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Report on the habilitation thesis of Piotr Achinger

Deformations, degenerations and homootopy type of algebraic varieties

Dr. Piotr Achinger habilitation thesis ("habilitation achievment") is based on five papers

[Hab 1] P. Achinger, M. Zdanowicz, *Some elementary examples of non-liftable varieties*. Proc. Amer. Math. Soc. 145 (2017), no. 11, 4717–4729.

[Hab 2] P. Achinger, M. Zdanowicz, *Serre-Tate theory for Calabi-Yau varieties*. J. Reine Angew. Math. 780 (2021), 139–196.

[Hab 3] P. Achinger, J. Witaszek, M. Zdanowicz, *Global Frobenius liftability I*. J. Eur. Math. Soc. (JEMS) 23 (2021), no. 8, 2601–2648.

[Hab 4] P. Achinger, *Wild ramification and $K(\pi, 1)$ spaces*. Invent. Math. 210 (2017), no. 2, 453–499.

[Hab 5] P. Achinger, M. Talpo, *Betti realization of varieties defined by formal Laurent series*. Geom. Topol. 25 (2021), no. 4, 1919–1978.

Four of the papers are joint with Mattia Talpo, Jakub Witaszek and Maciej Zdanowicz (all coauthors declared aproximately equal efforts).

Below I will briefly review these papers.

Some elementary examples of non-liftable varieties. In this paper two classes of examples of non-liftable varieties are constructed.

Definition. A proper scheme X over a field k of characteristic $p > 0$ is *liftable to characteristic zero* if there exists a proper flat scheme $\mathcal{X}$ over a local domain R of characteristic 0 with a residue field k such that

$$\mathcal{X} \otimes_R k \cong X.$$

First example of a smooth projective variety in characteristic $p > 0$ non-liftable to characteristic zero was constructed by Serre. Serre's example is *strongly* non-liftable: if it lifts over a local domain R then $pR = 0$.

In this paper the authors construct two types of examples;

- blow-up of the product $Y \times Y$ along the graph of Frobenius for a homogeneous space Y satisfying some extra conditions
- blow-up projective 3-space $\mathbb{P}^3_k$ over an algebraically closed field of characteristic p at the set $P = \mathbb{P}^3_k(\mathbb{F}_p)$ of $\mathbb{F}_p$-rational points and then at (strict transforms) of lines containing at least two points of P

Constructed non-liftable varieties satisfy five conditions demonstrating their good cohomological behaviour, these varieties avoid typical characteristic p pathologies. The proof of non-liftability in both cases is based on deformation arguments that can be rephrased as lifting of a blow-up is a blow-up of a lifting under some natural assumptions.

For the first type of examples liftability would produce a homogeneous space with non-trivial endomorphism, while for the second type – an impossible configuration of lines and points in projective 3-space in characteristic zero.

Serre-Tate theory for Calabi-Yau varieties. An abelian variety A over a perfect field k of characteristic $p > 0$ is *ordinary* iff the group of p-torsion is maximal $A[p] \cong (\mathbb{Z}/p)^g$, $g = \dim A$. Classical Serre-Tate theorem asserts that for an ordinary abelian variety A over k there is equivalence deformations of A and p-divisible points $A[p^\infty]$.

As a consequence deformations $\mathrm{Def}_{A/W}(W)$ of A over Witt vectors has a structure of a group. In particular we get the *canonical lift* $\tilde{A}$.. The goal of this paper is to extend the Serre-Tate theory to the case of Calabi-Yau manifolds. In fact the Authots present two versions of the Serre-Tate theory.

First version is a generalization of Nygaard theorem for K3 surfaces to an arbitrary dimension. Under certain assumptions for a 2-ordinary Calabi-Yau manifold X there is an isomorphism

$$\mathrm{Def}_{X/W_m} \cong T \otimes W_m,$$

where T is a certain formal torus. Integer m is fixed such that X has unobstructed deformations over $W_m(k)$. This theorem contains the assumption $h^0(X, T_X) = 0$, which should be compared with the classical proof of liftability of $K3$ surfaces.

The second version of the Serre-Tate theory yields a canonical lifting modulo p^2 of Calabi-Yau threefolds with Frobenius splitting.

Global Frobenius liftability I. The main goal of this paper is to propose a conjectural characterization of F-liftability and provide some evidences.

Definition. A scheme X over a field k of characteristic $p > 0$ is F–liftable iff there is a lifting $\tilde{X}$ of X over $W_2(k)$ together with a morphism $\tilde{F} : \tilde{X} \longrightarrow \tilde{X}$ such that $\tilde{F} \otimes_{W_2(k)} k \cong F$, where $F : X \longrightarrow X$ is the Frobenius morphism.

Conjecture. *If a smooth and projective variety over an algebraically closed field k of characteristic $p > 0$ is F-liftable then there is an étale Galois cover $f : Y \longrightarrow X$ such that the Albanese morphism of Y is a toric fibration.*

The converse implication to this conjecture is not true in general but it holds under some extra conditions (rem. 3.1.7).

An F-liftable smooth and proper scheme has some strong properties: is ordinary in the sense of Bloch and Kato, the Bott vanishing holds for X. Using deformation theory (obstruction class for Frobenius lifting) the authors proves a descent result for F-liftability.

Ix X is a homogeneous space with a transitive automorphism group, the authors prove that the F-liftability is equivalent to a decomposition as a product of an ordinary abelian variety and projective spaces and these happen exactly when the Albanese of X is an ordinary abelian variety A and the fibers of $X \longrightarrow A$ are toric varieties.

In a recent (accepted for publication) preprint (again with Witaszek and Zdanowicz) they gave characterization of F-liftable surfaces and a result that every F-liftable Fano threefold is toric.

These establishes the Conjecture in three important cases: surfaces, homogeneous spaces and Fano threefolds. Finally, the authors study the case of Calabi-Yau varieties.

Wild ramification and $K(\pi, 1)$ spaces. The main result of this paper is the following

Theorem. *Every connected affine $\mathbb{F}_p$–scheme $K(\pi, 1)$.*

In algebraic topology a topological space is $K(\pi, 1)$ iff the higher homotopy groups of X vanish. In the case of schemes in positive characteristic the definition of $K(\pi, 1)$ is more technical. The proof of the above theorem consists of a several reductions, and then an inductive proof in case of an affine space. Most important element of the $\mathbb{A}^n_k$ case is the Bertini theorem and the main difficulty in the proof of the latter is existence of wild ramifications.

From the main theorem the author deduces versions in rigid-analytic spaces and proves that $\pi_1(\mathbb{A}^n_k) \cong \pi_1(\mathbb{A}^m_k)$ only for $n = m$.

Betti realization of varieties defined by formal Laurent series. For a complete discretely valued field K with a fixed embedding $i : \mathcal{O}_K/m =: k \hookrightarrow \mathbb{C}$ the authors construct two functors.

First: there exists a functor Ψ from the category of schemes locally of finite type over K to the ∞-category of spaces over $\mathbb{S}^1$ that satisfies six properties: Finiteness, Sheaf property, Good reduction, Semistable reduction with boundary, Étale comparison and De Rham comparison.

Second: there exists a functor Ψ_{rig} from the category of smooth rigid analytic spaces over K to the ∞-category of spaces over $\mathbb{S}^1$ that satisfies five properties: Finiteness, Sheaf property, Good reduction, Semistable reduction, Comparison for schemes.

This paper was motivated by a question Treumann. The proofs use in an essential way on weak factorization theorem and log geometry.

Conclusion

Habilitation thesis presented by Piotr Achinger contains an impressive collection of deep and important results, proofs are very advanced and uses various areas of algebraic geometry. Piotr Achinger is an author of 11 published papers, from the six that do not belong to the habilitation five papers are prior to his PhD and one was published after PhD. Moreover two papers are accepted for publication and three are in the review process. Most of Achinger's papers are published in top journals including: Inventiones Mathematicae, Crelle's Journal, Journal of Algebraic Geometry and Compositio Mathematica.

Piotr Achinger is the Principal Investigator of an ERC Starting Grant *Homotopy theory of algebraic varieties* and NCN Sonata Grant *Deformacje i degeneracje rozmaitości algebraicznych*. Presented documentation includes also an impressive list of scientific visits and invited talks.

Piotr Achinger was a co-advisor of two concluded PhD thesis (P. Lewulis and M. Zdanowicz). He was an organizer of two editions of GAeL, conference *Wild Ramification and Irregular Singularities* in IMPAN and Simon's minisemester *Varieties: Arithmetic and Transformations*.

I strongly recommend that the works presented for habilitation achievement should be accepted.

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