On regular periodic solutions to the Navier-Stokes equations. Case B

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Abstract

We find a global a priori estimate for solutions to the Navier-Stokes equations with periodic boundary conditions guaranteeing in view of the Serrin type condition the existence of global regular solutions. We derive the following estimate

$$\|V(t)\|_{H^1(\Omega)} \le c, \tag{1}$$

where V is the velocity of the fluid.
The estimate (1) is proved in two steps. First we derive a global estimate guaranteeing the existence of global regular solutions to weakly compressible Navier-Stokes equations with large second viscosity, density close to a constant and gradient part of velocity small. Next we show that solutions to the Navier-Stokes equations remain close to solutions to the weakly compressible Navier-Stokes equations if the corresponding initial data and external forces are sufficiently close.

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1 Introduction

We are going to prove existence of global regular periodic solutions to the Navier-Stokes equations in a box $\Omega \subset \mathbb{R}^3$

$$
\begin{aligned}
&a(V_t + V\cdot\nabla V) - \mu\Delta V + \nabla P = aF \quad && \text{in} \ \ \Omega\times\mathbb{R}_+,\\
&\operatorname{div} V = 0 && \text{in} \ \ \Omega\times\mathbb{R}_+,\\
&V|_{t=0} = V_0 && \text{in} \ \ \Omega,
\end{aligned}
\tag{1.1}
$$

where $V = (V_1(x,t), V_2(x,t), V_3(x,t)) \in \mathbb{R}^3$ is the velocity of the fluid, $x = (x_1, x_2, x_3)$ are the Cartesian coordinates, $P = P(x,t) \in \mathbb{R}$ is the pressure, $\mu > 0$ is the viscosity coefficient, $F = F(x,t) = (F_1(x,t), F_2(x,t), F_3(x,t)) \in \mathbb{R}^3$ is the external force and a is positive constant.

Since the existence of global regular solutions to weakly compressible (second viscosity coefficient large, density close to a constant and divergence of velocity small) Navier-Stokes equations is known (see [Z1]) we are looking for solutions to (1.1) as for stability of these regular solutions. Therefore, the weakly compressible barotropic motions are described by the following problem

$$
\begin{aligned}
&\varrho v_t + \varrho v\cdot\nabla v - \mu\Delta v - \nu\nabla\operatorname{div} v + \nabla p = \varrho f \quad && \text{in} \ \ \Omega\times\mathbb{R}_+,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in} \ \ \Omega\times\mathbb{R}_+,\\
&v|_{t=0} = v_0, \quad \varrho|_{t=0} = \varrho_0 && \text{in} \ \ \Omega,
\end{aligned}
\tag{1.2}
$$

where $\varrho = \varrho(x,t) \in \mathbb{R}_+$ is the density of the fluid, $v = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is velocity, $p = A\varrho^{\varkappa}$, $\varkappa > 1$, A- constant, is the pressure.
By weakly compressible motions we mean such motions that

$$
\varrho = a + \eta,
\tag{1.3}
$$

where a is the constant from (1.1) and η is small. Moreover, ν is large and $\operatorname{div} v(0)$ is as small as we want. Then problem (1.2) takes the form

$$
\begin{aligned}
&(a+\eta)(v_t + v\cdot\nabla v) - \mu\Delta v - \nu\nabla\operatorname{div} v + a_0\nabla\eta\\
&\ = (p_\varrho(a) - p_\varrho(a+\eta))\nabla\eta + (a+\eta)f,\\
&\eta_t + v\cdot\nabla\eta + a\operatorname{div} v + \eta\operatorname{div} v = 0,\\
&v|_{t=0} = v_0, \quad \eta|_{t=0} = \eta_0,
\end{aligned}
\tag{1.4}
$$

where $p_\varrho = \frac{dp}{d\varrho}$, $a_0 = p_\varrho(a)$.
To show stability of incompressible motions in the set of weakly compressible barotropic motions we introduce the quantities

$$
u = v - V, \quad q = p - P, \quad g = f - F
\tag{1.5}
$$

Then the quantities u, q, g satisfy

$$
\begin{aligned}
&au_t + \eta(v_t + v\cdot\nabla v) + a(v\cdot\nabla u + u\cdot\nabla V) - \mu\Delta u - \nu\nabla\operatorname{div} v + \nabla q\\
&= ag + \eta f,\\
&u|_{t=0} = v_0 - V_0 \equiv u_0,
\end{aligned}
\tag{1.6}
$$

where we used that $v\cdot\nabla v - V\cdot\nabla V = v\cdot\nabla u + u\cdot\nabla V = v\cdot\nabla u + u\cdot\nabla(v-u)$. Moreover, η is a solution to the problem

$$
\begin{aligned}
&\eta_t + v\cdot\nabla\eta + a\operatorname{div} v + \eta\operatorname{div} v = 0\\
&\eta|_{t=0} = \eta_0.
\end{aligned}
\tag{1.7}
$$

We introduce potentials φ and ψ such that

$$
v = \operatorname{rot}\psi + \nabla\varphi + G,
\tag{1.8}
$$

where

$$
G = \frac{1}{a|\Omega|}\bigg[-\int\limits_\Omega \eta v dx + \int\limits_{\Omega^t}(a+\eta) f dx dt' + \int\limits_\Omega (a+\eta_0) v_0 dx\bigg].
$$

Then problems (1.6) and (1.7) take the form

$$
\begin{aligned}
&au_t + \eta(\operatorname{rot}\psi_t + \nabla\varphi_t + G_t + (\operatorname{rot}\psi + \nabla\varphi + G)\cdot\nabla(\operatorname{rot}\psi + \nabla\varphi))\\
&\quad + a((\operatorname{rot}\psi + \nabla\varphi + G)\cdot\nabla u + u\cdot\nabla V) - \mu\Delta u - \nu\nabla\Delta\varphi + \nabla q\\
&= ag + \eta f,\\
&u|_{t=0} = \nabla\varphi_0 + \operatorname{rot}\psi_0 - V_0,
\end{aligned}
\tag{1.9}
$$

and

$$
\begin{aligned}
&\eta_t + v\cdot\nabla\eta + a\Delta\varphi + \eta\Delta\varphi = 0\\
&\eta|_{t=0} = \eta_0.
\end{aligned}
\tag{1.10}
$$

The aim of this paper is deriving such estimate for solutions to the Navier-Stokes equations (1.1) that regularity of weak solutions can be proved. We are not able to do it for solutions to (1.1) directly.

However we proved in [Z1] the existence of global regular solutions to weakly compressible Navier-Stokes system (1.4). Having the result from [Z1] we construct system (1.6) for differences (1.5) with coefficients dependent on regular global solutions to (1.4). Therefore solutions to (1.1) are approximated by solutions to (1.4). Hence we have the system with small data so global estimates for regular solutions are easily derived. We restrict our considerations to derive the estimate for $\|u\|_{L_\infty(\mathbb{R}_+;H^1(\Omega))}$. Having the same estimate for v we obtain in (3.23) that $\|V\|_{L_\infty(\mathbb{R}_+;H^1(\Omega))}$ is bounded by data.

To formulate the main result we first recall the theorem on existence of global regular solutions to problem (1.4) from [Z1].

Theorem A. *Let $\nu > 0$, $T > 0$ be given. Let f_g be the gradient part of f. Let $v = \nabla\varphi + \operatorname{rot}\psi$, $\varrho = a + \eta$, a-positive constant be a solution to (1.4). Let $\eta(0)$, $\nabla\varphi(0)$, $\operatorname{rot}\psi(0) \in \Gamma_1^2(\Omega)$, $|\eta|_\infty < a/2$, $\|\nabla\varphi(0)\|_{\Gamma_1^2(\Omega)} \le c/\sqrt{\nu}$, $\|\operatorname{rot}\psi(0)\|_{\Gamma_1^2(\Omega)} \le c$, $\|\eta(0)\|_{\Gamma_1^2(\Omega)} \le c/\nu$, $f \in L_2(0,T;\Gamma_1^1(\Omega))$, $|f_g|_{L_2(0,T;L_{6/5}(\Omega))} \le c/\nu$, $f \in L_6(0,T;L_3(\Omega)) \cap L_1(0,T;L_\infty(\Omega))$. Assume that there exist positive constants φ_* and c_1 such that $c_1/\nu^\varkappa \le \varphi_* \le \varphi(0)$, where $\varkappa \in (1/2,1)$. Then for ν sufficiently large and $T < \nu$ there exists a regular solution to problem (1.4) such that*

$$\begin{aligned}&\sqrt{\nu}\nabla\varphi, \operatorname{rot}\psi \in L_\infty(0,T;\Gamma_1^2(\Omega)) \cap L_2(0,T;\Gamma_1^3(\Omega)),\\ &\nu\nabla\varphi \in L_2(0,T;\Gamma_1^3(\Omega)), \quad \nu\eta \in L_\infty(0,T;\Gamma_1^2(\Omega)).\end{aligned}$$

Hence $v \in \mathfrak{N}(\Omega^t)$, $t \le T$ and the estimate holds

$$\begin{aligned}&\|v\|_{\mathfrak{N}(\Omega^t)} \le \phi(\|\nu\eta(0), \sqrt{\nu}\nabla\varphi(0), \operatorname{rot}\psi(0)\|_{\Gamma_1^2(\Omega)}, \nu|f_g|_{L_2(0,T;L_{6/5}(\Omega))},\\ &\|f\|_{L_2(0,T;\Gamma_1^1(\Omega))}, \|f\|_{L_6(0,T;L_3(\Omega))\cap L_1(0,T;L_\infty(\Omega))}) \equiv D(0), \quad t \le T,\end{aligned} \tag{1.11}$$

where ϕ is an increasing positive function.
Assuming the decay $\|f(t)\|_1 \le f_0 e^{-\alpha t}$, $f_0 = \mathrm{const}$, $\alpha > 0$, and that $D(kT)$ is finite, where interval $(0,T)$ is replaced lby $(kT,(k+1)T)$, $k \in \mathbb{N}_0$ and assuming that T is such that

$$-\frac{a_*}{2}T + c\int_0^T (|v(t)|_{3,1}^2 + |\Delta\varphi(t)|_\infty^2 + \|v(t)\|_1^4 + \|f(t)\|_1^2)dt \le 0,$$

where $a_* = \min\{a_0, \mu/a\}$, $a_0 = p_\varrho(a)$, we obtain that

$$\|v\|_{\mathfrak{N}(\Omega\times(kT,(k+1)T))} \le D(kT).$$

Assumption A. *Let (v,η) be a solution to problem (1.4) described by Theorem A. Let v be described by potential φ and ψ by (1.8). Let $T > 0$ be given and*

$$B_2^2(t) = \|v(t)\|_2^2 + \|\nabla\varphi(t)\|_2^2$$

and

$$\int_{kT}^{(k+1)T} B_2^2(t)dt \le c\bigg(D^2(kT) + \frac{D^2(kT)}{\nu^2} + A_1^2A_2^2\bigg), \quad k \in \mathbb{N}_0,$$

where $D(kT)$ is defined in Theorem A and A_1 is defined in Lemma 2.1 in [Z1],

$$A_2 = |f|_{18/7,6,\Omega\times(kT,(k+1)T)} + |\varrho_0|_\infty^{1/6}|v_0|_6.$$

Theorem B. *Let Assumption 1 hold. Let (u, η) satisfy problem (1.9), (1.10). Assume that*

$$|u_x(0)|_2^2 \le \gamma \in (0, \gamma_*],$$

where γ_* is so small that

$$c \exp\left(2c \int_{kT}^{(k+1)T} B_2^2(t)dt\right)\gamma_*^2 \le \mu/2,$$

where T is the time of local solutions for any finite interval $[kT, (k+1)T]$. Assume that

$$\|g_r(t)\|_1^2 \le \gamma_0^2 \exp(-\alpha t),$$

where g is defined in (1.5) and g_r is the rotational part of g, γ_0, α are some constants.
Then for sufficiently small γ, γ_0 and sufficiently large T we have

$$\|u(t)\|_1^2 \le \gamma \exp\left[2c \int_{kT}^{(k+1)T} B_2^2(t)dt\right], \quad k \in \mathbb{N}_0, \quad t \in [kT, (k+1)T].$$

Then for solutions to problem (1.1) we have

$$\|V(t)\|_1^2 \le \gamma \exp\left[2c \int_{kT}^{t} B_2^2(t')dt'\right] + D^2(kT),$$

for $t \in [kT, (k+1)T]$, where T is defined in Theorem A.

Remark C. We hope that the paper meets one of the statements from [F].

There is a huge literature concerning the regularity problem of weak solutions to the Navier-Stokes equations. Therefore we are not able to present all papers devoted to this problem. Moreover, we are not able to close the list of mathematicians trying to solve it. Hence, we concdentrate the presentation on some directions and recall mathematicians working in these areas.

1. Formulation of sufficient conditions guaranteeing regularity of weak solutions.
   The first who formulated such conditions was J. Serrin [S]. This approach was continued by D. Chae, H.J. Choe, H. Kozono, H. Sohr, J. Neustupa, P. Penel and the references of their papers are cited in [Z3, Z4]. We have to recall results of G. Seregin, V. Šverák, T. Shilkin, A. Mikhaylov (see [S1, S2, S3, S4, SSS, SS1, ESS, MS]).

2. Local regularity theory.
   The direction of examining regularity of weak solutions of the Navier-Stokes equations was initiated in the celebrated paper of L. Caffarelli, R. Kohn, L. Nirenberg (see [CKN]). The famous mathematicians working in this directions are G. Seregin [S1, S2, S3, S4], V. Šverák [SS1].

3. Rotating Navier-Stokes equations.
   The existence of global regular solutions to the rotating Navier-Stokes equations was strongly examined by A. Babin, A. Mahalov, B. Nicolaenko (see [BMN1, BMN2, BMN3, MN]).

4. Global regular solutions to the Navier-Stokes equations with some special properties. We have to distinguish the following directions

   a. Thin domains (see [RS1, RS2, RS3]).
   b. Small variations of vorticity (see [CF]).
   c. Motions in cylindrical domains with data close to data of 2d solutions (see [Z5, Z6, Z7, Z8, NZ]).
   d. Motions in axisymmetric domains with data close to data of axisymmetric solutions (see [Z3, Z4]).

2 Notation and auxiliary results

We introduce the following simplified notation:

$$\|u\|_{L_p(\Omega)} = |u|_p, \quad p \in [1,\infty], \quad \|u\|_{H^s(\Omega)} = \|u\|_s, \quad s \in \mathbb{R}_+.$$

Moreover, we introduce

$$|u|_{k,l}^2 = \sum_{i=0}^{l} \|\partial_t^i u\|_{k-i}, \qquad |u|_{k,l,r,\Omega^t} = \left(\int_0^t |u(t')|_{k,l}^r dt'\right)^{1/r},$$

$$|u|_{r,q,\Omega^t} = \left(\int_0^t |u(t')|_r^q\right)^{1/q}, \quad \|u\|_{L_r(0,t;H^s(\Omega))} = \|u\|_{s,r,\Omega^t}.$$

Let v be defined in the form (1.8). We say that $v \in \mathfrak{M}(\Omega^T)$ if

$$\|v\|_{\mathfrak{M}(\Omega^T)} = \operatorname*{esssup}_{t\le T}\left(|\nabla\varphi(t)|_{2,1}^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi(t)|_{2,1}^2\right)^{1/2}$$
$$+ \left(\int_0^T \left(\nu|\nabla\varphi(t)|_{3,1}^2 + \frac{1}{\nu}|\mathrm{rot}\,\psi(t)|_{3,1}^2\right)dt\right)^{1/2} < \infty.$$

Next we say that $v\in\mathfrak{N}(\Omega^T)$ if

$$\|v\|_{\mathfrak{N}(\Omega^T)}=\operatorname*{esssup}_{t\le T}(\nu|\nabla\varphi(t)|_{2,1}^2+|\operatorname{rot}\psi(t)|_{2,1}^2)^{1/2}+\bigg(\int_0^T(\nu^2|\nabla\varphi(t)|_{3,1}^2+|\operatorname{rot}\psi(t)|_{3,1}^2)dt\bigg)^{1/2}<\infty.$$

If $\|v\|_{\mathfrak{M}(\Omega^T)}\le D_0$ then $\|v\|_{\mathfrak{N}(\Omega^T)}\le(1+\sqrt{\nu})D_0$.
Let

$$\Gamma_l^k(\Omega)=\{u:|u|_{k,l}<\infty\},\quad l\le k,\quad l,k\in\mathbb{N}_0.$$

To apply the Poincaré inequality we need

Remark 2.1. (see [Z1, Lemma 2.1]) Let (ϱ,v) be a solution to problem (1.2). Assume that $p=p(\varrho)=A\varrho^\varkappa$, $\varkappa>1$, $\varrho_0\in L_1(\Omega)$, $f\in L_{\infty,1}(\Omega^t)$, $\int_\Omega\big(\frac{1}{2}\varrho_0v_0^2+\frac{A}{\varkappa-1}\varrho_0^\varkappa\big)dx<\infty$. Then

$$\begin{aligned}&\int_\Omega\bigg(\frac{1}{2}\varrho v^2+\frac{A}{\varkappa-1}\varrho^\varkappa\bigg)dx+\mu|\nabla v|_{2,\Omega^t}^2+\nu|\operatorname{div}v|_{2,\Omega^t}^2\\&\le 2|\varrho_0|_1|f|_{\infty,1,\Omega^t}^2+\frac{3}{2}\int_\Omega\bigg(\frac{1}{2}\varrho_0v_0^2+\frac{A}{\varkappa-1}\varrho_0^\varkappa\bigg)dx\equiv\bar{A}_1^2.\end{aligned}\tag{2.1}$$

From (2.1) and for $|\eta|\le a/2$ we obtain

$$|v|_2^2+\nu|\Delta\varphi|_{2,\Omega^t}^2\le c(a)\bar{A}_1^2.\tag{2.2}$$

Our aim is to find an estimate for $\int_\Omega udx$.
Multiply $(1.2)_2$ by v, add to $(1.2)_1$ and integrate over Ω. Using the periodic boundary conditions we have

$$\frac{d}{dt}\int_\Omega\varrho vdx=\int_\Omega\varrho fdx.\tag{2.3}$$

Consider problem (1.1). Using the periodic boundary conditions we obtain

$$\frac{d}{dt}\int_\Omega Vdx=\int_\Omega Fdx.\tag{2.4}$$

Equations (2.3) and (2.4) imply

$$\frac{d}{dt}\int_\Omega(\varrho v-aV)dx=\int_\Omega(\varrho f-aF)dx.\tag{2.5}$$

Hence, it follows

$$\frac{d}{dt}\int_\Omega (au+\eta v)dx=\int_\Omega (ag+\eta f)dx. \tag{2.6}$$

Integrating (2.6) with respect to time yields

$$\begin{aligned}\int_\Omega udx&=-\frac{1}{a}\int_\Omega \eta vdx+\frac{1}{a}\int_{\Omega^t}(ag+\eta f)dxdt'\\&\quad+\int_\Omega u_0dx+\frac{1}{a}\int_\Omega \eta_0v_0dx.\end{aligned} \tag{2.7}$$

Hence

$$\begin{aligned}\bigg|\int_\Omega udx\bigg|&\le\frac{1}{a}(|\eta|_2|v|_2+|\eta|_{2,\infty,\Omega^t}|f|_{2,1,\Omega^t}+|\eta_0|_2|v_0|_2)\\&\quad+\bigg|\int_{\Omega^t}gdxdt'\bigg|+\bigg|\int_\Omega u_0dx\bigg|\le\frac{c}{a}(|\eta|_{2,\infty,\Omega^t}+|\eta_0|_2)\bar A_1\\&\quad+\bigg|\int_{\Omega^t}gdxdt'\bigg|+\bigg|\int_\Omega u_0dx\bigg|\end{aligned} \tag{2.8}$$

From Lemma 2.1 [Z1],

$$|v|_2^2+\mu\int_{kT}^t(\|v\|_1^2+\nu|\operatorname{div}v|_2^2)dt'\le cA_1^2, \tag{2.9}$$

where $t\in[kT,(k+1)T]$, $k\in\mathbb{N}_0$.

3 Estimates

First we obtain the energy estimate for solutions to problems (1.9), (1.10).

Lemma 3.1. *Assume that $\nabla\varphi\in H^1(\Omega)$, $v\in L_6(\Omega)$, $\eta\in H^1(\Omega)$, $v_t\in L_2(\Omega)$, $g_r\in L_2(\Omega)$, $f\in L_3(\Omega)$ and $\operatorname{div}g_r=0$. Assume that A_1 is the bound of the energy inequality for solutions to problem (1.2), $\eta\in L_\infty(\Omega^t)$,*

$\Delta\varphi \in L_2(\Omega^t)$, $v_t \in L_2(\Omega^t)$, $g \in L_2(\Omega^t)$ *and* $\left|\int_\Omega u_0 dx\right| < \infty$. *Then*

(3.1)
$$\begin{aligned}\frac{d}{dt}|u|_2^2 + \mu\|u\|_1^2 &\le c|u|_3^2(\|\nabla\varphi\|_1^2 + |v|_6^2) + c(|\nabla\varphi_t|_2^2 + \|\nabla\varphi\|_1^2\\ &\quad + \|\nabla\varphi\|_1^4) + c[|v|_3^2\|\nabla\varphi\|_1^2 + |v|_6^4\|\eta\|_1^2\\ &\quad + |v|_6^2(\|\eta\|_1^2\|\nabla\varphi\|_1^2 + \|\eta\|_1^2 + \|\nabla\varphi\|_1^2) + c|v_t|_2^2|\eta|_3^2\\ &\quad + c|g_r|_2^2 + c|\eta|_6^2|f|_3^2] + c\bigg[(|\eta|_2^2 + |\eta_0|_2^2)A_1^2 + \bigg|\int_{\Omega^t} g dx dt'\bigg|^2 + \bigg|\int_\Omega u_0 dx\bigg|^2\bigg].\end{aligned}$$

Proof. Let $\bar u = u - \nabla\varphi$. Then $\operatorname{div}\bar u = 0$. Multiply (1.6) by $\bar u$ and integrate over Ω. Then we have

$$\begin{aligned}&a\int_\Omega u_t\cdot\bar u dx + \int_\Omega \eta(v_t + v\cdot\nabla v)\cdot\bar u dx + a\int_\Omega v\cdot\nabla u\cdot\bar u dx\\ &\quad + a\int_\Omega u\cdot\nabla V\cdot\bar u dx - \mu\int_\Omega \Delta u\cdot\bar u dx = a\int_\Omega g\cdot\bar u dx + \int_\Omega \eta f\cdot\bar u dx.\end{aligned}\tag{3.2}$$

Now, we examine the particular terms in (3.2). The first term on the l.h.s. of (3.2) equals

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx - \int_\Omega u_t\cdot\nabla\varphi dx,$$

where integration by parts in the second term implies

$$-\int_\Omega u_t\cdot\nabla\varphi dx = \int_\Omega \Delta\varphi_t\varphi dx = -\int_\Omega \nabla\varphi_t\cdot\nabla\varphi dx.$$

The third term on the l.h.s. of (3.2) takes the form

$$\int_\Omega v\cdot\nabla u\cdot u dx - \int_\Omega v\cdot\nabla u\cdot\nabla\varphi dx \equiv I_1 + I_2,$$

where

$$I_1 = \frac{1}{2}\int_\Omega v\cdot\nabla u^2 dx = -\frac{1}{2}\int_\Omega \operatorname{div} v u^2 dx = -\frac{1}{2}\int_\Omega \Delta\varphi u^2 dx.$$

Hence

$$|I_1| \le \varepsilon|u|_6^2 + c/\varepsilon|u|_3^2|\Delta\varphi|_2^2.$$

Next,

$$I_2 = -\int_\Omega v \cdot \nabla(u \cdot \nabla\varphi)dx + \int_\Omega v \cdot \nabla\nabla\varphi \cdot udx \equiv I_{21} + I_{22},$$

where

$$I_{21} = \int_\Omega \Delta\varphi u \cdot \nabla\varphi dx$$

and

$$|I_{21}| \le \varepsilon|u|_6^2 + c/\varepsilon|\nabla\varphi|_3^2|\Delta\varphi|_2^2.$$

Finally,

$$|I_{22}| \le \varepsilon|u|_6^2 + c/\varepsilon|\nabla^2\varphi|_2^2|v|_3^2.$$

Consider the fourth term on the l.h.s. of (3.2). It takes the form

$$\begin{aligned}
&\int_\Omega u \cdot \nabla(v-u) \cdot \bar{u}dx = \int_\Omega u \cdot \nabla v \cdot \bar{u}dx - \int_\Omega u \cdot \nabla u \cdot \bar{u}dx \\
&= \int_\Omega u \cdot \nabla v(u - \nabla\varphi)dx - \int_\Omega u \cdot \nabla u \cdot (u - \nabla\varphi)dx \\
&= \int_\Omega u \cdot \nabla v \cdot udx - \int_\Omega u \cdot \nabla v \cdot \nabla\varphi dx - \int_\Omega u \cdot \nabla u \cdot udx \\
&\quad + \int_\Omega u \cdot \nabla u \cdot \nabla\varphi dx \equiv J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Integration by parts in J_1 yields

$$J_1 = \int_\Omega u \cdot \nabla(v \cdot u)dx - \int_\Omega u \cdot \nabla uvdx \equiv J_{11} + J_{12},$$

where

$$J_{11} = -\int_\Omega \Delta\varphi v \cdot udx$$

and

$$|J_{11}| \le \varepsilon|u|_6^2 + c/\varepsilon|\Delta\varphi|_2^2|v|_3^2.$$

Next,

$$|J_{12}| \le \varepsilon|\nabla u|_2^2 + c/\varepsilon|u|_3^2|v|_6^2.$$

Integration by parts in J_2 implies

$$\begin{aligned} J_2 &= \int\limits_\Omega u\cdot\nabla(v\cdot\nabla\varphi)dx - \int\limits_\Omega u\cdot\nabla\nabla\varphi\cdot vdx \\ &\equiv -\int\limits_\Omega \Delta\varphi v\cdot\nabla\varphi dx - \int\limits_\Omega u\cdot\nabla\nabla\varphi\cdot vdx \\ &\equiv J_{21}+J_{22}, \end{aligned}$$

where

$$|J_{21}| \le \varepsilon|\Delta\varphi|_2^2 + c/\varepsilon|v|_3^2|\nabla\varphi|_6^2$$

and

$$|J_{22}| \le \varepsilon|u|_6^2 + c/\varepsilon|\nabla^2\varphi|_2^2|v|_3^2.$$

Next, we consider J_3

$$J_3 = -\frac{1}{2}\int\limits_\Omega u\cdot\nabla u^2 dx = \frac{1}{2}\int\limits_\Omega \Delta\varphi u^2 dx.$$

Hence

$$|J_3| \le \varepsilon|u|_6^2 + c/\varepsilon|u|_3^2|\Delta\varphi|_2^2.$$

Finally,

$$|J_4| \le \varepsilon|\nabla u|_2^2 + c/\varepsilon|u|_3^2|\nabla\varphi|_6^2.$$

The last term on the l.h.s. of (3.2) equals

$$\mu|\nabla u|_2^2 - \mu\int\limits_\Omega \nabla u\cdot\nabla^2\varphi dx,$$

where the second integral is bounded by

$$\varepsilon|\nabla u|_2^2 + c/\varepsilon|\nabla^2\varphi|_2^2.$$

Consider the first term on the r.h.s. of (3.2).
Introducing potentials ψ_g, φ_g such that $g_r = \operatorname{rot}\psi_g$, $g_g = \nabla\varphi_g$ we have

$$g = f - F = f_r + f_g - F = f_r - F + f_g \equiv g_r + g_g.$$

Then

$$\int\limits_\Omega g\cdot\bar{u}dx = \int\limits_\Omega g_r\cdot\bar{u}dx = \int\limits_\Omega g_r\cdot udx - \int\limits_\Omega g_r\cdot\nabla\varphi dx \equiv K_1 + K_2,$$

where

$$|K_1| \le \varepsilon |u|_6^2 + c/\varepsilon |g_r|_{6/5}^2,$$
$$|K_2| \le |g_r|_2^2 + |\nabla\varphi|_2^2.$$

The second integral on the r.h.s. of (3.2) takes the form

$$\int_\Omega \eta f \cdot (u - \nabla\varphi) dx = \int_\Omega \eta f \cdot u dx - \int_\Omega \eta f \cdot \nabla\varphi dx \equiv K_3 + K_4,$$

where

$$|K_3| \le \varepsilon |u|_6^2 + c/\varepsilon |\eta|_2^2 |f|_3^2,$$
$$|K_4| \le |f|_3 |\eta|_6 |\nabla\varphi|_2.$$

Finally, we examine the second term on the l.h.s. of (3.2)

$$\begin{aligned}
&\int_\Omega \eta(v_t + v \cdot \nabla v) \cdot (u - \nabla\varphi) dx \\
&= \int_\Omega \eta(v_t + v \cdot \nabla v) \cdot u dx - \int_\Omega \eta(v_t + v \cdot \nabla v) \cdot \nabla\varphi dx \\
&= \int_\Omega \eta v_t \cdot u dx + \int_\Omega \eta v \cdot \nabla v \cdot u dx - \int_\Omega \eta v_t \cdot \nabla\varphi dx - \int_\Omega \eta v \cdot \nabla v \cdot \nabla\varphi dx \\
&\equiv L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

Continuing, we have

$$|L_1| \le \varepsilon |u|_6^2 + c/\varepsilon |v_t|_2^2 |\eta|_3^2.$$

To estimate L_2 we integrate by parts to get

$$\begin{aligned}
L_2 &= \int_\Omega v \cdot \nabla v \cdot u\eta dx \\
&= \int_\Omega v \cdot \nabla (v \cdot u\eta) dx - \int_\Omega v \cdot \nabla u \cdot v\eta dx - \int_\Omega v \cdot \nabla\eta v \cdot u dx \\
&\equiv L_{21} + L_{22} + L_{23}.
\end{aligned}$$

Integrating by parts in L_{21} gives

$$L_{21} = -\int_\Omega \Delta\varphi v \cdot u\eta dx.$$

Hence

$$|L_{21}| \le \varepsilon|u|_6^2 + c/\varepsilon|v|_6^2|\eta|_6^2|\Delta\varphi|_2^2,$$
$$|L_{22}| \le \varepsilon|\nabla u|_2^2 + c/\varepsilon|v|_6^4|\eta|_6^2,$$
$$|L_{23}| \le \varepsilon|u|_6^2 + c/\varepsilon|v|_6^4|\nabla\eta|_2^2.$$

Next

$$|L_3| \le |v_t|_2^2|\eta|_3^2 + \|\nabla\varphi\|_1^2.$$

Integrating by parts in L_4 yields

$$\begin{aligned} L_4 &= -\int_\Omega v\cdot\nabla v\cdot\nabla\varphi\eta dx = -\int_\Omega v\cdot\nabla(v\cdot\nabla\varphi\eta)dx \\ &\quad + \int_\Omega v\cdot\nabla\nabla\varphi\cdot v\eta dx + \int_\Omega v\cdot\nabla\eta v\cdot\nabla\varphi dx \\ &\equiv L_{41} + L_{42} + L_{43}, \end{aligned}$$

where

$$L_{41} = \int_\Omega \operatorname{div} vv\cdot\nabla\varphi\eta dx = \int_\Omega \Delta\varphi v\cdot\nabla\varphi\eta dx.$$

Therefore,

$$|L_{41}| \le |\Delta\varphi|_2^2 + |\nabla\varphi|_6^2|\eta|_6^2|v|_6^2.$$

Similarly,

$$|L_{42}| \le |v|_6^2(|\nabla^2\varphi|_2^2 + |\eta|_6^2), \quad |L_{43}| \le |v|_6^2(|\nabla\eta|_2^2 + |\nabla\varphi|_6^2).$$

Summarizing the above estimates and assuming that ε is sufficiently small implies the inequality

$$\begin{aligned} &\frac{d}{dt}|u|_2^2 + \mu|\nabla u|_2^2 \le \varepsilon|u|_6^2 + c\left(\frac{1}{\varepsilon}+1\right)|u|_3^2(\|\nabla\varphi\|_1^2 + |v|_6^2) \\ &\quad + c\left(\frac{1}{\varepsilon}+1\right)(|\nabla\varphi_t|_2^2 + \|\nabla\varphi\|_1^2 + \|\nabla\varphi\|_1^4) \\ &\quad + c\bigg[|v|_3^2\|\nabla\varphi\|_1^2 + |v|_6^4\|\eta\|_1^2 + |v|_6^2(\|\eta\|_1^2\|\nabla\varphi\|_1^2 + \|\eta\|_1^2 + \|\nabla\varphi\|_1^2) \\ &\quad + c|v_t|_2^2|\eta|_3^2 + c|g_r|_2^2 + c|\eta|_6^2|f|_3^2\bigg]. \end{aligned}$$

Applying the Poincaré inequality and using that ε is sufficiently small the above inequality implies

(3.3)
$$\begin{aligned}&\frac{d}{dt}|u|_2^2+\mu\|u\|_1^2\le c|u|_3^2(\|\nabla\varphi\|_1^2+|v|_6^2)+c(|\nabla\varphi_t|_2^2+\|\nabla\varphi\|_1^2\\&\quad+\|\nabla\varphi\|_1^4)+c\bigg[|v|_3^2\|\nabla\varphi\|_1^2+|v|_6^4\|\eta\|_1^2+|v|_6^2\|\eta\|_1^2\|\nabla\varphi\|_1^2\\&\quad+|v|_6^2(\|\nabla\varphi\|_1^2+\|\eta\|_1^2)+(|v_t|_2^2|\eta|_3^2+|g_r|_2^2+|\eta|_6^2|f|_3^2)+\bigg|\int_\Omega udx\bigg|^2\bigg].\end{aligned}$$

Using (2.8) in (3.3) implies (3.1) and concludes the proof. □

Lemma 3.2. *Assume that* $v\in\Gamma_1^2(\Omega)$, $\nabla\varphi\in\Gamma_1^2(\Omega)$, $\eta\in H^2(\Omega)$, $g_r\in H^1(\Omega)$, $f\in H^1(\Omega)$.
Then

(3.4)
$$\begin{aligned}&\frac{d}{dt}|u_x|_2^2+\mu\|\nabla u\|_1^2\le c|u_x|_2^6+c\|u\|_1^2(\|v\|_2^2+\|\nabla\varphi\|_2^2)+|v|_6^4\|\eta\|_2^2\\&\quad+c|v|_6^2(\|\nabla\varphi\|_2^2+\|\eta\|_2^2+\|\nabla\varphi\|_2^2\|\eta\|_2^2+\|\eta\|_2^2\|v\|_2^2+c\|v\|_2^2\|\nabla\varphi\|_2^2+\|\eta\|_2^2)\\&\quad+\|v_t\|_1^2\|\eta\|_2^2+c(\|\nabla\varphi_t\|_1^2+\|\nabla\varphi\|_2^2+\|g_r\|_1^2\\&\quad+\|f\|_1^2\|\eta\|_1^2).\end{aligned}$$

Proof. Differentiate (1.6) with respect to x, multiply the result by $\bar u_x$ and integrate over Ω. Then we have

$$\begin{aligned}&a\int_\Omega u_{xt}\cdot\bar u_xdx+\int_\Omega[\eta(v_t+v\cdot\nabla v)]_{,x}\cdot\bar u_xdx+a\int_\Omega[v\cdot\nabla u]_{,x}\cdot\bar u_xdx\\&\quad+a\int_\Omega[u\cdot\nabla(v-u)]_{,x}\cdot\bar u_xdx-\mu\int_\Omega\Delta u_x\cdot\bar u_xdx\\&=a\int_\Omega g_x\cdot\bar u_xdx+\int_\Omega(\eta f)_{,x}\bar u_xdx.\end{aligned}\tag{3.5}$$

Next, we examine the particular terms in (3.5). In these considerations we omit a. The first term on the l.h.s. of (3.5) equals

$$\int_\Omega u_{xt}\cdot u_xdx-\int_\Omega u_{xt}\cdot\nabla\varphi_xdx\equiv I_1+I_2,$$

where $I_1=\frac12\frac{d}{dt}|u_x|_2^2$ and we integrate by parts in I_2 to derive

$$I_2=\int_\Omega\Delta\varphi_{xt}\varphi_xdx=-\int_\Omega\nabla\varphi_{xt}\cdot\nabla\varphi_xdx.$$

Hence,

$$|I_2| \le |\nabla\varphi_{xt}|_2^2 + |\nabla\varphi_x|_2^2.$$

The second term on the l.h.s. of (3.5) equals

$$\begin{aligned}
&\int_\Omega \eta_x(v_t + v\cdot\nabla v)\cdot\bar u_x dx + \int_\Omega \eta(v_{xt} + v_x\cdot\nabla v + v\cdot\nabla v_x)\cdot\bar u_x dx\\
&= \int_\Omega \eta_x v_t\cdot u_x dx - \int_\Omega \eta_x v_t\cdot\nabla\varphi_x dx\\
&\quad + \int_\Omega \eta_x v\cdot\nabla v\cdot u_x dx - \int_\Omega \eta_x v\cdot\nabla v\cdot\nabla\varphi_x dx\\
&\quad + \int_\Omega \eta v_{xt}\cdot u_x dx - \int_\Omega \eta v_{xt}\cdot\nabla\varphi_x dx\\
&\quad + \int_\Omega \eta v_x\cdot\nabla v\cdot u_x dx - \int_\Omega \eta v_x\cdot\nabla v\cdot\nabla\varphi_x dx\\
&\quad + \int_\Omega \eta v\cdot\nabla v_x\cdot u_x dx - \int_\Omega \eta v\cdot\nabla v_x\cdot\nabla\varphi_x dx \equiv \sum_{i=1}^{10} J_i.
\end{aligned}$$

Now, we estimate the terms J_i, $i = 1,\dots,10$. First we have

$$\begin{aligned}
&|J_1| \le \varepsilon|u_x|_6^2 + c/\varepsilon|\eta_x|_3^2|v_t|_2^2,\\
&|J_2| \le |\eta_x|_6|v_t|_3|\nabla\varphi_x|_2 \le |v_t|_3^2\|\eta\|_1^2 + \|\nabla\varphi\|_1^2.
\end{aligned}$$

To examine J_3 we integrate by parts. Then we obtain

$$\begin{aligned}
J_3 &= \int_\Omega v\cdot\nabla v\cdot u_x\eta_x dx = \int_\Omega v\cdot\nabla(v\cdot u_x\eta_x)dx - \int_\Omega v\cdot\nabla u_x\cdot v\eta_x dx\\
&\quad - \int v\cdot\nabla\eta_x v\cdot u_x dx = J_{31} + J_{32} + J_{33}.
\end{aligned}$$

Integration by parts in J_{31} implies

$$J_{31} = -\int_\Omega \Delta\varphi v\cdot u_x\eta_x dx.$$

Hence

$$|J_{31}| \le \varepsilon|u_x|_6^2 + c/\varepsilon|\eta_x|_6^2|\Delta\varphi|_2^2|v|_6^2.$$

Next,

$$\begin{aligned}
&|J_{32}| \le \varepsilon|\nabla u_x|_2^2 + c/\varepsilon|v|_6^4|\eta_x|_6^2,\\
&|J_{33}| \le \varepsilon|u_x|_6^2 + c/\varepsilon|v|_6^4|\nabla\eta_x|_2^2.
\end{aligned}$$

To examine J_4 we integrate by parts. Then we have

$$\begin{aligned} J_4 = -\int\limits_\Omega v\cdot\nabla v\cdot\nabla\varphi_x\eta_x dx &= -\int\limits_\Omega v\cdot\nabla(v\cdot\nabla\varphi_x\eta_x)dx \\ &+\int\limits_\Omega v\cdot\nabla\nabla\varphi_x\cdot v\eta_x dx+\int\limits_\Omega v\cdot\nabla\eta_x\nabla\varphi_x\cdot v dx \\ &\equiv J_{41}+J_{42}+J_{43}. \end{aligned}$$

Integrating by parts in J_{41} implies

$$J_{41}=\int\limits_\Omega \Delta\varphi v\cdot\nabla\varphi_x\eta_x dx.$$

Therefore,

$$|J_{41}|\le|\Delta\varphi|_2^2+|v|_6^2|\eta_x|_6^2|\nabla\varphi_x|_6^2.$$

Continuing

$$|J_{42}|\le|v|_6^2(|\nabla^2\varphi_x|_2|\eta_x|_6)\le|v|_6^2(|\nabla\varphi_{xx}|_2^2+\|\eta\|_2^2)$$

and

$$|J_{43}|\le|v|_6^2|\nabla\eta_x|_2|\nabla\varphi_x|_6\le|v|_6^2(\|\eta\|_2^2+\|\nabla\varphi\|_2^2).$$

Estimating J_5 yields

$$|J_5|\le\varepsilon|u_x|_6^2+c/\varepsilon|\eta|_3^2|v_{xt}|_2^2.$$

Next

$$|J_6|\le|\eta|_\infty^2|v_{xt}|_2^2+|\nabla\varphi_x|_2^2\le\|v_t\|_1^2\|\eta\|_2^2+\|\nabla\varphi\|_1^2.$$

Integration by parts in J_7 implies

$$\begin{aligned} J_7=\int\limits_\Omega v_x\cdot\nabla(v\cdot u_x\eta)dx-\int\limits_\Omega v_x\cdot\nabla u_x\cdot v\eta dx-\int\limits_\Omega v_x\cdot\nabla\eta u_x\cdot v dx \\ \equiv J_{71}+J_{72}+J_{73}, \end{aligned}$$

where

$$J_{71}=-\int\limits_\Omega \operatorname{div} v_x v\cdot u_x\eta dx=-\int\limits_\Omega \Delta\varphi_x v\cdot u_x\eta dx.$$

Hence

$$|J_{71}|\le\varepsilon|u_x|_6^2+c/\varepsilon|\Delta\varphi_x|_2^2|v|_6^2|\eta|_6^2$$

and

$$|J_{72}| \le \varepsilon |\nabla u_x|_2^2 + c/\varepsilon |v_x|_6^2 |\eta|_6^2 |v|_6^2,$$
$$|J_{73}| \le \varepsilon |u_x|_6^2 + c/\varepsilon |\nabla \eta|_6^2 |v_x|_2^2 |v|_6^2.$$

Continuing, we integrate by parts in J_8 to derive

$$J_8 = -\int_\Omega v_x \cdot \nabla(v \cdot \nabla\varphi_x \eta) dx + \int_\Omega v_x \cdot \nabla\nabla\varphi_x \cdot v\eta dx + \int_\Omega v_x \cdot \nabla\eta\nabla\varphi_x \cdot v dx$$
$$\equiv J_{81} + J_{82} + J_{83}.$$

Integration by parts in J_{81} yields

$$J_{81} = \int_\Omega \Delta\varphi_x v \cdot \nabla\varphi_x \eta dx,$$

so

$$|J_{81}| \le |\Delta\varphi_x|_2^2 + |v|_6^2 |\nabla\varphi_x|_6^2 |\eta|_6^2.$$

Estimations of other terms in J_8 implies

$$|J_{82}| \le |\nabla^2\varphi_x|_2^2 + |v_x|_3^2 |\eta|_\infty^2 |v|_6^2,$$
$$|J_{83}| \le |\nabla\varphi_x|_6^2 + |v_x|_2^2 |\nabla\eta|_6^2 |v|_6^2.$$

Finally,

$$|J_9| \le \varepsilon |u_x|_6^2 + c/\varepsilon |\nabla v_x|_2^2 |v|_6^2 |\eta|_6^2,$$
$$|J_{10}| \le |\nabla\varphi_x|_6^2 + c|\nabla v_x|_2^2 |v|_6^2 |\eta|_6^2.$$

The third term on the l.h.s. of (3.5) takes the form

$$\int_\Omega v_x \cdot \nabla u \cdot u_x dx + \int_\Omega v \cdot \nabla u_x \cdot u_x dx - \int_\Omega v_x \cdot \nabla u \cdot \nabla\varphi_x dx$$
$$- \int_\Omega v \cdot \nabla u_x \cdot \nabla\varphi_x dx \equiv K_1 + K_2 + K_3 + K_4.$$

Integrating by parts with respect to x in K_1 yields

$$K_1 + K_2 = -\int_\Omega v \cdot \nabla u \cdot u_{xx} dx.$$

Then

$$|K_1 + K_2| \le \varepsilon |u_{xx}|_2^2 + c/\varepsilon |\nabla u|_2^2 |v|_\infty^2.$$

Similarly,

$$K_3 + K_4 = \int_\Omega v \cdot \nabla u \cdot \nabla\varphi_{xx} dx$$

so

$$|K_3+K_4|\le\varepsilon|\nabla u|_6^2+c/\varepsilon|\nabla\varphi_{xx}|_2^2|v|_3^2.$$

The fourth term on the l.h.s. of (3.5) has the form

$$\begin{aligned}&\int_\Omega(u\cdot\nabla v)_x\cdot(u_x-\nabla\varphi_x)dx-\int_\Omega(u\cdot\nabla u)_x(u_x-\nabla\varphi_x)dx\\&=\int_\Omega(u\cdot\nabla v)_x\cdot u_xdx-\int_\Omega(u\cdot\nabla v)_x\cdot\nabla\varphi_xdx-\int_\Omega(u\cdot\nabla u)_x\cdot u_xdx\\&\quad+\int_\Omega(u\cdot\nabla u)_x\cdot\nabla\varphi_xdx\equiv L_1+L_2+L_3+L_4.\end{aligned}$$

First we consider

$$L_1=\int_\Omega u_x\cdot\nabla v\cdot u_xdx+\int_\Omega u\cdot\nabla v_x\cdot u_xdx\equiv L_{11}+L_{12},$$

where

$$\begin{aligned}|L_{11}|&\le\varepsilon|u_x|_6^2+c/\varepsilon|u_x|_2^2|\nabla v|_3^2,\\|L_{12}|&\le\varepsilon|u_x|_6^2+c/\varepsilon|\nabla v_x|_2^2|u|_3^2.\end{aligned}$$

Next, we examine

$$L_2=-\int_\Omega u_x\cdot\nabla v\cdot\nabla\varphi_xdx-\int_\Omega u\cdot\nabla v_x\cdot\nabla\varphi_xdx\equiv L_{21}+L_{22},$$

where

$$\begin{aligned}|L_{21}|&\le\varepsilon|u_x|_6^2+c/\varepsilon|\nabla v|_2^2|\nabla\varphi_x|_3^2,\\|L_{22}|&\le\varepsilon|u|_\infty^2+c/\varepsilon|\nabla v_x|_2^2|\nabla\varphi_x|_2^2.\end{aligned}$$

Next, we examine L_3. We express it in the form

$$L_3=-\int_\Omega u\cdot\nabla u_x\cdot u_xdx-\int_\Omega u_x\cdot\nabla u\cdot u_xdx\equiv L_{31}+L_{32},$$

where

$$L_{31}=-\frac{1}{2}\int_\Omega u\cdot\nabla|u_x|^2dx=\frac{1}{2}\int_\Omega\Delta\varphi u_x^2dx.$$

Hence

$$|L_{31}|\le\varepsilon|u_x|_6^2+c/\varepsilon|\Delta\varphi|_3^2|u_x|_2^2$$

and

$$|L_{32}|\le|u_x|_3^3\le c|u_{xx}|_2^{3/2}|u_x|_2^{3/2}\le\varepsilon|u_{xx}|_2^2+c/\varepsilon|u_x|_2^6.$$

Finally, we examine

$$L_4=\int_\Omega u\cdot\nabla u_x\cdot\nabla\varphi_x dx+\int_\Omega u_x\cdot\nabla u\cdot\nabla\varphi_x dx\equiv L_{41}+L_{42}.$$

Continuing, we have

$$\begin{aligned}|L_{41}|&\le\varepsilon|\nabla u_x|_2^2+c/\varepsilon|\nabla\varphi_x|_3^2|u|_6^2,\\ |L_{42}|&\le\varepsilon|\nabla u|_6^2+c/\varepsilon|u_x|_2^2|\nabla\varphi_x|_3^2.\end{aligned}$$

The last term on the l.h.s. of (3.5) equals

$$\mu\int_\Omega|\nabla u_x|^2dx+\mu\int_\Omega\Delta u_x\nabla\varphi_x dx,$$

where the second term is treated in the way

$$\begin{aligned}\mu\int_\Omega\Delta u_x\cdot\nabla\varphi_x dx&=-\mu\int_\Omega\Delta\operatorname{div}u_x\varphi_x dx=-\mu\int_\Omega\Delta^2\varphi_x\varphi_x dx\\ &=-\int_\Omega|\Delta\varphi_x|^2dx.\end{aligned}$$

The first term on the r.h.s. of (3.5) equals

$$\int_\Omega g_{rx}(u-\nabla\varphi)_x dx=\int_\Omega g_{rx}u_x dx-\int_\Omega g_{rx}\nabla\varphi_x dx\equiv M_1+M_2,$$

where

$$\begin{aligned}|M_1|&\le\varepsilon|u_x|_2^2+c/\varepsilon|g_{rx}|_2^2,\\ |M_2|&\le|g_{rx}|_2^2+|\nabla\varphi_x|_2^2.\end{aligned}$$

Finally, the last term on the r.h.s. of (3.5) assumes the form

$$\int_\Omega(\eta_x f+\eta f_x)\cdot u_x dx-\int_\Omega(\eta_x f+\eta f_x)\cdot\nabla\varphi_x dx\equiv N_1+N_2,$$

where

$$\begin{aligned}|N_1|&\le\varepsilon|u_x|_6^2+c/\varepsilon(|\eta_x|_2^2|f|_3^2+|\eta|_3^2|f_x|_2^2),\\ |N_2|&\le|\nabla\varphi_x|_6\|\eta\|_1\|f\|_1.\end{aligned}$$

Employing the above estimates in (3.5) and using that ε is sufficiently small we obtain the inequality

(3.6)

$$\begin{aligned}\frac{d}{dt}|u_x|_2^2+\mu\|u_x\|_1^2&\le c|u_x|_2^6+c\|u\|_1^2(\|v\|_2^2+\|\nabla\varphi\|_2^2)+|v|_6^4\|\eta\|_2^2\\ &\quad+c\|v\|_2^2(\|\nabla\varphi\|_2^2+\|\eta\|_2^2)+c|v|_6^2(\|\nabla\varphi\|_2^2+\|\eta\|_2^2+\|\nabla\varphi\|_2^2\|\eta\|_2^2\\ &\quad+\|\eta\|_2^2\|v\|_2^2)+c\|v_t\|_1^2\|\eta\|_2^2+c(\|\nabla\varphi_t\|_1^2+\|\nabla\varphi\|_2^2+\|g_r\|_1^2+\|f\|_1^2\|\eta\|_1^2).\end{aligned}$$

This implies (3.4) and concludes the proof. $\square$

Let

$$
\begin{aligned}
B_1(t) &= \|\nabla\varphi(t)\|_1^2 + |v(t)|_6^2,\\
G_1^2(t) &= |\nabla\varphi_t|_2^2 + \|\nabla\varphi\|_1^2 + \|\nabla\varphi\|_1^4 + |v|_6^2(\|\nabla\varphi\|_1^2\\
&\quad + \|\eta\|_1^2 + \|\eta\|_1^2\|\nabla\varphi\|_1^2) + |v|_6^4\|\eta\|_1^2 + |v_t|_2^2|\eta|_3^2 + |g_r|_2^2\\
&\quad + |\eta|_6^2|f|_3^2 + c\bigg[(|\eta|_2^2 + |\eta_0|_2^2)A_1^2 + \bigg|\int_{\Omega^t} g dx dt'\bigg|^2\bigg] + \bigg|\int_\Omega u_0 dx\bigg|^2.
\end{aligned}
$$

Remark 3.3. In view of the above notation we express (3.1) in the form

$$
\frac{d}{dt}|u|_2^2 + \mu\|u\|_1^2 \le c|u|_3^2 B_1(t) + cG_1^2(t) \tag{3.7}
$$

In view of interpolation

$$
|u|_3 \le c|u_x|_2^{1/2}|u|_2^{1/2}
$$

we derive from (3.7) the inequality

$$
\frac{d}{dt}|u|_2^2 + \mu\|u\|_1^2 \le c|u|_2^2 B_1^2 + cG_1^2(t). \tag{3.8}
$$

Consider (3.8) in the time interval $[kT,(k+1)T]$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Moreover, we assume that in each time interval $[kT,(k+1)T]$ the first part of Theorem B holds which is proved step by step in time.
We use that

$$
\begin{aligned}
\int_{kT}^{(k+1)T} B_1^2(t)dt &\le \sup_{t\in[kT,(k+1)T]}\|\nabla\varphi(t)\|_1^2 \int_{kT}^{(k+1)T}\|\nabla\varphi(t)\|_1^2 dt\\
&\quad + \sup_{t\in[kT,(k+1)T]}|v(t)|_6^2 \int_{kT}^{(k+1)T}|v(t)|_6^2 dt \le \frac{c}{\nu^4} + A_1^2\bigg(\frac{c}{\nu^{2/3}} + \frac{c}{\nu^{2\alpha/3}} + A_2^2\bigg),
\end{aligned}
$$

where we used estimate for $\sup_t|v(t)|_6$ from (2.95) in [Z1], $\alpha>0$ and $A_2 = |f|_{18/7,6,\Omega^t} + |\varrho_0|_\infty^{1/6}|v_0|_6$.
Moreover

$$
\int_{kT}^{(k+1)T} G_1^2(t)dt \le \frac{c}{\nu} + ce^{-\alpha kT},
$$

where we used that $|\eta(0)|_{2,1} \le \frac{c}{\nu}$, $|g(t)|_2 \le \gamma_0 e^{-\alpha t}$. We express (3.8) in the form

$$\frac{d}{dt}|u|_2^2 + \mu_1|u|_2^2 + \mu_2\|u\|_1^2 \le c|u|_2^2 B_1^2 + cG_1^2,$$

where $\mu = \mu_1 + \mu_2$, $\mu_i > 0$, $i = 1, 2$. Then we obtain

$$\begin{aligned}&\frac{d}{dt}\left[|u|_2^2 \exp\left(\mu_1(t-kT) - c\int_{kT}^{t} B_1^2(t')dt'\right)\right]\\ &\quad + \mu_2\|u\|_1^2 \exp\left(\mu_1(t-kT) - c\int_{kT}^{t} B_1^2(t')dt'\right)\\ &\le cG_1^2 \exp\left(\mu_1(t-kT) - c\int_{kT}^{t} B_1^2(t')dt'\right).\end{aligned}$$

Integrating the above inequality from $t = kT$ to $t \in (kT, (k+1)T]$ we derive

$$\begin{aligned}&|u(t)|_2^2 + \mu_2\int_{kT}^{t} \|u(t')\|_1^2 \exp[\mu_1(t'-t)]dt'\\ &\le c\exp\left(c\int_{kT}^{t} B_1^2(t')dt'\right)\int_{kT}^{t} G_1^2(t')dt'\\ &\quad + c|u(kT)|_2^2 \exp\left[-\mu_1(t-kT) + c\int_{kT}^{t} B_1^2(t')dt'\right].\end{aligned} \tag{3.9}$$

Setting $t = (k+1)T$ we obtain

$$\begin{aligned}|u((k+1)T)|_2^2 &\le c\exp\left(c\int_{kT}^{(k+1)T} B_1^2(t)dt\right)\int_{kT}^{(k+1)T} G_1^2(t)dt\\ &\quad + c\exp\left(-\mu_1 T + c\int_{kT}^{(k+1)T} B_1^2(t)dt\right)\cdot |u(kT)|_2^2.\end{aligned}$$

Since

$$\int_{kT}^{(k+1)T} B_1^2(t)dt \le \frac{c}{\nu^2} + A_1^2A_2^2,$$

T close to ν and ν large we have that there exists a constant $\mu_0 > 0$ such that

$$-\mu_1 T + c\left(\frac{1}{\nu} + A_1^2 A_2^2\right) \leq -\mu_0 T.$$

Therefore, we have

$$|u((k+1)T)|_2^2 \leq c \exp\left(c\left(\frac{1}{\nu^2} + A_1^2 A_2^2\right)\right)\left(\frac{1}{\nu} + \gamma_0 \exp(-\alpha kT)\right) + \exp(-\mu_0 T)|u(kT)|_2^2.$$

Hence, by iteration we get

$$|u(kT)|_2^2 \leq \frac{c \exp\left(c\left(\frac{1}{\nu^2} + A_1^2 A_2^2\right)\right)\left(\frac{1}{\nu} + \gamma_0 \exp(-\alpha kT)\right)}{1 - e^{-\mu_0 T}} + \exp(-\mu_0 kT)|u(0)|_2^2 \equiv D_1^2(k). \tag{3.10}$$

Therefore for small $|u(0)|_2$, large ν and T close to ν we get that

$$|u(kT)|_2 \leq D_1(0)$$

which is also small and bounded. From (3.9) and (3.10) we have

$$|u(t)|_2^2 \leq c \exp\left(c\left(\frac{1}{\nu^2} + A_1^2 A_2^2\right)\right)\left(\frac{1}{\nu} + \gamma_0 \exp(-\alpha kT)\right) + \exp(-\mu_0 T) D_1^2(0) \equiv D_2^2, \quad t \in [kT, (k+1)T]. \tag{3.11}$$

Integrating (3.8) with respect to time from kT to $t \in (kT, (k+1)T]$, $k \in \mathbb{N}_0$, we obtain

$$|u(t)|_2^2 + \mu \int_{kT}^{t} \|u(t')\|_1^2 dt' \leq c D_2^2 \int_{kT}^{(k+1)T} B_1^2(t) dt + c \int_{kT}^{(k+1)T} G_1^2(t) dt + D_1^2(0). \tag{3.12}$$

This ends the Remark.

Next we obtain similar estimates to (3.11) and (3.12) for $\|u(t)\|_1$. For this purpose we add (3.1) and (3.4). Then we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|u\|_1^2+\mu\|u\|_2^2\le c\|u\|_1^6+c\|u\|_1^2(\|v\|_2^2+\|\nabla\varphi\|_2^2)\\
&\quad+c|v|_6^4\|\eta\|_2^2+c|v|_6^2(\|\nabla\varphi\|_2^2+\|\eta\|_2^2+\|\nabla\varphi\|_2^2\|\eta\|_2^2+\|\eta\|_2^2\|v\|_2^2)\\
&\quad+c\|v_t\|_1^2\|\eta\|_2^2+c\|v\|_2^2(\|\nabla\varphi\|_2^2+\|\eta\|_2^2)+c(\|\nabla\varphi_t\|_1^2\\
&\quad+\|\nabla\varphi\|_2^2+\|g_r\|_1^2+\|\eta\|_1^2\|f\|_1^2)+c\bigg[(|\eta|_2^2+|\eta_0|_2^2)A_1^2\\
&\quad+\bigg|\int_{\Omega^t}gdxdt'\bigg|^2+\bigg|\int_\Omega u_0dx\bigg|^2\bigg]\equiv c\|u\|_1^6+cB_2^2(t)\|u\|_1^2\\
&\quad+cG_2^2(t).
\end{aligned}
\tag{3.13}
$$

Lemma 3.4. *Assume that (v,ϱ) is a solution to problem (1.2), (1.3). Assume that (v,ϱ) is described by Theorem A. Assume that ν,T are sufficiently large, $\|g(t)\|_1\le\gamma_0\exp(-\alpha t)$, $\alpha>0$. Let constant A be introduced in Lemma 4.1 from [Z1]. Assume that $\|u(0)\|_1\le\gamma$, where γ is sufficiently small. Then*

$$
\|u(t)\|_1\le\gamma\exp A. \tag{3.14}
$$

Proof. To obtain estimate (3.14) for $\|u(t)\|_1$ we consider (3.13) in the form

$$
\frac{d}{dt}\|u\|_1^2\le-(\mu-c\|u\|_1^4)\|u\|_1^2+cB_2^2\|u\|_1^2+cG_2^2. \tag{3.15}
$$

Consider (3.15) in the time interval $[kT,(k+1)T]$. Introduce the quantities

$$
\begin{aligned}
&X^2(t)=\exp\bigg(-c\int_{kT}^t B_2^2(t')dt'\bigg)\|u(t)\|_1^2,\\
&K^2(t)=c\exp\bigg(-c\int_{kT}^t B_2^2(t')dt'\bigg)G_2^2(t).
\end{aligned}
\tag{3.16}
$$

Then (3.15) takes the form

$$
\frac{d}{dt}X^2\le-\bigg[\mu-c\exp\bigg(2c\int_{kT}^t B_2^2(t')dt'\bigg)X^4\bigg]X^2+K^2. \tag{3.17}
$$

From Lemma 4.1 form[Z1] and the considerations of Section 5 from [Z1] we have

$$
\int_{kT}^{(k+1)T}B_2^2(t)dt\le cA^2, \tag{3.18}
$$

where T is proportional to some increasing function of ν.
In view of Lemma 4.1 [Z1] we also have

$$K^2(t) \le c\left(\frac{A}{\nu^2} + \gamma_0^2 e^{-2\alpha t}\right). \tag{3.19}$$

Now we want to estimate $X(t)$ for $t \in [kT, (k+1)T]$. Assume that

$$X^2(kT) = \|u(kT)\|_1^2 \le \gamma. \tag{3.20}$$

Suppose that

$$t_* = \inf\{t \in [kT, (k+1)T] : X^2(t) > \gamma\}$$

Let $\gamma \in (0, \gamma_*]$, where γ_* is so small that

$$\mu - c\exp\left(2c\int_{kT}^{(k+1)T} B_2^2(t')dt'\right)\gamma_*^2 \ge \frac{\mu}{2} \tag{3.21}$$

In view of (3.18) condition (3.21) takes the form

$$\mu - c\exp(cA^2)\gamma_*^2 \ge \frac{\mu}{2} \tag{3.22}$$

Hence, for $t \le t_*$ we derive from (3.17) the inequality

$$\frac{d}{dt}X^2 \le -\frac{\mu}{2}X^2 + K^2. \tag{3.23}$$

Assume that $\frac{1}{\nu}$, γ_0 are so small that

$$K^2(t) \le c\left(\frac{A}{\nu^2} + \gamma_0^2 e^{-2\alpha t}\right) \le \frac{\mu}{4}\gamma \quad \text{for} \quad t \in [kT, (k+1)T].$$

Then

$$\frac{d}{dt}X^2\Big|_{t=t_*} \le -\left(\frac{\mu}{2} - \frac{\mu}{4}\right)\gamma < 0,$$

so t_* does not exist in $[kT, (k+1)T]$. Hence (3.14) holds for $t \in [kT, (k+1)T]$ under assumption that (3.20) holds. Now we have to show (3.20) for any $k \in \mathbb{N}$.
Since we showed that

$$\|u(t)\|_1^2 \le \gamma \exp\left[c\int_{kT}^{t} B_2^2(t')dt'\right], \quad t \in [kT, (k+1)T],$$

we obtain from (3.15) the inequality

$$\|u((k+1)T)\|_1^2 \le \exp\Bigg[-\mu(k+1)T + c\int\limits_{kT}^{(k+1)T} B_2^2(t')dt'$$

$$+ cT\gamma^2 \exp\Bigg[2c\int\limits_{kT}^{(k+1)T} B_2^2(t')dt'\Bigg]\Bigg] \cdot \int\limits_{kT}^{(k+1)T} G_2^2(t')\cdot$$

$$\cdot \exp\Bigg[\mu t' - c\int\limits_{kT}^{t'} B_2^2(t'')dt'' - cT\gamma^2 \exp\Bigg[2c\int\limits_{kT}^{t'} B_2^2(t'')dt''\Bigg]\Bigg]dt'$$

$$+ \exp\Bigg(-\mu T + c\int\limits_{kT}^{(k+1)T} B_2^2(t')dt' + cT\gamma^2 \exp\Bigg[2c\int\limits_{kT}^{(k+1)T} B_2^2(t')dt'\Bigg]\Bigg)\cdot$$

$$\cdot \|u(kT)\|_1^2.$$

Simplifying, we get

$$\|u((k+1)T)\|_1^2 \le \exp[cA^2 + cT\gamma^2 \exp A^2] \int\limits_{kT}^{(k+1)T} G_2^2(t)dt + \exp[-\mu T + cA^2 + cT\gamma^2 \exp A^2]\|u(kT)\|_1^2. \tag{3.24}$$

Using that

$$\int\limits_{kT}^{(k+1)T} G_2^2(t)dt \le \frac{c}{\nu^2} + \gamma_0 \exp(-\alpha kT)$$

and assuming that γ is so small that $T\gamma^2 \le c$ we obtain from (3.24) the inequality

$$\|u((k+1)T)\|_1^2 \le \exp[\phi(A)]\Big(\frac{c}{\nu^2} + \gamma_0 \exp(-\alpha kT)\Big) + \exp(-\mu T + \phi(A))\|u(kT)\|_1^2, \tag{3.25}$$

where $\phi(A) = c(A + \exp(cA^2))$.
Since $\|u(kT)\|_1^2 \le \gamma$ then for a given A, sufficiently large T, ν and sufficiently small γ_0 we have that

$$\|u((k+1)T)\|_1^2 \le \gamma.$$

Repeating the considerations for any $k \in \mathbb{N}$ we prove the lemma. □

Proof of Theorem B.

$$\|V(t)\|_1^2 \le \|u\|_1^2 + \|v\|_1^2 \le \gamma \exp\left(c\int\limits_{kT}^{t} B_2^2 dt'\right) + \|v\|_{\mathfrak{N}(\Omega\times(kT,t))}, \tag{3.26}$$

$t \in [kT, (k+1)T]$, $k \in \mathbb{N}_0$ and T is the time of local solution introduced in Theorem A. □

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