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Report on Wojciech Lubawski's thesis: Maps between classifying spaces of unitary groups

Main results of the thesis

If G is a compact connected Lie group then the set $[BG, BG]$ of homotopy endomorphisms of G is fairly well known (thanks to work of the thesis advisor among others). It is quite another matter to determine the set $[BG, BH]$ of homotopy homomorphisms of G to another compact Lie group H. This thesis is concerned with the challenging computation of the set of homotopy morphisms $[\mathrm{BU}(n), \mathrm{BU}(m)]$ between unitary groups. It is not stated explicitly in the Introduction exactly where to find the main results of the thesis but my guess is that we should look at Theorem B from p 59 and Theorem A from the very last page. (The Polish summary gave me a much better overview of the thesis than the English Introduction even though I don't read Polish.) Theorem A proves the existence of some new maps of $\mathrm{BU}(n)$ to $\mathrm{BU}(m)$. In the words of the first paragraph, 'we obtain certain classification of such maps'. Precisely what this certain classification amounts to is left unspecified, however.

Content the of thesis

A virtual character of $\mathrm{U}(n)$, an element of the representation ring $R(\mathrm{U}(n)))$, is a Π-character if it restricts to an actual character on the p-normalizer of the maximal torus in $\mathrm{U}(n)$ for all primes $p \mid n$. The semi-ring of Π-characters is denoted $R^+_\Pi(\mathrm{U}(n))$. These concepts are introduced and investigated in Chapters 1 and 2. (The formulation of Theorem 2.4.2, essential for what follows, is incorrect.)

In Chapter 3 the author produces a long list of explicit Π-characters of $\mathrm{U}(n)$. In Theorem B, from Chapter 4, he describes *all* Π-characters $\mu \in R^+_\Pi(\mathrm{U}(n))$ when

$n > 18$ and $\mu(1) \leq \frac{1}{2}n(n-1)(n+2)$ in terms of the long list of explicitly given Π-characters. This is a statement in representation theory.

In the final Chapter 5, the author turns to homotopy theory. There is character map $[\mathrm{BU}(n), \mathrm{BU}(m)] \to R(\mathrm{U}(n))$ from m-dimensional homotopy representations of $\mathrm{U}(n)$ to its representation ring. The homotopy representation semi-ring $R^h(\mathrm{U}(n))$ is the sub-semi-ring of $R(\mathrm{U}(n))$ obtained as m on the left hand side varies. The key property, and this is the reason behind our interest in Π-characters, is that $R^h(\mathrm{U}(n)) \subseteq R^+_\Pi(\mathrm{U}(n))$. (I must have overlooked the place in the thesis where this is explicitly stated.) Theorem A, proved using results from the the author's joint paper with the co-advisor, is a statement about the reverse inclusion. Theorem A simply says, I think, that the Π-characters of Theorem B are homotopy characters: They are actually realized by maps $\mathrm{BU}(n) \to \mathrm{BU}(m)$. (This is not the way the author states Theorem A and I may be wrong in my interpretation.)

Presentation of the results

I have the impression that there are places in the thesis where the formulations are not as precise as desirable. This is no doubt in some cases due to my incomplete understanding. In other cases, maybe the author should have paid more attention to details. As an illustration, let me comment on the pages 13–14, the first two pages of Chapter 2:

p 2, first paragraph The p-normalizer is here called N_p^n but on p 17 it is called both N_p and N_p^n

p 2, first paragraph The function $e(x)$ is defined in passing here. The reader may have forgotten this when he gets to Definition 2.5.1.

Proposition 2.1.2 The reference to Adams' book is not precise. This proposition is both a proposition and also a definition of the weights $\lambda_{(k_1,\dots,k_n)}$ which are absolutely essential to what follows. The formulation of the Proposition is grammatically terribly involved in my opinion. What is the role of the λ_i here? Are the z_i in the definition of λ_i the same as the z_i in the definition of $\lambda_{(k_1,\dots,k_n)}$?

Definition 2.3 It would be good with a reference to a place in the literature where this construction is taken from. The group $\Sigma_{(k_1,\dots,k_n)}$ is defined in Definition 2.1.7 but used already here. I think that the number $\#(\Sigma)_{(k_1,\dots,k_n)}$ is the order of this group, but this is not immediately clear. If this is so, why not write $\#\Sigma_{(k_1,\dots,k_n)}$? The entity $[k_1,\dots,k_n]$ is undefined but it is referred to in Corollary 2.1.5 (where it is called a sequence - but is it a sequence?).

Definition 2.1.7 Would it be advantageous to use the word 'partition' in this context? Again I think this Definition is a hybrid between a definition and a lemma.

Conclusion

I think that this thesis clearly shows that the author has an impressive knowledge of mathematics, an admirable perseverance, masters long arguments on a sophisticated level, and has reached a general mathematical maturity. It is quite impressive that it is possible in Theorem B to find an exhaustive list of Π-characters. Based on this thesis, I strongly recommend that the author be awarded the phd degree in mathematics.

Sincerely

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Professor