INTRODUCTION TO THE STACKS OF SHTUKAS

NGO DAC TUAN

These notes are an attempt to provide an introduction to the stacks of shtukas and their compactifications. The notion of shtukas (or F-bundles) was first defined by Drinfeld [Dri87, Dri89] in his proof of the Langlands correspondence for the general linear group GL_2 over function fields. It is recently used in the Lafforgue's proof of the Langlands correspondence for the general linear group of higher rank GL_r over function fields, *cf.* [Laf02].

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Notations

For the rest of these notes, let X be a smooth projective geometrically connected curve over a finite field $\mathbb{F}_q$ of q elements. The field F of $\mathbb{F}_q$-valued rational functions over X is called *the function field of* X. One can identify the places of F with the set $|X|$ of closed points of X.

For example, we could take the projective line $X = \mathbb{P}^1$. Then the function field of X is just the field of rational functions in one variable. Its elements are fractions $P(t)/Q(t)$, where $P(t)$ and $Q(t)$ are polynomials over $\mathbb{F}_q$ without common factors, and $Q(t) \neq 0$. The set $|X|$ corresponds to the set of irreducible polynomials over $\mathbb{F}_q$.

We will use freely the theory of schemes and stacks. For a reader unfamiliar with such matters, the book of Hartshorne [Har77] and that of Moret-Bailly and Laumon [Lau-MB99] would be good references. All schemes (or stacks) will be defined over $\operatorname{Spec} \mathbb{F}_q$ and we denote by $Y \times Z$ the fiber product $Y \times_{\operatorname{Spec} \mathbb{F}_q} Z$ for any such schemes (or stacks) Y and Z.

Suppose that S is a scheme over $\operatorname{Spec} \mathbb{F}_q$. We denote by $\mathrm{Frob}_S : S \longrightarrow S$ the Frobenius morphism which is identity on points, and is the Frobenius map $t \mapsto t^q$ on functions. For any $\mathcal{O}_{X \times S}$-module $\mathcal{F}$, we denote by $\mathcal{F}^\sigma$ the pull back $(\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{F}$.

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We fix also an algebraic closure k of $\mathbb{F}_q$. We define

$$\overline{X} = X \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec} k.$$

Finally, fix once for all an integer $r \geq 1$.

1. Modifications. Hecke stacks

1.1. Let $\overline{T}$ be a finite closed subscheme of the geometric curve $\overline{X}$. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are two vector bundles of rank r over $\overline{X}$. By definition, a $\overline{T}$-*modification* from $\mathcal{E}$ into $\mathcal{E}'$ is an isomorphism between the restrictions of $\mathcal{E}$ and $\mathcal{E}'$ to the open subscheme $\overline{X} - \overline{T}$ of $\overline{X}$:

$$\varphi : \mathcal{E}_{|\overline{X}-\overline{T}} \xrightarrow{\sim} \mathcal{E}'_{|\overline{X}-\overline{T}}.$$

Suppose that φ is a $\overline{T}$-modification from $\mathcal{E}$ into $\mathcal{E}'$ as above. If x is a geometric point in $\overline{T}$, $\mathcal{O}_x$ denotes the completion of $\mathcal{O}_{\overline{X}}$ at x, and F_x denotes the fraction field of $\mathcal{O}_x$. The isomorphism φ induces an isomorphism between the generic fibers V and V' of $\mathcal{E}$ and $\mathcal{E}'$, respectively:

$$\varphi : V \xrightarrow{\sim} V',$$

hence an isomorphism

$$\varphi : V_x \xrightarrow{\sim} V'_x.$$

The completion of $\mathcal{E}$ (resp. $\mathcal{E}'$) at x defines an $\mathcal{O}_x$-lattice $\mathcal{E}_x$ in V_x (resp. E'_x in V'_x). Then it follows from the theorem of elementary divisors that the relative position of two lattices $\mathcal{E}_x$ and $\mathcal{E}'_x$ is given by a sequence of integers

$$\lambda_x = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r) \in \mathbb{Z}^r.$$

This sequence is called *the invariant of the modification* φ *at* x. One remarks immediately that the invariant of the modification φ^{-1} at x is the sequence $(-\lambda_r \geq -\lambda_{r-1} \geq \ldots \geq -\lambda_1)$.

Consider the simplest example of all, the case that $\overline{T}$ contains only one geometric point x of $\overline{X}$ and $\lambda = (1, 0, \ldots, 0)$. Then an x-modification from a vector bundel $\mathcal{E}$ of rank r over $\overline{X}$ into another vector bundle $\mathcal{E}'$ of same rank whose invariant at x is λ is just an injection $\mathcal{E} \hookrightarrow \mathcal{E}'$ such that the quotient $\mathcal{E}'/\mathcal{E}$ is supported by x and of length 1. It is well known as *an (elementary) upper modification at* x.

Similarly, we keep $\overline{T} = x \in \overline{X}$ and modify $\lambda = (0, 0, \ldots, -1)$. Then an x-modification from $\mathcal{E}$ into $\mathcal{E}'$ whose invariant at x is λ is just an injection $\mathcal{E}' \hookrightarrow \mathcal{E}$ such that the quotient $\mathcal{E}/\mathcal{E}'$ is supported by x and of length 1. It is well known as *an (elementary) lower modification at* x.

1.2. We are ready to define (elementary) Hecke stacks Hecke^r. Over $\operatorname{Spec} k$, $\mathrm{Hecke}^r(\operatorname{Spec} k)$ is the category whose objects consist of

- a vector bundle $\mathcal{E}$ of rank r over $\overline{X}$,
- two geometric points $\infty, 0 \in \overline{X}$,
- an upper modification $\mathcal{E} \hookrightarrow \mathcal{E}'$ at ∞ and a lower modification $\mathcal{E}' \hookleftarrow \mathcal{E}''$ at 0,

and whose morphisms are isomorphisms between such data.

More generally, for a scheme S over $\mathbb{F}_q$, $\mathrm{Hecke}^r(S)$ is the category whose objects are the data consisting of

- a vector bundle $\mathcal{E}$ of rank r over $X \times S$,
- two morphisms $\infty, 0 : S \longrightarrow X$,
- a modification consisting of two injections

  $$\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''$$

  of $\mathcal{E}$ by two vector bundles $\mathcal{E}'$ and $\mathcal{E}''$ of rank r over $X \times S$ such that the quotients $\mathcal{E}'/\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}''$ are supported by the graphs of ∞ and 0 respectively, and they are invertible on their support,

and whose morphisms are isomorphisms between such data.

Assume that $f : S' \longrightarrow S$ is a morphism of schemes. Then the pull-back operator induces a functor between the two categories $\mathrm{Hecke}^r(S)$ and $\mathrm{Hecke}^r(S')$:

$$\begin{gathered} f^* : \mathrm{Hecke}^r(S) \longrightarrow \mathrm{Hecke}^r(S') \\ (\mathcal{E}, \infty, 0, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'') \mapsto (f^*\mathcal{E}, f \circ \infty, f \circ 0, f^*\mathcal{E} \hookrightarrow f^*\mathcal{E}' \hookleftarrow f^*\mathcal{E}'') \end{gathered}$$

Hence the collection of categories $\mathrm{Hecke}^r(S)$ when S runs through the category of schemes over $\mathbb{F}_q$ defines *the so-called (elementary) Hecke stack* Hecke^r.

We consider the stack Bun^r classifying vector bundles of rank r over X. It means that this stack associates to each scheme S the groupoid of vector bundles of rank r over the product $X \times S$. It is well known that Bun^r is an algebraic Deligne-Mumford stack, *cf.* [Lau-MB99].

Proposition 1.3. *The morphism of stacks*

$$\begin{gathered} \mathrm{Hecke}^r \longrightarrow X \times X \times \mathrm{Bun}^r \\ (\mathcal{E}, \infty, 0, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'') \mapsto (\infty, 0, \mathcal{E}) \end{gathered}$$

is representable, projective and smooth of relative dimension $2r-2$.

2. Stacks of Drinfeld shtukas

2.1. We have introduced the stacks Bun^r and Hecke^r classifying vector bundles of rank r and elementary Hecke modifications of rank r, respectively. We want to use these stacks to define the main object of these notes: the stack Sht^r of Drinfeld shtukas of rank r. It is simply the fiber product:

$$\begin{array}{ccc} \mathrm{Sht}^r & \longrightarrow & \mathrm{Bun}^r \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hecke}^r & \longrightarrow & \mathrm{Bun}^r \times \mathrm{Bun}^r, \end{array}$$

where the lower horizontal map is

$$\big(\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''\big) \mapsto (\mathcal{E}, \mathcal{E}''),$$

and the right vertical map is

$$\mathcal{E} \mapsto (\mathcal{E}, \mathcal{E}^\sigma).$$

In other words, the stack Sht^r of Drinfeld shtukas (or F-bundles, F-sheaves) of rank r associates to any scheme S the data $\widetilde{\mathcal{E}} = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \overset{\sim}{\longleftarrow} \mathcal{E}^\sigma)$ consisting of

- a vector bundle $\mathcal{E}$ of rank r over $X \times S$,
- two morphisms $\infty, 0 : S \longrightarrow X$,
- a modification consisting of two injections

  $$\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''$$

  of $\mathcal{E}$ by two vector bundles $\mathcal{E}'$ and $\mathcal{E}''$ of rank r over $X \times S$ such that the quotients $\mathcal{E}'/\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}''$ are supported by the graphs of ∞ and 0 respectively, and they are invertible on their support,
- an isomorphism $\mathcal{E}^\sigma \overset{\sim}{\longrightarrow} \mathcal{E}''$.

The morphisms 0 and ∞ are called *the zero and the pole* of the shtuka $\widetilde{\mathcal{E}}$.

In the case $r = 1$, a shtuka of rank 1 over a scheme S consists of a line bundle $\mathcal{L}$ over the fiber product $X \times S$ together with an isomorphism

$$\mathcal{L}^\sigma \otimes \mathcal{L}^{-1} \overset{\sim}{\longrightarrow} \mathcal{O}_{X\times S}(\Gamma_\infty - \Gamma_0)$$

where $\infty, 0 : S \longrightarrow X$ are two morphisms, and Γ_∞, Γ_0 are the graphs of ∞ and 0, respectively.

From this observation, it is easy to construct shtukas of higher rank. For higher rank $r \geq 2$, suppose that

$$\widetilde{\mathcal{L}} = (\mathcal{L}, \mathcal{L} \overset{j}{\hookrightarrow} \mathcal{L}' \overset{t}{\hookleftarrow} \mathcal{L}'' \overset{\sim}{\longleftarrow} \mathcal{L}^\sigma)$$

is a shtuka of rank 1 over a scheme S with the pole ∞ and the zero 0, and $\mathcal{F}$ is a vector bundle of rank $r-1$ over the curve X. Then one sees immediately that the direct sum

$$(\mathcal{L} \oplus \mathcal{F}, \mathcal{L} \oplus \mathcal{F} \overset{j\oplus \mathrm{id}}{\hookrightarrow} \mathcal{L}' \oplus \mathcal{F} \overset{t\oplus \mathrm{id}}{\hookleftarrow} \mathcal{L}' \oplus \mathcal{F} \overset{\sim}{\longleftarrow} \mathcal{L}^\sigma \oplus \mathcal{F})$$

is a shtuka of rank r over S with the same pole ∞ and the same zero 0.

Theorem 2.2 (Drinfeld)**.** *The stack* Sht^r *is an algebraic Deligne-Mumford stack and the characteristic map*

$$(\infty, 0) : \mathrm{Sht}^r \longrightarrow X \times X$$

is smooth of relative dimension $2r-2$*. Furthermore, it is locally of finite type.*

Thanks to Proposition 1.3, this theorem is a direct corollary of the following lemma applying to $\mathcal{W} = \mathrm{Sht}^r$, $\mathcal{M} = \mathrm{Hecke}^r$, $\mathcal{U} = \mathrm{Bun}^r$ and $Y = X \times X$.

Lemma 2.3. *Consider a cartesian diagram of stacks*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{U} \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{Frob}_{\mathcal{U}}, \mathrm{id}_{\mathcal{U}})} \\ \mathcal{M} & \xrightarrow{(\alpha,\beta)} & \mathcal{U} \times \mathcal{U} \\ {\scriptstyle \pi}\big\downarrow & & \\ Y & & \end{array}$$

where

- *Y is a scheme,*
- *$\mathcal{U}$ is algebraic and locally of finite type,*
- *$\mathcal{M}$ is algebraic and locally of finite type over Y,*
- *the morphism $(\pi, \alpha) : \mathcal{M} \longrightarrow Y \times U$ is representable.*

Then $\mathcal{W}$ is algebraic and locally of finite type. Moreover, the diagonal map $\mathcal{W} \longrightarrow \mathcal{W} \times \mathcal{W}$ which is representable, separated and of finite type is unramified everywhere, hence quasi-finite.

If $\mathcal{U}$ is smooth, and the morphism $(\pi, \alpha) : \mathcal{M} \longrightarrow Y \times U$ is smooth of relative dimension n, then the morphism $\mathcal{W} \longrightarrow Y$ is also smooth of relative dimension n.

The stack Sht^r has an infinite number of connected components $\mathrm{Sht}^{r,d}$ indexed by the degree of the associated vector bundle $\mathcal{E}$, i.e. one requires that $\deg \mathcal{E} = d$. Then

$$\mathrm{Sht}^r = \coprod_{d \in \mathbb{Z}} \mathrm{Sht}^{r,d}.$$

2.4. We consider the stack Triv_X^r which associates to each scheme S the data consisting of a vector bundle $\mathcal{E}$ of rank r over $X \times S$ together with an isomorphism $\mathcal{E}^\sigma \xrightarrow{\sim} \mathcal{E}$. As observed by Drinfeld, this stack is not interesting :

$$\mathrm{Triv}_X^r = \coprod_E \mathrm{Spec}\, \mathbb{F}_q / \mathrm{Aut}(E).$$

Here E runs through the set of vector bundles of rank r over the curve X and $\mathrm{Aut}(E)$ denotes the automorphism group of E.

Similarly, let I be a level, i.e. a finite closed subscheme of X. We consider the stack Triv_I^r which associates to each scheme S the data consisting of a vector bundle $\mathcal{E}$ of rank r over $I \times S$ together with an isomorphism $(\mathrm{id}_I \times \mathrm{Frob}_S)^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$. One can show that

$$\mathrm{Triv}_I^r = \mathrm{Spec}\, \mathbb{F}_q / \mathrm{GL}_r(\mathcal{O}_I).$$

For the rest of this section, we fix a level I of X. Suppose that $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \xleftarrow{\sim} \mathcal{E}^\sigma)$ is a shtuka of rank r over a scheme S such that the graphs of $\infty : S \longrightarrow X$ and $0 : S \longrightarrow X$ do not meet $I \times S$. Under this hypothesis, one obtains an isomorphism of restriction to $I \times S$:

$$\psi : \mathcal{E}_{\big| I\times S} \xrightarrow{\sim} \mathcal{E}^\sigma_{\big| I\times S}.$$

By definition, a *level structure of* $\widetilde{\mathcal{E}}$ *on* I is an isomorphism

$$\varphi : \mathcal{E}_{\big| I\times S} \xrightarrow{\sim} \mathcal{O}^r_{I\times S}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E}_{\big| I\times S} & \xrightarrow{\psi} & \mathcal{E}^\sigma_{\big| I\times S} \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^\sigma} \\ \mathcal{O}^r_{I\times S} & = & \mathcal{O}^r_{I\times S}. \end{array}$$

We denote by $\mathrm{Sht}^r_I(S)$ the category whose objects are shtukas of rank r over S together with a level structure on I and whose morphisms are isomorphisms between these objects.

Then the collection of categories $\mathrm{Sht}^r_I(S)$ when S runs over the category of schemes S over $\mathbb{F}_q$ defines a stack Sht^r_I called *the stack of shtukas of rank r with level structure on I.*

It fits into the cartesian diagram

$$\begin{array}{ccc} \mathrm{Sht}^r_I & \longrightarrow & \mathrm{Spec}\,\mathbb{F}_q \\ \big\downarrow & & \big\downarrow \\ \mathrm{Sht}^r \times_{X^2} (X - I)^2 & \longrightarrow & \mathrm{Triv}^r_I \end{array}$$

where

- the left arrow corresponds to the morphism forgetting the level structure,
- the right arrow corresponds to the trivial object $(\mathcal{O}^r_I, \mathrm{id})$ of $\mathrm{Triv}^r_I(\mathrm{Spec}\,\mathbb{F}_q)$,
- the lower arrow is given by the restriction to the level I which with the above notations sends $(\widetilde{\mathcal{E}}, \varphi : \mathcal{E}_{\big| I\times S} \xrightarrow{\sim} \mathcal{O}^r_{I\times S})$ to $(\mathcal{E}_{\big| I\times S}, \psi : \mathcal{E}_{\big| I\times S} \xrightarrow{\sim} \mathcal{E}^\sigma_{\big| I\times S})$.

It follows immediately:

Proposition 2.5 (Drinfeld)**.** *The forgetful morphism*

$$\mathrm{Sht}^r_I \longrightarrow \mathrm{Sht}^r \times_{X^2} (X - I)^2$$

is representable, finite, étale and Galois with Galois group $\mathrm{GL}_r(\mathcal{O}_I)$.

3. Harder-Narasimhan polygons

We have given a simple description of shtukas of rank 1. It implies that the stack $\mathrm{Sht}^{1,d}$ is of finite type for every integer d. However, this statement is not true for higher rank $r \geq 2$: the stack $\mathrm{Sht}^{r,d}$ is only locally of finite type but no longer of finite type. In order to construct interesting open substacks of finite type of $\mathrm{Sht}^{r,d}$, we digress for a moment to review the notion of Harder-Narasimhan polygons.

For the rest of these notes, a polygon is a map

$$p : [0, r] \longrightarrow \mathbb{R}$$

such that

- $p(0) = p(r) = 0$,
- p is affine on each interval $[i-1, i]$ for $0 < i \leq r$.

A polygon p is called

- *rational* if all the numbers $\{p(i)\}_{1<i<r}$ are rational,
- *convex* if for every integer i with $0 < i < r$, we have

$$-p(i+1) + 2p(i) - p(i-1) \geq 0,$$

- *big enough with respect to a real number* μ if all the terms $2p(i) - p(i-1) - p(i+1)$, with $0 < i < r$, are big enough with respect to μ,
- *integral with respect to an integer* d if all the terms $p(i) + \frac{i}{r}d$, with $0 < i < r$, are integers.

Suppose that $\mathcal{E}$ is a vector bundle over the geometric curve $\overline{X}$. Then the slope $\mu(E)$ of $\mathcal{E}$ is defined as the quotient

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E}.$$

With this defintion, $\mathcal{E}$ is called *semistable* (resp. *stable*) if for any proper subbundle $\mathcal{F}$ of $\mathcal{E}$, we have

$$\mu(\mathcal{F}) \leq \mu(E) \quad (resp. <).$$

It is obvious that

a) Any stable vector bundle over $\overline{X}$ is semistable.
b) Every line bundle over the geometric curve $\overline{X}$ is stable, hence semistable.

Suppose that $X = \mathbb{P}^1$. It is well known that every vector bundle over $\mathbb{P}^1$ is totally decomposable, i.e. there exist integers $n_1 \geq n_2 \geq \ldots \geq n_r$ such that

$$\mathcal{E} = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \ldots \oplus \mathcal{O}(n_r).$$

Then one sees immediately

a) $\mathcal{E}$ is semistable if and only if $n_1 = n_2 = \ldots = n_r$,
b) $\mathcal{E}$ is stable if and only if $r = 1$.

Harder and Narasimhan proved, *cf.* [Har-Nar75]:

Theorem 3.1 (Harder-Narasimhan). *Let $\mathcal{E}$ be a vector bundle over $\overline{X}$. Then there exists a unique filtration of maximal subbundles*

$$\mathcal{E}_0 = 0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_n = \mathcal{E}$$

of $\mathcal{E}$ which satisfies the following properties:

- $\mathcal{E}_i/\mathcal{E}_{i-1}$ *is semistable,*
- $\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \ldots > \mu(\mathcal{E}_n/\mathcal{E}_{n-1})$.

This filtration is called the (canonical) Harder-Narasimhan filtration associated to $\mathcal{E}$.

We define a polygon $p : [0, r] \longrightarrow \mathbb{R}^+$ as follows:

- $p(0) = p(r) = 0$,
- p *is affine on each interval* $[\mathrm{rk}\,\mathcal{E}_i, \mathrm{rk}\,\mathcal{E}_{i+1}]$, *for* $0 \leq i < r$,
- $p(\mathrm{rk}\,\mathcal{E}_i) = \deg \mathcal{E}_i - \frac{\mathrm{rk}\,\mathcal{E}_i}{r} \deg \mathcal{E}$.

Then it satisfies the following properties:

i) *p is convex.*
ii) *For any subbundle $\mathcal{F}$ of $\mathcal{E}$, we have*

$$\deg \mathcal{F} \leq \frac{\mathrm{rk}\,\mathcal{F}}{r} \deg \mathcal{E} + p(\mathrm{rk}\,\mathcal{F}). \tag{3.1.1}$$

This polygon is called the canonical Harder-Narasimhan filtration associated to $\mathcal{E}$.

We will give a full proof of this theorem in the case $r = 2$. For the general case, see [Har-Nar75]. Suppose that $\mathcal{E}$ is a vector bundle of rank 2 over $\overline{X}$.

If $\mathcal{E}$ is semistable, one verifies easily that

- $0 \subsetneq \mathcal{E}$ is the Harder-Narasimhan filtration of $\mathcal{E}$,
- the nil polygon $p = 0$ is the canonical Harder-Narasimhan polygon of $\mathcal{E}$.

If $\mathcal{E}$ is not semistable, then there exists a maximal line bundle $\mathcal{L}$ of $\mathcal{E}$ with $\deg \mathcal{L} > \deg \mathcal{E}/2$. We claim that

- $0 \subsetneq \mathcal{L} \subsetneq \mathcal{E}$ is the Harder-Narasimhan filtration of $\mathcal{E}$,
- the polygon $p(0) = 0, p(1) = \deg \mathcal{L} - \deg \mathcal{E}/2, p(2) = 0$ is the canonical Harder-Narasimhan polygon of $\mathcal{E}$.

In fact, one needs to check that if $\mathcal{L}'$ is a line bundle of $\mathcal{E}$, then $\deg \mathcal{L}' \leq \deg \mathcal{L}$. If $\mathcal{L}' \subseteq \mathcal{L}$, then it is obvious. Otherwise, the fact that $\mathcal{L}' \cap \mathcal{L} = 0$ implies an injection $\mathcal{L}' \hookrightarrow \mathcal{E}/\mathcal{L}$. Hence $\deg \mathcal{L}' \leq \deg \mathcal{E}/\mathcal{L} < \deg \mathcal{L}$. We are done.

Let us return to the projective line $X = \mathbb{P}^1$. Suppose that

$$\mathcal{E} = \mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \ldots \oplus \mathcal{O}(n_r)$$

with $n_1 = \ldots = n_{i_1} > n_{i_1+1} = \ldots = n_{i_2} > \ldots > n_{i_{n-1}+1} = \ldots = n_r$. Then

$$0 \subsetneq \mathcal{O}(n_{i_1})^{\oplus n_{i_1}} \subsetneq \mathcal{O}(n_1)^{\oplus n_{i_1}} \oplus \mathcal{O}(n_{i_2})^{\oplus n_{i_2} - n_{i_1}} \subsetneq \ldots \subsetneq \mathcal{E}$$

is the canonical Harder-Narasimhan filtration of $\mathcal{E}$.

Using the notion of canonical Harder-Narasimhan polygon, Lafforgue has introduced an interesting family of open substacks of finite type of $\mathrm{Sht}^{r,d}$, *cf.* [Laf98, théorème II.8].

Proposition 3.2 (Lafforgue)**.** *Let $p : [0, r] \longrightarrow \mathbb{R}$ be a convex polygon which is big enough with respect to the genus of X (or X for short) and r. Then there exists a unique open substack $\mathrm{Sht}^{r,d,p}$ of $\mathrm{Sht}^{r,d}$ such that a geometric point $\widetilde{\mathcal{E}}$ lies in this open if and only if the canonical Harder-Narasimhan polygon associated to $\mathcal{E}$ is bounded by p.*

The stack $\mathrm{Sht}^{r,d,p}$ is of finite type. Moreover, $\mathrm{Sht}^{r,d}$ is the union of these open substacks $\mathrm{Sht}^{r,d,p}$.

One can prove that these substacks $\mathrm{Sht}^{r,d,p}$ verify the valuative criterion for separatedness. However, they are not proper. This raises the compactification problem:

Problem 3.3. Find an algebraic proper stack $\mathcal{X}$ containing $\mathrm{Sht}^{r,d,p}$ as an open dense substack.

In the case $r = 2$, Drinfeld has constructed such a proper stack $\mathcal{X}$. For higher rank, Lafforgue has given a solution to this problem generalizing Drinfeld's construction. Let us pause for a moment to describe what needs to be done in order to construct such compactifications.

Step 1. *Introduce the stacks of degenerated and iterated shtukas which extends that of shtukas.*

This step is based on the well-studied scheme of complete homomorphisms of rank r which is obtained from the scheme of non-zero $n \times n$ matrices by a series of blow-ups. Roughly speaking, the last condition $\mathcal{E}^{\sigma} \xrightarrow{\sim} \mathcal{E}''$ will be replaced by a complete homomorphism $\mathcal{E}^{\sigma} \Rightarrow \mathcal{E}''$.

Step 2. *Stratify $\overline{\mathrm{Sht}^{r}}$ and give a modular description of each stratum.*

This step allows us to see how an iterated shtuka in each stratum can be "decomposed" as a product of shtukas.

Step 3. *Truncate $\overline{\mathrm{Sht}^{r,d}}$ to define proper compactifications $\overline{\mathrm{Sht}^{r,d,p}}$.*

This step is the hardest one: given a convex polygon p, one needs to truncate $\overline{\mathrm{Sht}^{r,d}}$ to obtain a proper substack $\overline{\mathrm{Sht}^{r,d,p}}$ containing $\mathrm{Sht}^{r,d,p}$. Lafforgue has used the semistable reduction à la Langton.

To simplify the exposition, from now on we will suppose $r = 2$ and refer any curious reader to the article of Lafforgue [Laf98] for the higher rank case.

4. Degenerated shtukas of rank 2

4.1. Drinfeld has remarked that to construct the desired compactifications, one needs to generalize the notion of shtukas. A simple way to do that is to loosen the last condition in the definition of a shtuka by replacing an isomorphism by a so-called *complete pseudo-homomorphism* or a *complete homomorphism*.

Suppose that S is a scheme over $\mathbb{F}_q$, and $\mathcal{E}$ and $\mathcal{F}$ are two vector bundles of rank 2 over S. Let $\mathcal{L}$ be a line bundle over S together with a global section $l \in \mathrm{H}^0(S, \mathcal{L})$. By definition, a *complete pseudo-homomorphism* of type $(\mathcal{L}, l)$

$$\mathcal{E} \Rightarrow \mathcal{F}$$

from $\mathcal{E}$ to $\mathcal{F}$ is a collection of two morphisms

$$\begin{aligned} u_1 :& \mathcal{E} \longrightarrow \mathcal{F} \\ u_2 :& \det \mathcal{E} \otimes \mathcal{L} \longrightarrow \det \mathcal{F} \end{aligned}$$

verifying

i) $\det u_1 = l u_2$,
ii) u_2 is an isomorphism.

A *complete homomorphism* of type $(\mathcal{L}, l)$

$$\mathcal{E} \Rightarrow \mathcal{F}$$

from $\mathcal{E}$ to $\mathcal{F}$ is a complete pseudo-homomorphism of type $(\mathcal{L}, l)$

$$\mathcal{E} \Rightarrow \mathcal{F}$$

from $\mathcal{E}$ to $\mathcal{F}$ satisfying the following additional condition:

iii) u_1 vanishes nowhere.

Suppose that the global section l is invertible. Hence the couple $(\mathcal{L}, l)$ can be identified with $(\mathcal{O}_S, 1)$. With this identification, a complete pseudo-homomorphism $\mathcal{E} \Rightarrow \mathcal{F}$ of type $(\mathcal{L}, l)$ is just an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathcal{F}$.

Now assume that $l = 0$. Then a complete pseudo-homomorphism (resp. a complete homomorphism) $\mathcal{E} \Rightarrow \mathcal{F}$ of type $(\mathcal{L}, l)$ consists of the following data:

- a maximal line bundle $\mathcal{E}_1$ of $\mathcal{E}$, a maximal line bundle $\mathcal{F}_1$ of $\mathcal{F}$ and an injection
$$\mathcal{E}/\mathcal{E}_1 \hookrightarrow \mathcal{F}_1$$
(resp. an isomorphism $\mathcal{E}/\mathcal{E}_1 \xrightarrow{\sim} \mathcal{F}_1$),
- an isomorphism
$$\det \mathcal{E} \otimes \mathcal{L} \xrightarrow{\sim} \det \mathcal{F}.$$

4.2. The stack DegSht^2 of degenerated shtukas of rank 2 associates to any scheme S the data $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{L}, l, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ consisting of

- a vector bundle $\mathcal{E}$ of rank 2 over $X \times S$,
- two morphisms $0, \infty : S \longrightarrow X$,
- a modification consisting of two injections
$$\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''$$
of $\mathcal{E}$ by two vector bundles $\mathcal{E}'$ and $\mathcal{E}''$ of rank 2 over $X \times S$ such that the quotients $\mathcal{E}'/\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}''$ are supported by the graphs of ∞ and 0 respectively, and are invertible on their support,

- a line bundle $\mathcal{L}$ over S and a global section $l \in \mathrm{H}^0(S, \mathcal{L})$,
- a complete pseudo-homomorphism of type $(\mathcal{L}, l)^{\otimes(q-1)}$

$$\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$$

  consisting of a morphism $u_1 : \mathcal{E}^\sigma \longrightarrow \mathcal{E}''$ and an isomorphism $u_2 : \det \mathcal{E}^\sigma \otimes \mathcal{L}^{q-1} \longrightarrow \det \mathcal{E}''$ such that
  i) $\det u_1 = l^{q-1} u_2$.
  ii) Generically, one can identify $\mathcal{E}''$ with $\mathcal{E}$ and with this identification, one requires that u_1 is not nilpotent.

In the above definition, the condition *i)* is exactly the condition appeared in the definition of a complete pseudo-homomorphism. However, the condition *ii)* is an additional condition.

This stack is algebraic in the sense of Artin, *cf.* [Laf98]. As before, we denote by $\mathrm{DegSht}^{2,d}$ the stack classifying degenerated shtukas of rank 2 and of degree d, i.e one requires that $\deg \mathcal{E} = d$. Hence

$$\mathrm{DegSht}^2 = \coprod_{d \in \mathbb{Z}} \mathrm{DegSht}^{2,d}.$$

Suppose that $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{L}, l, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ is a degenerated shtuka of rank 2 over k. Then we can identify $\mathcal{L}$ with the trivial line bundle $\mathcal{O}_{\mathrm{Spec}\, k}$ and l with an element of k.

a) If $l \neq 0$, then l is invertible. We have seen that the complete pseudo-homomorphism $\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$ is just an isomorphism $\mathcal{E}^\sigma \xrightarrow{\sim} \mathcal{E}''$ and the condition *ii)* in the above definition is automatically verified. This degenerated shtuka is in fact a Drinfeld shtuka.

b) Otherwise, $l = 0$. We have seen that the complete pseudo-homomorphism $\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$ is the data of

- a maximal line bundle $\overline{\mathcal{E}}_1$ of $\overline{\mathcal{E}} = \mathcal{E}^\sigma$ and a maximal line bundle $\mathcal{E}''_1$ of $\mathcal{E}''$,
- an injection $w : \overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \hookrightarrow \mathcal{E}''_1$.

We denote by $\mathcal{E}_1$ the maximal line bundle of $\mathcal{E}$ induced by $\mathcal{E}''_1$ using the elementary modification $\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''$. Then the condition *ii)* in the above definition is equivalent to the condition that $\overline{\mathcal{E}}_1 \cap \mathcal{E}_1^\sigma = 0$ in $\overline{\mathcal{E}}$. Consequently, $\overline{\mathcal{E}}_1 \oplus \mathcal{E}_1^\sigma$ is a subbundle of same rank as $\overline{\mathcal{E}}$ and so the quotient $\overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \oplus \mathcal{E}_1^\sigma$ is of finite length.

A geometric point $x \in \overline{X}$ is called a *degenerator* of the degenerated shtuka $\widetilde{\mathcal{E}}$ if one of the following conditions is verified:

- x is in the support of $\overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \oplus \mathcal{E}_1^\sigma$,
- x is in the support of the injection $w : \overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \hookrightarrow \mathcal{E}''_1$.

Proposition 4.3. *With the above notations, suppose that the canonical Harder-Narasimhan polygon of $\mathcal{E}$ is bounded by a polygon p_0. Then the number of degenerators of $\widetilde{\mathcal{E}}$ is bounded by a function of p_0.*

To prove this proposition, we observe that the injection $w : \overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \hookrightarrow \mathcal{E}_1''$ implies that $\deg \mathcal{E} \leq \deg \overline{\mathcal{E}}_1 + \deg \mathcal{E}_1''$. Since $\deg \mathcal{E}_1'' \leq \deg \mathcal{E}_1 + 1$, then $\deg \mathcal{E} \leq \deg \overline{\mathcal{E}}_1 + \deg \mathcal{E}_1 + 1$. Hence the support of $\overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \oplus \mathcal{E}_1^\sigma$ is of length at most 1.

On the other hand, we have just seen that the support of the injection w is of length bounded by $\deg \overline{\mathcal{E}}_1 + \deg \mathcal{E}_1 + 1 - \deg \mathcal{E}$, hence bounded by $2p_0(1) + 1$. We are done

5. Iterated shtukas of rank 2

Roughly speaking, iterated shtukas are degenerated shtukas satisfying several additional conditions. We will see that iterated shtukas can be "decomposed" as a product of shtukas.

The stack PreSht^2 *of pre-iterated shtukas of rank 2* associates to any scheme S the data $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{L}, l, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ consisting of

- a vector bundle $\mathcal{E}$ of rank 2 over $X \times S$,
- two morphisms $\infty, 0 : S \longrightarrow X$,
- a modification consisting of two injections

$$\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}''$$

  of $\mathcal{E}$ by two vector bundles $\mathcal{E}'$ and $\mathcal{E}''$ of rank 2 over $X \times S$ such that the quotients $\mathcal{E}'/\mathcal{E}$ and $\mathcal{E}'/\mathcal{E}''$ are supported by the graphs of ∞ and 0 respectively, and are invertible on their support,
- a line bundle $\mathcal{L}$ over S and a global section l of $\mathcal{L}$,
- a complete homomorphism of type $(\mathcal{L}, l)^{\otimes(q-1)}$

$$\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$$

  consisting of a morphism $u_1 : \mathcal{E}^\sigma \longrightarrow \mathcal{E}''$ and an isomorphism $u_2 : \det \mathcal{E}^\sigma \otimes \mathcal{L}^{q-1} \longrightarrow \det \mathcal{E}''$ such that
  - i) $\det u_1 = l^{q-1} u_2$.
  - i') u_1 vanishes nowhere.
  - ii) Generically, one can identify $\mathcal{E}''$ with $\mathcal{E}$ and with this identification, one requires that u_1 is not nilpotent.

As before, this stack has a stratification containing two strata as follows:

a) The open stratum $\mathrm{PreSht}^{2,\mathrm{open}}$ corresponds pre-iterated shtukas with an invertible global section l. One can then identify $(\mathcal{L}, l)$ with $(\mathcal{O}_S, 1)$ and show that the pre-iterated shtuka is in fact a shtuka. The open stratum is just the stack Sht^2 of shtukas of rank 2.

b) The closed stratum $\mathrm{PreSht}^{2,\mathrm{closed}}$ corresponds to the condition that the global section l is zero. Under this condition, the complete homomorphism $\mathcal{E}^\sigma \Rightarrow \mathcal{E}''$ consists of

- a maximal line bundle $\overline{\mathcal{E}}_1$ of $\overline{\mathcal{E}} = \mathcal{E}^\sigma$ and a maximal line bundle $\mathcal{E}_1''$ of $\mathcal{E}''$,
- an isomorphism $w : \overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \xrightarrow{\sim} \mathcal{E}_1''$.

We consider the substack $\overline{\mathrm{Sht}^{2,\mathrm{closed}}}$ of the closed stratum by imposing the following conditions:

i) If we set $\mathcal{E}_1' = \mathcal{E}_1''$, then $\mathcal{E}'/\mathcal{E}_1'$ is torsion-free, hence locally free of rank 1 over $X \times S$.
ii) The natural morphism $\mathcal{E}_1' \longrightarrow \mathcal{E}'/\mathcal{E}$ is surjective, hence the kernel $\mathcal{E}_1$ of this morphism is locally free of rank 1 over $X \times S$.

Drinfeld proved, *cf.* [Dri89]:

Proposition 5.1 (Drinfeld)**.** *There exists a unique substack* $\overline{\mathrm{Sht}^2}$ *of* $\overline{\mathrm{PreSht}^2}$ *such that* $\overline{\mathrm{Sht}^2} \cap \mathrm{PreSht}^{2,\mathrm{open}} = \mathrm{Sht}^2$ *and* $\overline{\mathrm{Sht}^2} \cap \mathrm{PreSht}^{2,\mathrm{closed}} = \overline{\mathrm{Sht}^{2,\mathrm{closed}}}$.

This stack is called *the stack of iterated shtukas of rank 2*. It is an algebraic stack in the sense of Artin. We denote by $\overline{\mathrm{Sht}^{2,d}}$ the stack classifying iterated shtukas of rank 2 and of degree d, i.e $\deg \mathcal{E} = d$. Hence

$$\overline{\mathrm{Sht}^2} = \coprod_{d \in \mathbb{Z}} \overline{\mathrm{Sht}^{2,d}}.$$

Observe that if $\widetilde{\mathcal{E}}$ is an iterated shtuka of rank 2 over k in the closed stratum, it has exactly one degenerator.

5.2. The goal of this section is to show how to decompose an iterated shtuka into shtukas. Suppose that $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{L}, l, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ is an iterated shtuka of rank 2 over a scheme S with the pole ∞ and the zero 0.

a) If $\widetilde{\mathcal{E}}$ lies in the open stratum Sht^2, then $\widetilde{\mathcal{E}}$ is in fact a Drinfeld shtuka.

b) If $\widetilde{\mathcal{E}}$ lies in the closed stratum $\overline{\mathrm{Sht}^{2,\mathrm{closed}}}$, we have constructed different maximal line bundles $\mathcal{E}_1$, $\mathcal{E}_1'$, $\mathcal{E}_1''$, $\overline{\mathcal{E}}_1$ of $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$, $\overline{\mathcal{E}}$, respectively.

We claim that $(\mathcal{E}_1 \hookrightarrow \mathcal{E}_1' \hookleftarrow \mathcal{E}_1^\sigma)$ is a shtuka of rank 1 over S with the pole ∞. For the injection $\mathcal{E}_1 \hookrightarrow \mathcal{E}_1'$, one takes the natural one. Since $\mathcal{E}_1^\sigma \cap \overline{\mathcal{E}}_1 = 0$, the composition

$$\mathcal{E}_1^\sigma \hookrightarrow \overline{\mathcal{E}}/\overline{\mathcal{E}}_1 \xrightarrow{\sim} \mathcal{E}_1'' = \mathcal{E}_1'$$

is in fact an injection. As $\mathcal{E}_1'/\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}'/\mathcal{E}$, this shtuka of rank 1 has the same pole ∞ as the iterated shtuka $\widetilde{\mathcal{E}}$.

Next, we claim that $(\mathcal{E}/\mathcal{E}_1 \otimes \mathcal{L} \hookleftarrow \overline{\mathcal{E}}_1 \otimes \mathcal{L}^q \hookrightarrow \overline{\mathcal{E}}/\mathcal{E}_1^\sigma \otimes \mathcal{L}^q)$ is a shtuka of rank 1 with the zero 0. In fact, the injection $\overline{\mathcal{E}}_1 \otimes \mathcal{L}^q \hookrightarrow \mathcal{E}/\mathcal{E}_1 \otimes \mathcal{L}$ is the composition

$$\overline{\mathcal{E}}_1 \otimes \mathcal{L}^q \xrightarrow{\sim} \mathcal{E}''/\mathcal{E}_1'' \otimes \mathcal{L} \hookrightarrow \mathcal{E}'/\mathcal{E}_1' \otimes \mathcal{L} \xrightarrow{\sim} \mathcal{E}/\mathcal{E}_1 \otimes \mathcal{L}.$$

The injection $\overline{\mathcal{E}}_1 \otimes \mathcal{L}^q \hookrightarrow \overline{\mathcal{E}}/\mathcal{E}_1^\sigma \otimes \mathcal{L}^q$ follows from the fact that $\mathcal{E}_1^\sigma \cap \overline{\mathcal{E}}_1 = 0$ in $\overline{\mathcal{E}}$. It is easy to see that this shtuka has the same zero 0 as that of the iterated shtuka $\widetilde{\mathcal{E}}$.

5.3. We observe that to give a polygon $p : [0,2] \longrightarrow \mathbb{R}$ is equivalent to give the real number $p(1)$. Suppose that d is an integer and $p : [0,2] \longrightarrow \mathbb{R}$ a convex polygon which is integral with respect to d. We will define a substack $\overline{\mathrm{Sht}^{2,d,p}}$ of $\overline{\mathrm{Sht}^{2,d}}$ containing $\mathrm{Sht}^{2,d,p}$ as an open dense substack as follows:

Suppose that $\widetilde{\mathcal{E}} = (\mathcal{E}, \mathcal{L}, l, \mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \Leftarrow \mathcal{E}^\sigma)$ is an iterated shtuka of rank 2 and of degree d over k. It lies in $\overline{\mathrm{Sht}^{2,d,p}}(k)$ if and only if the following condition is verified:

- If $\widetilde{\mathcal{E}}$ lies in the open stratum $\mathrm{Sht}^{2,d}$, one requires that the canonical Harder-Narasimhan polygon of $\mathcal{E}$ is bounded by p.
- If $\widetilde{\mathcal{E}}$ lies in the closed stratum $\overline{\mathrm{Sht}^{2,d,\mathrm{closed}}}$, one denotes by $\mathcal{E}_1$ (resp. $\overline{\mathcal{E}}_1$) be the maximal line bundle of $\mathcal{E}$ (resp. $\overline{\mathcal{E}}$) defined as before, then one requires

$$\deg \mathcal{E}_1 = \frac{d}{2} + p(1),$$
$$\deg \overline{\mathcal{E}}_1 = \frac{d}{2} - p(1) - 1.$$

Drinfeld proved that there exists a unique substack $\overline{\mathrm{Sht}^{2,d,p}}$ of $\overline{\mathrm{Sht}^{2,d}}$ which associates to each scheme S the data of iterated shtukas $\widetilde{\mathcal{E}}$ of rank 2 and of degree d over S such that for every geometric point s of S, the induced iterated shtuka is in $\overline{\mathrm{Sht}^{2,d,p}}(k)$. It contains $\mathrm{Sht}^{2,d,p}$ as an open dense substack.

Here is the desired theorem, *cf.* [Dri89]:

Theorem 5.4 (Drinfeld)**.** *Suppose that d is an integer and $p : [0,2] \longrightarrow \mathbb{R}$ a polygon which is big enough with respect to X and integral with respect to d. Then the natural morphism*

$$\overline{\mathrm{Sht}^{2,d,p}} \longrightarrow X \times X$$

is proper. In particular, $\overline{\mathrm{Sht}^{2,d,p}}$ is proper.

6. Valuative criterion of properness

In this section, we will try to sketch Drinfeld's proof of Theorem 5.4 using the semistable reduction à la Langton, *cf.* [Lan75].

6.1. Suppose that A is a discrete valuation ring over $\mathbb{F}_q$. We denote by K its fraction field, κ its residual field, π an uniformizing element and $\mathrm{val} : K \longrightarrow \mathbb{Z} \cup \{\infty\}$ the valuation map.

We denote by X_A (resp. X_K, X_κ) the fiber product $X \times \mathrm{Spec}\, A$ (resp. $X \times \mathrm{Spec}\, K$, $X \times \mathrm{Spec}\, \kappa$). Let A_X be the local ring of the scheme X_A at the generic point of the special fiber X_κ. It is a discrete valuation ring containing π as a uniformizing element. The fraction field K_X of A_X can be identified with the fraction field of $F \otimes K$ and the residual field of A_X can be identified with the fraction field of $F \otimes \kappa$. Finally, A_X

(resp. K_X) is equipped with a Frobenius endomorphism Frob induced by $\mathrm{Id}_F \otimes \mathrm{Frob}_A$ (resp. $\mathrm{Id}_F \otimes \mathrm{Frob}_K$).

Since X_A is a regular surface, it is well known that the category of vector bundles of rank r over X_A is equivalent to the category of vector bundles of rank r over the generic fiber X_K equipped with a lattice in its generic fiber. Suppose that E is a vector bundle of rank r over X_A, it correponds to the couple $(\mathcal{E}, M)$ where $\mathcal{E}$ is the restriction of E to the generic fiber X_K, and M is the local ring of E at the generic point of the special fiber X_κ. Let us denote by

$$(\mathcal{E}, M) \mapsto \mathcal{E}(M)$$

the quasi-inverse functor.

In order to prove the above theorem, we use the valuative criterion of properness, i.e. given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \overline{\mathrm{Sht}^{2,d,p}} \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} A & \longrightarrow & X \times X, \end{array}$$

one needs to show that up to a finite extension of A, there exists a unique morphism $\operatorname{Spec} A \longrightarrow \overline{\mathrm{Sht}^{2,d,p}}$ such that after putting it in the above diagram, the two triangles are still commutative. In the rest of this section, we will explain how to contruct this map. Its uniqueness which is easier to prove will not be discussed

For simplicity, suppose that the morphism $\operatorname{Spec} K \longrightarrow \overline{\mathrm{Sht}^{2,d,p}}$ factors through the open dense substack $\mathrm{Sht}^{2,d,p}$. Then we write $\widetilde{\mathcal{E}} = (\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \overset{\sim}{\longleftarrow} \mathcal{E}^\sigma)$ for the corresponding shtuka. One needs to extend it to a degenerated shtuka in $\overline{\mathrm{Sht}^{2,d,p}}(A)$.

6.2. Suppose that V is the generic fiber of $\mathcal{E}$. Then it is a vector space of dimension r over K_X equipped with an isomorphism $\varphi : V^\sigma \overset{\sim}{\longrightarrow} V$ or equivalently an injective semi-linear map

$$\varphi : V \longrightarrow V.$$

This means

- For u and v in V, $\varphi(u+v) = \varphi(u) + \varphi(v)$.
- For $t \in K$ and $v \in V$, $\varphi(tv) = \mathrm{Frob}(t) \cdot \varphi(v)$.

Such a couple (V, φ) is called a *φ-space*.

Suppose that M is a lattice in V. We have seen that it induces vector bundles $\mathcal{E}(M)$, $\mathcal{E}'(M)$ and $\mathcal{E}''(M)$ of rank 2 over X_A extending $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$, respectively. Furthermore, one has an induced modification $\mathcal{E}(M) \hookrightarrow \mathcal{E}'(M) \hookleftarrow \mathcal{E}''(M)$ whose quotients are automatically supported by the graphs of the pole and the zero $\infty, 0 : \operatorname{Spec} A \longrightarrow X$.

However, the isomorphism $\mathcal{E}^\sigma \overset{\sim}{\longrightarrow} \mathcal{E}''$ does not always extend to a complete pseudo-homomorphism $(\mathcal{E}(M))^\sigma \Rightarrow \mathcal{E}''(M)$. This can be done if and only if M is a so-called degenerated lattice. By definition, a lattice M of V is called *degenerated* if the following conditions are satisfied:

- $\varphi(M) \subseteq M$.
- The reduction $\overline{\varphi} : \overline{M} \longrightarrow \overline{M}$ is not nilpotent, where $\overline{M} = M/\pi M$.

If $\varphi(M) = M$, M is called a *φ-lattice*. Otherwise, the degenerated lattice M is called an *iterated lattice*.

Suppose that M is a degenerated lattice. Then it induces a degenerated shtuka

$$\mathcal{E}(M) \hookrightarrow \mathcal{E}'(M) \hookleftarrow \mathcal{E}''(M) \Rightarrow (\mathcal{E}(M))^\sigma$$

over X_A which extends the shtuka $\mathcal{E} \hookrightarrow \mathcal{E}' \hookleftarrow \mathcal{E}'' \xleftarrow{\sim} \mathcal{E}^\sigma$ over the generic fiber X_K.

The restriction to the special fiber X_κ gives a degenerated shtuka

$$\mathcal{E}^M \hookrightarrow \mathcal{E}'^M \hookleftarrow \mathcal{E}''^M \Leftarrow (\mathcal{E}^M)^\sigma = \overline{\mathcal{E}}^M$$

whose generic fiber is the quotient $M/\pi M$. If M is a φ-lattice, it is a Drinfeld shtuka. If M is an iterated lattice, it is a degenerated shtuka but not a Drinfeld shtuka. One has then maximal line bundle $\mathcal{E}_1^M$, $\overline{\mathcal{E}}_1^M$ of $\mathcal{E}^M$, $\overline{\mathcal{E}}^M$, respectively with $(\mathcal{E}_1^M)^\sigma \cap \overline{\mathcal{E}}_1^M = 0$. One remarks that it is an iterated shtuka if and only if $\deg \mathcal{E}_1^M + \deg \overline{\mathcal{E}}_1^M = \deg \mathcal{E}^M - 1$.

Proposition 6.3. *a) After a finite extension of A, there exists a degenerated lattice in V. Among these lattices, there is a maximal one noted M_0, i.e. every degenerated lattice is contained in M_0.*

b) Suppose that M_0 is iterated, hence the image $\overline{\varphi}(M_0/\pi M_0)$ is the unique $\overline{\varphi}$-invariant line of $M_0/\pi M_0$, says l. Then the degenerated lattices in V form a descending chain

$$M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_i \supsetneq M_{i+1} \dots$$

such that for any $i \in \mathbb{N}$, $M_0/M_i \simeq A_X/\pi^i A_X$ and the image of the natural map $M_i/\pi M_i \longrightarrow M_0/\pi M_0$ is the line l. In particular, every degenerated lattice is iterated.

c) Suppose that M_0 is a φ-lattice. Then for each $\overline{\varphi}$-invariant line l in $M_0/\pi M_0$, there exists a descending chain

$$M_0 \supsetneq M_{1,l} \supsetneq \dots \supsetneq M_{i,l} \supsetneq M_{i+1,l} \dots$$

such that for any $i \in \mathbb{N}$, $M_0/M_{i,l} \simeq A_X/\pi^i A_X$ and the image of the natural map $M_{i,l}/\pi M_{i,l} \longrightarrow M_0/\pi M_0$ is the line l.

These families cover all degenerated lattices when l runs through the set of invariant lines of $M_0/\pi M_0$. In particular, every degenerated lattice except M_0 is iterated.

6.4. We are now ready to prove the valuative criterion of properness. First, suppose that after a finite extension of A, the φ-space V admits a φ-space M_0. If the shtuka

$$\mathcal{E}^{M_0} \hookrightarrow \mathcal{E}'^{M_0} \hookleftarrow \mathcal{E}''^{M_0} \xleftarrow{\sim} (\mathcal{E}^{M_0})^\sigma = \overline{\mathcal{E}}^{M_0}$$

associated with M_0 verifies: for every line bundle $\mathcal{L}$ of $\mathcal{E}^{M_0}$, one has the inequality:

$$\deg \mathcal{L} \leq \frac{\deg \mathcal{E}^{M_0}}{2} + p(1),$$

then we are done. Otherwise, there exists a maximal line bundle $\mathcal{L}$ of $\mathcal{E}^{M_0}$ such that

$$\deg \mathcal{L} > \frac{\deg \mathcal{E}^{M_0}}{2} + p(1).$$

For the polygon p is convex enough, i.e. $p(1)$ is large enough, one can show that the generic fiber l of $\mathcal{L}$ is a $\overline{\varphi}$-invariant line in $M_0/\pi M_0$. One considers the descending chain corresponding to this invariant line l:

$$M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_i \supsetneq M_{i+1} \ldots$$

Since M_i is always iterated for $i \geq 1$, one has maximal line bundles $\mathcal{E}_1^{M_i}$ of $\mathcal{E}^{M_i}$. One shows that they form a descending chain:

$$\mathcal{E}_1^{M_0} := \mathcal{L} \supseteq \mathcal{E}_1^{M_1} \supseteq \mathcal{E}_1^{M_2} \supseteq \ldots \supseteq \mathcal{E}_1^{M_i} \supseteq \mathcal{E}_1^{M_{i+1}} \supseteq \ldots$$

with $\deg \mathcal{E}_1^{M_{i+1}} = \deg \mathcal{E}_1^{M_i}$ or $\deg \mathcal{E}_1^{M_{i+1}} = \deg \mathcal{E}_1^{M_i} - 1$. One can show that in the latter case, the degenerated shtuka associated to M_i is iterated.

Using the fact that the canonical Harder-Narasimhan polygon of the generic vector bundle $\mathcal{E}$ is bounded by p, one proves that there exists a positive integer i such that $\deg \mathcal{E}_1^{M_i} = \frac{\deg \mathcal{E}^{M_0}}{2} + p(1)$ and $\deg \mathcal{E}_1^{M_{i+1}} = \frac{\deg \mathcal{E}^{M_0}}{2} + p(1) - 1$. The lattice M_i is the desired one.

We can then suppose that for every finite extension of A, the maximal degenerated lattice in V is always iterated. The key observation is that under this hypothesis, we can suppose that the maximal line bundle $\mathcal{E}_1^{M_0}$ satisfies:

$$\deg \mathcal{E}_1^{M_0} > \frac{\deg \mathcal{E}^{M_0}}{2} + p(1).$$

This claim is in fact the hardest part of the proof. Admitting this result, one can then repeat the above arguments to find the desired lattice. We are done.

7. Another proof using the Geometric Invariant Theory

The previous proof is known as the semistable reduction à la Langton. In fact, Langton [Lan75] used a similar strategy to prove the properness of the moduli space of semistable vector bundles over a smooth projective variety.

Another well-known proof of the above result in the case of smooth projective curves is due to Seshadri: he used the Geometric Invariant Theory. We refer a unfamiliar reader to the excellent book [Mum-For-Kir94] for more details about this theory. Roughly speaking, given a quasi-projective scheme Y equipped with an action of a reductive group G, the Geometric Invariant Theory (or GIT for short) gives G-invariant open subschemes U of Y such that the quotient $U//G$ exists. The fundamental theorem of this theory is as follows: suppose that the action of G can be lifted to an ample line bundle of Y which will be called a *polarization*, one can define the open subsets Y^{s} and Y^{ss} of stable and semistable points of Y with respect to this polarization: $Y^s \subseteq Y^{\mathrm{ss}} \subseteq Y$. Then the quotient $Y^{\mathrm{ss}}//G$ exists and it is quasi-projective. The geometric points of the quotient $Y^{\mathrm{s}}//G$ are in bijection with the G-orbits of Y^{s}. Moreover, if Y is projective, the quotient $Y^{\mathrm{ss}}//G$ is also projective.

The definitions of stable and semistable points are quite abstract. Fortunately, in practice, one has a powerful tool to determine these points called *the Hilbert-Mumford numerical criterion.*

When the polarization varies, one gets different GIT quotients. However, one can show that given a couple (Y, G) as above, there is only a finite number of GIT quotients, *cf.* [Dol-Hu98, Tha96].

The strategy suggested by Sheshadri to prove the properness of the moduli space of semistable vector bundles of fixed rank and fixed degree over a smooth projective curve is to realize this moduli space as a GIT quotient of a projective scheme by a reductive group. As an immediate corollary, it is projective, hence proper.

Following a suggestion of L. Lafforgue, the author has successfully applied the GIT method to rediscover the Drinfeld compactifications $\overline{\mathrm{Sht}^{2,d,p}}$ and proved that these compactifications are proper over the product $X \times X$.

To do that, we fix a sufficiently convex polygon p_0. Suppose that N is a finite closed subscheme of the curve X and d is an integer. One first defines *the stack* $\mathrm{DegSht}_N^{2,d}$ *classifying degenerated shtukas of rank 2 and degree d with level structure N*. Next, one introduces the open quasi-projective substack $\mathrm{DegSht}_N^{2,d,p_0}$ by the same truncation process presented in the previous sections. It is equipped with different polarizations of a reductive group, says G, indexed by convex polygons $p \leq p_0$. Under several mild conditions on N and d, the different quotients $\mathrm{DegSht}_N^{2,d,p_0}//G$ are exactly the fiber products $\overline{\mathrm{Sht}^{2,d,p}} \times_{X^2} (X-N)^2$ $(p \leq p_0)$. Moreover, as a by-product of the Geometric Invariant Theory, one can show that these quotients are proper over $(X-N)^2$. One varies the level N to obtain the desired result. For more details, see [NgoDac04].

8. Discussion

Both of the previous approaches can be extended to the higher rank case: the semistable reduction à la Langton is done by Lafforgue [Laf98] and the GIT approach is done in [NgoDac04]. The latter one requires a technical condition that the cardinal of the finite field $\mathbb{F}_q$ is big enough with respect to the rank r.

Recently, for any split reductive group G over $\mathbb{F}_q$ and any sequence of dominant coweights $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, Ngo [Ngo03] and Varshavsky [Var04] have introduced the stack $\mathrm{Sht}_{G,\underline{\mu}}$ of G-shtukas associated to $\underline{\mu}$: it classifies the following data:

- a G-torsor $\mathcal{E}_0$ over X,
- n points $x_1, x_2, \ldots, x_n \in X$,
- modifications $\mathcal{E}_0 \rightsquigarrow \mathcal{E}_1 \rightsquigarrow \ldots \rightsquigarrow \mathcal{E}_n$ such that, for each integer $1 \leq i \leq n$, the modification $\mathcal{E}_i \rightsquigarrow \mathcal{E}_{i+1}$ is of type μ_i at x_i,
- an isomorphism $\mathcal{E}_0^\sigma \xrightarrow{\sim} \mathcal{E}_n$.

One can raise the question of compactifying certain interesting open substacks of finite type of these stacks $\mathrm{Sht}_{G,\underline{\mu}}$. The semistable reduction à la Langton seems to be difficult to generalize. However, the GIT method can be adapted without difficulties, *cf.* [NgoDac04, NgoDac06a, NgoDac06b].

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CNRS - Université de Paris Nord, LAGA - Département de Mathématiques, 99 avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: ngodac@math.univ-paris13.fr