:

$$\begin{aligned}\limsup_{n\to\infty} \int_0^t \int_\Omega A(s,x,\nabla u_n) \cdot \nabla\big(T_k(u_n)\big)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}s \leq \\ \leq \int_0^t \int_\Omega \alpha(s,x) \cdot \nabla\big(T_k(u)\big)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}s.\end{aligned} \tag{8.6.5}$$

Now, let $\alpha_k = \alpha \mathbb{1}_{|u(t,x)|<k}$. We claim that for any $k \in \mathbb{N}$, $\eta \in L^\infty\left(\Omega_T; \mathbb{R}^d\right)$, $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1$ and a.e. $t \in (0,T)$:

$$\int_\Omega \int_0^t (\alpha_k - A(s,x,\eta)) \cdot (\nabla T_k(u) - \eta)\, \psi(x)\, \mathrm{d}s\, \mathrm{d}x \geq 0. \tag{8.6.6}$$

Indeed, by monotonicity ((A3) in Assumption 8.2.4) we have that

$$\int_\Omega \int_0^t (A(s,x,\nabla T_k(u_n)) - A(s,x,\eta)) \cdot (\nabla T_k(u_n) - \eta)\, \psi(x)\, \mathrm{d}s\, \mathrm{d}x \geq 0.$$

By denoting $\Omega_t = (0,t) \times \Omega$, we see that:

- $\int_{\Omega_t} A(s,x,\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, \psi\, \mathrm{d}s\, \mathrm{d}x = \int_{\Omega_T} A(s,x,\nabla u_n) \cdot \nabla T_k(u_n)\, \psi\, \mathrm{d}s\, \mathrm{d}x$ since we have $\nabla\left[T_k(u_n)\right] = \nabla u_n\, \mathbb{1}_{|u_n|<k}$,
- $\int_{\Omega_t} A(s,x,\nabla T_k(u_n)) \cdot \eta\, \psi\, \mathrm{d}s\, \mathrm{d}x \to \int_{\Omega_t} \alpha \mathbb{1}_{|u(s,x)|<k} \cdot \eta\, \psi\, \mathrm{d}s\, \mathrm{d}x = \int_{\Omega_t} \alpha_k \cdot \eta\, \psi\, \mathrm{d}s\, \mathrm{d}x$. Indeed, we can write $A(s,x,\nabla T_k(u_n)) = A(s,x,\nabla u_n) \mathbb{1}_{|u_n(s,x)|<k}$ and pass to the limit with n using $A(s,x,\nabla u_n) \overset{*}{\rightharpoonup} \alpha(s,x)$ and $u_n \to u$ a.e.,
- $\int_{\Omega_t} A(s,x,\eta) \cdot \nabla T_k(u_n)\, \psi\, \mathrm{d}s\, \mathrm{d}x \to \int_{\Omega_t} A(s,x,\eta) \cdot \nabla T_k(u)\, \psi\, \mathrm{d}s\, \mathrm{d}x$ due to $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ and $u_n \to u$ a.e.

Therefore, (8.6.6) follows. By monotonicity trick (Lemma 8.4.6) we obtain $\alpha_k(t,x) = A(t,x,\nabla T_k(u))$ for any $k \in \mathbb{N}$ and this finally implies $\alpha = A(t,x,u)$ concluding the proof of existence.

Finally, to establish global energy inequality (8.2.4), we note that for a.e. $t \in (0,T)$

$$\int_\Omega u^2(t,x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega (u^n(t,x))^2\,\mathrm{d}x \tag{8.6.7}$$

as L^2 norm is weakly lower semicontinuous (since it is strongly continuous and convex). We claim that

$$\begin{aligned}\liminf_{n\to\infty} \int_0^t \int_\Omega A(s,x,\nabla u_n(s,x)) \cdot \nabla u_n(s,x)\ \mathrm{d}x\,\mathrm{d}s \geq \\ \geq \int_0^t \int_\Omega A(s,x,\nabla u(s,x)) \cdot \nabla u(s,x)\ \mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{8.6.8}$$

Indeed, let $k \in \mathbb{N}$. We can write

$$\begin{aligned}&A(s,x,\nabla u_n(s,x)) \cdot \nabla u_n(s,x) = \\ &\quad= \Big[A(s,x,\nabla u_n(s,x)) - A(s,x,\nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k})\Big] \cdot \Big[\nabla u_n(s,x) - \nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k}\Big] \\ &\quad\quad + A(s,x,\nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k}) \cdot \Big[\nabla u_n(s,x) - \nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k}\Big] \\ &\quad\quad + A(s,x,\nabla u_n(s,x)) \cdot \nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k},\end{aligned}$$

where the first term is nonnegative due to (A3) in Assumption 8.2.4. Recall that we already know that $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ weakly-$*$ in $L_M(\Omega_T)$ and $A(\cdot,\cdot,\nabla u_n) \overset{*}{\rightharpoonup} A(\cdot,\cdot,\nabla u)$ weakly-$*$ in $L_{M^*}(\Omega_T)$. Lemma 8.2.7 implies that the map $(s,x) \mapsto A(s,x,\nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k})$ is bounded. Therefore,

$$\begin{aligned}\liminf_{n\to\infty} \int_0^t \int_\Omega A(s,x,\nabla u_n(s,x)) \cdot \nabla u_n(s,x)\ \mathrm{d}x\,\mathrm{d}s &\geq \\ \geq \int_0^t \int_\Omega A(s,x,\nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k}) \cdot \nabla u(s,x)\mathbb{1}_{|\nabla u|\geq k}\ \mathrm{d}x\,\mathrm{d}s \\ + \int_0^t \int_\Omega A(s,x,\nabla u(s,x)) \cdot \nabla u(s,x)\mathbb{1}_{|\nabla u|\leq k},\end{aligned}$$

where the first term vanished due to presence of two characteristic functions $\mathbb{1}_{|\nabla u|\geq k}$ and $\mathbb{1}_{|\nabla u|\leq k}$. Finally, we let $k \to \infty$ and deduce (8.6.8).

By energy equality for the regularized problem (8.4.3), we have:

$$\begin{aligned}\int_\Omega \left[(u_n(t,x))^2 - (u_0(x))^2\right]\mathrm{d}x = -\int_0^t \int_\Omega A(s,x,\nabla u_n(s,x)) \cdot \nabla u_n(s,x)\ \mathrm{d}x\,\mathrm{d}s \\ - \int_0^t \int_\Omega \theta_n \nabla_\xi m(|\nabla u_n|) \cdot \nabla u_n(s,x)\ \mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega f(s,x)\,u_n(s,x)\ \mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

We note that $\int_0^t \int_\Omega \theta_n \nabla_\xi m(|\nabla u_n|) \cdot \nabla u_n(s,x)\, \mathrm{d}x\, \mathrm{d}s \geq 0$ so that

$$\int_\Omega \left[(u_n(t,x))^2 - (u_0(x))^2\right] \mathrm{d}x \leq - \int_0^t \int_\Omega A(s,x,\nabla u_n(s,x)) \cdot \nabla u_n(s,x)\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_\Omega f(s,x)\, u_n(s,x)\, \mathrm{d}x\, \mathrm{d}s.$$

Using (8.6.7) and (8.6.8), we let $n \to \infty$ and conclude the proof of the energy inequality (8.2.4) for a.e. $t \in (0,T)$. Finally, as the map $[0,T) \ni t \mapsto u(t,\cdot) \in L^2(\Omega)$ is weakly continuous, energy inequality holds for all $t \in [0,T)$. □

Lemma 8.6.1. *Let $1 \leq p < \infty$ and $\{u_n\}$ be a sequence such that $u_n \to u$ in $L^p((0,T)\times\Omega)$. Then, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$, such that*

$$\text{for a.e. } t \in (0,T) \qquad u_{n_k}(t,x) \to u(t,x) \text{ in } L^p(\Omega).$$

Moreover, if $\{u_{n_k}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ we have for a.e. $t \in (0,T)$

$$\int_\Omega |u(t,x)|^2\, \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |u_{n_k}(t,x)|^2\, \mathrm{d}x.$$

Proof. Consider sequence of functions $\{f_n\}$ defined with

$$f_n(t) := \left(\int_\Omega |u_n(t,x) - u(t,x)|^p\, \mathrm{d}x\right)^{1/p}.$$

Then, $f_n \to 0$ in $L^p(0,T)$ so it has a subsequence f_{n_k} that converges a.e. on $(0,T)$. It follows that $u_{n_k}(t,x) \to u(t,x)$ in $L^p(\Omega)$ for a.e. $t \in (0,T)$.

Concerning the second statement, let $\mathcal{N}$ be a subset of $(0,T)$ consisting of times for which the latter convergence holds and let $\mathcal{M}$ be a subset of $(0,T)$ for which

$$\|u_{n_k}(t,\cdot)\|_{L^2_x} \leq \sup_{k\in\mathbb{N}} \|u_{n_k}\|_{L^\infty_t L^2_x}.$$

Clearly, $(0,T) \setminus (\mathcal{N}\cap\mathcal{M})$ is a null set. For $t \in \mathcal{N}$ we have $u_{n_k}(t,x) \rightharpoonup u(t,x)$ in $L^p(\Omega)$. Moreover, standard subsequence argument combined with Banach-Alaoglu shows that for $t \in \mathcal{N}\cap\mathcal{M}$ we have $u_{n_k}(t,x) \rightharpoonup u(t,x)$ in $L^2(\Omega)$. The conclusion follows by weak lower semicontinuity of the (squared) $L^2(\Omega)$ norm. □

8.7 Proof of uniqueness result (Theorem 8.2.9)

To obtain the uniqueness of a weak solution, it is standard in the theory of parabolic equations to test the equation for the difference of solutions with the difference of solutions itself. In the Musielak-Orlicz framework, it is unfortunately not so straightforward. In fact, we want to improve the result of Lemma 8.5.3, where we showed the local energy equality, to the global energy equality, i.e. we want to remove the presence of the cut-off function. Next lemma shows that under the additional structural hypothesis on M (the radial symmetry), such procedure can be made rigorous.

Lemma 8.7.1. *Let Ω be a Lipschitz domain. Suppose that the N-function M is isotropic (as in assumptions of Theorem 8.2.9) and Assumption 8.2.1 is satisfied. Then, there is a family of functions $\{\psi_j\}$ compactly supported in Ω and fulfilling $\psi_j \to 1$ as $j \to \infty$, such that if $u \in L^\infty(0,T;L^\infty(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ with $\nabla u \in L_M(\Omega_T)$, we have*

$$\int_0^t \int_\Omega M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s \to 0 \text{ as } j \to \infty,$$

where the constant C_u can be chosen as $C_u = C\|\nabla u\|_{L_M}$ where C depends only on Ω.

Proof. Since Ω is a Lipschitz domain, we can flatten the boundary locally with bi-Lipschitz homeomorphisms so that using appropriate partition of unity, $\partial\Omega$ can be assumed to be flat. This argument relies heavily on the isotropy of M as otherwise it is not clear if $\nabla u \in L_M(\Omega_T)$ implies $\nabla(u \circ \Psi) \in L_M(\Omega_T)$ for some bi-Lipschitz homeomorphism Ψ.

Let $\Omega_j = \left\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \frac{1}{j}\right\}$ so that $\Omega_j \nearrow \Omega$ as $j \to \infty$. Moreover, let $\psi_j \in C_0^\infty(\Omega)$ such that $\psi_j = 1$ on Ω_j. Note that $\nabla\psi_j = 0$ on Ω_j and $|\nabla\psi_j| \le Cj$ for some constant C. We cover $\Omega \setminus \Omega_j$ with the family of disjoint cubes $\{Q_m^j\}_{m=1}^{N_j}$ with edge of length $\frac{1}{j}$. Then, we write for some constant C_u to be chosen later:

$$\begin{aligned}
&\int_0^t \int_{\Omega} M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s = \\
&= \int_0^t \int_{\Omega\setminus\Omega_j} M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s \\
&\leq \sum_{m=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega\setminus\Omega_j} M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s \\
&\leq \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega\setminus\Omega_j} \frac{M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right)}{M^{**}_{Q_m^j}\left(s, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right)} M^{**}_{Q_m^j}\left(s, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s.
\end{aligned} \tag{8.7.1}$$

Note that $\left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right| \leq \frac{j\|u\|_\infty}{C_u}$ so that we can apply Assumption 8.2.1 and deduce:

$$\limsup_{j\to\infty} \frac{M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right)}{M^{**}_{Q_m^j}\left(s, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right)} \leq C.$$

Therefore, (8.7.1) reads:

$$\begin{aligned}
&\int_0^t \int_{\Omega} M\left(s, x, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s \leq \\
&\qquad \leq C \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega\setminus\Omega_j} M^{**}_{Q_m^j}\left(s, \left|\frac{\nabla\psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\, \mathrm{d}s.
\end{aligned} \tag{8.7.2}$$

Now, for any $x = (x_1, x_2, ..., x_d) \in Q_m^j$, we write $x^* = (x_1, x_2, ..., 0)$ for its projection on the face of the cube Q_m^j sticking to the boundary $\partial\Omega$ (see Figure 8.1). Note that we assumed that the face of the cube is perpendicular to the axis of the last variable x_d which is not restrictive and can be obtained by choosing appropriate straightening bi-Lipschitz homeomorphisms.

For a.e. $x \in Q_m^j$, using absolute continuity on lines for Sobolev maps (cf. Theorem 4.21 in [131]), we can write:

$$u(s,x) = \int_0^{x_d} u_{x_d}(s, x_1, x_2, ..., r)\, \mathrm{d}r$$

where u_{x_d} denotes derivative with respect to the last variable. Note that since $|x_d| \leq \frac{1}{j}$, $|u(s,x)|$ can be bounded as

$$|u(s,x)| \leq \int_0^{x_d} |u_{x_d}(s, x_1, x_2, ..., x_{d-1}, r)|\, \mathrm{d}r \leq \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, ..., x_{d-1}, r)|\, \mathrm{d}r.$$

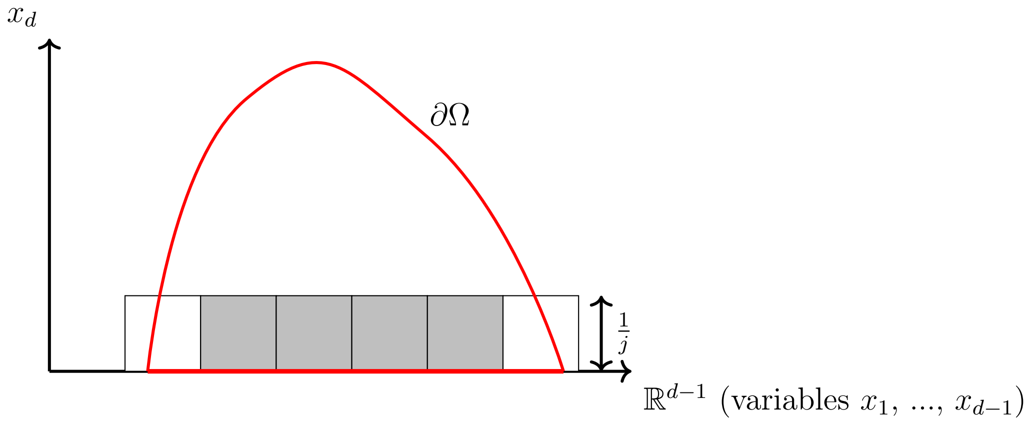

Figure 8.1: The boundary $\partial\Omega$ with some part of it flattened after change of coordinates. Gray cubes from the family $\{Q^j_m\}_{m=1}^{N_j}$ correspond to the area that is relevant for further computations after application of partition of unity.

Using this inequality in (8.7.2), we can continue as follows:

$$\begin{aligned}
&\sum_{i=1}^{N_j} \int_0^t \int_{Q^j_m \cap \Omega \setminus \Omega_j} M^{**}_{Q^j_m}\left(s, \left|\frac{\nabla \psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x \, \mathrm{d}s \leq \\
&\quad \leq \sum_{i=1}^{N_j} \int_0^t \int_{Q^j_m \cap \Omega \setminus \Omega_j} M^{**}_{Q^j_m}\left(s, \frac{|\nabla \psi_j(x)|}{C_u} \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, ..., x_{d-1}, r)| \, \mathrm{d}r\right) \mathrm{d}x \, \mathrm{d}s \\
&\quad = \sum_{i=1}^{N_j} \int_0^t \int_{Q^j_m \cap \Omega \setminus \Omega_j} M^{**}_{Q^j_m}\left(s, \frac{C}{C_u}\, j \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, ..., x_{d-1}, r)| \, \mathrm{d}r\right) \mathrm{d}x \, \mathrm{d}s \\
&\quad \leq \sum_{i=1}^{N_j} \int_0^{\frac{1}{j}} \int_0^t \int_{Q^j_m \cap \Omega \setminus \Omega_j} j\, M^{**}_{Q^j_m}\left(s, \frac{C}{C_u} |\nabla u(s, x_1, x_2, ..., x_{d-1}, r)|\right) \mathrm{d}x \, \mathrm{d}s \, \mathrm{d}r
\end{aligned}$$

where we used the bound $|\nabla \psi_j(x)| \leq Cj$ and Jensen's inequality. Note that the integrand does not depend on x_d and so, the integral over this variable cancels with the factor j. Finally, as cube has edge of length $\frac{1}{j}$, Fubini's theorem implies

$$\begin{aligned}
&\sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M^{**}_{Q_m^j}\left(s, \left|\frac{\nabla \psi_j(x)\, u(s,x)}{C_u}\right|\right) \mathrm{d}x\,\mathrm{d}s \leq \\
&\qquad = \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M^{**}_{Q_m^j}\left(s, \frac{C}{C_u}\left|\nabla u(s, x_1, x_2, ..., x_{d-1}, x_d)\right|\right) \mathrm{d}x\,\mathrm{d}s \\
&\qquad \leq \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M\left(s, x, \frac{C}{C_u}\left|\nabla u(s, x_1, x_2, ..., x_{d-1}, x_d)\right|\right) \mathrm{d}x\,\mathrm{d}s \\
&\qquad = \int_0^t \int_{\Omega \setminus \Omega_j} M\left(s, x, \frac{C}{C_u}\left|\nabla u(s,x)\right|\right) \mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Now, as $\nabla u \in L_M(\Omega_T)$, we can choose $C_u = C\|\nabla u\|_{L_M}$ so that the integral

$$\int_0^t \int_{\Omega} M\left(s, x, \frac{C}{C_u}\left|\nabla u(s,x)\right|\right) \mathrm{d}x\,\mathrm{d}s$$

is finite and the conclusion follows by $\Omega_j \nearrow \Omega$ as $j \to \infty$. $\square$

Lemma 8.7.2 (Global energy equality)**.** *Under assumptions of Lemma 8.5.3 and Theorem 8.2.9, the following energy equality is satisfied for a.e.* $t \in (0,T)$*:*

$$\begin{aligned}
&\int_\Omega \left[G_k(u(t,x)) - G_k(u_0(x))\right] \mathrm{d}x = \\
&= -\int_0^t \int_\Omega \alpha(s,x)\cdot \nabla\left[T_k(u(s,x))\right] \mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega f(s,x)\, T_k(u(s,x))\, \mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{8.7.3}$$

Proof. The main idea is to consider local energy equality (8.5.5) with a sequence of cut-off functions $\{\psi_j\}_{j\in\mathbb{N}}$ from Lemma 8.7.1.

Note that, as $j \to \infty$, $\psi_j \to 1$ in Ω. Therefore, to conclude (8.7.3) from (8.5.5) we only have to establish:

$$\int_0^t \int_\Omega \alpha(s,x)\cdot \nabla\psi_j(x)\, T_k(u(s,x))\, \mathrm{d}x\,\mathrm{d}s \to 0 \text{ as } j \to \infty.$$

Since $\nabla\psi_j(x) = 0$ for $x \in \Omega_j$, we write for some constant C_α and C_u to be chosen later:

$$\begin{aligned}
&\int_0^t \int_{\Omega\setminus\Omega_j} \alpha(s,x)\cdot \nabla\psi_j(x)\, T_k(u(s,x))\, \mathrm{d}x\,\mathrm{d}s \leq \\
&C_\alpha C_u \int_0^t \int_{\Omega\setminus\Omega_j} \left(M^*\left(s,x,\left|\frac{\alpha(s,x)}{C_\alpha}\right|\right) + M\left(s,x,\left|\frac{\nabla\psi_j(x)\, T_k(u(s,x))}{C_u}\right|\right)\right) \mathrm{d}x\,\mathrm{d}s,
\end{aligned} \tag{8.7.4}$$

where we have applied Young's inequality (Lemma 7.2.3). Since $\alpha \in L_{M*}(\Omega_T)$, there is C_α so that $M^*\left(s, x, \left|\frac{\alpha(s,x)}{C_\alpha}\right|\right)\, \mathrm{d}x\, \mathrm{d}s < \infty$. Choosing such C_α, the first integral on the (RHS) of (8.7.4) tends to 0 as $j \to \infty$ due to integrability of the integrand. Moreover, the second integral on the (RHS) of (8.7.4) converges to 0 due to Lemma 8.7.1. $\square$

Remark 8.7.3. Using Dominated Convergence Theorem, (8.7.3) implies that for a.e. $t \in (0,T)$:

$$\begin{aligned}\frac{1}{2}\int_\Omega \left[u^2(t,x) - u_0^2(x)\right] \mathrm{d}x = \\ = -\int_0^t \int_\Omega \alpha(s,x)\cdot \nabla u(s,x)\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_\Omega f(s,x)\, u(s,x)\, \mathrm{d}x\, \mathrm{d}s.\end{aligned} \tag{8.7.5}$$

Proof of Theorem 8.2.9. Energy equality (8.2.5) follows from Remark 8.7.3. Now, suppose there are two solutions u and v to (8.1.3). Then, their difference satisfies weak formultion for

$$(u-v)_t = \operatorname{div}\left[A(t,x,\nabla u) - A(t,x,\nabla v)\right]$$

with zero initial condition. Using (8.7.5) with $\alpha(t,x) = A(t,x,\nabla u) - A(t,x,\nabla v)$, we obtain for a.e. $t \in (0,T)$:

$$\begin{aligned}\frac{1}{2}\int_\Omega (u(t,x) - v(t,x))^2\, \mathrm{d}x = \\ = -\int_0^t \int_\Omega \Big[A(s,x,\nabla u) - A(s,x,\nabla v)\Big]\cdot \Big[\nabla u(s,x) - \nabla v(s,x)\Big]\, \mathrm{d}x\, \mathrm{d}s\end{aligned}$$

which due to weak monotonicity (A3) in Assumption 8.2.4 implies $u = v$ a.e. in Ω_T. $\square$

8.8 Appendix: Details on Examples 8.2.3 and 8.2.6

Example 8.2.3

Let $M(t,x,\xi) = |\xi|^{p(t,x)}$. We want to establish condition 8.2.2 in Remark 8.2.2. Fix $t \in (0,T)$ and $x, y \in \Omega$. When $|\xi| > 1$

$$\frac{M(t,x,\xi)}{M(t,y,\xi)} = |\xi|^{p(t,x)-p(t,y)} \leq |\xi|^{|p(t,x)-p(t,y)|} \leq |\xi|^{-\frac{C}{\log|x-y|}}$$

since $p(t,x) \in L^\infty(0,T;C_{\log}(\Omega))$. We let $\Theta(t,\delta,\xi) := |\xi|^{-\frac{C}{\log\delta}}$ so that for all $\widetilde{C} > 1$ we have

$$\Theta(t,\delta,\widetilde{C}\delta^{-1}) \le \left(\widetilde{C}\,\delta^{-1}\right)^{-\frac{C}{\log\delta}} = \left(\delta/\widetilde{C}\right)^{\frac{C}{\log\delta}} = e^{\frac{C}{\log\delta}\,\log(\delta/\widetilde{C})} = e^C\, e^{-\frac{C\,\log\widetilde{C}}{\log\delta}}$$

so that $\limsup_{\delta\to 0}\Theta(t,\delta,\widetilde{C}\delta^{-1})$ is bounded.

Similarly, let $M(t,x,\xi) = |\xi|^{p(t,x)} + a(t,x)\,|\xi|^{q(t,x)}$ and suppose that $q(t,y) - p(t,y) \le \alpha$. Then, for $|\xi| > 1$ we have

$$\begin{aligned}
\frac{M(t,x,\xi)}{M(t,y,\xi)} &= \frac{|\xi|^{p(t,x)} + a(t,x)\,|\xi|^{q(t,x)}}{|\xi|^{p(t,y)} + a(t,y)\,|\xi|^{q(t,y)}} = \frac{|\xi|^{q(t,x)}}{|\xi|^{q(t,y)}}\,\frac{|\xi|^{p(t,x)-q(t,x)} + a(t,x)}{|\xi|^{p(t,y)-q(t,y)} + a(t,y)} \le \\
&\le |\xi|^{q(t,x)-q(t,y)} \left[\frac{|\xi|^{p(t,x)-q(t,x)}}{|\xi|^{p(t,y)-q(t,y)}} + \frac{a(t,x)-a(t,y)}{|\xi|^{p(t,y)-q(t,y)}} + 1\right] \\
&\le |\xi|^{q(t,x)-q(t,y)} \left[|\xi|^{p(t,x)-p(t,y)}\,|\xi|^{q(t,y)-q(t,x)} + \frac{a(t,x)-a(t,y)}{|\xi|^{p(t,y)-q(t,y)}} + 1\right] \\
&\le |\xi|^{-\frac{C}{\log|x-y|}} \left[|\xi|^{-\frac{C}{\log|x-y|}}\,|\xi|^{-\frac{C}{\log|x-y|}} + |a|_\alpha\,|x-y|^\alpha\,|\xi|^{q(t,y)-p(t,y)} + 1\right] \\
&\le |\xi|^{-\frac{C}{\log|x-y|}} \left[|\xi|^{-\frac{C}{\log|x-y|}}\,|\xi|^{-\frac{C}{\log|x-y|}} + |a|_\alpha\,|x-y|^\alpha\,|\xi|^{\alpha} + 1\right]
\end{aligned}$$

where $|a|_\alpha$ is a constant such that $|a(t,x)-a(t,y)| \le |a|_\alpha\,|x-y|^\alpha$. Hence, we define

$$\Theta(t,\delta,\xi) = |\xi|^{-\frac{C}{\log\delta}} \left[|\xi|^{-\frac{C}{\log\delta}}\,|\xi|^{-\frac{C}{\log\delta}} + |a|_\alpha\,\delta^\alpha\,|\xi|^\alpha + 1\right].$$

We have already seen that $\left(\widetilde{C}\,\delta^{-1}\right)^{-\frac{C}{\log\delta}}$ is bounded when $\delta \to 0$. It follows that $\Theta(t,\delta,\widetilde{C}\delta^{-1})$ is bounded for such δ.

Example 8.2.6

In both examples, the only nontrivial condition in Assumption 8.2.4 is (A2) (growth and coercivity). For (F1), we study $A(t,x,\xi) = |\xi|^{p(t,x)-2}\,\xi$ with $M(t,x,\xi) = |\xi|^{p(t,x)} = A(t,x,\xi)\cdot\xi$ so that we only need to verify

$$M^*(t,x,A(t,x,\xi)) \le C\,A(t,x,\xi)\cdot\xi$$

for some numerical constant C. As we know that $M^*(t,x,\xi) \le C\,|\xi|^{p'(t,x)}$ where $p'(t,x)$ is Hölder conjugate of $p(t,x)$ (i.e. $\frac{1}{p(t,x)} + \frac{1}{p'(t,x)} = 1$) we have

$$\begin{aligned}
M^*(t,x,A(t,x,\xi)) &\le C\,\left||\xi|^{p(t,x)-2}\,\xi\right|^{p'(t,x)} = \\
&= C\,|\xi|^{(p(t,x)-1)\,p'(t,x)} = C\,|\xi|^{p(t,x)} = C\,A(t,x,\xi)\cdot\xi.
\end{aligned}$$

For (F2), we have $A(t,x,\xi) = |\xi|^{p(t,x)-2}\,\xi + a(t,x)\,|\xi|^{q(t,x)-2}\,\xi$ and $M(t,x,\xi) = |\xi|^{p(t,x)} + a(t,x)\,|\xi|^{q(t,x)}$. Again, since $A(t,x,\xi)\cdot\xi = M(t,x,\xi)$ we only have to prove

$$M^*(t,x,A(t,x,\xi)) \leq C\,A(t,x,\xi)\cdot\xi$$

for some numerical constant C. Using Definition 7.1.4,

$$\begin{aligned}
M^*(t,x,A(t,x,\xi)) &= \sup_{\eta\in\mathbb{R}^d}\{\eta\,A(t,x,\xi) - M(t,x,\xi)\}\\
&\leq \sup_{\eta\in\mathbb{R}^d}\left\{\eta\cdot\left(|\xi|^{p(t,x)-2}\,\xi + a(t,x)\,|\xi|^{q(t,x)-2}\,\xi\right) - \left(|\xi|^{p(t,x)} + a(t,x)\,|\xi|^{q(t,x)}\right)\right\}\\
&\leq \sup_{\eta\in\mathbb{R}^d}\left\{\eta\cdot\xi\,|\xi|^{p(t,x)-2} - |\xi|^{p(t,x)}\right\} + a(t,x)\sup_{\eta\in\mathbb{R}^d}\left\{\eta\cdot\xi\,|\xi|^{q(t,x)-2} - |\xi|^{q(t,x)}\right\}
\end{aligned}$$

We introduce auxillary notation $M_1(t,x,\xi) = |\xi|^{p(t,x)}$, $A_1(t,x,\xi) = |\xi|^{p(t,x)-2}\,\xi$ as well as $M_2(t,x,\xi) = |\xi|^{q(t,x)}$, $A_2(t,x,\xi) = |\xi|^{q(t,x)-2}\,\xi$ and we recognize that

$$\begin{aligned}
M^*(t,x,A(t,x,\xi)) &\leq M_1^*(t,x,A_1(t,x,\xi)) + a(t,x)\,M_2^*(t,x,A_2(t,x,\xi))\\
&\leq A_1(t,x,\xi)\cdot\xi + a(t,x)\,A_2(t,x,\xi)\cdot\xi = \mathcal{A}(t,x,\xi)\cdot\xi
\end{aligned}$$

which is justified by computations for the variable exponent case.

Chapter 9

Non-Newtonian fluids with discontinuous-in-time stress tensor

The results in this chapter have been published in:

- M. Bulíček, P. Gwiazda, J. Skrzeczkowski, J. Woźnicki. *Non-Newtonian fluids with discontinuous-in-time stress tensor.* Available at arXiv:2209.10695, cited as [54].

9.1 Introduction

We consider the system of partial differential equations

$$\begin{cases} \partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p(t,x) = \operatorname{div}_x S(t,x,Du) + f(t,x) \\ \operatorname{div}_x u = 0 \end{cases} \tag{9.1.1}$$

describing the flow of incompressible, homogeneous, non-Newtonian fluid. Here, $u = u(t,x)$ is the velocity of the fluid, p denotes pressure, S is the constitutively determined part of the Cauchy stress tensor depending on the symmetric gradient Du, f represents a given density of the external body forces and for simplicity, we consider that the density of the fluid is equal to one. The system of equations (9.1.1) is formulated on a space-time cylinder $\Omega_T := (0,T) \times \Omega$, where $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and d denotes the dimension. The above system is completed by the no-slip boundary conditions, i.e., u vanishes on $\partial\Omega$, and by the initial condition $u_0(x)$.

Most of the papers devoted to the analysis of non-Newtonian fluids assume Cauchy stress tensor to be of power type

$$S(t,x,Du) \sim (\nu_0 + \nu_1\,|Du|^{p-2})\,Du.$$

First existence results concerning (9.1.1) were proved for $p \geq \frac{11}{5}$ (in 3D) by Lions and Ladyzhenskaya in [184, 193]. Since then, many improvements have appeared, from the higher regularity method in [199] giving the bound $p \geq \frac{9}{5}$, followed by the L^∞-truncation method for $p \geq \frac{8}{5}$, see [140], the Lipschitz truncation method for $p > \frac{6}{5}$, see [55, 107, 141], up to a new definition of a solution in [1] leading to the theory for all $p > 1$. We refer to the extensive review in [35] in context of fluids with very complicated rheology. Nevertheless, let us remark that such equations are still a topic of research - recently they have been analyzed in the context of nonuniqueness and convex integration [58], extending the groundbreaking paper of Buckmaster and Vicol on Navier-Stokes equation [52].

In this chapter, we are interested in the case when S has the so-called non-standard growth. The iconic example is

$$S(t,x,Du) \sim (\nu_0 + \nu_1\,|Du|^{s(t,x)-2})\,Du, \tag{9.1.2}$$

where the exponent $s(t,x)$ depends on the time variable t and the spatial variable x. The motivation for considering (9.1.2) comes from the behaviour of electrorheological fluids whose mechanical properties dramatically change when an external electric field $\boldsymbol{E}$ is applied, see [235]. Then, the exponent $s(t,x)$ in (9.1.2) can be assumed to be a smooth function of $|\boldsymbol{E}|^2$ cf. [240, eq. (4.10)–(4.12)]. The topic has been extensively studied over the last years from the mathematical point of view, here we refer to [3, 4, 33, 103, 104, 106, 240]. These considerations have been recently generalized to the micropolar fluids [33, 128] and also chemically reacting fluids [108, 179, 180, 181].

A natural assumption on S, which reflects the structure (9.1.2) involves the growth and the coercivity formulated as the following inequality

$$c\,S(t,x,\xi) : \xi \geq |\xi|^{s(t,x)} + |S(t,x,\xi)|^{s'(t,x)} - h(t,x), \tag{9.1.3}$$

where c is some constant, $h \in L^1(\Omega_T)$ and $s'(t,x)$ is the Hölder conjugate exponent to $s(t,x)$, i.e., $s' := \frac{s}{s-1}$. Analysis of (9.1.1) with the Cauchy stress tensor of the form (9.1.3) requires the concept of generalized Lebesgue spaces $L^{s(t,x)}(\Omega_T)$. One can also generalize (9.1.3) by replacing power-type function with a generalized N-function. The resulting analysis requires application of the general Musielak–Orlicz spaces [77] as in [164, 166, 264, 265]. We remark that all results of this chapter can be formulated in this setting but we decided not to do so for the sake of clarity.

In this chapter we establish the existence of global-in-time and large-data solutions to (9.1.1) for exponents $s(t,x)$ being discontinuous in the time variable. The former approaches were based on the so-called log-Hölder continuity of the exponent $s(t,x)$, which allows one to use the density of smooth functions in the space $L^{s(t,x)}(\Omega_T)$, see [8, 84]. In fact, the log-Hölder continuity is necessary for the density to be true, see [84, Example 6.12]. Nevertheless, inspired by Chapter 8, where the following problem was treated

$$\partial_t u + \operatorname{div}\left(|\nabla u|^{s(t,x)-2}\nabla u\right) = f$$

with $s(t,x)$ being discontinuous in time and log-Hölder continuous in space, we do not require the smoothness of s with respect to the time variable here. In addition, we also do not require any relationship between the minimal and maximal values of $s(t,x)$. The only restriction is due to the convective term and has the form $s \geq \frac{3d+2}{d+2}$. Note that in case we consider a generalized Stokes problem only, i.e., we consider the system (9.1.1) without the term $\operatorname{div}_x(u(t,x) \otimes u(t,x))$, there is no restriction on s except the log-Hölder continuity with respect to the spatial variable.

When compared to Chapter 8, the main difficulty of the present work lies in the fact that (9.1.1) can be tested only with a divergence-free function. In particular, we cannot test it with the truncation of solution as it loses divergence-free property after applying truncation operator. Even when one recovers pressure p by the Nečas Theorem, one obtains terms which are not treatable as we have only weak convergence of both the pressure and derivatives of the solution. We remark that one can try to overcome this problem by applying approximation called Lipschitz truncation method, see [2,44,45,105,109]. Nevertheless, this approach does not seem to be applicable here as our work uses equation satisfied by the exploited approximation in the crucial way. Contrary to the mollification, as Lipschitz truncations are defined by the maximal function, it is not trivial to write equation satisfied by them.

9.2 Preliminaries and the main result

Functional analytic setting

We work in the variable exponent space $L^{s(t,x)}(\Omega_T)$. We refer to Chapter 7.4 for the definition and in particular, to Lemma 7.4.3 for basic inequalities.

Assumptions on data

Let us now state the needed assumptions on the exponent function $s(t,x)$:

Assumption 9.2.1. We assume that a measurable function $s(t,x) : \Omega_T \to [1,\infty)$ satisfies the following:

(A1) (continuity in space) $s(t,x)$ is a log-Hölder continuous functions on Ω uniformly in time, i.e. there is a constant C such that for all $x, y \in \Omega$ and all $t \in [0,T]$

$$|s(t,x) - s(t,y)| \le -\frac{C}{\log|x-y|},$$

(A2) (bounds) it holds that $\frac{3d+2}{d+2} =: s_{\min} \le s(t,x) \le s_{\max}$ for a.e. $(t,x) \in \Omega_T$.

For later purposes we also define an exponent s_0 as

$$s_0 := 3 + \frac{2}{d}. \tag{9.2.1}$$

We remark that the condition (A1) is somehow standard, as it guarantees good approximation properties (with respect to spatial variable x) in the variable exponent space. Assumption (A2) is related to the continuity of the three-linear form

$$v \longmapsto \int_{\Omega_T} v \otimes v : \nabla v \, \mathrm{d}x \, \mathrm{d}t$$

that appears in the analysis and is somehow "necessary" to obtain the so called energy equality. Note that the same problem appears in the classical Navier–Stokes equations.

Assumptions on the stress tensor

Concerning the stress tensor $S : (0, T) \times \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we assume the following:

Assumption 9.2.2. We assume that

(T1) $S(t, x, \xi)$ is a Carathéodory function and $S(t, x, 0) = 0$,

(T2) (coercivity and growth conditions) There exists a positive constant c and a non-negative, integrable function $h(t, x)$, such that for any $\xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and almost every $(t, x) \in \Omega_T$

$$c\, S(t, x, \xi) : \xi \geq |\xi|^{s(t,x)} + |S(t, x, \xi)|^{s'(t,x)} - h(t, x)$$

(T3) (monotonicity) S is monotone, i.e.:

$$(S(t, x, \xi_1) - S(t, x, \xi_2)) : (\xi_1 - \xi_2) \geq 0$$

for all $\xi_1 \neq \xi_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and almost every $(t, x) \in \Omega_T$.

Notice that the assumption (T1) is assumed just to simplify the approximating scheme. On the other hand, the monotonicity (T3) is the key property. One could consider a more general setting of maximal monotone graphs here without any problems, but to avoid the technical difficulties we consider S to be a Carathéodory mapping. Finally, the assumption (T2) is a natural setting to get bounds for gradient of an unknown velocity as well as bound on the stress tensor.

Main result

Having introduced all the needed notation, we may finally state the main theorem.

Theorem 9.2.3. *Let $S(t, x, \xi)$ satisfy the Assumption 9.2.2 with the exponent $s(t, x)$ satisfying Assumption 9.2.1. Let $f \in L^1(0, T; L^2(\Omega)) \cup L^{s'_{min}}(0, T; (W^{1,s_{min}}_0(\Omega))^*)$ and*

$u_0 \in L^2_{0,\mathrm{div}}(\Omega)$. Then, there exists $u \in L^\infty(0,T;L^2(\Omega)) \cap L^{s_{min}}(0,T;W_0^{1,s_{min}}(\Omega))$ with $Du \in L^{s(t,x)}(\Omega_T)$, such that $\operatorname{div} u = 0$ *almost everywhere in Ω_T and*

$$\begin{aligned} \int_{\Omega_T} -u \cdot \partial_t \phi - u \otimes u : \nabla \phi + S(t,x,Du) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ = \int_{\Omega_T} f \cdot \phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x) \cdot \phi(0,x)\,\mathrm{d}x \end{aligned} \tag{9.2.2}$$

for any $\phi \in C_c^\infty([0,T) \times \Omega)$ fulfilling $\operatorname{div} \phi = 0$ *almost everywhere in Ω_T.*

We would like to point out here that the assumption

$$f \in L^1(0,T;L^2(\Omega)) \cup L^{s'_{\min}}(0,T;(W_0^{1,s_{\min}}(\Omega))^*)$$

can be relaxed, namely the second part. But because we do not want to complicate the paper we do not consider it here. Even more, in the proof we consider only the case $f \in L^1(0,T;L^2(\Omega))$ to avoid difficulties. Finally, we want to emphasize again that the assumption (A2) is not needed in case of generalized Stokes system, i.e., if the term $\int_{\Omega_T} u \otimes u : \nabla \phi$ is omitted.

9.3 Approximation in variable exponent spaces

In this section we discuss a method to approximate functions u such that $Du \in L^{s(t,x)}(\Omega_T)$ in spatial variable. Moreover, we will guarantee that the convergence $Du^\varepsilon \to Du$ holds in $L^{s(t,x)}(\Omega_T)$. The main difficulty here is that we cannot use truncations as in Section 8.3 because truncation of a divergence-free vector field is not divergence-free in general. Therefore, we use embeddings to get sufficiently big integrability of u together with a suitable decomposition of Ω, cf. Lemma 9.3.2. Reader should recall Definitions 8.3.2 and 8.3.3.

The main result of this section reads:

Theorem 9.3.1. *Let $\Omega \subset \mathbb{R}^d$, and $\psi : \Omega \to \mathbb{R}$ be arbitrary such that $\psi \in C_c^\infty(\Omega)$. Let u satisfy $u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;W_0^{1,1}(\Omega))$ and $Du \in L^{s(t,x)}(\Omega_T)$. If we extend $u(t,x)$ by zero for all $x \notin \Omega$, then there exists $\varepsilon_0 > 0$ depending on ψ and Ω such that:*

(S1) $(u^\varepsilon\psi)^\varepsilon \in L^\infty(0,T;C_0^\infty(\Omega))$ *for all* $\varepsilon \in (0,\varepsilon_0)$,

(S2) $(u^\varepsilon\psi)^\varepsilon \to u\,\psi$ *a.e. in* Ω_T *and in* $L^1(0,T;L^1(\Omega))$ *as* $\varepsilon \to 0^+$,

(S3) $D\,(u^\varepsilon\psi)^\varepsilon \in L^{s(t,x)}(\Omega_T)$ *and* $D\,(u^\varepsilon\psi)^\varepsilon \to D\,(u\psi)$ *in* $L^{s(t,x)}(\Omega_T)$ *as* $\varepsilon \to 0^+$.

The rest of this section is devoted to the proof of Theorem 9.3.1.

We start with the following decomposition of Ω which will be used several times. The motivation for this decomposition is as follows. Suppose that we have a sequence $\{u_n\}$ such that $u_n \to u$ a.e. and we have uniform estimates on $\{Du_n\}$ in $L^{s(t,x)}(\Omega_T)$ and $\{u_n\}$ in $L^\infty(0,T;L^2(\Omega))$. Locally on some ball $B \subset \Omega$, we can use Korn's inequality and obtain estimate on $\{\nabla u_n\}$ in $L^{q(t)}((0,T)\times B)$ where $q(t) = \inf_{x\in B} s(t,x)$. By the parabolic interpolation (Lemma 9.8.1), we deduce an estimate on u in space $L^{q(t)\left(1+\frac{2}{d}\right)}((0,T)\times B)$. If we manage to guarantee $r(t) := \sup_{x\in B} s(t,x) < q(t)\left(1+\frac{2}{d}\right)$, we obtain $u_n \to u$ in $L^{s(t,x)}(B)$. In the decomposition below, we cover Ω with such balls B and so we overcome the problem of loosing integrability when applying embedding theorems/Korn's inequality in the variable exponent setting.

Lemma 9.3.2. *There exists* $r>0$ *and an open finite covering* $\{\mathcal{B}^i_r\}_{i=1}^N$ *of* Ω *by balls of radii* r *such that if we define*

$$q_i(t) := \inf_{x\in\mathcal{B}^i_{2r}} s(t,x), \qquad r_i(t) := \sup_{x\in\mathcal{B}^i_{2r}} s(t,x), \qquad R_i(t) := q_i(t)\left(1+\frac{2}{d}\right)$$

we have for all $i = 1,\ldots,N$

$$s_{min} \le q_i(t) \le s(t,x) \le r_i(t) < R_i(t) \qquad \text{on } (0,T)\times(\mathcal{B}^i_{2r}\cap\Omega)$$

and

$$R_i(t) - r_i(t) \ge \frac{s_{min}}{d}.$$

Proof. We cover Ω with balls $\mathcal{B}^i_r$ of equal radius r and the only problem is to find radius r satisfying assertions of the lemma. By (A1) in Assumption 9.2.1 we can choose r such that

$$\sup_{x\in\mathcal{B}^i_{2r}\cap\Omega} s(t,x) - \inf_{x\in\mathcal{B}^i_{2r}\cap\Omega} s(t,x) \le \frac{s_{\min}}{d}.$$

Since $s_{\min} \leq q_i(t)$, the conclusion follows. $\square$

Notation 9.3.3. In what follows, we always consider the covering constructed in Lemma 9.3.2. We also write ζ_i, $i = 1, \ldots, N$ for the partition of unity related to the open covering $\{\mathcal{B}_r^i\}$ of Ω, that is $\operatorname{supp} \zeta_i \subset \mathcal{B}_r^i$ and $\sum_{i=1}^N \zeta_i(x) = 1$ for all $x \in \Omega$.

Let us first observe that the fact $Du \in L^{s(t,x)}(\Omega_T)$ implies certain regularity properties.

Lemma 9.3.4. *Suppose that Assumption 9.2.1 holds true. Let u be a function such that $Du \in L^{s(t,x)}(\Omega_T)$, $u \in L^1(0,T; W_0^{1,1}(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. Then, u belongs to $L^{q_i(t)}(0,T; W^{1,q_i(t)}(\mathcal{B}_{2r}^i)) \cap L^{s_{min}}(0,T; W_0^{1,s_{min}}(\Omega))$ for all $i = 1, \ldots, N$. The norm of u in these spaces depends only on $\|Du\|_{L^{s(t,x)}}$, $\|u\|_{L_t^\infty L_x^2}$.*

Proof. As $q_i(t) \leq s(t,x)$ on $\mathcal{B}_{2r}^i$, we deduce $Du \in L^{q_i(t)}(0,T; L^{q_i(t)}(\mathcal{B}_{2r}^i))$. Then, the generalized Körn inequality implies that (for fixed t and i)

$$\|\nabla u\|_{L^{q_i(t)}(\mathcal{B}_{2r}^i)} \leq C(r, s_{\min}, s_{\max})(\|Du\|_{L^{q_i(t)}(\mathcal{B}_{2r}^i)} + \|u\|_2).$$

Then rasing the inequality to the $q_i(t)$ power, integrating over $t \in (0,T)$ and using the assumptions on u we have the first part of the statement. The second statement can proved exactly in the same way using that $s_{\min} \leq s(t,x)$ on Ω_T and the standard Körn/Poincare inequalities. $\square$

The most important tool is to approximate function $\xi \mapsto |\xi|^{s(t,x)}$ with functions independent of x or t. This is obtained in the following lemmas.

Lemma 9.3.5. *Suppose that Assumption 9.2.1 is satisfied. Then, for a.e. $t \in (0,T)$ and all balls $B_\gamma(x)$ such that $\overline{B_\gamma(x)} \cap \overline{\Omega}$ is nonempty, there exists $x^* \in \overline{B_\gamma(x)} \cap \overline{\Omega}$, $x^* = x^*(B_\gamma(x), t)$, such that for all ξ with $|\xi| \geq 1$, we have*

$$\inf_{y \in \overline{B_\gamma(x)} \cap \overline{\Omega}} |\xi|^{s(t,y)} = |\xi|^{s(t,x^*)}.$$

Remark 9.3.6. Note carefully that the minimizing point x^* is independent of ξ.

Proof of Lemma 9.3.5. First, we note that for a.e. $t \in (0,T)$, the map $x \mapsto s(t,x)$ is continuous and so is the map $x \mapsto |\xi|^{s(t,x)}$ (for fixed ξ). Using compactness of $\overline{B_\gamma(x)} \cap \overline{\Omega}$ and $|\xi| > 1$, we have

$$\inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} |\xi|^{s(t,y)} = |\xi|^{\inf_{y \in \overline{B_\gamma(x)} \cap \overline{\Omega}} s(t,y)}$$

and we choose x^* such that $\inf_{y \in \overline{B_\gamma(x)} \cap \overline{\Omega}} s(t,y) = s(t,x^*)$. $\square$

Lemma 9.3.7. *Let $E > 0$ be given. Then, there exists a constant $M = M(E)$, such that*

$$|\xi|^{s(t,y)} \le M \inf_{z \in \overline{B_\gamma(x)} \cap \overline{\Omega}} |\xi|^{s(t,z)} \le M\, |\xi|^{s(t,y)}$$

for all balls $B_\gamma(x)$, all $y \in \overline{B_\gamma(x)} \cap \overline{\Omega}$, all $\xi \in [1, E\,\gamma^{-(d+1)}]$ and all $\gamma \in \left(0, \frac{1}{2}\right)$.

Proof. Let $y_1, y_2 \in B_\gamma(x)$. As $|\xi| \ge 1$, we have

$$\frac{|\xi|^{s(t,y_1)}}{|\xi|^{s(t,y_2)}} = |\xi|^{s(t,y_1) - s(t,y_2)} \le |\xi|^{|s(t,y_1) - s(t,y_2)|}$$

Using log-Hölder continuity (9.2.1) in Assumption 9.2.1, we get

$$|\xi|^{|s(t,y_1) - s(t,y_2)|} \le |\xi|^{-\frac{C}{\log|y_1 - y_2|}} \le \left(E\,\gamma^{-(d+1)}\right)^{-\frac{C}{\log\gamma}} = E^{-\frac{C}{\log\gamma}}\,\gamma^{\frac{C\,(d+1)}{\log\gamma}} \le E^{\frac{C}{\log 2}}\,e^{C\,(d+1)}.$$

The conclusion follows from setting $M := E^{\frac{C}{\log 2}}\,e^{C\,(d+1)}$. $\square$

Proof of Theorem 9.3.1. Properties (S1) and (S2) follow from the standard properties of the convolutions. To see (S3), we first estimate $D\,(u^\varepsilon \psi)^\varepsilon$ in L^∞ norm. By product rule,

$$D\,(u^\varepsilon \psi)^\varepsilon = (Du^\varepsilon\,\psi)^\varepsilon + \frac{(u^\varepsilon \otimes \nabla\psi + \nabla\psi \otimes u^\varepsilon)^\varepsilon}{2}.$$

The first term on the right hand side can be estimated with the help of Young's inequality as

$$\begin{aligned} \|(Du^\varepsilon\,\psi)^\varepsilon\|_{L_t^\infty L_x^\infty} &\le \|Du^\varepsilon \psi\|_{L_t^\infty L_x^\infty} \le \|\psi\|_\infty\, \|Du^\varepsilon\|_{L_t^\infty L_x^\infty} \le \\ &\le \|\psi\|_\infty\, \|u\|_{L_t^\infty L_x^1}\, \|D\eta_\varepsilon\|_\infty \le \|\psi\|_\infty\, \|u\|_{L_t^\infty L_x^1}\, \|D\eta\|_\infty\, \frac{1}{\varepsilon^{d+1}}. \end{aligned}$$

Similarly, we have for the second term

$$\begin{aligned} \|(u^\varepsilon \otimes \nabla\psi)^\varepsilon\|_{L_t^\infty L_x^\infty} &\le \|u^\varepsilon \otimes \nabla\psi\|_{L_t^\infty L_x^\infty} \le \|D\psi\|_\infty\, \|u^\varepsilon\|_{L_t^\infty L_x^\infty} \le \\ &\le \|D\psi\|_\infty\, \|u\|_{L_t^\infty L_x^1}\, \|\eta\|_\infty\, \frac{1}{\varepsilon^d}. \end{aligned}$$

It follows that there exists a constant E depending only on $W^{1,\infty}$ norm of ψ and $L^\infty(0,T;L^2(\Omega))$ norm of u, such that

$$\|D\left(u^\varepsilon\psi\right)^\varepsilon\|_\infty \leq \frac{E}{\varepsilon^{d+1}}. \tag{9.3.1}$$

Now, we estimate $|D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x)}$. Clearly, when $|D\left(u^\varepsilon\psi\right)^\varepsilon| \leq 1$, we have

$$|D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x)} \leq 1. \tag{9.3.2}$$

Suppose that $|D\left(u^\varepsilon\psi\right)^\varepsilon| \geq 1$. We fix $x \in \Omega$ and consider a ball $B_{3\,\varepsilon}(x)$. Then, from Lemmas 9.3.5, 9.3.7 and the estimate (9.3.1) we obtain minimizing point $x^* \in B_{3\,\varepsilon}(x)$, such that

$$|D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x)} \leq M\,\, |D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x^*)}. \tag{9.3.3}$$

Combining two estimates (9.3.2) and (9.3.3), we deduce

$$|D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x)} \leq 1 + M\,\, |D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x^*)}. \tag{9.3.4}$$

Next, the function $v \mapsto |v|^{s(t,x^*)}$ is convex. Therefore, Jensen's inequality implies

$$\begin{aligned}
|D\left(u^\varepsilon\psi\right)^\varepsilon|^{s(t,x^*)}(t,x) &= \left|\int_{B_\varepsilon(0)} D\left(u^\varepsilon(t,x-y)\,\psi(x-y)\right)\,\eta_\varepsilon(y)\,\mathrm{d}y\right|^{s(t,x^*)} \\
&\leq \int_{B_\varepsilon(0)} |D\left(u^\varepsilon(t,x-y)\,\psi(x-y)\right)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\mathrm{d}y \\
&\leq C(s_{\max}) \int_{B_\varepsilon(0)} |Du^\varepsilon(t,x-y)\,\psi(x-y)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\mathrm{d}y + \\
&\quad + C(s_{\max}) \int_{B_\varepsilon(0)} |u^\varepsilon(t,x-y) \otimes \nabla\psi(x-y)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\mathrm{d}y \\
&\leq C(s_{\max})\,\|\psi\|_\infty^{s_{\max}} \int_{B_\varepsilon(0)} |Du^\varepsilon(t,x-y)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\mathrm{d}y + \\
&\quad + C(s_{\max})\,\|\nabla\psi\|^{s_{\max}} \int_{B_\varepsilon(0)} |u^\varepsilon(t,x-y)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\mathrm{d}y := A + B.
\end{aligned}$$

Concerning term A, we expand convolution, then we apply Jensen's inequality again and finally we observe that $x - y - z \in B_{3\,\varepsilon}(x)$, which allows us to apply Lemma 9.3.7 to get

$$\begin{aligned}
A &\leq C\,\|\psi\|_\infty^{s_{\max}} \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} |Du(t,x-y-z)|^{s(t,x^*)}\,\,\eta_\varepsilon(y)\,\eta_\varepsilon(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\leq C\,\|\psi\|_\infty^{s_{\max}} \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} |Du(t,x-y-z)|^{s(t,x-y-z)}\,\,\eta_\varepsilon(y)\,\eta_\varepsilon(z)\,\mathrm{d}y\,\mathrm{d}z =: a_\varepsilon(t,x)
\end{aligned} \tag{9.3.5}$$

Concerning term B, we proceed similarly, in the last step using partition of unity $\{\zeta_i\}$ from Notation 9.3.3:

$$\begin{aligned}
B &\leq C\, \|\nabla\psi\|_{\infty}^{s_{\max}} \int_{B_\varepsilon(0)} \int_{B_\varepsilon(0)} |u(t,x-y-z)|^{s(t,x^*)}\; \eta_\varepsilon(y)\, \eta_\varepsilon(z)\, \mathrm{d}y\, \mathrm{d}z \\
&\leq C\, \|\nabla\psi\|_{\infty}^{s_{\max}} \int_{B_\varepsilon(0)} \int_{B_\varepsilon(0)} \left(1 + |u(t,x-y-z)|^{s(t,x-y-z)}\right) \eta_\varepsilon(y)\, \eta_\varepsilon(z)\, \mathrm{d}y\, \mathrm{d}z \\
&\leq C \sum_{i=1}^{N} \zeta_i(x)\, \|\nabla\psi\|_{\infty}^{s_{\max}} \int_{B_\varepsilon(0)} \int_{B_\varepsilon(0)} \left(1 + |u(t,x-y-z)|^{r_i(t)}\right) \eta_\varepsilon(y)\, \eta_\varepsilon(z)\, \mathrm{d}y\, \mathrm{d}z \\
&=: C \sum_{i=1}^{N} b_\varepsilon^i(t,x).
\end{aligned} \tag{9.3.6}$$

where in the last line we used that when $x \in \mathcal{B}_r^i \cap \Omega$, we have $s(t,x-y-z) \leq r_i(t)$ as long as $2\varepsilon < r$. Combining (9.3.4), (9.3.5), (9.3.6) we obtain

$$0 \leq |D\,(u^\varepsilon \psi)^\varepsilon|^{s(t,x)} \leq C \left(1 + a_\varepsilon(t,x) + \sum_{i=1}^{N} b_\varepsilon^i(t,x)\right). \tag{9.3.7}$$

Now, the map $(t,x) \mapsto |Du(t,x)|^{s(t,x)} \in L^1(\Omega_T)$ so that due to Lemma 9.8.2, a_ε is convergent in $L^1(\Omega_T)$. Similarly, Lemma 9.3.4 together with interpolation result from Lemma 9.8.1 shows that the map $(t,x) \mapsto |u(t,x)|^{r_i(t)} \in L^1((0,T) \times \mathcal{B}_{2r}^i)$. Thanks to Lemma 9.8.2, b_ε^i is also convergent in $L^1((0,T) \times \mathcal{B}_r^i)$ so that taking into account the supports of ζ_i, b_ε^i is also convergent in $L^1(\Omega_T)$. Now, from (9.3.7), we deduce that the map $(t,x) \mapsto |D\,(u^\varepsilon \psi)^\varepsilon|^{s(t,x)}$ is uniformly integrable. Together with pointwise convergence as $\varepsilon \to 0$, this is sufficient to conclude $D\,(u^\varepsilon \psi)^\varepsilon \to D(u\,\psi)$ in $L^{s(t,x)}(\Omega_T)$. $\square$

9.4 Local energy equality

Following the presentation in Section 8.5, we first discuss a general abstract identity

$$\begin{aligned}
\int_{\Omega_T} -u \cdot \partial_t \phi - u \otimes u : \nabla\phi + (\alpha + \theta\,\beta) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\
= \int_{\Omega} u_0(x) \cdot \phi(0,x)\,\mathrm{d}x + \int_{\Omega_T} f \cdot \phi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{9.4.1}$$

which is required to be satisfied for any vector-valued $\phi \in C_c^\infty([0,T)\times\Omega)$ fulfilling $\operatorname{div}\phi = 0$ in Ω_T. Here, $\alpha, \beta : \Omega_T \to \mathbb{R}^{d\times d}_{\text{sym}}$, $\theta \geq 0$ and the term $\theta\,\beta$ can be seen as a regularizing term. In case $\theta = 0$ we are inspired by our setting and assume only

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^{R_i(t)}((0,T)\times\mathcal{B}^i_{2r}) \cap L^{s_0}(\Omega_T), \quad Du \in L^{s(t,x)}(\Omega_T), \quad (9.4.2)$$

$$\alpha \in L^{s'(t,x)}(\Omega_T) \text{ is a symmetric matrix}, \qquad f \in L^1(0,T;L^2(\Omega)), \qquad (9.4.3)$$

where, we recall (9.2.1), i.e., that $s_0 = 3 + \frac{2}{d}$. In case $\theta > 0$, we can additionally assume that

$$\beta \in L^{s'_{\max}}(\Omega_T) \text{ is a symmetric matrix}, \quad Du \in L^{s_{\max}}(\Omega_T), \qquad \theta > 0. \qquad (9.4.4)$$

Nevertheless, the uniform estimates can be obtained only in terms of (9.4.2)–(9.4.3) as we expect to lose (9.4.4) when $\theta \to 0$. This becomes more visible in the next section when we apply results obtained for the particular α and β. Notice also here, that $s_0 \leq R_i(t)$, which follows from the definition of $R_i(t)$ in Lemma 9.3.2, the definition of s_0 in (9.2.1) and the assumption $s_{\min} \geq \frac{3d+2}{d+2}$.

Our first target is to transform this identity to be satisfied by all test functions, not necessarily divergence-free. Here, the main difficulty is that we consider the problem with Dirichlet boundary condition. Hence, we cannot apply Hodge decomposition theorem. Instead, we apply Nečas theorem and use the method of harmonic pressure cf. [107, 139, 262].

First, we extend all functions to $\mathbb{R}^d$ with zero and we define p_1^i, p_2^i, p_3, p_4 as the unique functions satisfying for almost all $t \in (0,T)$

$$-\Delta p_1^i = \operatorname{div}\operatorname{div}(\alpha\,\zeta_i) \text{ in } \mathbb{R}^d, \qquad p_1^i(t,\cdot) \in L^{r_i'(t)}(\mathbb{R}^d), \qquad (9.4.5)$$

$$\Delta p_2^i = \operatorname{div}\operatorname{div}(u\otimes u\,\zeta_i) \text{ in } \mathbb{R}^d, \qquad p_2^i(t,\cdot) \in L^{R_i(t)/2}(\mathbb{R}^d), \qquad (9.4.6)$$

$$-\Delta p_3 = \operatorname{div} f \text{ in } \mathbb{R}^d, \qquad p_3(t,\cdot) \in W^{1,2}_{\text{loc}}(\mathbb{R}^d), \qquad (9.4.7)$$

$$-\Delta p_4 = \operatorname{div}\operatorname{div}(\theta\,\beta) \text{ in } \mathbb{R}^d, \qquad p_4(t,\cdot) \in L^{s'_{\max}}(\mathbb{R}^d). \qquad (9.4.8)$$

These functions have to be defined globally as we expect them to have only Lebesgue regularity so their trace is not well-defined. The main result of this section reads:

Theorem 9.4.1. *Let u be a solution of (9.4.1). Then, with p_i as above, there exists a uniquely determined function $p_h \in L^\infty(0,T;L^{s'_{\max}}(\Omega))$ (up to condition $\int_\Omega p_h(t,x)\,\mathrm{d}x = 0$ for almost all $t \in (0,T)$), such that for all $\varphi \in C^1_c([0,T)\times\Omega)$*

$$
\begin{aligned}
-\int_{\Omega_T} (u+\nabla p_h)\cdot\partial_t\varphi + (-u\otimes u+\alpha+\theta\beta):\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega u_0(x)\cdot\varphi(0,x)\,\mathrm{d}x \\
= \int_{\Omega_T} f\cdot\varphi - \sum_{i=1}^N (p_1^i+p_2^i)\,\operatorname{div}\varphi - (p_3+p_4)\,\operatorname{div}\varphi\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{9.4.9}
$$

Moreover, p_h is harmonic and so, it is locally smooth in the spatial variable, i.e., it satisfies

$$\Delta p_h = 0 \text{ in } \Omega \text{ for a.e. } t\in[0,T]. \tag{9.4.10}$$

In addition, we have the following estimate valid for all $\Omega' \Subset \Omega$

$$\|p_h\|_{L^\infty_t L^{s'_{\max}}_x} + \|p_h\|_{L^\infty(0,T;C^k(\Omega'))} \le C, \tag{9.4.11}$$

where the constant C depends on k, Ω', T and the norms $\|\alpha\|_{L^{s'(t,x)}_{t,x}}$, $\|Du\|_{L^{s(t,x)}_{t,x}}$, $\|\theta\,\beta\|_{L^{s'_{\max}}_{t,x}}$, $\|f\|_{L^1_t L^2_x}$, $\|u\|_{L^\infty_t L^2_x}$ and $\|u_0\|_{L^2_x}$. Finally, the following local energy equality holds: for any $\psi \in C^\infty_c(\Omega)$ and for a.e. $t\in(0,T)$

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega |u(t,x)+\nabla p_h(t,x)|^2\,\psi(x)\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u_0(x)|^2\,\psi(x)\,\mathrm{d}x \\
&+\int_0^t\int_\Omega (\alpha+\theta\beta):D(\psi(x)(u+\nabla p_h)(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau = \\
&=\int_0^t\int_\Omega (u\otimes u):\nabla(\psi(u+\nabla p_h))\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega f(u+\nabla p_h)\psi\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad -\int_0^t\int_\Omega \left(\sum_{i=1}^N (p_1^i+p_2^i)+p_3+p_4\right)(u+\nabla p_h)\nabla\psi\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}
\tag{9.4.12}
$$

The rest of this section is devoted to the proof of Theorem 9.4.1. First, we establish regularity of p_1^i, p_2^i, p_3, p_4.

Lemma 9.4.2. *There exists uniquely determined p_1^i, p_2^i, p_3, p_4 satisfying* (9.4.5)–(9.4.8). *Moreover, there exists a constant depending only on T and Ω such that*

$$\|p_1^i\|_{L_{t,x}^{r_i'(t)}} \le C\|\alpha\,\zeta_i\|_{L_{t,x}^{r_i'(t)}}, \qquad \|p_2^i\|_{L_{t,x}^{R_i(t)/2}} \le C\|u\,\sqrt{\zeta_i}\|^2_{L_{t,x}^{R_i(t)}},$$

$$\|p_1^i\|_{L_{t,x}^{s'_{\max}}} \le C\,\|\alpha\|_{L_{t,x}^{s'_{\max}}}, \qquad \|p_2^i\|_{L_{t,x}^{s_0/2}} \le C\,\|u\|^2_{L_{t,x}^{s_0}}$$

$$\|p_3\|_{L_t^1 W_x^{1,2}} \le C\,\|f\|_{L_1^1 L_x^2}, \qquad \|p_4\|_{L_{t,x}^{s'_{\max}}} \le C\,\|\theta\,\beta\|_{L_{t,x}^{s'_{\max}}}$$

Moreover, for each bounded $\Omega' \Subset \mathbb{R}^d \setminus \mathcal{B}_r^i$ we have

$$\|p_1^i\|_{L_t^{s'_{\max}}(0,T;C_x^k(\Omega'))} \le C\,(T,k,\Omega')\,\|\alpha\|_{L_{t,x}^{s'_{\max}}},$$

$$\|p_2^i\|_{L_t^{s_0/2}(0,T;C_x^k(\Omega'))} \le C\,(T,k,\Omega')\,\|u\|^2_{L_{t,x}^{s_0}}.$$

Proof. For p_1^i, p_2^i and p_4 this follows immiedately from Theorem 9.7.1 because we have $\alpha\,\zeta_i \in L^{r_i'(t)}(\Omega_T)$, $\alpha \in L^{s'_{\max}}(\Omega_T)$, $u{\otimes}u\,\zeta_i \in L^{R_i(t)/2}(\Omega_T)$ (with a norm controlled by $\|u\,\sqrt{\zeta_i}\|^2_{L_{t,x}^{R_i(t)}}$), $u{\otimes}u \in L_{t,x}^{s_0/2}$ and $\beta \in L_{t,x}^{s'_{\max}}$. To see the result for p_3, we fix $t \in (0,T)$ and solve the problem

$$\int_{\mathbb{R}^d} \nabla p_3 \cdot \nabla\varphi = -\int_{\mathbb{R}^d} f \cdot \nabla\varphi \quad \text{ for all } \varphi \in C_c(\mathbb{R}^d)$$

with the assumption $p_3(x) \to 0$ as $|x| \to \infty$. Therefore, the classical theory implies there exists a uniquely defined $p_3(t,x)$ fulfilling $\|\nabla p_3(t)\|_{L_x^2} \le \|f(t)\|_{L_x^2}$. Consequently the statement of the lemma for p_3 follows from the embedding theorem.

Finally, we obtain better estimates for p_1^i and p_2^i. We observe that these functions are harmonic in $\mathbb{R}^d\setminus\mathcal{B}_r^i$. Therefore, by Weyl's lemma, p_1^i and p_2^i are smooth in the spatial variable. To obtain uniform local estimate, we first consider p_1^i, we fix $\Omega' \Subset \mathbb{R}^d \setminus \mathcal{B}_r^i$ and apply [75, Theorem 4.2] to deduce

$$\|p_1^i\|_{L_t^{s'_{\max}}(0,T;W_x^{2,s'_{\max}}(\Omega'))} \le C\,(T,\Omega')\;\|p_1^i\|_{L_{t,x}^{s'_{\max}}} \le C\,(T,\Omega')\,\|\alpha\|_{L_{t,x}^{s'_{\max}}}.$$

As $\Delta p_1^i = 0$ in $\mathbb{R}^d \setminus \mathcal{B}_r^i$, we may iterate to get

$$\|p_1^i\|_{L_t^{s'_{\max}}(0,T;W_x^{2k,s'_{\max}}(\Omega'))} \le C\,(T,k,\Omega')\,\|\alpha\|_{L_{t,x}^{s'_{\max}}}$$

and this concludes the proof by the Sobolev embedding. The proof for p_2^i is completely analogous. □

Lemma 9.4.3. *There exists uniquely determined function* p_h *satisfying* (9.4.9) *and* $\int_\Omega p_h(t,x)\,\mathrm{d}x = 0$. *Moreover,* (9.4.10) *and* (9.4.11) *hold true.*

Proof. We divide the proof for several steps.

Existence of p_h. For fixed time $t > 0$, we consider functional

$$\mathcal{G}(\varphi) := \int_\Omega (u(t,x) - u_0(x)) \cdot \varphi(x) + \int_0^t \int_\Omega (-u \otimes u + \alpha + \theta\beta) : \nabla\varphi - \int_0^t \int_\Omega f \cdot \varphi$$
$$+ \sum_{i=1}^N \int_0^t \int_\Omega (p_1^i + p_2^i)\, \operatorname{div} \varphi + \int_0^t \int_\Omega (p_3 + p_4)\, \operatorname{div} \varphi$$

acting on vector-valued functions $\varphi : \Omega \to \mathbb{R}^d$. Now, we establish regularity of functional $\mathcal{G}$ by estimating terms appearing in its definition. First, thanks to (9.4.2)–(9.4.4) we have

$$\left| \int_\Omega (u(t,x) - u_0(x)) \cdot \varphi(x) \right| \leq \left(\|u\|_{L_t^\infty L_x^2} + \|u_0\|_{L_x^2} \right) \|\varphi\|_{L_x^2},$$

$$\left| \int_0^t \int_\Omega (-u \otimes u + \alpha + \theta\beta) : \nabla\varphi \right| \leq$$
$$\leq C(T) \left(\|u\|_{L_{t,x}^{s_0}}^2 \|\nabla\varphi\|_{L_x^{(s_0/2)'}} + \left(\|\alpha\|_{L_{t,x}^{s'_{\max}}} + \|\theta\beta\|_{L_{t,x}^{s'_{\max}}} \right) \|\nabla\varphi\|_{L_x^{s_{\max}}} \right).$$

Here, the constant $C(T)$ comes from applying Hölder's inequality in the time variable. Next,

$$\left| \int_0^t \int_\Omega f \cdot \varphi \right| \leq C(T)\, \|f\|_{L_t^1 L_x^2} \|\varphi\|_{L_x^2}.$$

For the terms with p_1^i, p_2^i, p_3, p_4 we apply Lemma 9.4.2 to obtain

$$\left| \int_0^t \int_\Omega p_1^i \, \operatorname{div} \varphi \right| \leq C(T, \|\alpha\|_{L_{t,x}^{s'_{\max}}}) \, \|\nabla\varphi\|_{L_x^{s_{\max}}},$$
$$\left| \int_0^t \int_\Omega p_2^i \, \operatorname{div} \varphi \right| \leq C(T, \|u\|_{L_{t,x}^{s_0}}^2) \, \|\nabla\varphi\|_{L_x^{(s_0/2)'}},$$
$$\left| \int_0^t \int_\Omega p_3 \, \operatorname{div} \varphi \right| \leq C(T, \|f\|_{L_t^1 L_x^2}) \, \|\nabla\varphi\|_{L_x^2},$$
$$\left| \int_0^t \int_\Omega p_4 \, \operatorname{div} \varphi \right| \leq C(T, \|\theta\,\beta\|_{L_{t,x}^{s'_{\max}}}) \, \|\nabla\varphi\|_{L_x^{s_{\max}}}.$$

We conclude that $\mathcal{G}$ is a bounded functional on $W_0^{1,s_{\max}}(\Omega)$ because $(s_0/2)' = \frac{3d+2}{d+2} \leq s_{\max}$. Moreover, if $\operatorname{div} \varphi = 0$, we obtain from (9.4.1) that $\mathcal{G}(\varphi) = 0$. Hence, applying the Nečas theorem, we know that there exists a distribution $p_h(t)$ fulfilling $\mathcal{G}(\varphi) = -\langle \nabla p_h(t), \varphi \rangle$ in the sense of distributions. Using then the above estimates and the Nečas theorem about negative norms, cf. Lemma 9.8.4, we get that for a.e. $t \in (0,T)$ there exists uniquely (up to function depending only on time) defined function $p_h(t,x)$ fulfilling

$$\int_\Omega (u(t,x) - u_0(x))\,\varphi(x) + \int_0^t \int_\Omega (-u \otimes u + \alpha + \theta\beta) \cdot \nabla\varphi + \int_0^t \int_\Omega f \,\cdot \varphi + \\ + \sum_{i=1}^N \int_0^t \int_\Omega (p_1^i + p_2^i) \cdot \operatorname{div}\varphi + \int_0^t \int_\Omega (p_3 + p_4) \cdot \operatorname{div}\varphi - \int_\Omega p_h(t,x)\, \operatorname{div}\varphi = 0. \tag{9.4.13}$$

Moreover, if we require that $\int_\Omega p_h(t,x) = 0$ for $t \in (0,T)$, then $\|p_h(t,\cdot)\|_{L_x^{s'_{\max}}} \le \widetilde{C}$, where $\widetilde{C}$ is a constant

$$\widetilde{C}\left(T, \|Du\|_{L_{t,x}^{s(t,x)}}, \|\alpha\|_{L_{t,x}^{s(t,x)}}, \|\theta\,\beta\|_{L_{t,x}^{s'_{\max}}}, \|f\|_{L_t^1 L_x^2}, \|u\|_{L_t^\infty L_x^2}, \|u_0\|_{L_x^2}\right).$$

Here, we used the fact that $\|u\|_{L_{t,x}^{s_0}} \le C(\|Du\|_{L_{t,x}^{s(t,x)}}, \|u\|_{L_t^\infty L_x^2})$. Furthermore, we included $\|\alpha\|_{L_{t,x}^{s'_{\max}}}$ into $\|\alpha\|_{L_{t,x}^{s(t,x)}}$. Taking supremum over all $t \in (0,T)$, we deduce that $\|p_h\|_{L_t^\infty L_x^{s'_{\max}}} \le \widetilde{C}$.

Function p_h is harmonic. Let $\varphi = \nabla\phi$ in (9.4.13) where $\phi \in C_c^\infty(\Omega)$. From (9.4.5)–(9.4.8) we deduce

$$\begin{aligned}\sum_{i=1}^N \int_0^t \int_\Omega (p_1^i + p_2^i)\, \operatorname{div}\varphi + \int_0^t \int_\Omega (p_3 + p_4)\, \operatorname{div}\varphi = \\ = \sum_{i=1}^N \int_0^t \int_\Omega (p_1^i + p_2^i)\, \Delta\phi + \int_0^t \int_\Omega (p_3 + p_4)\, \Delta\phi = \\ = \int_0^t \int_\Omega (-\alpha - \theta\beta + u \otimes u) : \nabla^2\phi + f \cdot \nabla\phi\end{aligned}$$

because $\sum_{i=1}^N \zeta_i = 1$ on Ω. Furthermore, the incompressibility condition yields

$$\int_\Omega (u(t,x) - u_0(x))\,\varphi(x)\,\mathrm{d}x = 0.$$

Therefore, (9.4.13) implies

$$\int_\Omega p_h(t,x)\,\Delta\phi(x)\,\mathrm{d}x = 0$$

so that $\Delta p_h(t,x) = 0$ in Ω in the sense of distributions.

Estimate on p_h. By Weyl's lemma, $p_h(t,x)$ is smooth in the spatial variable. To obtain uniform local estimate, we fix $\Omega' \Subset \Omega$ and apply [75, Theorem 4.2] to equation to deduce

$$\|p_h\|_{L^\infty_t(0,T;W^{2,s'_{\max}}_x(\Omega'))} \le C\, \|p_h\|_{L^\infty_t(0,T;L^{s'_{\max}}_x(\Omega))} \le C(\widetilde{C}, \Omega').$$

As the (RHS) of equation $\Delta p_h = 0$ is zero, we may iterate to prove

$$\|p_h\|_{L^\infty_t(0,T;W^{2k,s'_{\max}}_x(\Omega'))} \le C(\widetilde{C}, \Omega', k).$$

The conclusion follows by Sobolev embeddings.

Weak formulation for p_h. Multiplying (9.4.13) by an arbitrary $\partial_t \tau$, where $\tau \in C^1_c[0,T)$, integrating over $t \in (0,T)$ and using integration by parts with respect to time and space, using also the fact that p_h is smooth in the interior of Ω, we deduce with the help of (9.4.13) that for all smooth $\psi \in C^\infty_c(\Omega)$

$$\begin{aligned}
&\int_{\Omega_T} \partial_t\tau\,(u + \nabla p_h)\cdot\varphi = \int_{\Omega_T} \partial_t\tau\, u\cdot\varphi - \int_{\Omega_T} \partial_t\tau\, p_h \operatorname{div}\varphi \\
&= \int_{\Omega_T} \partial_t\tau\, u_0\cdot\varphi - \int_0^T \left(\partial_t\tau \int_0^t \int_\Omega (-u\otimes u + \alpha + \theta\beta) : \nabla\varphi + \int_0^t \int_\Omega f\,\cdot\varphi\right) \mathrm{d}t \,+ \\
&\quad - \sum_{i=1}^N \int_0^T \partial_t\tau \left(\int_0^t \int_\Omega (p_1^i + p_2^i) \operatorname{div}\varphi + \int_0^t \int_\Omega (p_3 + p_4) \operatorname{div}\varphi\right) \mathrm{d}t \\
&= -\int_\Omega u_0(x)\cdot\varphi(x)\tau(0)\,\mathrm{d}x + \int_{\Omega_T} \tau\,(-u\otimes u + \alpha + \theta\beta) : \nabla\varphi + \tau\, f\,\cdot\varphi \,+ \\
&\quad + \sum_{i=1}^N \int_{\Omega_T} \tau\,(p_1^i + p_2^i) \operatorname{div}\varphi + \tau\,(p_3 + p_4) \operatorname{div}\varphi
\end{aligned}$$

The relation (9.4.9) then follows directly. □

To prove Theorem 9.4.1, we need to extend the equation for negative times in a way that keeps in mind the initial condition u_0. We also include here a simple yet important regularity assertion: a solution mollified in spatial variable gains Sobolev regularity in time.

Lemma 9.4.4. *Let u be as in (9.4.9) and let us extend u to $\overline{u}$ by the following:*

$$\overline{u}(t,x) = \begin{cases} 0 & \text{when } t > T, \\ u(t,x) & \text{when } t \in (0,T], \\ u_0(x) & \text{when } t \leq 0. \end{cases}$$

In addition, let $\overline{f}$, $\overline{p_1^i}$, $\overline{p_2^i}$, $\overline{p_3}$, $\overline{p_4}$, $\overline{p_h}$, $\overline{u \otimes u}$, $\overline{\alpha}$, $\theta\overline{\beta}$ denote the extension by 0 outside of the time interval $(0,T)$ for the quantities f, p_1^i, p_2^i, p_3, p_4, p_h, $u \otimes u$, α, $\theta\beta$ respectively. Then

$$\begin{aligned}\int_{-T}^{T}\int_\Omega -(\overline{u} + \nabla\overline{p_h})\cdot\partial_t\phi - \overline{u\otimes u} : \nabla\phi + (\overline{\alpha} + \theta\overline{\beta}) : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \\ = -\sum_{i=1}^N \int_{-T}^{T}\int_\Omega (\overline{p_1^i} + \overline{p_2^i}) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t - \int_{-T}^{T}\int_\Omega (\overline{p_3} + \overline{p_4}) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{-T}^{T}\int_\Omega \overline{f}\cdot\phi \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{9.4.14}$$

holds for any $\phi \in C_c^\infty((-T,T)\times\Omega)$. Moreover,

$$\partial_t(\overline{u} + \nabla\overline{p_h}) \in L^1(-T,T;(W_0^{1,s_{\max}}(\Omega))^*). \tag{9.4.15}$$

Proof. From (9.4.9) we obtain (using also the zero extensions)

$$\begin{aligned}&\int_{-T}^{T}\int_\Omega -(\overline{u} + \nabla\overline{p_h})\,\partial_t\phi - \overline{u\otimes u} : \nabla\phi + (\overline{\alpha} + \theta\,\overline{\beta}) : \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \\ &\int_{-T}^{0}\int_\Omega -u_0\,\partial_t\phi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega -(u + \nabla p_h)\,\partial_t\phi - u\otimes u : \nabla\phi + (\alpha + \theta\beta) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ &-\int_\Omega u_0(x)\,\phi(0,x)\,\mathrm{d}x + \int_0^T\int_\Omega -(u + \nabla p_h)\,\partial_t\phi - u\otimes u : \nabla\phi + (\alpha + \theta\beta) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ &-\sum_{i=1}^N \int_{\Omega_T} (p_1^i + p_2^i) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} (p_3 + p_4) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} f\,\phi \,\mathrm{d}x\,\mathrm{d}t = \\ &-\sum_{i=1}^N \int_{\Omega_T} (\overline{p_1^i} + \overline{p_2^i}) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T} (\overline{p_3} + \overline{p_4}) \operatorname{div}\phi \,\mathrm{d}x\,\mathrm{d}t + \int_{-T}^{T}\int_\Omega \overline{f}\,\phi \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

which completes the proof of (9.4.14). The assertion (9.4.15) follows directly from (9.4.14) and the assumptions on data. □

Proof of Theorem 9.4.1. Most of the statements of Theorem 9.4.1 have been proven in Lemma 9.4.2–9.4.4. The only missing, but essential point, is the energy equality

(9.4.12). Hence, we focus on it in what follows. We use the following approximation of the indicator function $\mathbf{1}_{(\eta,\varrho)}$:

$$\gamma^\tau_{\eta,\varrho}(t) = \begin{cases} 0 \text{ for } t \le \eta - \tau \text{ or } t \ge \varrho + \tau \\ 1 \text{ for } \eta \le t \le \varrho \\ \text{affine for } t \in [\eta-\tau,\eta] \cup [\varrho, \varrho+\tau] \end{cases} \tag{9.4.16}$$

Here, η might be both negative and positive. When $\eta = 0$, we will mean the function

$$\gamma^\tau_{0,\varrho}(t) = \begin{cases} 0 \text{ on } [\varrho+\tau, 1] \\ 1 \text{ on } [0,\varrho] \\ \text{affine otherwise} \end{cases}$$

as the needed approximations.

For $\varrho, \eta \in (0,T)$ and $\delta, \tau, \varepsilon$ nonnegative and sufficiently small we define

$$\phi^{\delta,\tau,\varepsilon}_{\eta,\varrho}(t,x) = (\mathcal{R}^\delta(u^\varepsilon(t,x) + \nabla p^\varepsilon_h(t,x))\,\psi(x)\,\gamma^\tau_{-\eta,\varrho}(t)))^\varepsilon$$

and we use it as a test function in (9.4.14). We recall here that mollification operator $\mathcal{R}^\delta$ is defined in Definition 8.3.3. We obtain five different parts and we will study their limits as $\tau \to 0$, $\delta \to 0$ and $\varepsilon \to 0$ separately.

Term $(\overline{u} + \nabla\overline{p_h}) \cdot \partial_t\phi$. First, thanks to (9.4.15), we can deduce for the distributional derivative that $\partial_t(\overline{u} + \nabla\overline{p_h}) \in L^{s'_{\max}}(0,T;(W^{1,s_{\max}}_0(\Omega))^*)$, consequently, mollification with respect to the spatial variable leads to the fact

$$\partial_t(\overline{u} + \nabla\overline{p_h})^\varepsilon \in L^1(-T,T;L^1(\Omega')) \qquad \text{for any } \Omega' \Subset \Omega.$$

Therefore, we can use integration by parts and Fubini theorem applied to mollifier to deduce

$$\begin{aligned}
&-\int_{-T}^T\int_\Omega (\overline{u} + \nabla\overline{p_h}) \cdot \partial_t\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{-T}^T\int_\Omega (\overline{u} + \nabla\overline{p_h}) \cdot \partial_t(\mathcal{R}^\delta(u^\varepsilon(t,x) + \nabla p^\varepsilon_h(t,x))\,\psi(x)\,\gamma^\tau_{-\eta,\varrho}(t)))^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_{-T}^T\int_\Omega (\overline{u} + \nabla\overline{p_h})^\varepsilon \cdot \partial_t(\mathcal{R}^\delta(u^\varepsilon(t,x) + \nabla p^\varepsilon_h(t,x))\,\psi(x)\,\gamma^\tau_{-\eta,\varrho}(t))) \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{-T}^T\int_\Omega \partial_t(\overline{u} + \nabla\overline{p_h})^\varepsilon(t,x) \cdot \mathcal{R}^\delta(\overline{u}^\varepsilon(t,x) + \nabla\overline{p_h}^\varepsilon)\,\psi(x)\,\gamma^\tau_{-\eta,\varrho}(t)) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We observe that $u \in L^\infty(0,T;L^2(\Omega))$ and $p_h \in L^\infty(0,T;L^\infty_{loc}(\Omega))$ so that we have $(\overline{u}+\nabla\overline{p_h})^\varepsilon \in L^\infty(0,T;L^\infty_{loc}(\Omega))$. Moreover, $\partial_t(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x) \in L^1(\Omega_T)$ and so we can apply Lebesgue's dominated convergence to converge with $\tau \to 0$, $\delta \to 0$ and obtain convergence to

$$\int_{-\eta}^{\varrho}\int_\Omega \partial_t(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)\cdot(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Now since $(\overline{u}+\nabla\overline{p_h})^\varepsilon \in L^\infty(0,T;L^\infty_{loc}(\Omega))$, we may apply chain rule for Sobolev functions, similarly as in Lemma 8.5.3, to obtain

$$\int_{-\eta}^{\varrho}\int_\Omega (\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)\,\partial_t(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t = \\ = \frac{1}{2}\int_{-\eta}^{\varrho}\int_\Omega \partial_t|(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)|^2\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

By the absolute continuity of Sobolev functions on lines, we get that for all time arguments $\eta, \varrho \in (0,T)$

$$\begin{aligned}
&\int_{-\eta}^{\varrho}\int_\Omega \partial_t|(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x)|^2\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega |(\overline{u}+\nabla\overline{p_h})^\varepsilon(\varrho,x)|^2\,\psi(x)\,\mathrm{d}x - \int_\Omega |(\overline{u}+\nabla\overline{p_h})^\varepsilon(-\eta,x)|^2\,\psi(x)\,\mathrm{d}x \\
&= \int_\Omega |(u+\nabla p_h)^\varepsilon(\varrho,x)|^2\,\psi(x)\,\mathrm{d}x - \int_\Omega |u_0^\varepsilon(x)|^2\,\psi(x)\,\mathrm{d}x.
\end{aligned}$$

Recall, we used the fact that $\overline{p_h}(t,x) = 0$ and $u(t,x) = u_0(x)$ for negative t's. Now, we want to let $\varepsilon \to 0$. Note that for a.e. $\varrho \in (0,T)$ we have $u(\varrho,x)+\nabla p_h(\varrho,x) \in L^2_{loc}(\Omega)$ and $u_0 \in L^2(\Omega)$. Thanks to Lemma 9.8.3, this implies

$$\int_\Omega |(u+\nabla p_h)^\varepsilon(\varrho,x)|^2\,\psi(x)\,\mathrm{d}x - \int_\Omega |u_0^\varepsilon(x)|^2\,\psi(x)\,\mathrm{d}x \to \\ \to \int_\Omega |(u+\nabla p_h)(\varrho,x)|^2\,\psi(x)\,\mathrm{d}x - \int_\Omega |u_0(x)|^2\,\psi(x)\,\mathrm{d}x.$$

<u>Terms $(\alpha+\theta\beta) : D\phi$.</u> As $(\overline{u}+\nabla\overline{p_h})^\varepsilon \in L^\infty(0,T;L^\infty_{loc}(\Omega))$ and time derivatives are not involved, convergence results with $\delta \to 0$ and $\tau \to 0$ are trivial. Therefore, we focus on convergence $\varepsilon \to 0$. We first write

$$\int_{-\eta}^{\varrho}\int_\Omega (\overline{\alpha}+\theta\overline{\beta})(t,x) : D(\psi(x)(\overline{u}+\nabla\overline{p_h})^\varepsilon(t,x))^\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \\ = \int_0^{\varrho}\int_\Omega (\alpha+\theta\beta)(t,x) : D(\psi(x)(u+\nabla p_h)^\varepsilon(t,x))^\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

When $\theta > 0$, we can simply use that $\alpha + \theta\beta \in L^{s'_{\max}}(\Omega_T)$ and $Du, u \in L^{s_{\max}}(\Omega_T)$ (the latter by Körn's inequality) to obtain

$$\int_0^{\varrho} \int_\Omega (\alpha + \theta\beta)(t,x) : D(\psi(x)(u + \nabla p_h)^\varepsilon(t,x))^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to$$
$$\to \int_0^{\varrho} \int_\Omega (\alpha + \theta\beta)(t,x) : D(\psi(x)(u + \nabla p_h)(t,x)) \,\mathrm{d}x\,\mathrm{d}t.$$

For the case $\theta = 0$ we apply Theorem 9.3.1 to obtain

$$D(\psi(x)(u + \nabla p_h)^\varepsilon(t,x))^\varepsilon \to D(\psi(x)(u(t,x) + \nabla p_h) \text{ modularly in } L^{s(t,x)}(\Omega_T).$$

Then, since $\alpha \in L^{s'(t,x)}(\Omega_T)$, we may apply Theorem 7.4.2 and conclude

$$\int_0^{\varrho} \int_\Omega \alpha(t,x) : D(\psi(x)(u + \nabla p_h)^\varepsilon(t,x))^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to$$
$$\to \int_0^{\varrho} \int_\Omega \alpha(t,x) : D(\psi(x)(u(t,x) + \nabla p_h)) \,\mathrm{d}x\,\mathrm{d}t.$$

<u>Term $\overline{f}\,\phi$.</u> We have

$$\int_{-\eta}^{\varrho} \int_\Omega \overline{f} \cdot ((\overline{u} + \nabla\overline{p_h})^\varepsilon \psi)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^{\varrho} \int_\Omega f^\varepsilon \cdot ((u + \nabla p_h)^\varepsilon \psi) \,\mathrm{d}x\,\mathrm{d}t.$$

Next, thanks to the assumption on f, we know that

$$f^\varepsilon \to f \text{ strongly in } L^1(0,T;L^2(\Omega)).$$

On the other hand, we also have the weak* convergence result

$$(u + \nabla p_h)^\varepsilon \psi \rightharpoonup^* (u + \nabla p_h)\psi \text{ weakly}^* \text{ in } L^\infty(0,T;L^2_{loc}(\Omega)).$$

Hence, we get

$$\int_0^{\varrho} \int_\Omega f \cdot ((u + \nabla p_h)^\varepsilon \psi)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \to \int_0^{\varrho} \int_\Omega f \cdot (u + \nabla p_h)\,\psi \,\mathrm{d}x\,\mathrm{d}t.$$

<u>Terms $\overline{p_j^i}$ $\operatorname{div}\phi$ for $i = 1, ..., N$ and $j = 1, 2$.</u> Due to incompressibility of u and thanks to the fact that p_h is harmonic, we can write

$$\int_{-\eta}^{\varrho} \int_\Omega \overline{p_j^i}(t,x) \operatorname{div}((\overline{u} + \nabla\overline{p_h})^\varepsilon \psi)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^{\varrho} \int_\Omega (p_j^i)^\varepsilon (u + \nabla p_h)^\varepsilon \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}t.$$

Next, we decompose the integration domain onto $\mathcal{B}^i_{3r/2}$ and $\Omega \setminus \mathcal{B}^i_{3r/2}$. When $\varepsilon < \frac{r}{4}$, we can write

$$\begin{aligned}\int_0^\varrho \int_\Omega (p_j^i)^\varepsilon (u + \nabla p_h)^\varepsilon \nabla \psi \,\mathrm{d}x\,\mathrm{d}t = &\int_0^\varrho \int_{\Omega\cap\mathcal{B}^i_{3r/2}} (p_j^i)^\varepsilon (u\,\mathbb{1}_{\mathcal{B}^i_{2r}} + \nabla p_h)^\varepsilon \nabla \psi \,\mathrm{d}x\,\mathrm{d}t + \\ &+ \int_0^\varrho \int_{\Omega\setminus\mathcal{B}^i_{3r/2}} (p_j^i\,\mathbb{1}_{\Omega\setminus\mathcal{B}^i_{5r/4}})^\varepsilon (u + \nabla p_h)^\varepsilon \nabla \psi \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We discuss now the cases $j = 1, 2$ separately. For $j = 1$, we have $p_1^i \in L^{r_i'(t)}(\Omega_T)$ from Lemma 9.4.2 and $u \in L^{r_i(t)}((0,T)\times\mathcal{B}^i_{2r})$ from (9.4.2) (recall that $r_i \le R_i$), so we can pass to the limit in the first term. For the second one, we use harmonic regularity of p_j^i outside $\mathcal{B}^i_r$. Namely, we have $u \in L^\infty(0,T;L^2(\Omega))$ and $\nabla p_h \in L^\infty(0,T;L^\infty_{loc}(\Omega))$ so that $(u + \nabla p_h) \in L^\infty(0,T;L^2(\operatorname{supp}\psi))$. As $p_1^i \in L^{s'_{\max}}(0,T;L^2(\Omega\setminus\mathcal{B}^i_{5r/4}))$, cf. Lemma 9.4.2, the convergence is clear.

For $j = 2$ the proof is similar. More precisely, for the first term we observe that since $p_2^i \in L^{R_i(t)/2}(\Omega_T)$, it is sufficient that $u \in L^{(R_i(t)/2)'}((0,T)\times\mathcal{B}^i_{3r/2})$. This is the case because

$$(R_i(t)/2)' \le R_i(t) \iff R_i(t) \ge 3.$$

Recalling the definition of R_i in Lemma 9.3.2, using the fact that $q_i(t) \ge s_{\min} \ge \frac{3d+2}{d+2}$, we see that

$$R_i(t) \ge s_{\min}\left(1 + \frac{2}{d}\right) \ge \frac{3d+2}{d+2}\left(\frac{d+2}{d}\right) = s_0 = 3 + \frac{2}{d} > 3.$$

The second term is controlled in exactly the same way as for $j = 1$.

Terms $\overline{p_j}\operatorname{div}\phi$ for $j = 3, 4$. Similarly as above, it is sufficient to study the integral $\int_0^\varrho \int_\Omega p_j^\varepsilon \cdot (u + \nabla p_h)^\varepsilon \nabla\psi \,\mathrm{d}x\,\mathrm{d}t$. For $j = 3$ the proof is simple as $p_3 \in L^1(0,T;L^2(\Omega))$ so it is sufficient that $(u + \nabla p_h) \in L^\infty(0,T;L^2_{loc}(\Omega))$ which is of course the case. For $j = 4$, we observe that $\theta > 0$ so that we can use additional regularity from (9.4.4) which yields $u \in L^{s_{\max}}(\Omega_T)$ by Körn's and Poincaré's inequalities. As $p_4 \in L^{s'_{\max}}(\Omega_T)$, the conclusion is clear.

Term $\overline{u \otimes u} : \nabla \phi$. We write as in the previous step

$$\begin{aligned}\int_{-\eta}^{\varrho} \int_{\Omega} \overline{u \otimes u} : \nabla(\psi(\overline{u} + \nabla \overline{p_h})^{\varepsilon})^{\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \\ = \int_{0}^{\varrho} \int_{\Omega} (u \otimes u)^{\varepsilon} : (\nabla \psi \otimes (u + \nabla p_h)^{\varepsilon}) \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{0}^{\varrho} \int_{\Omega} (u \otimes u)^{\varepsilon} : (\psi\,(\nabla(u + \nabla p_h))^{\varepsilon}) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

First, $u \otimes u \in L^{s_0/2}(\Omega_T)$ and ∇u and u belong to $L^{s_{\min}}(\Omega_T)$. As $\nabla p_h \in L^{\infty}(0, T; L^{\infty}_{loc}(\Omega))$, it is sufficient that $(s_0/2)' \leq s_{\min}$. In fact, we have $s_{\min} = (s_0/2)'$. Indeed, $s_0 = s_{\min}\left(1 + \frac{2}{d}\right)$ so that

$$(s_0/2)' = \frac{s_0}{s_0 - 2} = \frac{s_{\min}(d+2)}{s_{\min}(d+2) - 2d} = \frac{3d+2}{3d+2-2d} = s_{\min}$$

and this concludes the proof. □

9.5 The approximating problem

The crucial step of the existence proof is the approximation of the stress tensor S. Namely, we set

$$S^{\theta}(t, x, \xi) := S(t, x, \xi) + \theta \nabla_{\xi} m(|\xi|), \qquad m(|\xi|) := |\xi|^{s_{\max}}. \tag{9.5.1}$$

The advantage of such an approximation lies in the fact that function S^{θ} satisfies Assumption 9.2.2 with $s(t, x) \equiv s_{\max}$, see Lemma 9.5.3 below. In particular, the analysis of problems with function S^{θ} is substantially easier and can be performed in usual Lebesgue spaces.

Now, we formulate the result concerning existence of solutions to the approximation problem

$$\begin{aligned}\partial_t(u^{\theta} + \nabla p_h^{\theta}) + \operatorname{div}(u^{\theta} \otimes u^{\theta}) = \operatorname{div} S^{\theta}(t, x, Du^{\theta}) + f + \sum_{i=1}^{N} \nabla(p_1^{i,\theta} + p_2^{i,\theta}) + \nabla(p_3 + p_4^{\theta}), \\ \operatorname{div} u^{\theta} = 0.\end{aligned} \tag{9.5.2}$$

Theorem 9.5.1. *Let S satisfy Assumption 9.2.2 and S^θ be defined in* (9.5.1)*. Then, for any $f \in L^1(0,T;L^2(\Omega))$ and any initial condition $u_0 \in L^2_{0,\mathrm{div}}(\Omega)$, there exists a function $u^\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^{s_{\max}}(0,T;W_0^{1,s_{\max}}(\Omega))$, $Du^\theta \in L^{s_{\max}}(\Omega_T)$, such that $S^\theta(t,x,Du^\theta) \in L^{s'_{\max}}(\Omega_T)$ and*

$$\begin{aligned}\int_{\Omega_T} -u^\theta \cdot \partial_t \phi - u^\theta \otimes u^\theta : \nabla\phi + S^\theta(t,x,Du^\theta) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ = \int_{\Omega_T} f \cdot \phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x) \cdot \phi(0,x)\,\mathrm{d}x\end{aligned} \tag{9.5.3}$$

for any vector-valued $\phi \in C_c^\infty([0,T)\times\Omega)$ fulfilling $\operatorname{div}\phi = 0$. Moreover, the following global energy equality is satisfied for all $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\int_\Omega |u^\theta(t,x)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u_0(x)|^2\,\mathrm{d}x + \int_0^t \int_\Omega S^\theta(\tau,x,Du^\theta) : Du^\theta \,\mathrm{d}x\,\mathrm{d}\tau = \\ = \int_0^t \int_\Omega f \cdot u^\theta \,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned} \tag{9.5.4}$$

In the construction of a solution to (9.2.2), we want to let $\theta \to 0$ in (9.5.2) and (9.5.3). To this end, we need certain estimates independent of θ, which is the content of the next result.

Theorem 9.5.2. *Let $\{u^\theta\}$ be the sequence of solutions to* (9.5.2) *constructed in Theorem 9.5.1. Let $\{p_1^{i,\theta}\}$, $\{p_2^{i,\theta}\}$, $\{p_3\}$, $\{p_4^\theta\}$, $\{p_h^\theta\}$ be the sequences of pressures obtained by Theorem 9.4.1 with $\alpha = S(t,x,Du^\theta)$ and $\beta = \nabla_\xi m(|Du^\theta|)$. Then,*

(B1) $\{u^\theta\}$ *is bounded in* $L^\infty(0,T;L^2(\Omega))$,

(B2) $\{Du^\theta\}$ *is bounded in* $L^{s(t,x)}(\Omega_T)$,

(B3) $\{u^\theta\}$ *is bounded in* $L^{s_{\min}}(0,T;W_0^{1,s_{\min}}(\Omega))$ *and* $L^{q_i(t)}(0,T;W^{1,q_i(t)}(\mathcal{B}^i_{2r}))$,

(B4) $\{u^\theta\}$ *is bounded in* $L^{s_0}(\Omega_T)$ *and* $L^{R_i(t)}((0,T)\times\mathcal{B}^i_{2r})$,

(B5) $\{S(t,x,Du^\theta)\}$ *is bounded in* $L^{s'(t,x)}(\Omega_T)$,

(B6) $\{\theta\,|Du^\theta|^{s_{\max}}\}$ *is bounded in* $L^1(\Omega_T)$,

(B7) $\{\theta^{1-s'_{\max}}\left|\theta\,\nabla_\xi m(|Du^\theta|)\right|^{s'_{\max}}\}$ *is bounded in* $L^1(\Omega_T)$,

(B8) $\{p_1^{i,\theta}\}$ *is bounded in* $L^{r_i'(t)}(\Omega_T)$ *and* $L^{s'_{\max}}(0,T;L^\infty_{loc}(\mathbb{R}^d\setminus\mathcal{B}^i_r))$,

(B9) $\{p_2^{i,\theta}\}$ *is bounded in* $L^{R_i(t)/2}(\Omega_T)$ *and* $L^{s_0/2}(0,T;L^\infty_{loc}(\mathbb{R}^d\setminus\mathcal{B}^i_r))$,

(B10) $\{\theta^{-1/s_{\max}} p_4^\theta\}$ *is bounded in* $L^{s'_{\max}}(\Omega_T)$,

(B11) $\{p_h^\theta\}$ *is bounded in* $L^\infty(0,T;L^{s'_{\max}}(\Omega))$,

(B12) $\{p_h^\theta\}$ *is bounded in* $L^\infty(0,T;W^{2,\infty}_{loc}(\Omega))$,

(B13) $\{\partial_t u^\theta\}$ *is bounded in* $L^1(0,T;V^*_{2,d})$,

(B14) $\{\partial_t(u^\theta + \nabla p_h^\theta)\}$ *is bounded in* $L^1(0,T;(W_0^{1,s_{\max}}(\Omega))^*)$,

where $V_{2,d}$ *is closure of* $\{\phi \in C_c^\infty(\Omega) |\operatorname{div}\phi = 0\}$ *in* $W^{2,d}(\Omega)$.

The rest of this section is devoted to the proofs of Theorems 9.5.1 and 9.5.2. We begin by establishing certain properties of function S^θ.

Lemma 9.5.3. *Function* S^θ *satisfies the following:*

(R1) $S^\theta(t,x,\xi)$ *is a Carathéodory function and* $S(t,x,0) = 0$,

(R2) (coercitivity and growth in $L^{s(t,x)}$*) there exists a positive constant* c *and a non-negative, integrable function* $h(t,x)$*, such that for any* $\xi \in \mathbb{R}^{d\times d}_{sym}$ *and almost every* $(t,x) \in (0,T)\times\Omega$

$$c\, S^\theta(t,x,\xi) : \xi \geq |\xi|^{s(t,x)} + |S(t,x,\xi)|^{s'(t,x)} + \theta \nabla_\xi m(|\xi|)\cdot\xi - h(t,x);$$

the constant c *and function* h *can be chosen independently of* θ,

(R3) (coercitivity and growth in $L^{s_{\max}}$*) there exists a positive constant* c^θ *and a non-negative, integrable function* $h^\theta(t,x)$*, such that for any* $\xi \in \mathbb{R}^{d\times d}_{sym}$ *and almost every* $(t,x) \in (0,T)\times\Omega$

$$c^\theta\, S^\theta(t,x,\xi) : \xi \geq |\xi|^{s_{\max}} + |S^\theta(t,x,\xi)|^{s'_{\max}} - h^\theta(t,x)$$

(R4) (monotonicity) S *is strictily monotone, i. e.:*

$$(S^\theta(t,x,\xi_1) - S^\theta(t,x,\xi_2)) : (\xi_1 - \xi_2) > 0$$

for all $\xi_1 \neq \xi_2 \in \mathbb{R}^{d\times d}_{sym}$ *and almost every* $(t,x) \in (0,T)\times\Omega$.

Proof. Properties (R1) and (R4) are fairly obvious. To see (R2) and (R3), we first note that by the definition of the convex conjugate we have

$$\nabla_\xi m(|\xi|) \cdot \xi = |\xi|^{s_{\max}} + C_* \, |\nabla_\xi m(|\xi|)|^{s'_{\max}}, \qquad C_* := \frac{1}{s'_{\max} \, s_{\max}^{s'_{\max}-1}}. \tag{9.5.5}$$

As S satisfies (T2) in Assumption 9.2.2, we have

$$c\, S^\theta(t,x,\xi) \cdot \xi \geq |\xi|^{s(t,x)} + |S(t,x,\xi)|^{s'(t,x)} - h(t,x) + c\,\theta \nabla_\xi m(|\xi|) \cdot \xi$$

so that we obtain (R2). To see (R3), we estimate term $S^\theta(t,x,\xi)\cdot\xi$ more carefully using (9.5.5):

$$\begin{aligned} c\, S^\theta(t,x,\xi) \cdot \xi &\geq |\xi|^{s(t,x)} + |S(t,x,\xi)|^{s'(t,x)} - h(t,x) + c\,\theta \nabla_\xi m(|\xi|) \cdot \xi \\ &\geq 0 + |S(t,x,\xi)|^{s'_{\max}} - 1 - h(t,x) + c\,\theta\, |\xi|^{s_{\max}} + c\,\theta\, C_* \, |\nabla_\xi m(|\xi|)|^{s'_{\max}}, \end{aligned}$$

where we estimated $|S(t,x,\xi)|^{s'(t,x)} \geq |S(t,x,\xi)|^{s'_{\max}} - 1$ which follows from the inequality $s'(t,x) \geq s'_{\max}$. Applying Jensen's inequality, we obtain with $h = h(t,x)$

$$\begin{aligned} |S(t,x,&\xi)|^{s'_{\max}} - h + c\,\theta\, |\xi|^{s_{\max}} + c\,\theta\, C_* \, |\nabla_\xi m(|\xi|)|^{s'_{\max}} \geq \\ &\geq 2 \min(1, c\, C_*) \left(\frac{1}{2} |S(t,x,\xi)|^{s'_{\max}} + \frac{1}{2} |\theta \nabla_\xi m(|\xi|)|^{s'_{\max}} \right) + c\,\theta |\xi|^{s_{\max}} - h - 1 \\ &\geq 2 \min(1, c\, C_*) \left| \frac{1}{2} S^\theta(t,x,\xi) \right|^{s'_{\max}} + c\,\theta |\xi|^{s_{\max}} - h - 1 \\ &\geq \min(\min(1, c\, C_*)\, 2^{1-s'_{\max}}, c\,\theta)(|S^\theta(t,x,\xi)|^{s'_{\max}} + |\xi|^{s_{\max}}) - h - 1 \end{aligned}$$

Thus, taking

$$c^\theta := \frac{c}{\min(\min(1, c\, C) 2^{1-s'_{\max}}, c\,\theta)}, \qquad h^\theta(t,x) := \frac{h(t,x) + 1}{\min(\min(1, c\, C) 2^{1-s'_{\max}}, c\,\theta)}$$

concludes the proof of (R3). □

Proof of Theorem 9.5.1. The proof of this theorem follows the lines of the proof in [163]. The only difference is the dependence of stress tensor S on time variable. □

Proof of Theorem 9.5.2. We combine the energy equality (9.5.4) and coercivity estimate (R2) in Lemma 9.5.3 to deduce

$$\begin{aligned} \frac{1}{2} \int_\Omega |u^\theta(t,x)|^2 \, \mathrm{d}x + \frac{1}{c} \int_{\Omega_t} \left(|Du|^{s(\tau,x)} + |S(\tau,x,Du)|^{s'(\tau,x)} + c\,\theta \nabla_\xi m(|Du^\theta|) \cdot Du^\theta \right) \mathrm{d}x \, \mathrm{d}\tau \\ = \frac{1}{2} \int_\Omega |u_0(x)|^2 \, \mathrm{d}x + \int_{\Omega_t} f \cdot u^\theta \, \mathrm{d}x \, \mathrm{d}\tau + \int_{\Omega_t} h(\tau,x) \, \mathrm{d}x \, \mathrm{d}\tau. \end{aligned}$$

Using the Hölder inequality, we can estimate $\int_\Omega f \cdot u^\theta \leq \|f\|_2(\|u^\theta\|_2^2+1)$ on the right hand side. Using the Grönwall lemma and also the assumptions on f, u_0 and h, we deduce the right hand side is bounded independently of θ and consequently, we conclude the proof of (B1), (B2) and (B5). Moreover, it shows that $\int_{\Omega_T} \theta \nabla_\xi m(|Du^\theta|) \cdot Du^\theta \,\mathrm{d}x\,\mathrm{d}\tau$ is bounded uniformly in $\theta \in (0,1)$. But then, using (9.5.5) we deduce

$$\int_{\Omega_T} \left(\theta\,|Du|^{s_{\max}} + \theta\, C_r\, |\nabla_\xi m(|Du|)|^{s'_{\max}}\right) \mathrm{d}x\,\mathrm{d}\tau = \int_{\Omega_T} \theta \nabla_\xi m(|Du^\theta|) \cdot Du^\theta \,\mathrm{d}x\,\mathrm{d}\tau.$$

This implies (B6) and (B7).

To see (B3), we observe that (B2) implies that $\{Du^\theta\}$ is bounded in $L^{s_{\min}}(\Omega_T)$ and $L^{q_i(t)}_{t,x}((0,T) \times \mathcal{B}^i_{2r})$ (because we have $s_{\text{min}} \leq s(t,x)$ on Ω_T and $q_i(t) \leq s(t,x)$ on $(0,T) \times \mathcal{B}^i_{2r}$). Then, Körn's inequality implies that $\{\nabla u^\theta\}$ is bounded in $L^{s_{\min}}(\Omega_T)$ and $L^{q_i(t)}_{t,x}((0,T) \times \mathcal{B}^i_{2r})$. To conclude the estimate, we note that the $\int_\Omega u^\theta(t,x)\,\mathrm{d}x$ is controlled in $L^\infty(0,T)$ by (B1) so that the claim follows by the Poincaré inequality. Next, estimate (B4) follows from (B1) and (B3) together with Lemma 9.8.1.

To obtain estimates on the pressures we apply Theorem 9.4.1 and Lemma 9.4.2 with

$$\alpha = S^\theta(t,x,Du^\theta), \qquad \beta = \nabla_\xi m(|Du^\theta|)$$

so that (B8)–(B12) follows from (B4), (B5) and (B7).

The bound (B13) can be obtained by the following argument. We have (for divergence-free distributional formulation):

$$\partial_t u^\theta = \underbrace{-\operatorname{div}(u^\theta \otimes u^\theta) + \operatorname{div} S^\theta(t,x,Du^\theta) + f}_{:= A^\theta}$$

We want to prove that A^θ defines a functional on $L^\infty(0,T;V_{2,d})$. This is clear because functions in $L^\infty(0,T;V_{2,d})$ have spatial derivatives in $L^\infty(0,T;L^z(\Omega))$ for all $z < \infty$ and all the functions $u^\theta \otimes u^\theta$, $S^\theta(t,x,Du^\theta)$ and f belong at least to some $L^1(0,T;L^a(\Omega))$ with $a > 1$, with the norm independent of $\theta \in (0,1)$.

Now, we move to establishing the regularity of the time derivative $\partial_t(u^\theta + \nabla p_h^\theta)$ as in (B14). In view of (9.5.1) and (9.5.2), we can write (in the sense of distributions)

$$\partial_t(u^\theta+\nabla p_h^\theta) = -\operatorname{div}(u^\theta \otimes u^\theta) + \operatorname{div} S^\theta(t,x,Du^\theta) + f + \sum_{i=1}^{N} \nabla(p_1^{i,\theta}+p_2^{i,\theta})+\nabla(p_3+p_4^\theta).$$

We observe that all of the functions $u^\theta \otimes u^\theta$, $S^\theta(t,x,Du^\theta)$, f, p_i^θ are uniformly bounded at least in $L^1(0,T;L^{s'_{\max}}(\Omega))$ (this uses inequalities $s'_{\max} \leq s'_{\min}$ and equality $s_0/2 = s'_{\min}$) so that $\partial_t(u^\theta+\nabla p_h^\theta)$ is bounded in $L^1(0,T;(W_0^{1,s'_{\max}}(\Omega))^*)$, hence (B14) holds. □

9.6 Proof of existence result via the monotonicity method

Proof of Theorem 9.2.3. The proof is divided into four steps.

Step 1: Approximating problem and compactness. Let u^θ be a solution to (9.5.2) constructed in Theorem 9.5.1. Let $p_1^{i,\theta}$, $p_2^{i,\theta}$, p_3, p_4^θ, p_h^θ be the sequences of pressures obtained in Theorem 9.4.1. First, thanks to Theorem 9.5.2, we can extract appropriate subsequences such that

(C1) $u^\theta \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;L^2(\Omega))$,

(C2) $u^\theta \to u$ a.e. in Ω_T and in $L^c(\Omega_T)$ for all $c < s_0$,

(C3) $u^\theta \to u$ in $L^{c_1}(0,T;L^{c_2}(\Omega))$ for all $c_1 < \infty$ and $c_2 < 2$,

(C4) $u^\theta \to u$ in $L^{R_i(t)-\delta}((0,T)\times\mathcal{B}_{2r}^i)$ for all $\delta > 0$,

(C5) $\theta^{1/s_{\max}} u^\theta \to 0$ in $L^{s_{\max}}(\Omega_T)$,

(C6) $S(t,x,Du^\theta) \rightharpoonup \chi$ in $L^{s'(t,x)}(\Omega_T)$ and $L_{t,x}^{r_i'(t)}((0,T)\times\mathcal{B}_{2r}^i)$ for some $\chi \in L^{s'(t,x)}(\Omega_T)$,

(C7) $Du^\theta \rightharpoonup Du$ weakly in $L^{s(t,x)}(\Omega_T)$,

(C8) $\theta\,\nabla_\xi m(|Du^\theta|) \to 0$ in $L^{s_{\max}}(\Omega_T)$,

(C9) $p_1^{i,\theta} \overset{*}{\rightharpoonup} \widetilde{p_1^i}$ in $L_{t,x}^{r_i'(t)}$ and $L^{s'_{\max}}(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}^d\setminus\mathcal{B}_r^i))$,

(C10) $p_2^{i,\theta} \overset{*}{\rightharpoonup} \widetilde{p_2^i}$ in $L_{t,x}^{R_i(t)/2}$ and $L^{s_0/2}(0,T;L^\infty_{\mathrm{loc}}(\mathbb{R}^d\setminus\mathcal{B}_r^i))$, (C11) $p_4^\theta \to 0$ in $L^{s'_{\max}}(\Omega_T)$,

(C12) $p_h^\theta \overset{*}{\rightharpoonup} \widetilde{p_h}$ in $L^{s'_{\max}}(\Omega_T)$ and $L^\infty(0,T;W^{2,\infty}_{loc}(\Omega))$,

(C13) $\nabla p_h^\theta \to \nabla \widetilde{p_h}$ in $L^c(0,T;L^c_{loc}(\Omega))$ for all $c < \infty$,

(C14) $\nabla^2 p_h^\theta \to \nabla^2 \widetilde{p_h}$ in $L^c(0,T;L^c_{loc}(\Omega))$ for all $c < \infty$,

(C15) $D(\nabla p_h^\theta) \to D(\nabla \widetilde{p_h})$ in $L^c(0,T;L^c_{loc}(\Omega))$ for all $c < \infty$,

(C16) $\int_\Omega |(u^\theta + \nabla p_h^\theta)(t,x)|^2 \psi(x)\,\mathrm{d}x \to \int_\Omega |(u + \nabla \widetilde{p_h})(t,x)|^2 \psi(x)\,\mathrm{d}x$ for a.e. $t \in [0,T]$ and $\psi \in C_c^\infty(\Omega)$,

Indeed, the strong convergence in $L^c(\Omega_T)$ in (C2) follows by interpolation: the sequence $\{u^\theta\}$ is bounded in $L^{s_0}(\Omega_T)$ (see (B4)) and $\{u^\theta\}$ is strongly compact in $L^1(\Omega_T)$ by Aubin–Lions lemma 9.8.5 (this uses (B3) and (B13)). Similarly, we obtain (C3) and (C4), this time exploiting uniform bounds of $\{u^\theta\}$ in $L^2(0,T;L^\infty(\Omega))$ and $L^{R_i(t)}((0,T) \times \mathcal{B}^i_{2r})$. To see (C5), we note that (B6) and Körn's inequality imply uniform bound $\{\theta^{1/s_{\max}} \nabla u^\theta\}$ in $L^{s_{\max}}(\Omega_T)$ so that by Sobolev embedding and Dirichlet boundary condition we have uniform bound $\{\theta^{1/s_{\max}} u^\theta\}$ in $L^c(\Omega_T)$ for some $c > s_{\max}$. As $\theta^{1/s_{\max}} u^\theta \to 0$ in $L^1(\Omega_T)$, we conclude by interpolation. Next, convergence results (C6)–(C12) follow from Banach–Alaoglu theorem and estimates (B5), (B2) and (B7)–(B12), respectively. Next, we can use (B3), (B12), (B14) and the Aubin–Lions Lemma (Lemma 9.8.5) to conclude that

$$(u^\theta + \nabla p_h^\theta) \to (u + \nabla \widetilde{p_h}) \qquad \text{in } L^1(0,T;L^1_{loc}(\Omega)).$$

Thus, (C13) follows from (B12). Finally, since p_h^θ is harmonic with respect to the spatial variable, we have that $\|p_h^\theta - \widetilde{p_h}\|_{W^{k,2}(\Omega'')} \leq C(k,\Omega'',\Omega')\|p_h^\theta - \widetilde{p_h}\|_{L^1(\Omega')}$ for all $\Omega'' \Subset \Omega' \subset \Omega$ and all k. Consequently, (C14) and (C15) follow from (C13). The last property (C16) holds true because of the presence of the function ψ having compact support in Ω and thus we can combine (C13) and (C3) and use the classical properties of the Lebesgue spaces.

Now, for each $\theta \in (0,1)$ we use Theorem 9.5.1 to have a distributional formulation without pressure:

$$\int_{\Omega_T} -u^\theta \cdot \partial_t \phi - u^\theta \otimes u^\theta : \nabla\phi + S^\theta(t,x,Du^\theta) : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ = \int_{\Omega_T} f\cdot\phi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\cdot\phi(0,x)\,\mathrm{d}x. \tag{9.6.1}$$

satisfied for all vector-valued $\phi \in C_c^\infty([0,T)\times\Omega)$ fulfilling $\operatorname{div}\phi = 0$. We can let $\theta \to 0$ in (9.6.1) to obtain

$$\int_{\Omega_T} -u \cdot \partial_t \phi - u \otimes u : \nabla\phi + \chi : D\phi \,\mathrm{d}x\,\mathrm{d}t = \\ = \int_{\Omega_T} f\cdot\phi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\cdot\phi(0,x)\,\mathrm{d}x. \tag{9.6.2}$$

The only nontrivial step in the passage to the limit above concerns the stress tensor $S^\theta(t,x,Du^\theta)$. However, by (9.5.1), we may write $S^\theta(t,x,Du^\theta) = S(t,x,Du^\theta) + \theta\nabla_\xi m(|Du^\theta|)$. Then, by (C8), we know that the regularizing term converges in $L^1(\Omega_T)$ which is sufficient to perform the desired passage to the limit $\theta \to 0$.

In view of (9.6.2), the proof of existence of solutions will be concluded if we prove $\chi(t,x) = S(t,x,Du)$.

Step 2: Local energy equalities. Applying Theorem 9.4.1, we also have a distributional formulation with pressure

$$\int_{\Omega_T} -(u^\theta + \nabla p_h^\theta)\cdot\partial_t\phi - u^\theta \otimes u^\theta : \nabla\phi \;+\; S^\theta(t,x,Du^\theta) : D\phi\,\mathrm{d}x\,\mathrm{d}t = \\ = \int_\Omega u_0(x)\cdot\phi(0,x)\,\mathrm{d}x - \int_{\Omega_T}\left(\sum_{i=1}^N (p_1^{i,\theta} + p_2^{i,\theta}) + p_3 + p_4^\theta\right)\operatorname{div}\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} f\cdot\phi\,\mathrm{d}x\,\mathrm{d}t \tag{9.6.3}$$

satisfied for all $\phi \in C_c^\infty([0,T)\times\Omega)$. We can let $\theta\to 0$ in (9.6.3) similarly as above to obtain

$$\int_{\Omega_T} -(u + \nabla\widetilde{p_h})\cdot\partial_t\phi - u \otimes u : \nabla\phi \;+\; \chi : D\phi\,\mathrm{d}x\,\mathrm{d}t = \\ = \int_\Omega u_0(x)\cdot\phi(0,x)\,\mathrm{d}x - \left(\int_{\Omega_T}\sum_{i=1}^N (\widetilde{p_1^i} + \widetilde{p_2^i}) + \widetilde{p_3} + \widetilde{p_4}\right)\operatorname{div}\phi\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} f\cdot\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{9.6.4}$$

On the other hand, we may apply Theorem 9.4.1 directly to (9.6.2). This yields pressures p_1^i, p_2^i, p_3, p_4 and p_h with a distributional formulation as (9.6.4) but with $\widetilde{p_j^i}$, $\widetilde{p_j}$ and $\widetilde{p_h}$ replaced by p_j^i, p_j and p_h respectively. By the uniqueness (linearity) in the Lemma 9.4.2, we obtain $\widetilde{p_j^i} = p_j^i$ and $\widetilde{p_j} = p_j$ almost everywhere. On the other hand, p_h is obtained from the Nečas theorem 9.8.4 uniquely up to the condition

$$\int_\Omega p_h(t,x)\,\mathrm{d}x = 0.$$

But from the weak convergence (C12), the strong convergence (C13) and the Poincaré inequality, we may deduce that for almost all $t \in (0,T)$

$$0 = \lim_{\theta\to 0}\int_\Omega p_h^\theta(t,x)\,\mathrm{d}x = \int_\Omega \widetilde{p_h}(t,x)\,\mathrm{d}x.$$

Hence $\widetilde{p_h} = p_h$. For further reference, we recall local energy equalities obtained from (9.6.1) and (9.6.2) by Theorem 9.4.1. There hold

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u^\theta(t,x) + \nabla p_h^\theta(t,x)|^2\,\psi(x)\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u_0(x)|^2\,\psi(x)\,\mathrm{d}x\\
&\quad + \int_0^t\int_\Omega S^\theta(\tau,x,Du^\theta) : D(\psi(x)(u^\theta + \nabla p_h^\theta)(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau\\
&= + \int_0^t\int_\Omega (u^\theta\otimes u^\theta) : \nabla(\psi(u^\theta + \nabla p_h^\theta)) + f\cdot(u^\theta + \nabla p_h^\theta)\psi\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad - \int_0^t\int_\Omega\left(\sum_{i=1}^N (p_1^{i,\theta} + p_2^{i,\theta}) + p_3 + p_4^\theta\right)(u^\theta + \nabla p_h^\theta)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}\tag{9.6.5}$$

and also

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u(t,x) + \nabla p_h(t,x)|^2\,\psi(x)\,\mathrm{d}x - \frac{1}{2}\int_\Omega |u_0(x)|^2\,\psi(x)\,\mathrm{d}x\\
&\quad + \int_0^t\int_\Omega \chi(\tau,x) : D(\psi(x)(u + \nabla p_h)(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau =\\
&= + \int_0^t\int_\Omega (u\otimes u) : \nabla(\psi(u + \nabla p_h))f\cdot(u + \nabla p_h)\psi\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad - \int_0^t\int_\Omega\left(\sum_{i=1}^N (p_1^i + p_2^i) + p_3\right)(u + \nabla p_h)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}\tau,
\end{aligned}\tag{9.6.6}$$

for a.e. $t \in (0,T)$ and all test functions $\psi \in C_c^\infty(\Omega)$. The idea is to compare (9.6.5) with (9.6.6) in the limit $\theta \to 0$ to identify χ via monotonicity arguments.

Step 3: Limits of the pressure terms $p_j^{i,\theta}$. In this step, we prove for $i = 1, ..., N$ and $j = 1, 2$ and that

$$\int_0^t \int_\Omega p_j^{i,\theta}(u^\theta + \nabla p_h^\theta) \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}\tau \to \int_0^t \int_\Omega p_j^i(u + \nabla p_h) \cdot \nabla\psi \, \mathrm{d}x \, \mathrm{d}\tau \tag{9.6.7}$$

Let $j = 1$ and $i \in \{1, ..., N\}$ be fixed. First, we split the integral for $\Omega \cap \mathcal{B}_{2r}^i$ and $\Omega \setminus \mathcal{B}_{2r}^i$. We treat the resulting terms separately.

- On $\mathcal{B}_{2r}^i$ we have the weak convergence of $p_1^{i,\theta}$ in $L^{r_i'(t)}(\Omega_T)$ so it is sufficient to have strong convergence of $u^\theta + \nabla p_h^\theta$ in $L^{r_i(t)}(0, T; L_{loc}^{r_i(t)}(\Omega))$ thanks to the compact support of ψ. This follows from (C4) and (C13) as $R_i(t) - r_i(t) \geq \frac{s_{\min}}{d}$.

- On $\Omega \setminus \mathcal{B}_{2r}^i$ we use the weak* convergence of $p_1^{i,\theta}$ in $L^{s_{\max}'}(0, T; L^\infty(\Omega))$ from (C9) and local strong convergence of $u^\theta + \nabla p_h^\theta$ in $L^{s_{\max}}(0, T; L_{loc}^1(\Omega))$ from (C3) and (C13).

Now, let $j = 2$ and $i \in \{1, ..., N\}$ be fixed. As above, we split the integral for $\Omega \cap \mathcal{B}_{2r}^i$ and $\Omega \setminus \mathcal{B}_{2r}^i$.

- On $\mathcal{B}_{2r}^i$ we have the weak convergence of $p_2^{i,\theta}$ in $L^{R_i(t)/2}(\Omega_T)$ so it is sufficient to have strong convergence of $u^\theta + \nabla p_h^\theta$ in $L^{(R_i(t)/2)'}(0, T; L_{loc}^{(R_i(t)/2)'}(\Omega))$. However, we have $\left(\frac{R_i(t)}{2}\right)' < R_i(t)$ because $R_i(t) > 3$ (note that we already checked this in Section 9.4 below (9.4.4)). Therefore, the required strong convergence follows from (C4) and (C13).

- On $\Omega \setminus \mathcal{B}_{2r}^i$ we use the weak* convergence of $p_2^{i,\theta}$ in $L^{s_0/2}(0, T; L^\infty(\Omega))$ from (C10) and local strong convergence of $u^\theta + \nabla p_h^\theta$ in $L^{s_0/2}(0, T; L_{loc}^1(\Omega))$ from (C3) and (C13).

Step 4: Limits of the other terms. First, we notice that a direct application of (C16), $f, p_3 \in L^1(0, T; L^2(\Omega))$ (Lemma 9.4.2), (C12), (C13), (C1) and (C2) yields for almost all $t \in (0, T)$

$$\frac{1}{2} \int_\Omega |u^\theta(t, x) + \nabla p_h^\theta(t, x)|^2 \psi(x) \, \mathrm{d}x \to \frac{1}{2} \int_\Omega |u(t, x) + \nabla p_h(t, x)|^2 \psi(x) \, \mathrm{d}x, \tag{9.6.8}$$

$$\int_0^t \int_\Omega f \cdot (u^\theta + \nabla p_h^\theta)\psi \,\mathrm{d}x\,\mathrm{d}\tau \to \int_0^t \int_\Omega f \cdot (u + \nabla p_h)\psi \,\mathrm{d}x\,\mathrm{d}\tau, \tag{9.6.9}$$

$$\int_0^t \int_\Omega p_3(u^\theta + \nabla p_h^\theta) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau \to \int_0^t \int_\Omega p_3(u + \nabla p_h) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau. \tag{9.6.10}$$

Similarly, we also have

$$\int_0^t \int_\Omega p_4^\theta(u^\theta + \nabla p_h^\theta) \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau \to 0 \tag{9.6.11}$$

because we can estimate

$$\left|\int_0^t \int_\Omega p_4^\theta(u^\theta + \nabla p_h^\theta)\nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau\right| \le \|p_4^\theta\,\theta^{-1/s_{\max}}\|_{L_{t,x}^{s'_{\max}}} \|\theta^{1/s_{\max}}(u^\theta + \nabla p_h^\theta)\nabla\psi\|_{L_{t,x}^{s_{\max}}} \to 0$$

due to estimate (B10) and convergences (C5) and (C13). Now we want to prove that

$$\int_0^t \int_\Omega (u^\theta \otimes u^\theta) : \nabla(\psi(u^\theta + \nabla p_h^\theta)) \,\mathrm{d}x\,\mathrm{d}\tau \to \int_0^t \int_\Omega (u \otimes u) : \nabla(\psi(u + \nabla p_h)) \,\mathrm{d}x\,\mathrm{d}\tau. \tag{9.6.12}$$

We split $\psi(u^\theta + \nabla p_h^\theta) = \psi u^\theta + \psi \nabla p_h^\theta$. The convergence for the term ∇p_h^θ is a simple consequence of (C13), (C14) and $u^\theta \to u$ in $L^2(\Omega_T)$ from (C2). Therefore, we focus on $\int_0^t \int_\Omega (u^\theta \otimes u^\theta) : \nabla(\psi u^\theta) \,\mathrm{d}x\,\mathrm{d}\tau$. We easily compute

$$\int_0^t \int_\Omega (u^\theta \otimes u^\theta) : \nabla(\psi u^\theta) \,\mathrm{d}x\,\mathrm{d}\tau = \\ = \frac{1}{2}\int_0^t \int_\Omega |u^\theta|^2\, u^\theta \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau - \frac{1}{2}\int_0^t \int_\Omega \psi \operatorname{div} u^\theta\, |u^\theta|^2 \,\mathrm{d}x\,\mathrm{d}\tau.$$

The first term converges to $\frac{1}{2}\int_0^t \int_\Omega |u|^2\, u \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau$ because $u^\theta \to u$ strongly in $L^3(\Omega_T)$ as in (C2) (note that $s_0 > 3$). The second term vanishes by the incompressibility condition so that we obtain (9.6.12).

Collecting (9.6.7)–(9.6.12), we conclude that for almost all $t \in (0, T)$

$$\limsup_{\theta \to 0} \int_0^t \int_\Omega S^\theta(\tau, x, Du^\theta) : D(\psi(x)(u^\theta + \nabla p_h^\theta)(\tau, x)) \,\mathrm{d}x\,\mathrm{d}\tau \le \\ \le \int_0^t \int_\Omega \chi(\tau, x) : D(\psi(x)(u + \nabla p_h)(\tau, x)) \,\mathrm{d}x\,\mathrm{d}\tau. \tag{9.6.13}$$

Step 5: monotonicity inequality. In this step, we will prove for a.e. $t \in (0,T)$ and $\psi \in C_c^\infty(\Omega)$ we have

$$\limsup_{\theta\to 0} \int_0^t \int_\Omega S(t,x,Du^\theta) : Du^\theta \psi(x) \,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t \int_\Omega \chi(t,x) : Du\, \psi(x)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{9.6.14}$$

We decompose term on the (LHS) of (9.6.13) into six parts X_1, X_2, X_3, X_4, X_5, X_6 as follows:

$$\begin{aligned}
&\int_0^t \int_\Omega S^\theta(\tau,x,Du^\theta) : D(\psi(x)(u^\theta + \nabla p_h^\theta)(\tau,x))\,\mathrm{d}x\,\mathrm{d}\tau = \\
&= \int_0^t \int_\Omega S(\tau,x,Du^\theta) : Du^\theta \psi(x)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega S(\tau,x,Du^\theta) : [D(\nabla p_h^\theta)\,\psi(x)]\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega S(\tau,x,Du^\theta) : [\nabla\psi(x) \otimes (u^\theta + \nabla p_h^\theta)]\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega \theta\, \nabla_\xi m(|Du^\theta|) : Du^\theta\, \psi(x)\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \theta\, \nabla_\xi m(|Du^\theta|) : D(\nabla p_h^\theta)\,\psi(x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad + \int_0^t \int_\Omega \theta\, \nabla_\xi m(|Du^\theta|) : (\nabla\psi(x) \otimes (u^\theta + \nabla p_h^\theta))\,\mathrm{d}x\,\mathrm{d}\tau \\
&=: X_1 + X_2 + X_3 + X_4 + X_5 + X_6.
\end{aligned}$$

Term X_1 is the one we want to estimate. For term X_2, we have

$$\int_0^t \int_\Omega S(\tau,x,Du^\theta) : [D(\nabla p_h^\theta)\psi(x)]\,\mathrm{d}x\,\mathrm{d}\tau \to \int_0^t \int_\Omega \chi(\tau,x) : [D(\nabla p_h)\psi(x)]\,\mathrm{d}x\,\mathrm{d}\tau \tag{9.6.15}$$

because $S(t,x,Du^\theta) \rightharpoonup \chi$ in $L^{s'(t,x)}(\Omega_T)$ so that $S(t,x,Du^\theta) \rightharpoonup \chi$ in $L^{s'_{\max}}(\Omega_T)$ and $D(\nabla p_h^\theta) \to D(\nabla p_h)$ in $L^{s_{\max}}(0,T; L^{s_{\max}}_{loc}(\Omega))$. For X_3 we claim that

$$\int_0^t \int_\Omega S(\tau,x,Du^\theta) : [\nabla\psi \otimes (u^\theta + \nabla p_h^\theta)]\,\mathrm{d}x\,\mathrm{d}\tau \to \int_0^t \int_\Omega \chi(\tau,x) : [\nabla\psi \otimes (u + \nabla p_h)]\,\mathrm{d}x\,\mathrm{d}\tau. \tag{9.6.16}$$

To prove this we write $1 = \sum_{i=1}^N \zeta_i$ where $\{\zeta_i\}$ is the partition of unity from Notation 9.3.3 so that we only need to study the integral

$$\int_0^t \int_\Omega \zeta_i\, S(\tau,x,Du^\theta) : [\nabla\psi \otimes (u^\theta + \nabla p_h^\theta)]\,\mathrm{d}x\,\mathrm{d}\tau.$$

As ζ_i is supported in $\mathcal{B}_r^i$ we can use weak convergence of $S(\tau,x,Du^\theta)$ in $L^{r_i'(t)}(\Omega_T)$ from (C6) and the strong convergence $u^\theta + \nabla p_h^\theta \to u + \nabla p_h$ in $L^{r_i(t)}(0,T; L^{r_i(t)}_{loc}(\Omega))$

from (C13) and (C4) (this uses also $R_i(t) - r_i(t) \geq \frac{s_{\min}}{d}$).

Next, for the terms X_4, X_5, X_6 we have

$$X_4 \geq 0, \qquad X_5 \to 0, \qquad X_6 \to 0. \tag{9.6.17}$$

where the convergence $X_5 \to 0$ follows immiedately from (C8) and estimate (B12). Concerning X_6, the argument is the same as in (9.6.11) because we have exactly the same integrability of $\theta\, \nabla_\xi m(|Du^\theta|)$ as of p_4^θ. Plugging (9.6.15)–(9.6.17) into (9.6.13) we obtain (9.6.14).

Step 6: conclusion by monotonicity trick. By the assumption (T3) we have

$$\int_{\Omega_T} (S(t,x,Du^\theta) - S(t,x,\eta)) : (Du^\theta - \eta)\, \psi(x) \geq 0 \tag{9.6.18}$$

for any $\eta \in L^\infty(\Omega_T)$. Now, let us study limits of two terms appearing in (9.6.18). First, we claim

$$\int_{\Omega_T} S(t,x,\eta) : Du^\theta\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t \to \int_{\Omega_T} S(t,x,\eta) : Du\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t. \tag{9.6.19}$$

Indeed, since $\eta \in L^\infty(\Omega_T)$, then $S(t,x,\eta) \in L^\infty(\Omega_T)$ so that (9.6.19) follows by weak convergence (C7). Second, as a direct consequence of (C6) we have

$$\int_{\Omega_T} S(t,x,Du^\theta) : \eta\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t \to \int_{\Omega_T} \chi(t,x) : \eta\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t. \tag{9.6.20}$$

Hence, using (9.6.14), (9.6.19) and (9.6.20), we may take $\limsup_{\theta \to 0}$ in (9.6.18) to deduce

$$\int_{\Omega_T} (\chi(t,x) - S(t,x,\eta)) : (Du - \eta)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t \geq 0.$$

Using Lemma 8.4.6 (Minty's monotonicity trick), we finally obtain $\chi(t,x) = S(t,x,Du)$ a.e. $\square$

9.7 Appendix A: Poisson equation in $L^p(\mathbb{R}^d)$

The classical theory (see [130, Theorem 1], p. 23) states, that given $f \in C_c^\infty(\mathbb{R}^d)$ the equation

$$-\Delta u = f \text{ on } \mathbb{R}^d, \qquad u(x) \to 0 \text{ as } |x| \to \infty.$$

admits the unique smooth solution given via Newtonian potential

$$u(x) = \Gamma * f, \qquad \Gamma(x) = \begin{cases} -\frac{1}{2\pi} \log|x| & \text{if } d = 2, \\ \frac{1}{d(d-2)\alpha(d)}|x|^{2-d} & \text{if } d > 3, \end{cases}$$

where $\alpha(d)$ is the volume of the unit ball. Here we focus on the theory for $L^p(\mathbb{R}^d)$ spaces:

Theorem 9.7.1. *Let $g \in L^p(\Omega)$ and consider its extension to $\mathbb{R}^d$ with 0. Then, there exists the unique distributional solution to*

$$-\Delta u = \operatorname{div}\operatorname{div} g \ \text{ in } \mathbb{R}^d, \qquad g \in L^p(\mathbb{R}^d).$$

Moreover, $\|u\|_{L^p(\mathbb{R}^d)} \leq C\, \|g\|_{L^p(\Omega)}$.

To prove the theorem, we will need a few simple lemmas.

Lemma 9.7.2 (decay estimates for the Poisson's equation)**.** *Let $f \in C_c^\infty(\mathbb{R}^d)$ and let R be such that supp$f \subset B_R$.*

(A) Let $u = \Gamma * \operatorname{div} f$. Then, there is a constant depending on $\|f\|_{L^1}$ such that for $|x| > 2R$

$$|u(x)| \leq C\,|x|^{1-d}, \qquad |\nabla u| \leq C\,|x|^{-d}.$$

(B) Let $u = \Gamma * f$. Then, there is a constant depending on $\|f\|_{L^1}$ such that for $|x| > 2R$

$$|u(x)| \leq \begin{cases} C\,|x|^{2-d} & \text{if } d > 2 \\ C\,\log||x| - R| & \text{if } d = 2. \end{cases} \qquad |\nabla u| \leq C\,|x|^{1-d}.$$

Proof. First, we consider (A). Let $|x| > 2R$. We observe that

$$\Gamma * \operatorname{div} f(x) = \int_{B_R} \Gamma(x-y)\operatorname{div} f(y)\,\mathrm{d}y = -\int_{B_R} \nabla\Gamma(x-y)\, f(y)\,\mathrm{d}y,$$

where integration by parts is justified because we are away from singularity of Γ and f is compactly supported. Now,

$$|x - y| \geq |x| - |y| \geq |x| - R \geq |x|/2.$$

The conclusion follows because $\nabla\Gamma(x-y)$ is of the form $\frac{C}{|x-y|^{d-1}}$. The estimate on ∇u is proved in exactly the same way: this time we note that second order derivatives of $\Gamma(x-y)$ are of the form $\frac{C}{|x-y|^{d}}$. Finally, the proof of (B) is completely analogous: the only difference is that we cannot pass the divergence operator from f to Γ which results in a worse decay estimates. □

Lemma 9.7.3 (L^p global estimate)**.** *Let $g \in C_c^\infty(\mathbb{R}^d)$ and let $u = \Gamma * \operatorname{div}\operatorname{div} g$. Then, $\|u\|_{L^p(\mathbb{R}^d)} \le C\,\|g\|_{L^p(\mathbb{R}^d)}$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\|\varphi\|_{L^{p'}(\mathbb{R}^d)} \le 1$. Let $\phi_\varphi := \Gamma * \varphi$. Then, $\|D^2\phi_\varphi\|_{L^{p'}} \le C$, cf. [75, Theorem 3.5]. We have

$$\int_{\mathbb{R}^d} u(x)\,\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d} u(x)\,\Delta\phi_\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \nabla u(x)\,\nabla\phi_\varphi(x)\,\mathrm{d}x.$$

The integration by parts is justified here as for large R:

$$\left|\int_{\partial B_R} u\,\nabla\phi_\varphi\cdot\mathbf{n}\,\mathrm{d}S\right| \le C\,R^{1-d}\,R^{1-d}\,R^{d-1} \to 0.$$

and the boundary term disappears. Furthermore,

$$\int_{\mathbb{R}^d} \nabla u(x)\,\nabla\phi_\varphi(x)\,\mathrm{d}x = -\int_{\mathbb{R}^d} \Delta u(x)\,\phi_\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \operatorname{div}\operatorname{div} g(x)\,\phi_\varphi(x)\,\mathrm{d}x.$$

Again, to justify the integration by parts we just compute

$$\left|\int_{\partial B_R} \phi_\varphi\,\nabla u\cdot\mathbf{n}\,\mathrm{d}S\right| \le \begin{cases} \log(R)\,R^{-2}\,R & (\text{ if } d=2) \\ R^{2-d}\,R^{-d}\,R^{d-1} & (\text{ if } d>2) \end{cases} \to 0.$$

Finally, by compact support of g, we can integrate by parts twice to deduce

$$\left|\int_{\mathbb{R}^d} \operatorname{div}\operatorname{div} g(x)\,\phi_\varphi(x)\,\mathrm{d}x\right| \le C\,\|g\|_{L^p(\mathbb{R}^d)}$$

uniformly in φ. The conclusion follows. □

We are in position to prove Theorem 9.7.1.

Proof of Theorem 9.7.1. To see existence, we consider usual mollification g_ε of g and define $u_\varepsilon = \Gamma * g_\varepsilon$. Then, Lemma 9.7.3 gives sufficient bounds to pass to the limit in the distributional formulation. The needed estimate follows from the weak lower-semicontinuity of the norm.

For the uniqueness part assume, that there are two solutions $u_1, u_2 \in L^p(\mathbb{R}^d)$. Then from Weyl's lemma (cf. [247]) $u := u_1 - u_2$ is a harmonic function and so, it is smooth. Then, the mean value property implies

$$|u(x)| \leq \frac{1}{\alpha(d)R^d} \int_{B_R(x)} |u(y)| \,\mathrm{d}y \leq \frac{\|u\|_{L^p(\mathbb{R}^d)}}{\alpha(d)^p} R^{-d/p}.$$

Sending $R \to \infty$ we deduce $u = 0$.

□

9.8 Appendix B: Useful results

Lemma 9.8.1 (Lemma 1.17, Chapter 5, [240])**.** *Suppose that* $v \in L^\infty(0,T;L^2(\Omega)) \cap L^q(0,T;W^{1,q}(\Omega))$ *and* $q \geq 2$*. Then,* $v \in L^{r_0}(\Omega_T)$ *where* $r_0 = q\left(1 + \frac{2}{d}\right)$ *and*

$$\|v\|_{L^{r_0}_{t,x}} \leq C(\|v\|_{L^\infty_t L^2_x}, \|v\|_{L^q_t W^{1,q}_x}).$$

Lemma 9.8.2. *Let* $v \in L^1(\Omega_T)$ *and* η_ε *be as in Definition 8.3.2. Then*

$$\int_{B_\varepsilon(0)} \int_{B_\varepsilon(0)} v(t, x - y - z)\, \eta_\varepsilon(y)\, \eta_\varepsilon(z) \,\mathrm{d}y \,\mathrm{d}z \to v(t,x) \text{ in } L^1(\Omega_T).$$

Lemma 9.8.3. *Let* $f \in L^2(\Omega)$ *and* $\psi \in L^\infty(\Omega)$*. Then*

$$\int_\Omega |f^\varepsilon(x)|^2\, \psi(x) \,\mathrm{d}x \to \int_\Omega |f(x)|^2\, \psi(x) \,\mathrm{d}x.$$

Lemma 9.8.4. (Nečas theorem about negative norms [250, Lemma 2.2.2]) *Let* $\Omega \subset \mathbb{R}^d$*,* $d \geq 2$*, be a bounded Lipchitz domain, let* $1 < q < \infty$*. Suppose* $f \in W^{-1,q}(\Omega)^d$ *satisfies*

$$f(v) = 0 \text{ for all } v \in C_0^\infty(\Omega),\ \operatorname{div} v = 0$$

Then there exists a unique $p \in L^q(\Omega)$ *satisfying*

$$\int_\Omega p \,\mathrm{d}x = 0, \qquad f = \nabla p$$

in the sense of distributions. Moreover,

$$\|p\|_{L^q_x} \leq C \|f\|_{W^{-1,q}(\Omega)^d} \tag{9.8.1}$$

with some constant $C = C(q, \Omega_0, \Omega) > 0$.

We remark that $W^{-1,q}(\Omega)$ is the dual space of $W^{1,q'}(\Omega)$.

Lemma 9.8.5. (Generalized Aubin–Lions lemma, [239, Lemma 7.7]) *Denote by*

$$W^{1,p,q}(I; X_1, X_2) := \left\{ u \in L^p(I; X_1); \frac{du}{dt} \in L^q(I; X_2) \right\}$$

Then if X_1 is a separable, reflexive Banach space, X_2 is a Banach space and X_3 is a metrizable locally convex Hausdorff space, X_1 embeds compactly into X_2, X_2 embeds continuously into X_3, $1 < p < \infty$ and $1 \leq q \leq \infty$, we have

$$W^{1,p,q}(I; X_1, X_3) \text{ embeds compactly into } L^p(I; X_2)$$

In particular any bounded sequence in $W^{1,p,q}(I; X_1, X_3)$ has a convergent subsequence in $L^p(I; X_2)$.

Chapter 10

New results on the absence of Lavrentiev phenomenon for double phase functionals

The results in this chapter have been published in:

- M. Bulíček, P. Gwiazda, J. Skrzeczkowski. *On a range of exponents for absence of Lavrentiev phenomenon for double phase functionals.* Archive for Rational Mechanics and Analysis, 246, 209–240, 2022, cited as [57].

10.1 Introduction and the main result

We conclude the thesis with the result in calculus of variations that is based on our methods in Chapter 8. While this area does not fit to the topic of the thesis, we want to show that the presented methods has much wider applications. To simplify the presentation and omit technical details, the main result will be formulated and proved only in the simplest case. For the most general case we refer to [57].

We consider a class of functionals with the so-called (p, q)-growth. The prominent example we have in mind is

$$\mathcal{G}(u) := \int_{\Omega} |\nabla u(x)|^p \,\mathrm{d}x + \int_{\Omega} a(x)\,|\nabla u(x)|^q \,\mathrm{d}x. \tag{10.1.1}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $u : \Omega \to \mathbb{R}$ is an argument of the functional $\mathcal{G}$, $a : \Omega \to [0, \infty)$ is a given nonnegative and continuous function and $1 \le p < q < \infty$ are given numbers. Functional $\mathcal{G}$ is an interesting toy model for studying minimisation of functionals with the so-called non-standard growth. Indeed, depending on whether $a = 0$ or $a > 0$, $\mathcal{G}$ exhibits either the p- or the q-growth.

Roughly speaking, when q is close to p, functional $\mathcal{G}$ enjoys all usual properties of the functionals with standard growth like p-Dirichlet energy $\int_\Omega |\nabla u|^p \,\mathrm{d}x$, for instance de Giorgi-Nash-Moser theory implying local Hölder regularity of the minimizers, see [29, Chapter 3] and [183]. One of the well-known new features that appear in the case of $\mathcal{G}$ is the so-called Lavrentiev phenomenon. For instance, there exists a function $a \in C^\alpha(\overline{\Omega})$ with $\alpha \in (0,1)$, exponents p, q fulfilling $p < d < d + \alpha < q$ and boundary data $u_0 \in W^{1,q}(\Omega)$ such that

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{G}(u) < \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{G}(u). \tag{10.1.2}$$

Results of this type are of great importance as they imply that minimizers are not smooth (they do not even belong to $W^{1,q}(\Omega)$). Consequently, all typical results of calculus of variations, including de Giorgi-Nash-Moser theory, are simply not available. They are also usually the first step to prove regularity of the minimizers as they allow to approximate the minimizer with a sequence of smooth functions and write the related Euler-Lagrange equation, see [82, 83].

On the other hand, it is known that if $q \le p + \alpha\, \frac{p}{d}$, the Lavrentiev phenomenon does not occur for the toy model (10.1.1), see [127]. Under the additional assumption $u_0 \in L^\infty(\Omega)$, the range of exponents has been improved to $q \le p + \alpha$ [82, Proposition 3.6, Remark 5]. The latter work heavily depends on the properties of minimizers and the L^∞ bound for the minimizer of the functional (10.1.1) form a nontrivial part of the result in [82].

In this chapter we prove that neither the assumption $u_0 \in L^\infty(\Omega)$ nor any additional property of minimizer (higher integrability, continuity) is irrelevant for the absence of Lavrentiev phenomenon. More precisely, in the case $\Omega = B$ (unit ball), we prove that one does not observe Lavrentiev phenomenon if

$$q \leq p + \alpha \max\left(1, \frac{p}{d}\right) \tag{10.1.3}$$

and boundary data $u_0 \in W^{1,q}(\Omega)$. In this case, we have

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{G}(u) = \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{G}(u) = \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{G}(u). \tag{10.1.4}$$

This significantly improves the available results for the case $p < d$. Moreover, our proof is elementary as it is based on a simple regularisation argument together with Young's convolution inequality. In particular, we do not use estimates on minimizers of functional (10.1.1). Consequently, our method easily extends to the vector-valued maps and cover variable-exponent functionals as well, see Section 10.4 and the full paper [57].

The question of whether (10.1.2) or (10.1.4) holds true is related to the density of $C_c^\infty(\Omega)$ in the Musielak–Orlicz–Sobolev space $W_0^{1,\psi}(\Omega)$ corresponding to the functional (10.1.1), see (10.2.2)–(10.2.4) for definitions. In this context, we prove that the density result hold true for p, q satisfying (10.1.3) which is again better then so-far known regime of exponents announced in [8].

Finally, we want to point out and emphasize the main novelties of the paper. Standard methods [83, 126] for proving (10.1.4) are based on regularization of arbitrary function $u \in W_0^{1,p}(\Omega)$ satisfying $\mathcal{G}(u) < \infty$ with a sequence of smooth functions $u^\varepsilon = u * \eta_\varepsilon$ and passing to the limit $\mathcal{G}(u^\varepsilon) \to \mathcal{G}(u)$ as $\varepsilon \to 0$. The latter is not trivial because the integrand in (10.1.1) is x-dependent. More precisely, if the integrand is convex and autonomous (i.e. it does not depend on x) one can use Jensen's inequality and Vitali convergence theorem to prove that $\mathcal{G}(u^\varepsilon) \to \mathcal{G}(u)$ whenever $\mathcal{G}(u) < \infty$. In particular, there is no Lavrentiev phenomenon in this case, see also [42].

The strategy to deal with the non-autonomous case is to approximate locally the integrand with autonomous function that does not depend on x (see Lemma 10.3.5) so that one can exploit Jensen's inequality. The approximation requires good estimate on $\|\nabla u^\varepsilon\|_\infty$ which results in constraint on exponents p and q. The estimate on gradient is obtained by writing $\nabla u^\varepsilon = \nabla u * \eta_\varepsilon$ and using the fact that $\nabla u \in L^p(\Omega)$. Our main contribution is an observation that it is sufficient to approximate only bounded functions u (i.e. $u \in L^\infty(\Omega)$). It turns out that for $p < d$, it is a better strategy to write $\nabla u^\varepsilon = u * \nabla \eta_\varepsilon$ and exploit the estimate $u \in L^\infty(\Omega)$ rather that $\nabla u \in L^p(\Omega)$.

Let us discuss our results within the context of previous works related to this topic. The first studies concerning functionals changing their ellipticity rate at each point have been carried out by Zhikov [269,270,271,272]. In particular, in [271] he observed that it may happen that (10.1.4) does not hold, extending thus similar observations made by Lavrentiev [188] and Mania [200]. As discussed in Chapter 1, the example of Mania shows that lack of Lavrentiev phenomenon is important for numerical approximation of minimizers.

Another related direction of research is the regularity of minimizers. Although the fundamental results for minimizers were obtained by Marcellini [203, 204, 205, 206] more than 20 years ago, it is in fact still an active topic of research and the results in this area are published in major mathematical journals, see for instance [21, 25, 26, 30, 43, 61, 82, 83, 94, 210, 211, 216, 226, 245].

Going back to the functional (10.1.1), the available results for boundary data $u_0 \in W^{1,q}(\Omega)$ provide both positive and negative answers to the question whether (10.1.4) holds true. On the one hand, if $q \le p + \frac{p\,\alpha}{d}$ then (10.1.4) is indeed valid [126, 127]. On the other hand, if $q > p + \alpha \max\left(1, \frac{p-1}{d-1}\right)$ then counterexample in [20, Theorem 34] shows that (10.1.4) is violated (see also [127, Lemma 7] for a weaker result concerning the case $p < d < d + \alpha < q$ obtained with more elementary methods). In this paper we establish (10.1.4) for $q \le p + \alpha \, \max\left(1, \frac{p}{d}\right)$ which partially fills the gap between currently known positive and negative results concerning the Lavrentiev phenomenon. Moreover, in view of [20, Theorem 34], our result is the first sharp result for $p \leq d$.

Next, we wish to address two issues that appeared in previous papers on this topic. First, in [83, Lemma 4.1] there is the following claim: for every $\varepsilon > 0$ and ball $B_r(x) \subset \Omega$, there exists $p_\varepsilon < q_\varepsilon$ satisfying

$$\varepsilon\, p_\varepsilon > q_\varepsilon - p_\varepsilon - \alpha_\varepsilon \frac{p_\varepsilon}{d} > 0, \tag{10.1.5}$$

a coefficient $a_\varepsilon \in C^\alpha(\overline{\Omega})$ and a boundary data $u_0 \in W^{1,q}(B_r(x)) \cap L^\infty(B_r(x))$ such that

$$\inf_{u \in u_0 + W_0^{1,p_\varepsilon}(B_r(x))} \mathcal{G}(u) < \inf_{u \in u_0 + W_0^{1,p_\varepsilon}(B_r(x)) \cap W^{1,q_\varepsilon}_{\mathrm{loc}}(B_r(x))} \mathcal{G}(u).$$

Although it is a very nice result, it does not prove that range of exponents $q \leq p + \alpha\, \frac{p}{d}$ is optimal for absence of the Lavrentiev phenomenon and it does not contradict our result about the range stated in (10.1.3). In fact, authors refer to the counterexample from [127] constructed for exponents satisfying $p < d < d + \alpha < q$ i.e. exponents that do not meet our range because the distance between p and q is greater than α. In fact, it is shown that there exists p_ε and q_ε but it follows also from the proof that they are constructed in the following way: for $\delta > 0$ to be specified later, we define $p_\varepsilon := d - \delta$, $q_\varepsilon := d + \alpha + \delta$ and find a proper counterexample constructed in [127]. Then, when $p_\varepsilon \geq 1$, we have

$$\varepsilon\, p_\varepsilon \geq \varepsilon, \qquad q_\varepsilon - p_\varepsilon - \alpha_\varepsilon \frac{p_\varepsilon}{d} = 2\,\delta + \alpha\, \frac{\delta}{d} = \delta\, \left(2 + \frac{\alpha}{d}\right)$$

so that (10.1.5) is satisfied if we let $\delta := \frac{\varepsilon}{2\,(2+\alpha/d)}$. Consequently, $p_\varepsilon \to d$ as $\varepsilon \to 0$, which is in perfect coincidence with (10.1.3).

Second, we also want compare our result with [82], where authors proved that the Lavrentiev phenomenon is not observed for $q \leq p + \alpha$ in the particular cases when minimizers of (10.1.1) are bounded, but this requires an extra assumption on the boundary data, namely that the boundary data u_0 is bounded and apply the maximum principle [190]. In addition, reasoning in [82] is based on the so-called Morrey type estimate on the gradient of minimizer which is not an obvious result itself. Comparing to our work, we prove that the Lavrentiev phenomenon does not occur independently of the properties of minimizers or boundedness of boundary data. Our methods are elementary and are based on simple estimates on convolutions. We point out that one could naively think that our result is a consequence of [82] and a simple approximation argument (boundary data $u_0 \in W^{1,q}(\Omega)$ is approximated with a sequence $\{u_{0,n}\} \subset W^{1,q}(\Omega) \cap L^{\infty}(\Omega)$) but it is not necessarily true that sequence of minimizers has then a subsequence converning again to a minimizer of the limit problem.

Finally, we want to point out that such functionals as (10.1.1) appear in mathematical hyperelasticity theory which studies the stress-strain behaviour of the hyperelastic materials. In this theory, the optimal transformation of the material $u : \Omega \to \Omega$ is given by the minimizer of the functional

$$\int_{\Omega} (W(\nabla u) - u\, f)\, \mathrm{d}x$$

where W is *the stored-energy density* and f is the external force-field. There are many choices for W but one possible could be of power-type $W(\nabla u) = |\nabla u|^p$ with p describing hardening properties of the material. In this context, functional $\mathcal{G}$ corresponds to the composite of two different materials with different hardening properties while the coefficient a describes the fraction of one material in the composite. For more details on calculus of variations in hyperelasticity we refer to [236, Chapter 1.7] and [81].

10.2 Musielak–Orlicz–Sobolev spaces and absence of Lavrentiev phenomenon

In this section we prove that the Lavrentiev phenomenon does not occur if smooth compactly supported functions are dense in the related Musielak-Orlicz-Sobolev space. As we will be interested in smaller class of N-functions than in Chapters

7 and 8, we introduce simplified notation. For $1 \leq p < q$, we define the N-function

$$\psi(x, \xi) = |\xi|^p + a(x)\,|\xi|^q. \tag{10.2.1}$$

For $f : \Omega \to \mathbb{R}^d$ such that $\int_\Omega \psi(x, |f(x)|)\,\mathrm{d}x < \infty$, we define the related Luxembourg norm with

$$\|f\|_\psi = \inf\left\{\lambda > 0 : \int_\Omega \psi\left(x, \frac{|f(x)|}{\lambda}\right)\mathrm{d}x \leq 1\right\}. \tag{10.2.2}$$

Finally, the Musielak–Orlicz–Sobolev spaces are defined as

$$W^{1,\psi}(\Omega) = \{w \in W^{1,1}(\Omega) : \|\nabla w\|_\psi < \infty\}, \qquad W_0^{1,\psi}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\psi}(\Omega), \tag{10.2.3}$$

the latter one corresponds to the space of functions vanishing at the boundary. These are normed spaces with norm

$$\|w\|_{1,\psi} = \|w\|_1 + \|\nabla w\|_\psi. \tag{10.2.4}$$

One can think of $W^{1,\psi}(\Omega)$ as the space of functions having gradient integrable with p or q power depending on whether $a = 0$ or not.

Remark 10.2.1. For $p = 1$, ψ is not an N-function as defined in Definition 7.1.2 as it does not satisfy (M4). However, this condition is only important for characterization of certain dual spaces (see Remark 7.1.6) and such results are not used in this chapter. The most important will be Δ_2 condition (7.3.1) and its consequences stated in Lemma 7.3.1 which does not require (M4).

Next two lemmas show that to prove the absence of the Lavrentiev phenomenon, it is sufficient to demonstrate that every $u \in W_0^{1,\psi}(\Omega) \cap L^\infty(\Omega)$ can be approximated in the topology of $W^{1,\psi}$ by smooth functions from $C_c^\infty(\Omega)$.

Lemma 10.2.2. *Space $W_0^{1,\psi}(\Omega) \cap L^\infty(\Omega)$ is dense in $W_0^{1,\psi}(\Omega)$.*

Proof. Let $u \in W_0^{1,\psi}(\Omega)$. Consider truncation of u defined as

$$T_k(u) = \begin{cases} u & \text{if } |u| \leq k, \\ k\,\frac{u}{|u|} & \text{if } |u| > k. \end{cases} \tag{10.2.5}$$

Clearly, $T_k(u) \in L^\infty(\Omega)$. Moreover, chain rule for Sobolev maps implies that $\nabla T_k(u) = \nabla u \, \mathbb{1}_{|u|\leq k}$ so that $\nabla T_k(u) \to \nabla u$ a.e. as $k \to \infty$. As $\psi(x,0) = 0$, we have

$$0 \leq \psi(x, |\nabla T_k(u)|) = \psi(x, |\nabla u|) \, \mathbb{1}_{|u|\leq k} \leq \psi(x, |\nabla u|)$$

so that the sequence $\{\psi(x, |\nabla T_k(u))|\}$ is uniformly integrable. Application of (C4) from Lemma 7.3.1 concludes the proof. $\square$

Lemma 10.2.3. *Suppose that for every $u \in W_0^{1,\psi}(\Omega) \cap L^\infty(\Omega)$ there exists a sequence $\{u^n\} \subset C_c^\infty(\Omega)$ such that $\|u^n - u\|_{1,\psi} \to 0$ as $n \to \infty$. Then, the space $C_c^\infty(\Omega)$ is dense in $W_0^{1,\psi}(\Omega)$ and the Lavrentiev phenomenon does not occur, i.e., for all $u_0 \in W^{1,q}(\Omega)$ we have*

$$\inf_{u \in u_0 + W_0^{1,p}(\Omega)} \mathcal{G}(u) = \inf_{u \in u_0 + W_0^{1,q}(\Omega)} \mathcal{G}(u) = \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{G}(u).$$

Proof. Thanks to Lemma 10.2.2, $C_c^\infty(\Omega)$ is dense in $W_0^{1,\psi}(\Omega)$. Let $u^* \in W^{1,p}(\Omega)$ be the minimizer of $\mathcal{G}$ i.e.

$$\inf_{u \in u_0 + W^{1,p}(\Omega)} \mathcal{G}(u) = \mathcal{G}(u^*).$$

The minimizer exists by a usual application of direct method in calculus of variations, cf. [236, Theorem 2.7]. Note that we always have

$$\mathcal{G}(u^*) = \inf_{u \in u_0 + W^{1,p}(\Omega)} \mathcal{G}(u) \leq \inf_{u \in u_0 + W^{1,q}(\Omega)} \mathcal{G}(u) \leq \inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{G}(u)$$

because $p < q$. To prove the reversed inequality, we write $u^* = u_0 + \overline{u}$ where $\overline{u} \in W_0^{1,p}$. Note that $u_0 \in W^{1,\psi}(\Omega)$ (because $W^{1,q}(\Omega) \subset W^{1,\psi}(\Omega)$) and $u^* \in W^{1,\psi}(\Omega)$ (because $\mathcal{G}(u^*) < \infty$ cf. Lemma 7.3.1 (C1)). It follows that $\overline{u} = u^* - u_0 \in W_0^{1,\psi}(\Omega)$. Now, consider the sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that $u_n \to \overline{u}$ in $W^{1,\psi}(\Omega)$ which exists due to the assumptions. It follows that $u_n + u_0 \to \overline{u} + u_0 = u^*$ in $W^{1,\psi}(\Omega)$. In particular, $\mathcal{G}(u_0 + u_n) \to \mathcal{G}(u^*)$ cf. Lemma 7.3.1 (C3). Note that $u_0 + u_n \in u_0 + C_c^\infty(\Omega)$. It follows that

$$\inf_{u \in u_0 + C_c^\infty(\Omega)} \mathcal{G}(u) \leq \mathcal{G}(u_0 + u_n) \to \mathcal{G}(u^*) \qquad \text{as } n \to \infty.$$

$\square$

10.3 Detailed proof in the special case

In this section we prove the result in the case when $\Omega = B$ (unit ball centered at 0). The corresponding functional then takes the following form

$$\mathcal{G}(u) := \int_B |\nabla u(x)|^p + a(x)\,|\nabla u(x)|^q\,\mathrm{d}x = \int_B \psi(x, \nabla u(x))\,\mathrm{d}x.$$

We start with introducing mollification that will be used to define the approximation of functions in $W_0^{1,\psi}(B)$.

Definition 10.3.1 (Mollification with squeezing)**.** *For $\varepsilon \in (0, 1/4)$ we set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right)$ where η is a usual mollification kernel. Then, for arbitrary $u : \mathbb{R}^d \to \mathbb{R}$, we define $u^\varepsilon : \mathbb{R}^d \to \mathbb{R}$ as*

$$u^\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(y)\, u\left(\frac{x}{1-2\varepsilon} - y\right) \mathrm{d}y.$$

The main result reads:

Theorem 10.3.2 (absence of Lavrentiev phenomenon)**.** *Let $u \in W_0^{1,\psi}(B) \cap L^\infty(B)$ with $a \in C^\alpha(\overline{B})$. Suppose that*

$$1 \le p < q \le p + \alpha \max\left(1, \frac{p}{d}\right).$$

Consider sequence u^ε as in Definition 10.3.1 with $\varepsilon \in \left(0, \frac{1}{4}\right)$. Then,

(E1) $u^\varepsilon \in C_c^\infty(B)$,

(E2) $\mathcal{G}(u^\varepsilon) \to \mathcal{G}(u)$ as $\varepsilon \to 0$,

(E3) $u^\varepsilon \to u$ in $W^{1,\psi}(B)$ as $\varepsilon \to 0$,

(E4) $C_c^\infty(B)$ is dense in $W_0^{1,\psi}(B)$ and Lavrentiev phenomenon does not occur, i.e. for all boundary data $u_0 \in W^{1,q}(B)$

$$\inf_{u \in u_0 + W_0^{1,p}(B)} \mathcal{G}(u) = \inf_{u \in u_0 + W_0^{1,q}(B)} \mathcal{G}(u) = \inf_{u \in u_0 + C_c^\infty(B)} \mathcal{G}(u).$$

To prove Theorem 10.3.2 we need a series of auxillary results.

Lemma 10.3.3. *Let $u \in W_0^{1,1}(B)$ and be extended by zero onto $\mathbb{R}^d$. Then, $u^\varepsilon \in C_c^\infty(B)$. Moreover, $\frac{x}{1-2\varepsilon} - y \in B_{5\varepsilon}(x)$ for all y such that $|y| \le \varepsilon$.*

Proof. Smoothness follows from standard properties of convolutions cf. [130, Appendix C.4]. To see the compact support, let $|x| \geq 1-\varepsilon$ and $|y| \leq \varepsilon$. Then,

$$\left|\frac{x}{1-2\,\varepsilon} - y\right| \geq \frac{1-\varepsilon}{1-2\,\varepsilon} - \varepsilon = \frac{1-\varepsilon}{1-2\,\varepsilon} - \frac{\varepsilon - 2\varepsilon^2}{1-2\,\varepsilon} = \frac{1-2\,\varepsilon+2\,\varepsilon^2}{1-2\,\varepsilon} = 1 + \frac{2\,\varepsilon^2}{1-2\,\varepsilon} > 1$$

so that $u\left(\frac{x}{1-2\varepsilon} - y\right) = 0$. It follows that u^ε is supported in $B_{1-\varepsilon}$. To see the second property, we estimate

$$\left|x - \frac{x}{1-2\varepsilon} + y\right| \leq |x|\frac{2\varepsilon}{1-2\varepsilon} + |y| \leq 4\varepsilon + \varepsilon = 5\varepsilon,$$

where we used $\frac{1}{1-2\,\varepsilon} \leq 2$, i.e. $\varepsilon \leq \frac{1}{4}$. □

Lemma 10.3.4. *Let $u \in W_0^{1,1}(B)$ be such that $\mathcal{G}(u) < \infty$ and consider its extension to $\mathbb{R}^d$. Then,*

(D1) $\psi\left(\frac{x}{1-2\,\varepsilon}, |\nabla u|\left(\frac{x}{1-2\,\varepsilon}\right)\right) \to \psi(x, |\nabla u(x)|)$ *in* $L^1(\mathbb{R}^d)$,

(D2) $\int_{\mathbb{R}^d} \psi\left(\frac{x}{1-2\,\varepsilon} - y, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right) \eta_\varepsilon(y)\,\mathrm{d}y \to \psi\left(x, |\nabla u|(x)\right)$ *in* $L^1(\mathbb{R}^d)$.

Proof. To see (D1), we note that the convergence holds in the pointwise sense. Moreover, the considered sequence is supported only for $x \in B_{1-2\varepsilon}$. Therefore, to establish convergence in $L^1(\mathbb{R}^d)$, it is sufficient to prove equiintegrability of the sequence $\left\{\psi\left(\frac{x}{1-2\,\varepsilon}, |\nabla u|\left(\frac{x}{1-2\,\varepsilon}\right)\right)\right\}_\varepsilon$ and apply the Vitali convergence theorem. To this end, we need to prove

$$\forall_{\eta>0}\,\exists_{\delta>0}\,\forall_{A\subset B, |A|\leq\delta} \qquad \int_A \psi\left(\frac{x}{1-2\,\varepsilon}, |\nabla u|\left(\frac{x}{1-2\,\varepsilon}\right)\right)\mathrm{d}x \leq \eta.$$

We fix η and arbitrary $A \subset B$. Using change of variables we have

$$\begin{aligned}\int_A \psi\left(\frac{x}{1-2\,\varepsilon}, |\nabla u|\left(\frac{x}{1-2\,\varepsilon}\right)\right)\mathrm{d}x &= \\ &= (1-2\,\varepsilon)^d \int_{A/(1-2\,\varepsilon)} \psi(x, |\nabla u|(x))\,\mathrm{d}x \leq \int_{2A} \psi(x, |\nabla u|(x))\,\mathrm{d}x,\end{aligned}$$

where for $c \in \mathbb{R}^+$, cA denotes a usual scaled set. By assumptions we have $\mathcal{G}(u) < \infty$, so that if we set

$$\omega(\tau) := \sup_{C\subset\mathbb{R}^d: |C|\leq\tau} \int_C \psi(x, |\nabla u|(x))\,\mathrm{d}x,$$

then $\omega(\tau)$ is a non-decreasing function, continuous at 0. Therefore, we may find τ such that $\omega(\tau) \le 2^{-q}\,\eta$. Then, we choose $\delta = 2^{-d}\,\tau$ to conclude the proof of (D1). Finally, the convergence result (D2) follows from Young's convolutional inequality and (D1). $\square$

Lemma 10.3.5. *Let φ be given by* (10.2.1)*. Then for all balls $B_\gamma(x)$ such that $\overline{B_\gamma(x)} \cap \overline{B}$ is nonempty, there exists $x^* \in \overline{B_\gamma(x)} \cap \overline{B}$ such that for all ξ*

$$\inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} \psi(y,\xi) = \psi(x^*,\xi).$$

Proof. Using continuity of a and compactness of $\overline{B_\gamma(x)} \cap \overline{B}$ we have

$$\inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} \psi(y,\xi) = \inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} [|\xi|^p + a(y)\,|\xi|^q] = |\xi|^p + |\xi|^q \inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} a(y)$$

and we choose y^* such that $\inf_{y \in \overline{B_\gamma(x)} \cap \overline{B}} a(y) = a(y^*)$. $\square$

Lemma 10.3.6. *Let $D > 0$. There exists constants M, N depending possibly on p, q and D such that for all $\gamma \in \left(0, \frac{1}{2}\right)$, all $\xi \in \left(0, D\,\gamma^{-\min\left(1,\frac{d}{p}\right)}\right)$, all $x \in \overline{B}$ and all $y, z \in \overline{B_\gamma(x)} \cap \overline{B}$ we have*

$$\psi(z,\xi) \le M\,\psi(y,\xi) + N.$$

Proof. First, we may assume that $\xi > 1$ as for $\xi \in [0,1]$ we have

$$\psi(x,\xi) \le 1 + \|a\|_\infty \le 1 + \|a\|_\infty + \psi(y,\xi) \tag{10.3.1}$$

so the assertion follows with $M = 1$ and $N = 1 + \|a\|_\infty$. Hence, we fix $\xi > 1$ and some ball $B_\gamma(x)$ such that $B_\gamma(x) \cap \overline{\Omega}$ is not empty. As $a \in C^\alpha(\overline{B})$ we have

$$|\psi(z,\xi) - \psi(y,\xi)| \le |a|_\alpha\,|\xi|^q\,|z-y|^\alpha$$

which implies $y, z \in \overline{B_\gamma(x)} \cap \overline{\Omega}$:

$$\psi(z,\xi) \ge \psi(y,\xi) - |a|_\alpha\,|\xi|^q\,\gamma^\alpha.$$

To bootstrap this estimate, we fix $\delta \in (0,1)$ and write

$$\psi(z,\xi) = \delta\,\psi(z,\xi) + (1-\delta)\,\psi(z,\xi) \ge \delta\,\psi(y,\xi) - \delta\,|a|_\alpha\,|\xi|^q\,\gamma^\alpha + (1-\delta)\,|\xi|^p, \tag{10.3.2}$$

where we used to estimate the first term and lower bound $\psi(z,\xi) \geq |\xi|^p$ to estimate the second term. Now, we may write

$$\delta\,|a|_\alpha\,|\xi|^q\,\gamma^\alpha = \delta\,|a|_\alpha\,|\xi|^{q-p}\,|\xi|^p\,\gamma^\alpha \leq \delta\,|a|_\alpha\,D^{q-p}\,\gamma^{\alpha-(q-p)\,\min\left(1,\,\frac{d}{p}\right)}\,|\xi|^p, \tag{10.3.3}$$

where we used $|\xi| \leq D\,\gamma^{-\min\left(1,\,\frac{d}{p}\right)}$. As $q-p \leq \alpha\,\max\left(1,\frac{p}{d}\right)$, we have

$$\alpha-(q-p)\,\min\left(1,\,\frac{d}{p}\right) \geq \alpha-\alpha\,\max\left(1,\frac{p}{d}\right)\,\min\left(1,\,\frac{d}{p}\right) = \alpha-\alpha = 0.$$

It follows that $\gamma^{\alpha-(q-p)\,\min\left(1,\,\frac{d}{p}\right)} \leq 1$ for $\gamma \in \left(0,\frac{1}{2}\right)$. Hence, coming back to (10.3.2) we obtain

$$\begin{aligned}\psi(z,\xi) \geq \delta\,\psi(y,\xi) - \delta\,|a|_\alpha\,D^{q-p}\,|\xi|^p + (1-\delta)\,|\xi|^p = \\ = \delta\,\psi(y,\xi) + \left((1-\delta)-\delta\,|a|_\alpha\,D^{q-p}\right)\,|\xi|^p.\end{aligned}$$

We choose $\delta = \frac{1}{1+|a|_\alpha\,D^{q-p}}$ so that $\left((1-\delta)-\delta\,|a|_\alpha\,D^{q-p}\right)\,|\xi|^p = 0$. Hence, for all $y,z \in \overline{B_\gamma(x)} \cap \overline{\Omega}$

$$\psi(z,\xi) \geq \delta\,\psi(y,\xi)$$

so combining with (10.3.1), the proof is concluded with $M = \max(1/\delta,1)$ and $N = C_2\,(1+|\xi_0|^q)$. □

Proof of Theorem 10.3.2. The first property follows from construction. To prove the convergence, we note that

$$\mathcal{G}\,(u^\varepsilon) = \int_B \psi(x,|\nabla u^\varepsilon|\,(x))\,\mathrm{d}x.$$

We would like to take mollification out of the function φ using its convexity and Jensen's inequality. However, this is not possible as function φ depends also on x explicitly. To overcome this problem we use Lemmata 10.3.5 and 10.3.6 to approximate $\psi(x,\xi) \approx \psi(x^*,\xi)$, apply Jensen's inequality. Finally, we use the fact that $\psi(x^*,\xi) \leq \psi(x,\xi)$ as the map $y \mapsto \psi(y,\xi)$ attains locally minimum at $y = x^*$.

Case 1: $p \leq d$. In this case we have $q \leq p+\alpha$. Using Young's convolution inequality we obtain:

$$\|\nabla u^\varepsilon\|_\infty \leq \|u\|_\infty\,\|\nabla\eta_\varepsilon\|_1 \leq D\,(5\varepsilon)^{-1}, \tag{10.3.4}$$

where we choose $D := 5\,\|u\|_\infty\,\|\nabla\eta\|_1$. Let $x \in B$. Applying Lemma 10.3.6 with $\gamma = 5\,\varepsilon$ and Lemma 10.3.5 we obtain $x^* \in \overline{B_{5\varepsilon}(x)} \cap \overline{B}$ and constants M, N such that

$$\psi(x, |\nabla u^\varepsilon|(x)) \leq M\,\psi(x^*, |\nabla u^\varepsilon|(x)) + N. \tag{10.3.5}$$

Note that

$$\begin{aligned}
\psi(x^*, |\nabla u^\varepsilon(x)|) &= \psi\left(x^*, \frac{1}{1-2\,\varepsilon}\left|\int_{B_\varepsilon} \nabla u\left(\frac{x}{1-2\,\varepsilon} - y\right)\,\eta_\varepsilon(y)\,\mathrm{d}y\right|\right) \leq \\
&\leq \left(\frac{1}{1-2\,\varepsilon}\right)^q \psi\left(x^*, \int_{B_\varepsilon} |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\,\eta_\varepsilon(y)\,\mathrm{d}y\right) \\
&\leq 2^q\,\psi\left(x^*, \int_{B_\varepsilon} |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\,\eta_\varepsilon(y)\,\mathrm{d}y\right),
\end{aligned}$$

where we used that φ is of the form (10.2.1). Then, Jensen's inequality implies

$$\begin{aligned}
&\psi\left(x^*, \int_{B_\varepsilon} |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\,\eta_\varepsilon(y)\,\mathrm{d}y\right) \leq \\
&\qquad\leq \int_{B_\varepsilon} \psi\left(x^*, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right)\,\eta_\varepsilon(y)\,\mathrm{d}y.
\end{aligned} \tag{10.3.6}$$

If $\frac{x}{1-2\,\varepsilon} - y$ does not belong to $\overline{B}$ then $\psi\left(x^*, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right) = 0$. Otherwise, Lemma 10.3.3 implies $\frac{x}{1-2\,\varepsilon} - y \in \overline{B} \cap \overline{B_{5\,\varepsilon}(x)}$ so that

$$\psi\left(x^*, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right) \leq \psi\left(\frac{x}{1-2\,\varepsilon} - y, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right)$$

due to the minimality of x^* and nonnegativity of a. As $x \in B$ was fixed, we obtain inequality

$$\psi(x, |\nabla u^\varepsilon|(x)) \leq 2^q\,M \int_{B_\varepsilon} \psi\left(\frac{x}{1-2\,\varepsilon} - y, |\nabla u|\left(\frac{x}{1-2\,\varepsilon} - y\right)\right)\,\eta_\varepsilon(y)\,\mathrm{d}y + N \tag{10.3.7}$$

valid for all $x \in B$. Now, we observe that $\varphi(x, |\nabla u^\varepsilon|(x))$ converges to $\varphi(x, |\nabla u|(x))$ a.e. Moreover, the (RHS) of (10.3.7) is convergent in $L^1(B)$ cf. Lemma 10.3.4 (D2) so that $\{\varphi(x, |\nabla u^\varepsilon|(x))\}_\varepsilon$ is uniformly integrable in $L^1(B)$. Therefore, Vitali convergence theorem (Corollary 7.3.3) implies

$$\psi(x, |\nabla u^\varepsilon|(x)) \to \psi(x, |\nabla u|(x)) \qquad \text{in } L^1(B) \text{ as } \varepsilon \to 0.$$

Thanks to triangle inequality we obtain (E2). To see (E3), we note a simple estimate $|a+b|^q \leq 2^{q-1}\left(|a|^q + |b|^q\right)$ so that

$$\psi\left(x, |\nabla u(x) - \nabla u^\varepsilon(x)|\right) \leq 2^{q-1}\psi\left(x, |\nabla u|(x)\right) + 2^{q-1}\psi\left(x, |\nabla u^\varepsilon|(x)\right).$$

It follows that the sequence $\{\psi\left(x, |\nabla u(x) - \nabla u^{\varepsilon}(x)|\right)\}_{\varepsilon}$ is again uniformly integrable and Vitali convergence theorem (Corollary 7.3.3) yields

$$\psi\left(x, |\nabla u(x) - \nabla u^{\varepsilon}(x)|\right) \to 0 \qquad \text{in } L^1(B) \text{ as } \varepsilon \to 0,$$

concluding the proof of (E3) due to (C2) in Lemma 7.3.1. This shows that any bounded function in $W_0^{1,\varphi}(B)$ can be approximated with smooth compactly supported functions so that (E4) follows from Lemma 10.2.3.

<u>**Case 2:** $p > d$**.**</u> In this case we have $q \leq p + \alpha\,\frac{p}{d}$. Note that

$$\nabla u^{\varepsilon}(x) = \frac{1}{1-2\,\varepsilon}\int_{B_\varepsilon} \nabla u\left(\frac{x}{1-2\,\varepsilon} - y\right)\eta_\varepsilon(y)\,\mathrm{d}y.$$

Therefore, instead of (10.3.4), we can compute

$$\|\nabla u^{\varepsilon}\|_{\infty} \leq \frac{1}{1-2\,\varepsilon}\left\|\nabla u\left(\frac{\cdot}{1-2\varepsilon}\right)\right\|_p \|\eta_\varepsilon\|_{p'} \leq 2\left\|\nabla u\left(\frac{\cdot}{1-2\varepsilon}\right)\right\|_p \|\eta_\varepsilon\|_{p'}\,, \tag{10.3.8}$$

where p' is the usual Hölder conjugate exponent. Using change of variables we obtain:

$$\|\eta_\varepsilon\|_{p'}^{p'} = \int_{B_\varepsilon} \frac{1}{\varepsilon^{d\,p'}}\left|\eta\left(\frac{x}{\varepsilon}\right)\right|^{p'} \mathrm{d}x = \varepsilon^{d\,(1-p')}\int_B |\eta(x)|^{p'}\,\mathrm{d}x = \varepsilon^{-p'\,\frac{d}{p}}\|\eta\|_{p'}^{p'},$$

so that $\|\eta_\varepsilon\|_{p'} = \varepsilon^{-\frac{d}{p}}\|\eta\|_{p'}$. Using change of variables again,

$$\left\|\nabla u\left(\frac{\cdot}{1-2\varepsilon}\right)\right\|_p \leq \|\nabla u\|_p$$

which is finite as $\mathcal{G}(u) < \infty$. Therefore, (10.3.8) boils down to

$$\|\nabla u^{\varepsilon}\|_{\infty} \leq D\,(5\varepsilon)^{-\frac{d}{p}},$$

where $D := 5^{\frac{d}{p}}\,\|\nabla u\|_p\,\|\eta\|_{p'}$. Using Lemma 10.3.6 we obtain estimate (10.3.5). The rest of the proof is exactly the same. $\square$

10.4 The general case

In this section we briefly explain how to adapt the reasoning in Section 10.3 to cover general bounded domain and more general functionals. The target here is not to present the complete reasoning but rather a general idea that leads to these generalizations.

General bounded Lipschitz domains

We can also consider general bounded Lipschitz domain Ω (that is, the boundary of Ω is locally a graph of Lipschitz function). Here, the main ingredient is the following concept of star-shaped domains and the decomposition theorem from [225, Lemma 3.14].

Definition 10.4.1. *(1) A bounded domain $U \subset \mathbb{R}^d$ is said to be star-shaped with respect to $\overline{x}$ if every ray starting from $\overline{x}$ intersects with ∂U at one and only one point.*
(2) A bounded domain $U \subset \mathbb{R}^d$ is said to be star-shaped with respect to the ball $B_\gamma(x_0)$ if U is star-shaped with respect to all $y \in B_\gamma(x_0)$.

Lemma 10.4.2. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Then, there exist domains $\{U_i\}_{i=1,\dots,n}$ such that*

$$\overline{\Omega} \subset \bigcup_{i=1}^{n} U_i.$$

and $\Omega \cap U_i$ is star-shaped with respect to some ball $B_{R_i}(x_i)$.

Furthermore, one can prove that if U is a star-shaped domain with respect to the ball, it can be uniformly shrinked in the following sense (for the proof, see [57, Lemma 6.4]):

Lemma 10.4.3. *Let $U \subset \mathbb{R}^d$ be a star-shaped domain with respect to the ball B_R. Let $\kappa_\varepsilon = 1 - \frac{4\varepsilon}{R}$. Then, $\mathrm{dist}(\kappa_\varepsilon U, \partial U) \geq 2\varepsilon$. In particular,*

$$\overline{\kappa_\varepsilon U + \varepsilon B} \subset U.$$

More generally, if U is star-shaped with respect to the ball $B_R(x_0)$,

$$\overline{\kappa_\varepsilon (U - x_0) + \varepsilon B} \subset (U - x_0).$$

This allows to define a smooth and compactly supported approximation as follows:

Definition 10.4.4 (Mollification with squeezing on star-shaped domain)**.** *Let U be a star-shaped domain with respect to the ball $B_R(x_0)$. Given $u \in W_0^{1,1}(U)$ we extend it with 0 to $\mathbb{R}^d$ and define*

$$\mathcal{S}_U^{\varepsilon} u(x) := \int_{\mathbb{R}^d} u\left(x_0 + \frac{x - x_0 - y}{\kappa_\varepsilon}\right) \eta_\varepsilon(y)\, \mathrm{d}y,$$

where $\kappa_\varepsilon = 1 - \frac{4\varepsilon}{R}$.

Now, let Ω be a Lipschitz bounded domain. From Lemma 10.4.2 we obtain a family of domains such that $\overline{\Omega} \subset \bigcup_{i=1}^n U_i$ where $\{\Omega \cap U_i\}_{i=1,\ldots,n}$ are star-shaped domains with respect to balls $B_R(x_i)$ (without loss of generality, we may assume that the radii of the balls are the same by taking $R := \min_{i=1,\ldots n} R_i$). In particular, $\{U_i\}_{i=1,\ldots,n}$ is an open covering of $\overline{\Omega}$ so there exists a partition of unity related to this covering: a family of functions $\{\theta_i\}_{i=1,\ldots,n}$ such that

$$\theta_i \in C_c^\infty(U_i), \qquad 0 \le \theta_i \le 1, \qquad \sum_{i=1}^n \theta_i = 1 \text{ on } \overline{\Omega}.$$

Given $u \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega)$ we extend it with 0 as above and we set

$$\mathcal{S}^\varepsilon u := \sum_{i=1}^n \mathcal{S}_{U_i}^\varepsilon (u\,\theta_i) = \sum_{i=1}^n \int_{B_\varepsilon} (u\,\theta_i)\left(x_i + \frac{x - x_i - y}{\kappa_\varepsilon}\right) \eta_\varepsilon(y)\, \mathrm{d}y \tag{10.4.1}$$

where $\kappa_\varepsilon = 1 - \frac{4\varepsilon}{R}$. We note that since u vanishes outside of Ω, function $u\,\theta_i$ is supported in $\Omega \cap U_i$ which is star-shaped.

General functionals

In fact, our results can be extended to functionals of the form

$$\mathcal{H}(u) = \int_\Omega \psi(x, |\nabla u(x)|)\, \mathrm{d}x,$$

where ψ satisfies the following Assumptions.

Assumption 10.4.5. We assume that $\psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(A1) (vanishing at 0) $\psi(x, \xi) = 0$ if and only if $\xi = 0$,

(A2) (convexity) for each x, the map $\mathbb{R}^+ \ni \xi \mapsto \psi(x, \xi)$ is convex, (A3) ($p-q$ growth) there exist exponents $1 \leq p < q < \infty$ and $\xi_0 \geq 1$ and constants C_1 and C_2 such that

$$C_1\,|\xi|^p \leq \psi(x,\xi) \text{ for } \xi \geq \xi_0, \qquad \psi(x,\xi) \leq C_2\,(1+|\xi|^q) \text{ for all } \xi \geq 0,$$

(A4) (Δ_2 condition) there exists a constant C_4 such that

$$\psi(x,2\xi) \leq C_4\,\psi(x,\xi).$$

(A5) (autonomous lower-bound) there is function $m_\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and ξ_0 such that for $\xi \geq \xi_0$ we have $m_\psi(\xi) \leq \psi(x,\xi)$ and $\frac{m_\psi(\xi)}{\xi} \to \infty$ as $\xi \to \infty$,

Assumption 10.4.6. We assume that for all $D > 1$, there are constants $M = M(p,q,D)$ and $N = N(p,q,D)$ such that

$$\psi(z,\xi) \leq M\,\psi(y,\xi) + N \tag{10.4.2}$$

for all balls $B_\gamma(x)$, all $y, z \in \overline{B_\gamma(x)} \cap \overline{\Omega}$, all $\xi \in \left[0, D\gamma^{-\min\left(1,\frac{d}{p}\right)}\right]$ and all $\gamma \in \left(0,\frac{1}{2}\right)$.

The main idea is to use Assumption 10.4.6 to mimic our strategy from Lemmas 10.3.5 and 10.3.6. Here, the main difficulty is that the function

$$\xi \mapsto \inf_{\overline{B_\gamma(x)} \cap \overline{\Omega}} \psi(x,\xi)$$

is not necessarily convex so Jensen's inequality cannot be applied as in (10.3.6). To illustrate the solution to this problem, let us assume that (10.4.2) is satisfied for all $\xi \in \mathbb{R}$ (with a natural extension with 0 for $\xi < 0$). Then, we take infimum over $y \in \overline{B_\gamma(x)} \cap \overline{\Omega}$ and we apply second conjugate (see [236, Section 2.6]). The idea is that the second conjugate is convex (in fact, it is the greatest convex minorant) and it preserves inequalities. Therefore,

$$\psi(z,\xi) \leq M\,\left(\inf\nolimits_{y \in \overline{B_\gamma(x)} \cap \overline{\Omega}} \psi(y,\xi)\right)^{**} + N \leq M\,\psi(x,\xi) + N \tag{10.4.3}$$

where we used that $\xi \mapsto \psi(x,\xi)$ is convex and $x, z \in \overline{B_\gamma(x)} \cap \overline{\Omega}$. It follows that one can use convex function $\xi \mapsto \left(\operatorname{essinf}_{y \in \overline{B_\gamma(x)} \cap \overline{\Omega}} \psi(y,\xi)\right)^{**}$ in place of $\psi(x^*,\xi)$ in the proof of Theorem 10.3.2.

The main difficulty in the method described above is that inequality (10.4.2) is satisfied only for ξ belonging to some interval while the second conjugate operation is in fact a nonlocal operator (that is, its value at point ξ depends in fact on values in some neighbourhood of ξ). Still, one can prove that inequality (10.4.3) is satisfied and for the proof we refer to [57, Lemma 6.2].

As Assumption 10.4.6 seems to be abstract, we provide here an example. We consider function

$$\phi(x,\xi) := |\xi|^{p(x)} + a(x)\,|\xi|^{q(x)} \tag{10.4.4}$$

and the related functional reads:

$$\mathcal{J}(u,\Omega) := \int_\Omega \left[|\nabla u|^{p(x)} + a(x)\,|\nabla u|^{q(x)}\right] \mathrm{d}x. \tag{10.4.5}$$

Assumption 10.4.7. We assume that:

(B1) ($p{-}q$ growth) there exist p, q with $1 < p \le q$ such that the functions $p(x), q(x) : \Omega \to [1,\infty)$ satisfy $p \le p(x) \le q(x) \le q$,

(B2) (log-Hölder continuity) there are constants C_p, C_q such that for all $x, y \in \Omega$ with $|x-y| \le \min\left(\operatorname{diam}\Omega, \frac{1}{2}\right)$ we have

$$|p(x)-p(y)| \le -\frac{C_p}{\log|x-y|}, \qquad |q(x)-q(y)| \le -\frac{C_q}{\log|x-y|}.$$

(B3) (α-Hölder continuity) $a \in C^\alpha(\overline{\Omega})$ with constant $|a|_\alpha$.

Lemma 10.4.8. *Under Assumption 10.4.7, function ϕ defined with* (10.4.4) *satisfies Assumption 10.4.6 for q and p such that $q \le p + \alpha\, \max\left(1, \frac{p}{d}\right)$.*

Proof. Inequality (10.4.2) is clear when $\xi \le 1$. For $\xi > 1$ one can apply directly reasoning from (E2) in Example 8.2.3. For the detailed proof see [57, Lemma 3.3]. $\square$

Bibliography

[1] A. Abbatiello and E. Feireisl. On a class of generalized solutions to equations describing incompressible viscous fluids. *Ann. Mat. Pura Appl. (4)*, 199(3):1183–1195, 2020.

[2] E. Acerbi and N. Fusco. An approximation lemma for $W^{1,p}$ functions. In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ., pages 1–5. Oxford Univ. Press, New York, 1988.

[3] E. Acerbi and G. Mingione. Regularity results for stationary electro-rheological fluids. *Arch. Ration. Mech. Anal.*, 164(3):213–259, 2002.

[4] E. Acerbi, G. Mingione, and G. A. Seregin. Regularity results for parabolic systems related to a class of non-Newtonian fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):25–60, 2004.

[5] R. A. Adams. *Sobolev spaces.* Pure and Applied Mathematics, Vol. 65. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[6] K. Adimurthi, S.-S. Byun, and J. Oh. Interior and boundary higher integrability of very weak solutions for quasilinear parabolic equations with variable exponents. *Nonlinear Anal.*, 194:111370, 54, 2020.

[7] K. Adimurthi, S.-S. Byun, and J.-T. Park. End point gradient estimates for quasilinear parabolic equations with variable exponent growth on nonsmooth domains. *Calc. Var. Partial Differential Equations*, 60(4):Paper No. 145, 67, 2021.

[8] Y. Ahmida, I. Chlebicka, P. Gwiazda, and A. Youssfi. Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces. *J. Funct. Anal.*, 275(9):2538–2571, 2018.

[9] J. J. Alibert and G. Bouchitté. Non-uniform integrability and generalized Young measures. *J. Convex Anal.*, 4(1):129–147, 1997.

[10] N. J. Alves and A. E. Tzavaras. The relaxation limit of bipolar fluid models. *Discrete Contin. Dyn. Syst.*, 42(1):211–237, 2022.

[11] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[12] G. Andrews and J. M. Ball. Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity. *J. Differential Equations*, 44(2):306–341, 1982. Special issue dedicated to J. P. LaSalle.

[13] P. Antonelli, L. E. Hientzsch, and P. Marcati. On the low Mach number limit for quantum Navier-Stokes equations. *SIAM J. Math. Anal.*, 52(6):6105–6139, 2020.

[14] P. Antonelli and P. Marcati. On the finite energy weak solutions to a system in quantum fluid dynamics. *Comm. Math. Phys.*, 287(2):657–686, 2009.

[15] P. Antonelli and S. Spirito. Global existence of weak solutions to the Navier-Stokes-Korteweg equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 39(1):171–200, 2022.

[16] S. Antontsev and V. Zhikov. Higher integrability for parabolic equations of $p(x,t)$-Laplacian type. *Adv. Differential Equations*, 10(9):1053–1080, 2005.

[17] D. G. Aronson and J. L. Vázquez. The porous medium equation as a finite-speed approximation to a Hamilton-Jacobi equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4(3):203–230, 1987.

[18] R. Arora and S. Shmarev. Strong solutions of evolution equations with $p(x,t)$-Laplacian: existence, global higher integrability of the gradients and second-order regularity. *J. Math. Anal. Appl.*, 493(1):124506, 31, 2021.

[19] C. Audiard and B. Haspot. Global well-posedness of the Euler-Korteweg system for small irrotational data. *Comm. Math. Phys.*, 351(1):201–247, 2017.

[20] A. K. Balci, L. Diening, and M. Surnachev. New examples on Lavrentiev gap using fractals. *Calc. Var. Partial Differential Equations*, 59(5):Paper No. 180, 34, 2020.

[21] A. K. Balci and M. Surnachev. Lavrentiev gap for some classes of generalized Orlicz functions. *Nonlinear Anal.*, 207:Paper No. 112329, 22, 2021.

[22] J. M. Ball. A version of the fundamental theorem for Young measures. In *PDEs and continuum models of phase transitions (Nice, 1988)*, volume 344 of *Lecture Notes in Phys.*, pages 207–215. Springer, Berlin, 1989.

[23] J. M. Ball and V. J. Mizel. One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. *Arch. Rational Mech. Anal.*, 90(4):325–388, 1985.

[24] C. Bardos and E. S. Titi. Euler equations for an ideal incompressible fluid. *Uspekhi Mat. Nauk*, 62(3(375)):5–46, 2007.

[25] P. Baroni, M. Colombo, and G. Mingione. Harnack inequalities for double phase functionals. *Nonlinear Anal.*, 121:206–222, 2015.

[26] P. Baroni, M. Colombo, and G. Mingione. Regularity for general functionals with double phase. *Calc. Var. Partial Differential Equations*, 57(2):Paper No. 62, 48, 2018.

[27] D. Basarić. Existence of dissipative (and weak) solutions for models of general compressible viscous fluids with linear pressure. *J. Math. Fluid Mech.*, 24(2):Paper No. 56, 22, 2022.

[28] E. Bäumle and M. Růžička. Existence of weak solutions for unsteady motions of micropolar electrorheological fluids. *SIAM J. Math. Anal.*, 49(1):115–141, 2017.

[29] L. Beck. *Elliptic regularity theory*, volume 19 of *Lecture Notes of the Unione Matematica Italiana*. Springer, Cham; Unione Matematica Italiana, Bologna, 2016. A first course.

[30] P. Bella and M. Schäffner. On the regularity of minimizers for scalar integral functionals with (p, q)-growth. *Anal. PDE*, 13(7):2241–2257, 2020.

[31] S. Benzoni-Gavage, R. Danchin, and S. Descombes. On the well-posedness for the Euler-Korteweg model in several space dimensions. *Indiana Univ. Math. J.*, 56(4):1499–1579, 2007.

[32] A. J. Bernoff and C. M. Topaz. Biological aggregation driven by social and environmental factors: a nonlocal model and its degenerate Cahn-Hilliard approximation. *SIAM J. Appl. Dyn. Syst.*, 15(3):1528–1562, 2016.

[33] L. C. Berselli, D. Breit, and L. Diening. Convergence analysis for a finite element approximation of a steady model for electrorheological fluids. *Numer. Math.*, 132(4):657–689, 2016.

[34] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.

[35] J. Blechta, J. Málek, and K. R. Rajagopal. On the classification of incompressible fluids and a mathematical analysis of the equations that govern their motion. *SIAM J. Math. Anal.*, 52(2):1232–1289, 2020.

[36] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007.

[37] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic equations with p, q-growth. *J. Math. Pures Appl. (9)*, 100(4):535–563, 2013.

[38] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic systems with p, q-growth: a variational approach. *Arch. Ration. Mech. Anal.*, 210(1):219–267, 2013.

[39] V. Bögelein and Q. Li. Very weak solutions of degenerate parabolic systems with non-standard $p(x, t)$-growth. *Nonlinear Anal.*, 98:190–225, 2014.

[40] D. Bothe, M. Pierre, and G. Rolland. Cross-diffusion limit for a reaction-diffusion system with fast reversible reaction. *Comm. Partial Differential Equations*, 37(11):1940–1966, 2012.

[41] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In *Optimal control and partial differential equations*, pages 439–455. IOS, Amsterdam, 2001.

[42] P. Bousquet. Non occurence of the Lavrentiev gap for multidimensional autonomous problems. *to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 2022.

[43] D. Breit. New regularity theorems for non-autonomous variational integrals with (p, q)-growth. *Calc. Var. Partial Differential Equations*, 44(1-2):101–129, 2012.

[44] D. Breit, L. Diening, and M. Fuchs. Solenoidal Lipschitz truncation and applications in fluid mechanics. *J. Differential Equations*, 253(6):1910–1942, 2012.

[45] D. Breit, L. Diening, and S. Schwarzacher. Solenoidal Lipschitz truncation for parabolic PDEs. *Math. Models Methods Appl. Sci.*, 23(14):2671–2700, 2013.

[46] Y. Brenier, C. De Lellis, and L. Székelyhidi, Jr. Weak-strong uniqueness for measure-valued solutions. *Comm. Math. Phys.*, 305(2):351–361, 2011.

[47] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[48] D. Bresch, M. Gisclon, and I. Lacroix-Violet. On Navier-Stokes-Korteweg and Euler-Korteweg systems: application to quantum fluids models. *Arch. Ration. Mech. Anal.*, 233(3):975–1025, 2019.

[49] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[50] H. Brezis and F. E. Browder. Strongly nonlinear parabolic initial-boundary value problems. *Proceedings of the National Academy of Sciences*, 76(1):38–40, 1979.

[51] F. Bubba, B. Perthame, C. Pouchol, and M. Schmidtchen. Hele–Shaw Limit for a System of Two Reaction-(Cross-)Diffusion Equations for Living Tissues. *Arch. Ration. Mech. Anal.*, 236(2):735–766, 2020.

[52] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math. (2)*, 189(1):101–144, 2019.

[53] M. Bulíček, P. Gwiazda, M. Kalousek, and A. Świerczewska-Gwiazda. Existence and homogenization of nonlinear elliptic systems in nonreflexive spaces. *Nonlinear Analysis*, 2019.

[54] M. Buliček, P. Gwiazda, J. Skrzeczkowski, and J. Woźnicki. Non-Newtonian fluids with discontinuous-in-time stress tensor. *arXiv preprint arXiv:2209.10695*, 2022.

[55] M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska Gwiazda. On unsteady flows of implicitly constituted incompressible fluids. *SIAM J. Math. Anal.*, 44(4):2756–2801, 2012.

[56] M. Bulíček, P. Gwiazda, and J. Skrzeczkowski. Parabolic equations in Musielak-Orlicz spaces with discontinuous in time N-function. *J. Differential Equations*, 290:17–56, 2021.

[57] M. Bulíček, P. Gwiazda, and J. Skrzeczkowski. On a range of exponents for absence of Lavrentiev phenomenon for double phase functionals. *Arch. Ration. Mech. Anal.*, 246(1):209–240, 2022.

[58] J. Burczak, S. Modena, and L. Székelyhidi. Non uniqueness of power-law flows. *Comm. Math. Phys.*, 388(1):199–243, 2021.

[59] H. Byrne and M. Chaplain. Growth of necrotic tumors in the presence and absence of inhibitors. *Mathematical Biosciences*, 135(2):187–216, 1996.

[60] H. Byrne and D. Drasdo. Individual-based and continuum models of growing cell populations: a comparison. *J. Math. Biol.*, 58(4-5):657–687, 2009.

[61] S.-S. Byun and J. Oh. Regularity results for generalized double phase functionals. *Anal. PDE*, 13(5):1269–1300, 2020.

[62] S.-S. Byun and J. Ok. Nonlinear parabolic equations with variable exponent growth in nonsmooth domains. *SIAM J. Math. Anal.*, 48(5):3148–3190, 2016.

[63] S.-S. Byun, J. Ok, and K. Song. Hölder regularity for weak solutions to nonlocal double phase problems. *J. Math. Pures Appl. (9)*, 168:110–142, 2022.

[64] J. Cahn. On spinodal decomposition. *Acta Metallurgica*, 9(9):795–801, 1961.

[65] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial free energy. *The Journal of Chemical Physics*, 28(2):258–267, 1958.

[66] J. A. Carrillo, Y.-P. Choi, M. Hauray, and S. Salem. Mean-field limit for collective behavior models with sharp sensitivity regions. *J. Eur. Math. Soc. (JEMS)*, 21(1):121–161, 2019.

[67] J. A. Carrillo, K. Craig, and Y. Yao. Aggregation-diffusion equations: dynamics, asymptotics, and singular limits. In *Active particles. Vol. 2. Advances in theory, models, and applications*, Model. Simul. Sci. Eng. Technol., pages 65–108. Birkhäuser/Springer, Cham, 2019.

[68] J. A. Carrillo, T. Dębiec, P. Gwiazda, and A. Świerczewska-Gwiazda. Dissipative measure-valued solutions to the Euler-Poisson equation. *arXiv preprint arXiv:2109.07536*, 2021.

[69] J. A. Carrillo, C. Elbar, and J. Skrzeczkowski. Degenerate Cahn-Hilliard systems: From nonlocal to local. 2023. In preparation.

[70] J. A. Carrillo, H. Murakawa, M. Sato, H. Togashi, and O. Trush. A population dynamics model of cell-cell adhesion incorporating population pressure and density saturation. *J. Theoret. Biol.*, 474:14–24, 2019.

[71] J. A. Carrillo, Y. Peng, and A. Wróblewska-Kamińska. Relative entropy method for the relaxation limit of hydrodynamic models. *Netw. Heterog. Media*, 15(3):369–387, 2020.

[72] J. A. Carrillo, A. Wróblewska-Kamińska, and E. Zatorska. Pressureless Euler with nonlocal interactions as a singular limit of degenerate Navier-Stokes system. *J. Math. Anal. Appl.*, 492(1):124400, 27, 2020.

[73] F. Charve. Convergence of a low order non-local Navier-Stokes-Korteweg system: the order-parameter model. *Asymptot. Anal.*, 100(3-4):153–191, 2016.

[74] F. Charve and B. Haspot. Convergence of capillary fluid models: from the non-local to the local Korteweg model. *Indiana Univ. Math. J.*, 60(6):2021–2059, 2011.

[75] Y.-Z. Chen and L.-C. Wu. *Second order elliptic equations and elliptic systems*, volume 174 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1998. Translated from the 1991 Chinese original by Bei Hu.

[76] I. Chlebicka, F. Giannetti, and A. Zatorska-Goldstein. Elliptic problems with growth in nonreflexive Orlicz spaces and with measure or L^1 data. *Journal of Mathematical Analysis and Applications*, 2019.

[77] I. Chlebicka, P. Gwiazda, A. Świerczewska Gwiazda, and A. Wróblewska-Kamińska. *Partial differential equations in anisotropic Musielak-Orlicz spaces*. Springer Monographs in Mathematics. Springer, Cham, [2021] ©2021.

[78] I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. Parabolic equation in time and space dependent anisotropic Musielak-Orlicz spaces in absence of Lavrentiev's phenomenon. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 36(5):1431–1465, 2019.

[79] I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein. Renormalized solutions to parabolic equations in time and space dependent anisotropic Musielak-

Orlicz spaces in absence of Lavrentiev's phenomenon. *J. Differential Equations*, 267(2):1129–1166, 2019.

[80] G. Cianfarani Carnevale and C. Lattanzio. High friction limit for Euler-Korteweg and Navier-Stokes-Korteweg models via relative entropy approach. *J. Differential Equations*, 269(12):10495–10526, 2020.

[81] P. G. Ciarlet. *Mathematical elasticity. Volume I. Three-dimensional elasticity*, volume 84 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, [2022] ©2022. Reprint of the 1988 edition [0936420].

[82] M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. *Arch. Ration. Mech. Anal.*, 218(1):219–273, 2015.

[83] M. Colombo and G. Mingione. Regularity for double phase variational problems. *Arch. Ration. Mech. Anal.*, 215(2):443–496, 2015.

[84] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces: Foundations and harmonic analysis*. Springer Science & Business Media, 2013.

[85] S. Cygan, A. Marciniak-Czochra, G. Karch, and K. Suzuki. Stable discontinuous stationary solutions to reaction-diffusion-ode systems. *In preparation*, 2021.

[86] S. Cygan, A. Marciniak-Czochra, G. Karch, and K. Suzuki. Instability of all regular stationary solutions to reaction-diffusion-ODE systems. *J. Differential Equations*, 337:460–482, 2022.

[87] S. Dai and Q. Du. Weak solutions for the Cahn-Hilliard equation with degenerate mobility. *Arch. Ration. Mech. Anal.*, 219(3):1161–1184, 2016.

[88] Y. Dai and P. Mu. Incompressible limit of the compressible primitive equations with gravity: well-prepared initial data. *J. Differential Equations*, 336:204–238, 2022.

[89] N. David and M. Schmidtchen. On the incompressible limit for a tumour growth model incorporating convective effects. *arXiv preprint arXiv:2103.02564, to appear in Comm. Pure Appl. Math.*, 2021.

[90] E. Davoli, H. Ranetbauer, L. Scarpa, and L. Trussardi. Degenerate nonlocal Cahn-Hilliard equations: well-posedness, regularity and local asymptotics. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 37(3):627–651, 2020.

[91] E. Davoli, L. Scarpa, and L. Trussardi. Local asymptotics for nonlocal convective Cahn-Hilliard equations with $W^{1,1}$ kernel and singular potential. *J. Differential Equations*, 289:35–58, 2021.

[92] E. Davoli, L. Scarpa, and L. Trussardi. Nonlocal-to-local convergence of Cahn-Hilliard equations: Neumann boundary conditions and viscosity terms. *Arch. Ration. Mech. Anal.*, 239(1):117–149, 2021.

[93] C. De Filippis. Gradient bounds for solutions to irregular parabolic equations with (p, q)-growth. *Calc. Var. Partial Differential Equations*, 59(5):Paper No. 171, 32, 2020.

[94] C. De Filippis and G. Mingione. On the regularity of minima of non-autonomous functionals. *J. Geom. Anal.*, 30(2):1584–1626, 2020.

[95] C. De Lellis. Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio [after Ambrosio, DiPerna, Lions]. Number 317, pages Exp. No. 972, viii, 175–203. 2008. Séminaire Bourbaki. Vol. 2006/2007.

[96] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math. (2)*, 170(3):1417–1436, 2009.

[97] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.

[98] C. De Lellis and L. Székelyhidi, Jr. Dissipative Euler flows and Onsager's conjecture. *J. Eur. Math. Soc. (JEMS)*, 16(7):1467–1505, 2014.

[99] M. G. Delgadino. Convergence of a one-dimensional Cahn-Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 50(4):4457–4482, 2018.

[100] F. Demengel and G. Demengel. *Functional spaces for the theory of elliptic partial differential equations.* Universitext. Springer, London; EDP Sciences, Les Ulis, 2012. Translated from the 2007 French original by Reinie Erné.

[101] S. Demoulini, D. M. A. Stuart, and A. E. Tzavaras. Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. *Arch. Ration. Mech. Anal.*, 205(3):927–961, 2012.

[102] R. DeVore and G. Petrova. The averaging lemma. *J. Amer. Math. Soc.*, 14(2):279–296, 2001.

[103] L. Diening, F. Ettwein, and M. Růžička. $C^{1,\alpha}$-regularity for electrorheological fluids in two dimensions. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):207–217, 2007.

[104] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*. Springer, 2011.

[105] L. Diening, J. Málek, and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM Control Optim. Calc. Var.*, 14(2):211–232, 2008.

[106] L. Diening and M. Růžička. An existence result for non-Newtonian fluids in non-regular domains. *Discrete Contin. Dyn. Syst. Ser. S*, 3(2):255–268, 2010.

[107] L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(1):1–46, 2010.

[108] L. Diening, T. Scharle, and E. Süli. Uniform Hölder-norm bounds for finite element approximations of second-order elliptic equations. *IMA J. Numer. Anal.*, 41(3):1846–1898, 2021.

[109] L. Diening, S. Schwarzacher, B. Stroffolini, and A. Verde. Parabolic Lipschitz truncation and caloric approximation. *Calc. Var. Partial Differential Equations*, 56(4):Paper No. 120, 27, 2017.

[110] R. J. DiPerna. Measure-valued solutions to conservation laws. *Arch. Rational Mech. Anal.*, 88(3):223–270, 1985.

[111] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.

[112] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[113] R. J. DiPerna, P.-L. Lions, and Y. Meyer. L^p regularity of velocity averages. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 8(3-4):271–287, 1991.

[114] R. J. DiPerna and A. J. Majda. Oscillations and concentrations in weak solutions of the incompressible fluid equations. *Comm. Math. Phys.*, 108(4):667–689, 1987.

[115] D. Donatelli, E. Feireisl, and P. Marcati. Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Comm. Partial Differential Equations*, 40(7):1314–1335, 2015.

[116] C. Düll, P. Gwiazda, A. Marciniak-Czochra, and J. Skrzeczkowski. *Spaces of measures and their applications to structured population models*, volume 36 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2022.

[117] J. E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Arch. Rational Mech. Anal.*, 88(2):95–133, 1985.

[118] T. Dębiec, B. Perthame, M. Schmidtchen, and N. Vauchelet. Incompressible limit for a two-species model with coupling through Brinkman's law in any dimension. *J. Math. Pures Appl.*, 145:204–239, 2021.

[119] N. El Ghani and N. Masmoudi. Diffusion limit of the Vlasov-Poisson-Fokker-Planck system. *Commun. Math. Sci.*, 8(2):463–479, 2010.

[120] C. Elbar, P. Gwiazda, J. Skrzeczkowski, and A. Świerczewska Gwiazda. From nonlocal Euler-Korteweg to local Cahn-Hilliard. 2023. In preparation.

[121] C. Elbar, M. Mason, B. Perthame, and J. Skrzeczkowski. From Vlasov equation to degenerate nonlocal Cahn-Hilliard equation. Published online in *Communications in Mathematical Physics*, doi: 10.1007/s00220-023-04663-3.

[122] C. Elbar, B. Perthame, and A. Poulain. Degenerate Cahn-Hilliard and incompressible limit of a Keller-Segel model. *Commun. Math. Sci.*, 20(7):1901–1926, 2022.

[123] C. Elbar and J. Skrzeczkowski. Degenerate Cahn-Hilliard equation: From nonlocal to local. *arXiv preprint arXiv:2208.08955*, 2022.

[124] C. M. Elliott and H. Garcke. On the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 27(2):404–423, 1996.

[125] A. Elmahi and D. Meskine. Parabolic equations in Orlicz spaces. *Journal of the London Mathematical Society*, 72(2):410–428, 2005.

[126] A. Esposito, F. Leonetti, and P. Vincenzo Petricca. Absence of Lavrentiev gap for non-autonomous functionals with (p, q)-growth. *Adv. Nonlinear Anal.*, 8(1):73–78, 2019.

[127] L. Esposito, F. Leonetti, and G. Mingione. Sharp regularity for functionals with (p, q) growth. *J. Differential Equations*, 204(1):5–55, 2004.

[128] F. Ettwein, M. Růžička, and B. Weber. Existence of steady solutions for micropolar electrorheological fluid flows. *Nonlinear Anal.*, 125:1–29, 2015.

[129] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990.

[130] L. C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 1998.

[131] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Chapman and Hall/CRC, 2015.

[132] L. C. Evans and M. Portilheiro. Irreversibility and hysteresis for a forward-backward diffusion equation. *Math. Models Methods Appl. Sci.*, 14(11):1599–1620, 2004.

[133] C. Falcó, R. Baker, and J. Carrillo. A local continuum model of cell-cell adhesion. *SIAM Journal on Applied Mathematics*, 2022. (accepted for publication, arXiv:2206.14461).

[134] E. Feireisl, P. Gwiazda, A. Świerczewska Gwiazda, and E. Wiedemann. Dissipative measure-valued solutions to the compressible Navier-Stokes system. *Calc. Var. Partial Differential Equations*, 55(6):Art. 141, 20, 2016.

[135] E. Feireisl, Y.-S. Kwon, and A. Novotný. On the long-time behavior of dissipative solutions to models of non-Newtonian compressible fluids. *Arch. Ration. Mech. Anal.*, 241(1):475–495, 2021.

[136] E. Feireisl and P. Laurençot. Non-isothermal Smoluchowski-Poisson equations as a singular limit of the Navier-Stokes-Fourier-Poisson system. *J. Math. Pures Appl. (9)*, 88(4):325–349, 2007.

[137] U. S. Fjordholm, S. Mishra, and E. Tadmor. On the computation of measure-valued solutions. *Acta Numer.*, 25:567–679, 2016.

[138] G. B. Folland. *Real analysis: Modern techniques and their applications.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999.

[139] J. Frehse, J. Málek, and M. Steinhauer. An existence result for fluids with shear dependent viscosity—steady flows. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996)*, volume 30, pages 3041–3049, 1997.

[140] J. Frehse, J. Málek, and M. Steinhauer. On existence results for fluids with shear dependent viscosity—unsteady flows. In *Partial differential equations (Praha, 1998)*, volume 406 of *Chapman & Hall/CRC Res. Notes Math.*, pages 121–129. Chapman & Hall/CRC, Boca Raton, FL, 2000.

[141] J. Frehse, J. Málek, and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.*, 34(5):1064–1083, 2003.

[142] S. Frigeri. On a nonlocal Cahn-Hilliard/Navier-Stokes system with degenerate mobility and singular potential for incompressible fluids with different densities. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 38(3):647–687, 2021.

[143] C. G. Gal, A. Giorgini, and M. Grasselli. The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property. *J. Differential Equations*, 263(9):5253–5297, 2017.

[144] C. G. Gal and M. Grasselli. Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 27(1):401–436, 2010.

[145] G. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems.* Springer Science & Business Media, 2011.

[146] D. Gallenmüller and E. Wiedemann. On the selection of measure-valued solutions for the isentropic Euler system. *J. Differential Equations*, 271:979–1006, 2021.

[147] H. Garcke, K. F. Lam, R. Nürnberg, and E. Sitka. A multiphase Cahn-Hilliard-Darcy model for tumour growth with necrosis. *Math. Models Methods Appl. Sci.*, 28(3):525–577, 2018.

[148] H. Garcke, K. F. Lam, E. Sitka, and V. Styles. A Cahn-Hilliard-Darcy model for tumour growth with chemotaxis and active transport. *Math. Models Methods Appl. Sci.*, 26(6):1095–1148, 2016.

[149] P. Germain and P. LeFloch. Finite energy method for compressible fluids: the Navier-Stokes-Korteweg model. *Comm. Pure Appl. Math.*, 69(1):3–61, 2016.

[150] B. Gess. Optimal regularity for the porous medium equation. *J. Eur. Math. Soc. (JEMS)*, 23(2):425–465, 2021.

[151] G. Giacomin and J. L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys.*, 87(1-2):37–61, 1997.

[152] G. Giacomin and J. L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. II. Interface motion. *SIAM J. Appl. Math.*, 58(6):1707–1729, 1998.

[153] J. Giesselmann, C. Lattanzio, and A. E. Tzavaras. Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics. *Arch. Ration. Mech. Anal.*, 223(3):1427–1484, 2017.

[154] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Springer, 2015.

[155] A. Giorgini, A. Miranville, and R. Temam. Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system. *SIAM J. Math. Anal.*, 51(3):2535–2574, 2019.

[156] F. Golse, P.-L. Lions, B. Perthame, and R. Sentis. Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.*, 76(1):110–125, 1988.

[157] F. Golse, B. Perthame, and R. Sentis. Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre

principale d'un opérateur de transport. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(7):341–344, 1985.

[158] F. Golse and L. Saint-Raymond. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.*, 155(1):81–161, 2004.

[159] T. Goudon. Hydrodynamic limit for the Vlasov-Poisson-Fokker-Planck system: analysis of the two-dimensional case. *Math. Models Methods Appl. Sci.*, 15(5):737–752, 2005.

[160] N. Guillen, I. Kim, and A. Mellet. A Hele-Shaw limit without monotonicity. *Arch. Ration. Mech. Anal.*, 243(2):829–868, 2022.

[161] P. Gwiazda. On measure-valued solutions to a two-dimensional gravity-driven avalanche flow model. *Math. Methods Appl. Sci.*, 28(18):2201–2223, 2005.

[162] P. Gwiazda, O. Kreml, and A. Świerczewska Gwiazda. Dissipative measure-valued solutions for general conservation laws. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 37(3):683–707, 2020.

[163] P. Gwiazda and A. Świerczewska Gwiazda. On non-Newtonian fluids with a property of rapid thickening under different stimulus. *Math. Models Methods Appl. Sci.*, 18(7):1073–1092, 2008.

[164] P. Gwiazda and A. Świerczewska Gwiazda. On steady non-Newtonian fluids with growth conditions in generalized Orlicz spaces. *Topol. Methods Nonlinear Anal.*, 32(1):103–113, 2008.

[165] P. Gwiazda, A. Świerczewska Gwiazda, and E. Wiedemann. Weak-strong uniqueness for measure-valued solutions of some compressible fluid models. *Nonlinearity*, 28(11):3873–3890, 2015.

[166] P. Gwiazda, A. Świerczewska Gwiazda, and A. Wróblewska. Monotonicity methods in generalized Orlicz spaces for a class of non-Newtonian fluids. *Math. Methods Appl. Sci.*, 33(2):125–137, 2010.

[167] P. Gwiazda, P. Wittbold, A. Wróblewska-Kamińska, and A. Zimmermann. Renormalized solutions to nonlinear parabolic problems in generalized Musielak-Orlicz spaces. *Nonlinear Anal.*, 129:1–36, 2015.

[168] P. Harjulehto and P. Hästö. *Orlicz spaces and generalized Orlicz spaces*, volume 2236 of *Lecture Notes in Mathematics*. Springer, Cham, 2019.

[169] P. A. Hästö. A fundamental condition for harmonic analysis in anisotropic generalized Orlicz spaces. *J. Geom. Anal.*, 33(1):Paper No. 7, 15, 2023.

[170] M. Heida and J. Málek. On compressible Korteweg fluid-like materials. *Internat. J. Engrg. Sci.*, 48(11):1313–1324, 2010.

[171] D. Hilhorst, M. Mimura, and H. Ninomiya. Fast reaction limit of competition-diffusion systems. In *Handbook of differential equations: evolutionary equations. Vol. V*, Handb. Differ. Equ., pages 105–168. Elsevier/North-Holland, Amsterdam, 2009.

[172] X. Huo, A. Jüngel, and A. E. Tzavaras. High-friction limits of Euler flows for multicomponent systems. *Nonlinearity*, 32(8):2875–2913, 2019.

[173] P.-E. Jabin. A review of the mean field limits for Vlasov equations. *Kinet. Relat. Models*, 7(4):661–711, 2014.

[174] P.-E. Jabin and L. Vega. Averaging lemmas and the X-ray transform. *C. R. Math. Acad. Sci. Paris*, 337(8):505–510, 2003.

[175] P.-E. Jabin and L. Vega. A real space method for averaging lemmas. *J. Math. Pures Appl. (9)*, 83(11):1309–1351, 2004.

[176] P.-E. Jabin and Z. Wang. Mean field limit and propagation of chaos for Vlasov systems with bounded forces. *J. Funct. Anal.*, 271(12):3588–3627, 2016.

[177] D. Jamet, O. Lebaigue, N. Coutris, and J. M. Delhaye. The second gradient method for the direct numerical simulation of liquid-vapor flows with phase change. *J. Comput. Phys.*, 169(2):624–651, 2001.

[178] I. Kim and O. Turanova. Uniform convergence for the incompressible limit of a tumor growth model. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 35(5):1321–1354, 2018.

[179] S. Ko. Existence of global weak solutions for unsteady motions of incompressible chemically reacting generalized Newtonian fluids. *J. Math. Anal. Appl.*, 513(1):Paper No. 126206, 24, 2022.

[180] S. Ko, P. Pustějovská, and E. Süli. Finite element approximation of an incompressible chemically reacting non-Newtonian fluid. *ESAIM Math. Model. Numer. Anal.*, 52(2):509–541, 2018.

[181] S. Ko and E. Süli. Finite element approximation of steady flows of generalized Newtonian fluids with concentration-dependent power-law index. *Math. Comp.*, 88(317):1061–1090, 2019.

[182] D. J. Korteweg. Sur la forme qui prennent les équations du mouvement des fluids si l'on tient compte des forces capillaires par des variations de densité considérables mais continues et sur la théorie de la capillarité dans l'hypothèse d'une variation continue. *Archives Néerlandaises des sciences exactes et naturelles*, 6:1–24, 1901.

[183] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by S. Smith.

[184] O. A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Mathematics and its Applications, Vol. 2. Gordon and Breach, Science Publishers, New York-London-Paris, 1969. Second English edition, revised and enlarged, Translated from the Russian by Richard A. Silverman and John Chu.

[185] R. Landes. On the existence of weak solutions for quasilinear parabolic initial-boundary value problems. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 89(3-4):217–237, 1981.

[186] C. Lattanzio and A. E. Tzavaras. Relative entropy in diffusive relaxation. *SIAM J. Math. Anal.*, 45(3):1563–1584, 2013.

[187] C. Lattanzio and A. E. Tzavaras. From gas dynamics with large friction to gradient flows describing diffusion theories. *Comm. Partial Differential Equations*, 42(2):261–290, 2017.

[188] M. Lavrentieff. Sur quelques problèmes du calcul des variations. *Ann. Mat. Pura Appl.*, 4(1):7–28, 1927.

[189] D. Lee, J.-Y. Huh, D. Jeong, J. Shin, A. Yun, and J. Kim. Physical, mathematical, and numerical derivations of the Cahn–Hilliard equation. *Computational Materials Science*, 81:216–225, 2014.

[190] F. Leonetti and F. Siepe. Maximum principle for vector valued minimizers. *J. Convex Anal.*, 12(2):267–278, 2005.

[191] Q. Li. Very weak solutions of subquadratic parabolic systems with non-standard $p(x,t)$-growth. *Nonlinear Anal.*, 156:17–41, 2017.

[192] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[193] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.

[194] P.-L. Lions and N. Masmoudi. From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II. *Arch. Ration. Mech. Anal.*, 158(3):173–193, 195–211, 2001.

[195] P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191, 1994.

[196] S. Lisini, D. Matthes, and G. Savaré. Cahn–Hilliard and thin film equations with nonlinear mobility as gradient flows in weighted-Wasserstein metrics. *J. Differential Equations*, 253(2):814–850, 2012.

[197] X. Liu and E. S. Titi. Zero Mach number limit of the compressible primitive equations: well-prepared initial data. *Arch. Ration. Mech. Anal.*, 238(2):705–747, 2020.

[198] J. S. Lowengrub, H. B. Frieboes, F. Jin, Y.-L. Chuang, X. Li, P. Macklin, S. M. Wise, and V. Cristini. Nonlinear modelling of cancer: bridging the gap between cells and tumours. *Nonlinearity*, 23(1):R1–R91, 2010.

[199] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[200] B. Mania. Sopra un esempio di Lavrentieff. *Boll. Un. Mat. Ital.*, 13:147–153, 1934.

[201] P. Marcati. Approximate solutions to conservation laws via convective parabolic equations. *Comm. Partial Differential Equations*, 13(3):321–344, 1988.

[202] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 84(1):129–147, 1990.

[203] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions. *Arch. Rational Mech. Anal.*, 105(3):267–284, 1989.

[204] P. Marcellini. Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations*, 90(1):1–30, 1991.

[205] P. Marcellini. Regularity for elliptic equations with general growth conditions. *J. Differential Equations*, 105(2):296–333, 1993.

[206] P. Marcellini. Everywhere regularity for a class of elliptic systems without growth conditions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(1):1–25, 1996.

[207] P. Marcellini. A variational approach to parabolic equations under general and p, q-growth conditions. *Nonlinear Anal.*, 194:111456, 17, 2020.

[208] A. Marciniak-Czochra, G. Karch, and K. Suzuki. Unstable patterns in reaction-diffusion model of early carcinogenesis. *J. Math. Pures Appl. (9)*, 99(5):509–543, 2013.

[209] A. Marciniak-Czochra, G. Karch, and K. Suzuki. Instability of Turing patterns in reaction-diffusion-ODE systems. *J. Math. Biol.*, 74(3):583–618, 2017.

[210] C. Mariconda and G. Treu. Non-occurrence of a gap between bounded and Sobolev functions for a class of nonconvex Lagrangians. *J. Convex Anal.*, 27(4):1247–1259, 2020.

[211] C. Mariconda and G. Treu. Non-occurrence of the Lavrentiev phenomenon for a class of convex nonautonomous Lagrangians. *Open Math.*, 18(1):1–9, 2020.

[212] C. Mascia, A. Terracina, and A. Tesei. Two-phase entropy solutions of a forward-backward parabolic equation. *Arch. Ration. Mech. Anal.*, 194(3):887–925, 2009.

[213] N. Masmoudi and M. L. Tayeb. Diffusion limit of a semiconductor Boltzmann-Poisson system. *SIAM J. Math. Anal.*, 38(6):1788–1807, 2007.

[214] D. Matthes, R. J. McCann, and G. Savaré. A family of nonlinear fourth order equations of gradient flow type. *Comm. Partial Differential Equations*, 34(10-12):1352–1397, 2009.

[215] S. Melchionna, H. Ranetbauer, L. Scarpa, and L. Trussardi. From nonlocal to local Cahn-Hilliard equation. *Adv. Math. Sci. Appl.*, 28(2):197–211, 2019.

[216] G. Mingione. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math.*, 51(4):355–426, 2006.

[217] G. J. Minty. on a "monotonicity" method for the solution of non-linear equations in Banach spaces. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1038–1041, 1963.

[218] A. Miranville. *The Cahn-Hilliard equation. Recent advances and applications*, volume 95 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2019. Recent advances and applications.

[219] Y. Morita and N. Shinjo. Reaction-diffusion models with a conservation law and pattern formations. *Josai Mathematical Monographs*, 9:177–190, 2016.

[220] A. Moussa. Some variants of the classical Aubin-Lions lemma. *J. Evol. Equ.*, 16(1):65–93, 2016.

[221] A. Moussa, B. Perthame, and D. Salort. Backward parabolicity, cross-diffusion and Turing instability. *J. Nonlinear Sci.*, 29(1):139–162, 2019.

[222] F. Murat. A survey on compensated compactness. In *Contributions to modern calculus of variations (Bologna, 1985)*, volume 148 of *Pitman Res. Notes Math. Ser.*, pages 145–183. Longman Sci. Tech., Harlow, 1987.

[223] J. Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1983.

[224] A. Novick-Cohen and R. L. Pego. Stable patterns in a viscous diffusion equation. *Trans. Amer. Math. Soc.*, 324(1):331–351, 1991.

[225] A. Novotný and I. Straškraba. *Introduction to the mathematical theory of compressible flow*, volume 27 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[226] J. Ok. Regularity for double phase problems under additional integrability assumptions. *Nonlinear Anal.*, 194:111408, 13, 2020.

[227] P. Pedregal. *Parametrized measures and variational principles*, volume 30 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1997.

[228] B. Perthame. *Kinetic formulation of conservation laws*, volume 21 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2002.

[229] B. Perthame, F. Quirós, and J. L. Vázquez. The Hele-Shaw asymptotics for mechanical models of tumor growth. *Arch. Ration. Mech. Anal.*, 212(1):93–127, 2014.

[230] B. Perthame and J. Skrzeczkowski. Fast reaction limit with nonmonotone reaction function. Published online in *Communications on Pure and Applied Mathematics*, doi: 10.1002/cpa.22042.

[231] B. Perthame and E. Tadmor. A kinetic equation with kinetic entropy functions for scalar conservation laws. *Comm. Math. Phys.*, 136(3):501–517, 1991.

[232] P. I. Plotnikov. Passage to the limit with respect to viscosity in an equation with a variable direction of parabolicity. *Differentsial' nye Uravneniya*, 30(4):665–674, 734, 1994.

[233] A. C. Ponce. An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc. (JEMS)*, 6(1):1–15, 2004.

[234] F. Poupaud and J. Soler. Parabolic limit and stability of the Vlasov-Fokker-Planck system. *Math. Models Methods Appl. Sci.*, 10(7):1027–1045, 2000.

[235] K. Rajagopal and M. Růžička. On the modeling of electrorheological materials. *Mechanics Research Communications*, 23(4):401–407, 1996.

[236] F. Rindler. *Calculus of variations*. Universitext. Springer, Cham, 2018.

[237] C. Rohde. On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions. *ZAMM Z. Angew. Math. Mech.*, 85(12):839–857, 2005.

[238] F. Rothe. *Global solutions of reaction-diffusion systems*, volume 1072 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[239] T. Roubíček. *Nonlinear partial differential equations with applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2013.

[240] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[241] L. Saint-Raymond. *Hydrodynamic limits of the Boltzmann equation*, volume 1971 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[242] E. Sandier and S. Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, 57(12):1627–1672, 2004.

[243] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[244] C. Schmeiser. Lecture notes on Kinetic Transport Theory. Available at: `https://homepage.univie.ac.at/christian.schmeiser/Kinetics-lecture-notes.pdf/`. [Online; accessed 26-December-2022].

[245] T. Schmidt. Regularity of minimizers of $W^{1,p}$-quasiconvex variational integrals with (p,q)-growth. *Calc. Var. Partial Differential Equations*, 32(1):1–24, 2008.

[246] S. Serfaty. Gamma-convergence of gradient flows on Hilbert and metric spaces and applications. *Discrete Contin. Dyn. Syst.*, 31(4):1427–1451, 2011.

[247] C. G. Simader and H. Sohr. *The Dirichlet problem for the Laplacian in bounded and unbounded domains*, volume 360 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996. A new approach to weak, strong and $(2+k)$-solutions in Sobolev-type spaces.

[248] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[249] J. Skrzeczkowski. Fast reaction limit and forward-backward diffusion: A Radon–Nikodym approach. *Comptes Rendus. Mathématique*, 360(G2):189–203, 2022.

[250] H. Sohr. *The Navier-Stokes equations.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2001. An elementary functional analytic approach, [2013 reprint of the 2001 original] [MR1928881].

[251] A. Świerczewska-Gwiazda. Nonlinear parabolic problems in Musielak–Orlicz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 98:48–65, 2014.

[252] L. Székelyhidi, Jr. From isometric embeddings to turbulence. In *HCDTE lecture notes. Part II. Nonlinear hyperbolic PDEs, dispersive and transport equations*, volume 7 of *AIMS Ser. Appl. Math.*, page 63. Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2013.

[253] A. Szepessy. An existence result for scalar conservation laws using measure valued solutions. *Comm. Partial Differential Equations*, 14(10):1329–1350, 1989.

[254] A. Szepessy. Measure-valued solutions of scalar conservation laws with boundary conditions. *Arch. Rational Mech. Anal.*, 107(2):181–193, 1989.

[255] E. Tadmor and T. Tao. Velocity averaging, kinetic formulations, and regularizing effects in quasi-linear PDEs. *Comm. Pure Appl. Math.*, 60(10):1488–1521, 2007.

[256] S. Takata and T. Noguchi. A simple kinetic model for the phase transition of the van der Waals fluid. *J. Stat. Phys.*, 172(3):880–903, 2018.

[257] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass.-London, 1979.

[258] J. L. Vázquez. Hyperbolic aspects in the theory of the porous medium equation. In *Metastability and incompletely posed problems (Minneapolis, Minn., 1985)*, volume 3 of *IMA Vol. Math. Appl.*, pages 325–342. Springer, New York, 1987.

[259] J. L. Vázquez. *The porous medium equation.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory.

[260] M. Westdickenberg. Some new velocity averaging results. *SIAM J. Math. Anal.*, 33(5):1007–1032, 2002.

[261] E. Wiedemann. Weak-strong uniqueness in fluid dynamics. In *Partial differential equations in fluid mechanics*, volume 452 of *London Math. Soc. Lecture Note Ser.*, pages 289–326. Cambridge Univ. Press, Cambridge, 2018.

[262] J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mech.*, 9(1):104–138, 2007.

[263] A. Wróblewska. Steady flow of non-Newtonian fluids - monotonicity methods in generalized Orlicz spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 72(11):4136–4147, 2010.

[264] A. Wróblewska-Kamińska. Unsteady flows of non-Newtonian fluids in generalized Orlicz spaces. *Discrete Contin. Dyn. Syst.*, 33(6):2565–2592, 2013.

[265] A. Wróblewska-Kamińska. Existence result for the motion of several rigid bodies in an incompressible non-Newtonian fluid with growth conditions in Orlicz spaces. *Nonlinearity*, 27(4):685–716, 2014.

[266] M. Yan. Extension of convex function. *arXiv preprint arXiv:1207.0944*, 2012.

[267] L. C. Young. Generalized surfaces in the calculus of variations. *Ann. of Math. (2)*, 43:84–103, 1942.

[268] L. C. Young. Generalized surfaces in the calculus of variations. II. *Ann. of Math. (2)*, 43:530–544, 1942.

[269] V. Zhikov. Lavrentiev phenomenon and homogenization for some variational problems. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(5):435–439, 1993.

[270] V. V. Zhikov. Problems of convergence, duality, and averaging for a class of functionals of the calculus of variations. *Dokl. Akad. Nauk SSSR*, 267(3):524–528, 1982.

[271] V. V. Zhikov. On Lavrentiev's phenomenon. *Russian J. Math. Phys.*, 3(2):249–269, 1995.

[272] V. V. Zhikov. On some variational problems. *Russian J. Math. Phys.*, 5(1):105–116 (1998), 1997.