IM PAN Preprint 682 (2007)

Michał Baran

Approximations for solutions of Lévy-type stochastic differential equations

Approximations for solutions of Lévy-type stochastic differential equations [*]

Michał Baran

Mathematics Department of Cardinal Stefan Wyszyński University in Warsaw, Poland

m.baran@uksw.edu.pl

Abstract

The problem of the construction of strong approximations with a given order of convergence for jump-diffusion equations is studied. General approximation schemes are constructed for Lévy type stochastic differential equation. In particular, the paper generalizes the results from [3] and [2]. The Euler and the Milstein schemes are shown for finite and infinite Lévy measure.

Key words: discrete approximations, Lévy-type differential equations

AMS Subject Classification: 60H10, 60G57.

1 Introduction

The problem of approximation construction for solution of stochastic differential equation is widely studied throughout many papers. The authors' attention is focused mainly on the equation of the form:

$$Y_t = Y_0 + \int_0^t f(Y_{s-})dZs, \tag{1.1}$$

where Y_0 is a random variable with known distribution, f-some regular function and Z-a driving process. There are many approximation methods for the solution of (1.1) depending on the driving process and the optimality criteria imposed on the approximating error. The case when Z is a Wiener process the problem is comprehensively studied in the book [3]. Variety schemes for the so called weak and strong approximations are presented, in particular their dependence on the mesh of the partition of the interval $[0, T]$. Denoting by $\bar{Y}$ the approximation, the optimality criteria for weak solutions have a form: $\mathbf{E}[g(Y_T) - g(\bar{Y}_T)] \longrightarrow \min$, where g is some regular function, while for strong solutions: $\mathbf{E} \sup_t |Y_t - \bar{Y}_t|^2 \longrightarrow \min$. The schemes use the increments of time, increments of the Wiener process and, for higher order of convergence, some normally distributed random variables correlated with the increments of the Wiener process. Thus for practical implementation we have to generate normally distributed, correlated random variables.

The simplest approximating scheme for the equation (1.1) is the Euler scheme which has the following structure:

$$\bar{Y}_0 = Y_0, \qquad \bar{Y}_{\frac{(i+1)T}{n}} = f(\bar{Y}_{\frac{iT}{n}})(Z_{\frac{(i+1)T}{n}} - Z_{\frac{iT}{n}}),$$

---

where $\{\frac{iT}{n}, i = 0, 1, ..., n\}$ is a partition of the interval $[0, T]$. In the case of the Wiener driving process it is easy to construct. However, for a general Lévy driving process it is no longer so simple. This is because of the difficulty of practical construction of the increments of Z when the Lévy measure is infinite, i.e. when the measure of a unit ball is infinite. If the increments can not be simulated, then they themselves have to be approximated in some sense and the accuracy of such construction should be studied. This way of approximating is presented for example in [7] and [5]. The main idea in these papers is to reduce the problem by replacing increments of Z by suitable increments of the compound Poisson process, which can be practically simulated. It should be pointed out that our approach is more general since a significant majority of papers consider approximation problem using different modifications of the Euler scheme.

In this paper we work with a stochastic differential equation of the form:

$$Y_t = Y_0 + \int_0^t b(Y_{s_-})ds + \int_0^t \sigma(Y_{s_-})dW_s + \int_0^t \int_{|x|<1} F(Y_{s_-}, x)\tilde{N}(ds, dx)$$

$$+ \int_0^t \int_{|x|\geq 1} G(Y_{s_-}, x)N(ds, dx),$$

where b, σ, F, G are some regular functions, W- a standard Wiener process and $N, \tilde{N}$ - a Poisson random measure and its compensated measure respectively. We focus on the strong approximations, i.e. the error is measured by $\mathbf{E}\sup_t |Y_t - \bar{Y}_t|^2$. The strong approximation is of order γ if $\mathbf{E}\sup_t |Y_t - \bar{Y}_t|^2 \leq \delta^{2\gamma}$, where δ is the mesh of partition of the interval $[0, T]$. Our aim is to construct the strong approximation for a previously fixed number $\gamma > 0$. The idea is to apply the Itô formula to the process Y many times, i.e. to the process Y and then to the coefficients in its expansion. The approximation is built of some of the coefficients which are chosen appropriately. The main result is Theorem 4.1 providing the description of the approximation. This theorem is a generalization of the results from [3] for diffusion processes and [2] for diffusion processes with jumps generated by a standard Poisson process. For $\gamma = \frac{1}{2}$ we obtain the Euler approximation but we can also built approximations of higher order. The approximation given by Theorem 4.1 has one limitation - in case when the Lévy measure of a unit ball is infinite, some ingredients are hard to simulate. This difficulty concerns the possibility of simulating integrals with respect to the compensated Poisson measure on unit balls. This problem hasn't appeared in [3] or [2] since there were no jumps or were equal to 1 only. To overcome this problem we modify the approximation by replacing all unit balls with ε- discs which are obtained by cutting ε- balls from unit balls. This procedure causes that the error depends not only on δ but on ε as well. Theorem 5.3 provides the error description. It is a sum of $\delta^{2\gamma}$ and some function of ε which tends to zero when $\varepsilon \longrightarrow 0$. The speed of convergence of this function depends on the behavior of the Lévy measure near 0. Concluding, if the Lévy measure is finite then the approximation is given by Theorem 4.1, if it is not - by Theorem 5.3, but then the error depends on ε also. Note that in the first case we are able to construct strong approximations of higher order than the Euler scheme.

The paper is organized as follows, in Section 2 we present known facts concerning Lévy-type stochastic differential equation and describe the procedure of solution expansion with the use of the Itô formula. Section 3 contains precise formulation of the problem which is beeing successively solved in Section 4. This section consists of three preceding lemmas which are used in the main Theorem 4.1. In this section we adopt some ideas and estimation from [3] to the present jump-diffusion settings. Section 5 is devoted to the modification of the approximation in the case where the Lévy measure is infinite. Section 6 consists of two examples of strong approximations schemes for $\gamma = \frac{1}{2}$ and $\gamma = 1$, i.e. the Euler and Milstein schemes.

2 Basic definitions and facts

Let $(\Omega, \mathcal{F}_t; t \in [0,T], P)$ be a probability space with filtration generated by two independent processes: a standard Wiener process W and a random Poisson measure N. The Poisson random measure defined on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ is assumed to have the intensity measure ν which is a Lévy measure. By $\tilde{N}$ we denote the compensated Poisson random measure. Since we will consider stochastic integrals of different types, the class of integrands should be specified. While the integrals with respect to time and the Poisson measure are well understood, the class of integrands with respect to W and $\tilde{N}$ should be precised.

Definition 2.1 *A mapping $g_1 : \Omega \times [0,T] \longrightarrow \mathbb{R}$ is integrable with respect to W if it is predictable and satisfies the integrability condition:* $\mathbf{E} \int_0^T g_1^2(s) ds < \infty.$

Definition 2.2 *Let E be a subset of $\mathbb{R}$. A mapping $g_2 : \Omega \times [0,T] \times E \longrightarrow \mathbb{R}$ is integrable with respect to $\tilde{N}$ if it is predictable and satisfies the integrability condition:* $\mathbf{E} \int_0^T \int_E g_2^2(s,x) \nu(dx) ds < \infty.$

In these classes of integrands both integrals are square-integrable martingales and the following *isometric formulas* hold:

$$\mathbf{E}\Big(\int_0^T g_1(s) dW_s \Big)^2 = \mathbf{E} \int_0^T g_1^2(s) ds$$

$$\mathbf{E}\Big(\int_0^T \int_E g_2(s,x) \tilde{N}(ds,dx) \Big)^2 = \mathbf{E} \int_0^T \int_E g_2^2(s,x) \nu(dx) ds.$$

Throughout all the paper we will work with a stochastic differential equation of the form:

$$Y_t = Y_0 + \int_0^t b(Y_{s_-}) ds + \int_0^t \sigma(Y_{s_-}) dW_s + \int_0^t \int_B F(Y_{s_-}, x) \tilde{N}(ds, dx)$$
$$+ \int_0^t \int_{B'} G(Y_{s_-}, x) N(ds, dx), \qquad (2.2)$$

where $t \in [0,T]$, $B = \{x : |x| < 1\}$, $B' = \{x : |x| \geq 1\}$. For simplicity the initial condition is assumed to be deterministic, i.e. $Y_0 \in \mathbb{R}$. Coefficients $b : \mathbb{R} \longrightarrow \mathbb{R}$, $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $G : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are measurable and satisfy the following conditions.

(A1) Lipschitz condition: there exists a constant $K_1 > 0$ such that:

$$| b(y_1) - b(y_2) |^2 + | \sigma(y_1) - \sigma(y_2) |^2 + \int_B | F(y_1, x) - F(y_2, x) |^2 \nu(dx)$$

$$+ \int_{B'} | G(y_1, x) - G(y_2, x) |^2 \nu(dx) \leq K_1 | y_1 - y_2 |^2 \quad \forall y_1, y_2 \in \mathbb{R}.$$

(A2) Growth condition: there exists a constant $K_2 > 0$ such that:

$$| b(y) |^2 + | \sigma(y) |^2 + \int_B | F(y, x) |^2 \nu(dx)$$

$$+ \int_{B'} | G(y, x) |^2 \nu(dx) \leq K_2 | 1 + y^2 | \quad \forall y \in \mathbb{R}.$$

Theorem 2.3 *Under assumptions (**A1**) and (**A2**) there exists a unique, adapted, càdlàg solution of (2.2). Moreover, the solution satisfies:*

$$\mathbf{E} | Y_t |^2 \leq C_1 (1 + Y_0^2) \quad \forall t \in [0,T], \qquad (2.3)$$

where $C_1 \geq 0$.

Theorem 2.3 is a consequence of Theorem 6.2.3, Theorem 6.2.9 and Corollary 6.2.4 in [1], where the Lipschitz and the growth conditions are imposed on the coefficients b, σ, F only and $G(\cdot, x)$ is assumed to be continuous. The estimation (2.3) itself is a consequence of Corollary 6.2.4 in [1] and the proof of Theorem 6.2.3, where the inequality:

$$\mathbf{E} \mid Y_t \mid^2 \leq C(t)(1 + Y_0^2) \quad \forall\, t \in [0, T], \tag{2.4}$$

is shown for equation (2.2) but without the term $\int_0^t \int_{B'} G(Y_{s_-}, x) N(ds, dx)$. Under assumptions $(\mathbf{A1}), (\mathbf{A2})$ the same estimation can be obtained for (2.2) with the use of similar arguments. Moreover, $C(\cdot)$ is a continuous function and as such it is bounded on the interval $[0, T]$ and thus (2.3) holds.

In the sequel the proposition below will be used and for the reader's convenience we provide the proof.

Proposition 2.4 *Under assumptions (**A1**) and (**A2**) the solution Y of (2.2) satisfies the estimation:*

$$\mathbf{E} \sup_{0 \leq s \leq T} \mid Y_s \mid^2 \leq C_2(1 + Y_0^2)$$

for some constant $C_2 \geq 0$.

Proof: We write the solution in the form:

$$Y_s = Y_0 + \int_0^s b(Y_{u-})du + \int_0^s \sigma(Y_{u-})dW_u + \int_0^s \int_B F(Y_{u-}, x)\tilde{N}(du, dx)$$

$$+ \int_0^s \int_{B'} G(Y_{u-}, x)\tilde{N}(du, dx) + \int_0^s \int_{B'} G(Y_{u-}, x)\nu(dx)du,$$

and thus:

$$Y_s^2 \leq 6\left(Y_0^2 + \left(\int_0^s b(Y_{u-})du \right)^2 + \left(\int_0^s \sigma(Y_{u-})dW_u \right)^2 + \left(\int_0^s \int_B F(Y_{u-}, x)\tilde{N}(du, dx) \right)^2 \right.$$

$$\left. + \left(\int_0^s \int_{B'} G(Y_{u-}, x)\tilde{N}(du, dx) \right)^2 + \left(\int_0^s \int_{B'} G(Y_{u-}, x)\nu(dx)du \right)^2 \right).$$

Using the Doob and Schwartz inequalities as well as isometric formulas for stochastic integrals we obtain:

$$\mathbf{E} \sup_{0 \leq s \leq T} Y_s^2 \leq 6\left(Y_0^2 + T\mathbf{E} \int_0^T b^2(Y_{u-})du + 4\mathbf{E} \int_0^T \sigma^2(Y_{u-})du + 4\mathbf{E} \int_0^T \int_B F^2(Y_{u-}, x)\nu(dx)du \right.$$

$$\left. + 4\mathbf{E} \int_0^T \int_{B'} G^2(Y_{u-}, x)\nu(dx)du + T\nu(B')\mathbf{E} \int_0^T \int_{B'} G^2(Y_{u-}, x)\nu(dx)du \right).$$

Using assumption $(\mathbf{A2})$ we obtain:

$$\mathbf{E} \sup_{0 \leq s \leq T} Y_s^2 \leq 6\left(Y_0^2 + K_2(T + 12 + T\nu(B')) \int_0^T (1 + \mathbf{E}Y_u^2)du \right).$$

By (2.3) we have:

$$\mathbf{E} \sup_{0 \leq s \leq T} Y_s^2 \leq 6\left(Y_0^2 + K_2(T + 12 + T\nu(B')) \int_0^T \left(1 + C_1(1 + Y_0) \right)du \right),$$

4

and finally we have the desired estimation:

$$\mathbf{E} \sup_{0 \leq s \leq T} Y_s^2 \leq C_2(1 + Y_0^2).$$

$\square$

For the process Y beeing a solution of (2.2) and for a real function f of class C^2 we have the following form of the Itô formula:

$$f(Y_t) = f(Y_0) + \int_0^t f'(Y_{s_-})b(Y_{s_-})ds + \int_0^t f'(Y_{s_-})\sigma(Y_{s_-})dW_s + \frac{1}{2}\int_0^t f''(Y_{s_-})\sigma^2(Y_{s_-})ds$$

$$+ \int_0^t \int_{B'} \left\{ f(Y_{s_-} + G(Y_{s_-}, x)) - f(Y_{s_-}) \right\} N(ds, dx)$$

$$+ \int_0^t \int_B \left\{ f(Y_{s_-} + F(Y_{s_-}, x)) - f(Y_{s_-}) \right\} \tilde{N}(ds, dx) \qquad (2.5)$$

$$+ \int_0^t \int_B \left\{ f(Y_{s_-} + F(Y_{s_-}, x)) - f(Y_{s_-}) - F(Y_{s_-}, x)f'(Y_{s_-}) \right\} \nu(dx)ds.$$

Introducing the following operators:

$$L^0 f(y) \quad := f'(y)b(y) + \frac{1}{2}f''(y)\sigma^2(y) + \int_B \{f(y + F(y, x)) - f(y) - F(y, x)f'(y)\}\nu(dx)$$

$$L^1 f(y) \quad := f'(y)\sigma(y)$$

$$L^2 f(y, x) \quad := f(y + F(y, x)) - f(y)$$

$$L^3 f(y, x) \quad := f(y + G(y, x)) - f(y),$$

we can write (2.5) in the operator form:

$$f(Y_t) = f(Y_0) + \int_0^t L^0 f(Y_{s-})ds + \int_0^t L^1 f(Y_{s-})dW_s$$

$$+ \int_0^t \int_B L^2 f(Y_{s-}, x)\tilde{N}(ds, dx) + \int_0^t \int_{B'} L^3 f(Y_{s-}, x)N(ds, dx).$$

We would like to apply the Itô formula not only to the function f, but to the coefficient functions: $L^0 f, L^1 f, L^2 f, L^3 f$ or in general to any function which is smooth enough as well. Since functions $L^2 f$ and $L^3 f$ depend on two arguments (x, y), we admit the following rules of acting operators on the multiargument real function $g(y, x_1, x_2, ..., x_l)$:

$$L^0 g(y, x_1, ..., x_l) := \frac{\partial}{\partial y}g(y, x_1, ..., x_l)b(y) + \frac{1}{2}\frac{\partial^2}{\partial y^2}g(y, x_1, ..., x_l)\sigma^2(y)$$

$$+ \int_B \left\{ g(y + F(y, x), x_1, ..., x_l) - g(y, x_1, ..., x_l) - F(y, x)\frac{\partial}{\partial y}g(y, x_1, ..., x_l) \right\}\nu(dx)$$

$$L^1 g(y, x_1, ..., x_l) := \frac{\partial}{\partial y}g(y, x_1, ..., x_l)\sigma(y)$$

$$L^2 g(y, x_1, ..., x_l, x_{l+1}) := g(y + F(y, x_{l+1}), x_1, ..., x_l) - g(y, x_1, ..., x_l)$$

$$L^3 g(y, x_1, ..., x_l, x_{l+1}) := g(y + G(y, x_{l+1}), x_1, ..., x_l) - g(y, x_1, ..., x_l).$$

5

To describe the higher order Itô expansion of f we will use the notion of *multiindices* and *multiple stochastic integrals*. A multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{l(\alpha)})$ is a finite sequence of elements such that $\alpha_i \in \{0, 1, 2, 3\}$ for $i = 1, 2, ..., l(\alpha)$. The number of all elements equal to

a) 0 will be denoted by $s(\alpha)$,

b) 1 will be denoted by $w(\alpha)$,

c) 2 will be denoted by $\tilde{n}(\alpha)$,

d) 3 will be denoted by $n(\alpha)$.

The length $l(\alpha)$ of α is thus given as $l(\alpha) = s(\alpha) + w(\alpha) + \tilde{n}(\alpha) + n(\alpha)$. For the sake of convenience we also define $k(\alpha) := \tilde{n}(\alpha) + n(\alpha)$. For technical reasons we also consider the empty index denoted by v with length 0, i.e. $l(v) = 0$. For a given multiindex $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{l(\alpha)})$ let us define:

$$\alpha- = (\alpha_1, \alpha_2, ..., \alpha_{l(\alpha)-1})$$

$$-\alpha = (\alpha_2, ..., \alpha_{l(\alpha)}).$$

Definition 2.5 *A set of multiindices $\mathcal{A}$ is called a* **hierarchical set** *if $\forall \alpha \in \mathcal{A}$:*

$$l(\alpha) < \infty \quad and \quad \alpha \in \mathcal{A} \backslash \{v\} \implies -\alpha \in \mathcal{A}.$$

A set of multiindices $\mathcal{B}(\mathcal{A})$, where $\mathcal{A}$ is a hierarchical set, is called a **remainder set** *of $\mathcal{A}$ if $\forall \alpha \in \mathcal{B}(\mathcal{A})$*

$$\alpha \notin \mathcal{A} \quad and \quad -\alpha \in \mathcal{A}.$$

Assume that $g(s, x_1, x_2, ..., x_l)$ is a regular stochastic process, i.e. such that all the stochastic integrals written below exist in the sense of Definitions 2.1 and 2.2. Let ρ and τ be fixed points in the interval $[0, T]$ s.t. $\rho \leq \tau$. A multiple stochastic integral on the interval $[\rho, \tau]$ with respect to any multiindex α s.t. $k(\alpha) \leq l$ is defined by the induction procedure. First, we define the integral with respect to the empty index:

$$I_v[g]_\rho^\tau(x_1, ..., x_l) = g(\tau, x_1, ..., x_l).$$

Now, assume that $I_{\alpha-}[g]_\rho^\tau(x_1, x_2, ..., x_k)$ depends on k parameters, where $0 \leq k \leq l$. Then we define the multiple integral as follows:

1) if $\alpha_{l(\alpha)} = 0$ then

$$I_\alpha[g]_\rho^\tau(x_1, ..., x_k) = \int_\rho^\tau I_{\alpha-}[g]_\rho^{s-}(x_1, ..., x_k)ds,$$

2) if $\alpha_{l(\alpha)} = 1$ then

$$I_\alpha[g]_\rho^\tau(x_1, ..., x_k) = \int_\rho^\tau I_{\alpha-}[g]_\rho^{s-}(x_1, ..., x_k)dW_s,$$

3) if $\alpha_{l(\alpha)} = 2$ and $k \geq 1$ then

$$I_\alpha[g]_\rho^\tau(x_1, ..., x_{k-1}) = \int_\rho^\tau \int_B I_{\alpha-}[g]_\rho^{s-}(x_1, ..., x_k)\tilde{N}(ds, dx_k),$$

4) if $\alpha_{l(\alpha)} = 3$ and $k \geq 1$ then

$$I_\alpha[g]_\rho^\tau(x_1, ..., x_{k-1}) = \int_\rho^\tau \int_{B'} I_{\alpha-}[g]_\rho^{s-}(x_1, ..., x_k)N(ds, dx_k).$$

6

Let us notice that it follows from the description above that $I_\alpha[g]$ depends on $l - k(\alpha)$ parameters, i.e. $I_\alpha[g]_\rho^\tau = I_\alpha[g]_\rho^\tau(x_1, x_2, ..., x_{l-k(\alpha)})$.

Example Let $g = g(s, x_1, x_2, x_3)$. Then:

$$I_{(1)}[g]_\rho^\tau(x_1, x_2, x_3) = \int_\rho^\tau g(s-, x_1, x_2, x_3) dW_s,$$

$$I_{(213)}[g]_0^t(x_1) = \int_0^t \int_{B'} \int_0^{s_1-} \int_0^{s_2-} \int_B g(s_3-, x_1, x_2, x_3) \, \tilde{N}(ds_3, dx_3) \, dW_{s_2} \, N(ds_1, dx_2).$$

The processes which serve as integrands in multiple integrals in the expansion of $f(Y)$ will be obtained with the use of *coefficient functions* f_α, where α is a multiindex. We define the coefficient function with respect to any multiindex α by the induction procedure:

$$f_v(y) = f(y),$$

$$f_\alpha(y, x_1, ..., x_{k(\alpha)}) = L^{\alpha_1}\left[f_{-\alpha}(y, x_1, ..., x_{k(-\alpha)}) \right](y, x_1, ..., x_{k(\alpha)}).$$

Example For a given function $f = f(y)$ we get:

$$f_{(10)}(y) = L^1 L^0 f,$$

$$f_{(2013)}(y, x_1, x_2) = L^2 L^0 L^1 L^3 f.$$

For simplicity we omit here the dependence on arguments on the right hand side.

Notice, that the coefficient function $f_\alpha = f_\alpha(y, x_1, ..., x_{k(\alpha)})$ depends on $k(\alpha)$ parameters, i.e. on $x_1, x_2, ..., x_{k(\alpha)}$. However, the multiple integral $I_\alpha[f_\alpha]_\rho^\tau = I_\alpha[f_\alpha(y, x_1, ..., x_{k(\alpha)})]_\rho^\tau$ does not depend on any parameter.

We have the following analogue of Theorem 5.5.1 in [3] which is also called the Itô - Taylor expansion. It is a consequence of the Itô formula and definitions of the hierarchical and remainder sets.

Theorem 2.6 *For any hierarchical set $\mathcal{A}$ and a smooth function f we have the following representation:*

$$f(Y_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(Y_\rho, x_1, ..., x_{k(\alpha)})]_\rho^\tau + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)})]_\rho^\tau, \qquad (2.6)$$

assuming that all the integrals above exist.

Notice that the first sum in (2.6) consists of all integrals for which the integrands do not depend on time while the second sum contains all integrals with the integrands dependent on time. Since we are interested in the approximation of the process Y itself, to the end of the paper we will consider the identity function only, i.e. $f(y) = y$.

In the sequel we use two auxiliary lemmas.

Lemma 2.7 (The Gronwall lemma) *Let $g, h : [0, T] \longrightarrow \mathbb{R}$ be integrable and satisfy:*

$$0 \leq g(t) \leq h(t) + L \int_0^t g(s) ds$$

for $t \in [0, T]$ and $L > 0$. Then:

$$g(t) \leq h(t) + L \int_0^t e^{L(t-s)} h(s) ds$$

for $t \in [0, T]$.

Lemma 2.8 *Let g be a cádlág function on the interval $[0, T]$. Then for any $(\rho, \tau] \subseteq [0, T]$ we have:*

$$\sup_{s \in (\rho, \tau]} g(s-) \leq \sup_{s \in (\rho, \tau]} g(s).$$

Proof: Let $(s_n)_{n=1,2,...}$ be a sequence such that $s_n \in (\rho, \tau]$ for $n = 1, 2, ...$ satisfying $g(s_n-) \longrightarrow \sup_{s \in (\rho, \tau]} g(s-) := K$. Since g is cádlág, for any $\varepsilon > 0$ there exits a sequence $(s_n^\varepsilon)_{n=1,2,...}$ such that $s_n^\varepsilon \in (\rho, \tau]$ for $n = 1, 2, ...$ and satisfies:

$$g(s_n^\varepsilon) \geq g(s_n-) - \varepsilon \quad \text{for } n = 1, 2, ...$$

and thus:

$$\varlimsup_{n \longrightarrow \infty} g(s_n^\varepsilon) \geq K - \varepsilon.$$

Letting $\varepsilon \longrightarrow 0$ we obtain $sup_{s \in (\rho, \tau]} g(s) \geq K$. $\qquad\qquad\square$

3 Problem formulation

Our approximation of the process Y, which is the solution of (2.2), will be based on a fixed partition

$$0 = \tau_0 < \tau_1 < ... < \tau_n = T$$

of the interval $[0, T]$. For the sake of simplicity all the partition points are assumed to be non-random. The diameter of this partition is assumed to be smaller than δ, i.e. $\max_{i=0,1,...,n-1}(\tau_{i+1} - \tau_i) < \delta$. The approximation denoted by Y^δ is obtained from the first sum of multiple integrals in the Itô-Taylor expansion (2.6). The procedure can be described as follows. Starting from the known value Y_0^δ, which can be equal to Y_0, we calculate the value Y_t^δ for $t \in (0, \tau_1]$ using the first sum in (2.6). Using value $Y_{\tau_1}^\delta$ we repeat the procedure for $t \in (\tau_1, \tau_2]$ and so on. Denoting $n_t = \max\{k : \tau_k \leq t\}$ we define process Y^δ as:

$$Y_t^\delta = \sum_{\alpha \in \mathcal{A}} I_\alpha [f_\alpha (Y_{\tau_{n_t}}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_{n_t}}^t. \tag{3.7}$$

The motivation for the form of the approximation is justified by the possibility of practical calculation multiple integrals for which integrands does not depend on time (at least for low order integrals). In fact, in the case of integrals with respect to the compensated Poisson measure additional difficulty occurs which is related to the property of Lévy measure. It is discussed in Section 5. We focus on the problem of finding a strong approximation of order $\gamma > 0$, i.e. such that

$$\mathbb{E} \sup_{t \in [0, T]} | Y_t - Y_t^\delta |^2 \leq C \delta^{2\gamma} \tag{3.8}$$

for some constant $C > 0$. The rate of convergence γ is fixed and in practical application it is the multiplicity of $\frac{1}{2}$, i.e. $\gamma = \frac{1}{2}, 1, \frac{3}{2}, ...$

Thus our goal can be summarized as follows: for a fixed $\gamma > 0$ find a hierarchical set $\mathcal{A}$ such that the approximation Y^δ defined by (3.7) satisfies (3.8).

8

4 Construction of the strong approximation

Before formulating the main theorem let us introduce the following notation. For any multiindex α s.t. $k(\alpha) > 0$ we denote by $\beta(\alpha)$ a multiindex which is obtained from α by deleting all the coordinates equal to 0 or 1. Then the sets B_i^α for $i = 1, 2, ..., k(\alpha)$ are defined as follows

$$B_i^\alpha := \begin{cases} B \text{ if } \beta(\alpha)_{k(\alpha)+1-i} = 2 \\ B^{'} \text{ if } \beta(\alpha)_{k(\alpha)+1-i} = 3. \end{cases}$$

The following result is a generalization of Theorem 10.6.3 in [3] and Theorem 7 in [2].

Theorem 4.1 *Let us assume that coefficients in equation (2.2) satisfy conditions* **(A1)**,**(A2)**. *Let Y^δ be the approximation of the form (3.7), for the solution Y of (2.2), constructed with the use of the hierarchical set $\mathcal{A}_\gamma$, where:*

$$\mathcal{A}_\gamma := \left\{ l(\alpha) + s(\alpha) \leq 2\gamma \quad or \quad l(\alpha) = s(\alpha) = \gamma + \frac{1}{2} \right\}. \tag{4.9}$$

Moreover, assume that coefficient functions f_α satisfy:
(A3) *for any $\alpha \in \mathcal{A}_\gamma$ holds:*

$$\int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} \mid f_\alpha(y_1, x_1, ..., x_{k(\alpha)}) - f_\alpha(y_2, x_1, ..., x_{k(\alpha)}) \mid^2 \nu(dx_{k(\alpha)})...\nu(dx_1)$$

$$\leq K_\alpha \mid y_1 - y_2 \mid^2,$$

(A4) *for any $\alpha \in \mathcal{A}_\gamma \cup \mathcal{B}(\mathcal{A}_\gamma)$ holds:*

$$\int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} \mid f_\alpha(y, x_1, x_2, ..., x_{k(\alpha)}) \mid^2 \nu(dx_{k(\alpha)})...\nu(dx_1) \leq L_\alpha(1 + y^2),$$

where K_α, L_α are some constants.
Then for $\delta \in (0, 1)$ the inequality:

$$\mathbf{E} \sup_{s \in [0,T]} \mid Y_s - Y_s^\delta \mid^2 \leq E_1(\gamma, T) \mid Y_0 - Y_0^\delta \mid^2 + E_2(\gamma, T, Y_0)\delta^{2\gamma}$$

holds.

The proof is presented at the end of this section. First we present three auxiliary lemmas and a proposition.

Lemma 4.2 *Let ρ, τ be two fixed points in the interval $[0, T]$ s. t. $\rho < \tau$, $\tau - \rho < \delta$. If all the integrals below exist then we have:*

$$\mathbf{E} \sup_{s \in (\rho, \tau]} \left\{ \int_\rho^s g(u, x_1, x_2, ..., x_l)du \right\}^2 \leq \delta^2 \mathbf{E} \left\{ \sup_{u \in (\rho, \tau]} g^2(u, x_1, x_2, ..., x_l) \right\}, \tag{4.10}$$

$$\mathbf{E} \sup_{s \in (\rho, \tau]} \left\{ \int_\rho^s g(u, x_1, x_2, ..., x_l)du \right\}^2 \leq \delta \int_\rho^\tau \mathbf{E} \left\{ g^2(u, x_1, x_2, ..., x_l) \right\}du, \tag{4.11}$$

$$\mathbf{E} \sup_{s \in (\rho, \tau]} \left\{ \int_\rho^s g(u, x_1, x_2, ..., x_l)dW_u \right\}^2 \leq 4\delta \mathbf{E} \left\{ \sup_{u \in (\rho, \tau]} g^2(u, x_1, x_2, ..., x_l) \right\}, \tag{4.12}$$

$$\mathbf{E} \sup_{s \in (\rho, \tau]} \left\{ \int_\rho^s g(u, x_1, x_2, ..., x_l)dW_u \right\}^2 \leq 4 \int_\rho^\tau \mathbf{E} \left\{ g^2(u, x_1, x_2, ..., x_l) \right\}du, \tag{4.13}$$

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s \int_B g(u,x_1,...,x_l)\tilde{N}(du,dx_l) \right\}^2 \leq 4\delta \int_B \mathbf{E}\left\{ \sup_{u\in(\rho,\tau]} g^2(u,x_1,...,x_l) \right\}\nu(dx_l),$$

(4.14)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s \int_B g(u,x_1,...,x_l)\tilde{N}(du,dx_l) \right\}^2 \leq 4 \int_\rho^\tau \int_B \mathbf{E}\{ g^2(u,x_1,x_2,...,x_l) \}\nu(dx_l)du,$$

(4.15)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s \int_{B'} g(u,x_1,...,x_l)N(du,dx_l) \right\}^2 \leq$$

$$2\delta(4+\delta\nu(B')) \int_{B'} \mathbf{E}\left\{ \sup_{u\in(\rho,\tau]} g^2(u,x_1,...,x_l) \right\}\nu(dx_l),$$

(4.16)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s \int_{B'} g(u,x_1,...,x_l)N(du,dx_l) \right\}^2 \leq$$

$$2(4+\delta\nu(B')) \int_\rho^\tau \int_{B'} \mathbf{E}\{ g^2(u,x_1,...,x_l) \}\nu(dx_l)du.$$

(4.17)

Note, that due to Lemma 2.8, the lemma above remains true if we replace the upper limit "s" in the left hand side integrals with "$s-$".

Proof: All these inequalities are proved with the use of the Schwartz and Doob inequalities, the isometric formula for stochastic integrals and Fubini's theorem.
(4.11)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s g(u,x_1,x_2,...,x_l)du \right\}^2 \leq \mathbf{E} \sup_{s\in(\rho,\tau]} \delta\left\{ \int_\rho^s g^2(u,x_1,x_2,...,x_l)du \right\}$$

$$\leq \delta\mathbf{E}\left\{ \int_\rho^\tau g^2(u,x_1,x_2,...,x_l)du \right\} = \delta \int_\rho^\tau \mathbf{E}\{ g^2(u,x_1,x_2,...,x_l) \}du$$

(4.10)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s g(u,x_1,x_2,...,x_l)du \right\}^2 \leq \delta \int_\rho^\tau \mathbf{E}\{ g^2(u,x_1,x_2,...,x_l) \}du$$

$$\leq \delta^2 \mathbf{E}\left\{ \sup_{u\in(\rho,\tau]} g^2(u,x_1,x_2,...,x_l) \right\}$$

(4.13)

$$\mathbf{E} \sup_{s\in(\rho,\tau]} \left\{ \int_\rho^s g(u,x_1,x_2,...,x_l)dW_u \right\}^2 \leq 4 \sup_{s\in(\rho,\tau]} \mathbf{E}\left\{ \int_\rho^s g(u,x_1,x_2,...,x_l)dW_u \right\}^2$$

$$= 4 \sup_{s\in(\rho,\tau]} \mathbf{E}\left\{ \int_\rho^s g^2(u,x_1,x_2,...,x_l)du \right\} \leq 4\mathbf{E}\left\{ \int_\rho^\tau g^2(u,x_1,x_2,...,x_l)du \right\}$$

$$= 4 \int_\rho^\tau \mathbf{E}\{ g^2(u,x_1,x_2,...,x_l) \}du$$

10

(4.12)

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s g(u,x_1,x_2,...,x_l)dW_u\right\}^2 \le 4\mathbf{E}\left\{\int_\rho^\tau g^2(u,x_1,x_2,...,x_l)du\right\}$$

$$\le 4\delta\,\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}g^2(u,x_1,x_2,...,x_l)\right\}$$

(4.15)

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_B g(u,x_1,...,x_l)\tilde{N}(du,dx_l)\right\}^2 \le 4\sup_{s\in(\rho,\tau]}\mathbf{E}\left\{\int_\rho^s\int_B g(u,x_1,...,x_l)\tilde{N}(du,dx_l)\right\}^2$$

$$= 4\sup_{s\in(\rho,\tau]}\mathbf{E}\left\{\int_\rho^s\int_B g^2(u,x_1,...,x_l)\nu(dx_l)du\right\} \le 4\int_\rho^\tau\int_B \mathbf{E}\{g^2(u,x_1,x_2,...,x_l)\}\nu(dx_l)du$$

(4.14)

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_B g(u,x_1,...,x_l)\tilde{N}(du,dx_l)\right\}^2 \le 4\,\mathbf{E}\left\{\int_\rho^\tau\int_B g^2(u,x_1,x_2,...,x_l)\nu(dx_l)du\right\}$$

$$\le 4\,\mathbf{E}\left\{\delta\sup_{u\in(\rho,\tau]}\int_B g^2(u,x_1,x_2,...,x_l)\nu(dx_l)\right\} \le 4\delta\mathbf{E}\left\{\int_B\sup_{u\in(\rho,\tau]}g^2(u,x_1,x_2,...,x_l)\nu(dx_l)\right\}$$

$$= 4\delta\int_B\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}g^2(u,x_1,...,x_l)\right\}\nu(dx_l)$$

(4.17)

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)\tilde{N}(du,dx_l)\right\}^2$$

$$= \mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)N(du,dx_l) + \int_\rho^s\int_{B'} g(u,x_1,...,x_l)\nu(dx_l)du\right\}^2$$

$$\le 2\left\{\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)\tilde{N}(du,dx_l)\right\}^2 + \mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)\nu(dx_l)du\right\}^2\right\}$$

(4.18)

The first component is bounded by analogous expression as in (4.15). For the second we have the following inequalities:

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)\nu(dx_l)du\right\}^2$$

$$\le \mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} 1\nu(dx_l)du \cdot \int_\rho^s\int_{B'} g^2(u,x_1,...,x_l)\nu(dx_l)du\right\}^2$$

$$\le \delta\nu(B')\int_\rho^\tau\int_{B'}\mathbf{E}\{g^2(u,x_1,...,x_l)\}\nu(dx_l)du.$$

As a consequence we obtain:

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'} g(u,x_1,...,x_l)N(du,dx_l)\right\}^2 \le 2(4+\delta\nu(B'))\int_\rho^\tau\int_{B'}\mathbf{E}\{g^2(u,x_1,...,x_l)\}\nu(dx_l)du.$$

11

(4.16)

For the second term in (4.18) we have the following inequalities:

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int_\rho^s\int_{B'}g(u,x_1,...,x_l)\nu(dx_l)du\right\}^2\leq\delta\nu(B')\mathbf{E}\left\{\int_\rho^\tau\int_{B'}g^2(u,x_1,...,x_l)\nu(dx_l)du\right\}$$

$$\leq\delta^2\nu(B')\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}\int_{B'}g^2(u,x_1,...,x_l)\nu(dx_l)\right\}\leq\delta^2\nu(B')\mathbf{E}\left\{\int_{B'}\sup_{u\in(\rho,\tau]}g^2(u,x_1,...,x_l)\nu(dx_l)\right\}$$

$$=\delta^2\nu(B')\int_{B'}\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}g^2(u,x_1,...,x_l)\right\}\nu(dx_l).$$

Taking into account (4.18), the inequality above and (4.14) we obtain:

$$\mathbf{E}\sup_{s\in(\rho,\tau]}\left\{\int\limits_\rho^s\int\limits_{B'}g(u,x_1,...,x_l)N(du,dx_l)\right\}^2\leq2\delta(4+\delta\nu(B'))\int\limits_{B'}\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}g^2(u,x_1,...,x_l)\right\}\nu(dx_l).$$

$\square$

Lemma 4.3 *Let ρ,τ be two fixed points in the interval $[0,T]$ s.t. $\rho<\tau$, $\tau-\rho<\delta$ and $\alpha\neq v$ be a fixed multiindex. If all the integrals below exist for the process $g=g(u,x_1,...,x_l)$, where $l\geq k(\alpha)$, then we have:*

$$\mathbf{E}\{\sup_{s\in(\rho,\tau]}I_\alpha^2[g]_\rho^s(x_1,...,x_{l-k(\alpha)})\}\leq\delta^{l(\alpha)+s(\alpha)-1}4^{w(\alpha)+\tilde{n}(\alpha)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha)}.$$

$$\cdot\int_\rho^\tau\mathbf{E}\int\limits_{B_1^\alpha}\int\limits_{B_2^\alpha}...\int\limits_{B_{k(\alpha)}^\alpha}g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha)+1})du.$$

(4.19)

Proof: We will apply the induction procedure with respect to the length of α. If $l(\alpha)=1$ then (4.19) follows from inequalities (4.11), (4.13), (4.15), (4.17) in Lemma 4.2 applied to $\alpha=0$, $\alpha=1$, $\alpha=2$, $\alpha=3$ respectively.

Now assume that (4.19) is true for $\alpha-$ and let us show that it is also true for α. We will consider several cases.

a) $\alpha_{l(\alpha)}=0$; In this case $k(\alpha-)=k(\alpha)$ and $B_i^{\alpha-}=B_i^\alpha$ for $i=1,2,...,k(\alpha)$. By (4.10), Lemma 2.8 and the inductive assumption we have:

$$\mathbf{E}\{\sup_{s\in(\rho,\tau]}I_\alpha^2[g]_\rho^s(x_1,...,x_{l-k(\alpha)})\}=\mathbf{E}\left\{\sup_{s\in(\rho,\tau]}\int_\rho^s I_{\alpha-}[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha-)})du\right\}^2$$

$$\leq\delta^2\mathbf{E}\left\{\sup_{u\in(\rho,\tau]}I_{\alpha-}^2[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha-)})\right\}$$

$$\leq\delta^2\delta^{l(\alpha-)+s(\alpha-)-1}4^{w(\alpha-)+\tilde{n}(\alpha-)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha-)}.$$

$$\cdot\int_\rho^\tau\mathbf{E}\int\limits_{B_1^{\alpha-}}\int\limits_{B_2^{\alpha-}}...\int\limits_{B_{k(\alpha-)}^{\alpha-}}g^2(u,x_1,x_2,...,x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha-)+1})du$$

12

$$= \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4+\delta\nu(B^{'})) \Big\}^{n(\alpha)} \cdot$$

$$\cdot \int_\rho^\tau \mathbf{E} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} g^2(u,x_1,x_2,...,x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha)+1})du.$$

b) $\alpha_{l(\alpha)} = 1$; In this case $k(\alpha-) = k(\alpha)$ and $B_i^{\alpha-} = B_i^\alpha$ for $i = 1,2,...,k(\alpha)$. By (4.12), Lemma 2.8 and the inductive assumption we have:

$$\mathbf{E}\{ \sup_{s\in(\rho,\tau]} I_\alpha^2[g]_\rho^s \}(x_1,...,x_{l-k(\alpha)}) = \mathbf{E}\Big\{ \sup_{s\in(\rho,\tau]} \int_\rho^s I_{\alpha-}[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha-)})dW_u \Big\}^2$$

$$\leq 4\delta\mathbf{E}\Big\{ \sup_{u\in(\rho,\tau]} I_{\alpha-}^2[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha-)}) \Big\}$$

$$\leq 4\delta\,\delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \Big\{ 2(4+\delta\nu(B^{'})) \Big\}^{n(\alpha-)} \cdot$$

$$\cdot \int_\rho^\tau \mathbf{E} \int_{B_1^{\alpha-}} \int_{B_2^{\alpha-}} ... \int_{B_{k(\alpha-)}^{\alpha-}} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha-)+1})du$$

$$= \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4+\delta\nu(B^{'})) \Big\}^{n(\alpha)} \cdot$$

$$\cdot \int_\rho^\tau \mathbf{E} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha)+1})du.$$

c) $\alpha_{l(\alpha)} = 2$; By (4.14), Lemma 2.8 and the inductive assumption we have:

$$\mathbf{E}\{ \sup_{s\in(\rho,\tau]} I_\alpha^2[g]_\rho^s(x_1,...,x_{l-k(\alpha)}) \} = \mathbf{E}\Big\{ \sup_{s\in(\rho,\tau]} \int_\rho^s \int_B I_{\alpha-}[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha)+1})\tilde{N}(du,dx_{l-k(\alpha)+1}) \Big\}^2$$

$$\leq 4\delta \int_B \mathbf{E}\Big\{ \sup_{u\in(\rho,\tau]} I_{\alpha-}^2[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha)+1}) \Big\}\nu(dx_{l-k(\alpha)+1})$$

$$\leq 4\delta\,\delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \Big\{ 2(4+\delta\nu(B^{'})) \Big\}^{n(\alpha-)} \cdot$$

$$\cdot \int_B \int_\rho^\tau \mathbf{E} \int_{B_1^{\alpha-}} \int_{B_2^{\alpha-}} ... \int_{B_{k(\alpha-)}^{\alpha-}} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha-)+1})\,du\,\nu(dx_{l-k(\alpha)+1})$$

$$= \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4+\delta\nu(B^{'})) \Big\}^{n(\alpha)} \cdot$$

$$\cdot \int_\rho^\tau \mathbf{E} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha)+1})\,du.$$

d) $\alpha_{l(\alpha)} = 3$; By (4.16), Lemma 2.8 and the inductive assumption we have:

$$\mathbf{E}\{\sup_{s\in(\rho,\tau]} I_\alpha^2[g]_\rho^s(x_1,...,x_{l-k(\alpha)})\} = \mathbf{E}\Big\{\sup_{s\in(\rho,\tau]}\int_\rho^s\int_{B'} I_{\alpha-}[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha)+1})N(du,dx_{l-k(\alpha)+1})\Big\}^2$$

$$\leq 2(4\delta + \delta^2\nu(B'))\int_{B'}\mathbf{E}\Big\{\sup_{u\in(\rho,\tau]} I_{\alpha-}^2[g]_\rho^{u-}(x_1,...,x_{l-k(\alpha)+1})\Big\}\nu(dx_{l-k(\alpha)+1})$$

$$\leq 2(4\delta + \delta^2\nu(B'))\,\delta^{l(\alpha-)+s(\alpha-)-1}4^{w(\alpha-)+\tilde{n}(\alpha-)}\Big\{2(4+\delta\nu(B'))\Big\}^{n(\alpha-)}\cdot$$

$$\cdot\int_{B'}\int\int_\rho^\tau\mathbf{E}\int_{B_1^{\alpha-}}\int_{B_2^{\alpha-}}...\int_{B_{k(\alpha-)}^{\alpha-}} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha-)+1})\,du\,\nu(dx_{l-k(\alpha)+1})$$

$$= \delta^{l(\alpha)+s(\alpha)-1}4^{w(\alpha)+\tilde{n}(\alpha)}\Big\{2(4+\delta\nu(B'))\Big\}^{n(\alpha)}\cdot$$

$$\cdot\int_\rho^\tau\mathbf{E}\int_{B_1^\alpha}\int_{B_2^\alpha}...\int_{B_{k(\alpha)}^\alpha} g^2(u,x_1,x_2,...x_l)\nu(dx_l)\,\nu(dx_{l-1})\,...\,\nu(dx_{l-k(\alpha)+1})\,du.$$

$$\square$$

For any multiindex $\alpha \neq v$ and a process $g = g(s,x_1,...,x_{k(\alpha)})$ we define two auxiliary functionals:

$$F_t^\alpha[g] := \mathbf{E}\sup_{s\in[0,t]}\left(\sum_{i=0}^{n_s-1} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} + I_\alpha[g]_{\tau_{n_s}}^s\right)^2, \tag{4.20}$$

$$G_{\rho,\tau}^\alpha[g] := \mathbf{E}\sup_{s\in[\rho,\tau]}\int_{B_1^\alpha}\int_{B_2^\alpha}...\int_{B_{k(\alpha)}^\alpha} g^2(s,x_1,...,x_{k(\alpha)})\nu(dx_{k(\alpha)})...\nu(dx_1). \tag{4.21}$$

Lemma 4.4 *For any multiindex $\alpha \neq \nu$ and a process g s.t. $G_{0,t}^\alpha[g] < \infty$ we have the following inequality:*

$$F_t^\alpha[g] \leq \begin{cases} t\,\delta^{l(\alpha)+s(\alpha)-2}\int_0^t G_{0,u}^\alpha[g]\,du & \text{if} \quad l(\alpha) = s(\alpha) \\[2mm] C(\alpha,t)\,\delta^{l(\alpha)+s(\alpha)-1}\int_0^t G_{0,u}^\alpha[g]\,du & \text{if} \quad l(\alpha) \neq s(\alpha)\,. \end{cases}$$

Proof: We consider several cases:

    a) $l(\alpha) = s(\alpha)$,

    b) { $w(\alpha) > 0$ or $\tilde{n}(\alpha) > 0$} and

        b1) $\alpha_{l(\alpha)} = 0$,

        b2) $\alpha_{l(\alpha)} = 1$,

        b3) $\alpha_{l(\alpha)} = 2$,

        b4) $\alpha_{l(\alpha)} = 3$,

    c) $n(\alpha) > 0$ and $w(\alpha) = \tilde{n}(\alpha) = 0$.

a) $l(\alpha) = s(\alpha)$

By the Schwartz inequality and Lemma 4.3 we have:

$$F_t^\alpha[g] = \mathbf{E} \sup_{s \in [0,t]} \left(\int_0^s I_{\alpha-}[g]_{\tau_{n_u}}^{u-} \, du \right)^2 \leq \mathbf{E} \sup_{s \in [0,t]} \left[s \int_0^s I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-} \, du \right] \leq t \int_0^t \mathbf{E} \left(I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-} \right) du$$

$$\leq t \int_0^t \mathbf{E} \sup_{s \in (\tau_{n_u}, u]} \left(I_{\alpha-}^2[g]_{\tau_{n_u}}^{s-} \right) du \leq t \int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} \int_{\tau_{n_u}}^u \mathbf{E} g^2(s) ds \, du$$

$$\leq t \int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} \int_{\tau_{n_u}}^u \mathbf{E} \sup_{w \in [\tau_{n_u}, s]} g^2(w) ds \, du \leq t \int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} \int_{\tau_{n_u}}^u G_{\tau_{n_u}, s}^\alpha[g] ds \, du$$

$$\leq t \int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} \delta G_{\tau_{n_u}, u}^\alpha[g] du \leq t \delta^{l(\alpha)+s(\alpha)-2} \int_0^t G_{0,u}^\alpha[g] du.$$

b1) $\{ w(\alpha) > 0$ or $\tilde{n}(\alpha) > 0 \}$ and $\alpha_{l(\alpha)} = 0$

The following inequality holds:

$$F_t^\alpha[g] \leq 2 \, \mathbf{E} \sup_{s \in [0,t]} \left(\sum_{i=0}^{n_s-1} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + 2 \, \mathbf{E} \sup_{s \in [0,t]} \left(I_\alpha[g]_{\tau_{n_s}}^s \right)^2.$$

Notice that the process $\sum_{i=0}^{n_s-1} I_\alpha[g]_{\tau_i}^{\tau_{i+1}}$ is a martingale because it contains integral with respect to the Wiener process or with respect to the compensated Poisson measure. First let us consider the first sum.

$$\mathbf{E} \sup_{s \in [0,t]} \left(\sum_{i=0}^{n_s-1} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 \leq 4 \sup_{s \in [0,t]} \mathbf{E} \left(\sum_{i=0}^{n_s-1} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2$$

$$= 4 \sup_{s \in [0,t]} \mathbf{E} \left(\sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} + I_\alpha[g]_{\tau_{n_s-1}}^{\tau_{n_s}} \right)^2$$

$$= 4 \sup_{s \in [0,t]} \mathbf{E} \left(\left(\sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + 2 \sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} I_\alpha[g]_{\tau_{n_s-1}}^{\tau_{n_s}} + I_\alpha^2[g]_{\tau_{n_s-1}}^{\tau_{n_s}} \right)$$

$$\leq 4 \sup_{s \in [0,t]} \left\{ \mathbf{E} \left(\sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + \mathbf{E} \sup_{u \in (\tau_{n_s-1}, \tau_{n_s}]} I_\alpha^2[g]_{\tau_{n_s-1}}^u \right\}$$

$$\leq 4 \sup_{s \in [0,t]} \left\{ \mathbf{E} \left(\sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha)} . \right.$$

$$\left. \cdot \int_{\tau_{n_s-1}}^{\tau_{n_s}} \mathbf{E} \int_{B_1^\alpha} \int_{B_2^\alpha} \dots \int_{B_{k(\alpha)}^\alpha} g^2(u, x_1, x_2, \dots x_{k(\alpha)}) \nu(dx_{k(\alpha)}) \, \nu(dx_{k(\alpha)-1}) \dots \nu(dx_1) du \right\}$$

15

$$\leq 4 \sup_{s \in [0,t]} \left\{ \mathbf{E} \left(\sum_{i=0}^{n_s-2} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \cdot \right.$$

$$\left. \cdot \int_{\tau_{n_s-1}}^{\tau_{n_s}} G_{\tau_{n_s-1},u}^\alpha du \right\}$$

$$\leq 4 \sup_{s \in [0,t]} \left\{ \mathbf{E} \left(\sum_{i=0}^{n_s-3} I_\alpha[g]_{\tau_i}^{\tau_{i+1}} \right)^2 + \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \cdot \right.$$

$$\left. \cdot \left(\int_{\tau_{n_s-2}}^{\tau_{n_s-1}} G_{\tau_{n_s-2},u}^\alpha du + \int_{\tau_{n_s-1}}^{\tau_{n_s}} G_{\tau_{n_s-1},u}^\alpha du \right) \right\}$$

$$\leq 4 \sup_{s \in [0,t]} \left\{ \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \cdot \right.$$

$$\left. \cdot \left(\int_{\tau_0}^{\tau_1} G_{\tau_0,u}^\alpha du + \int_{\tau_1}^{\tau_2} G_{\tau_1,u}^\alpha du + ... + \int_{\tau_{n_s-1}}^{\tau_{n_s}} G_{\tau_{n_s-1},u}^\alpha du \right) \right\}$$

$$\leq \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)+1} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \int_0^t G_{0,u}^\alpha du.$$

For the second sum we have the following inequalities:

$$\mathbf{E} \sup_{s \in [0,t]} \left(I_\alpha[g]_{\tau_{n_s}}^s \right)^2 \leq \delta \, \mathbf{E} \sup_{s \in [0,t]} \int_{\tau_{n_s}}^s I_{\alpha-}^2[g]_{\tau_{n_s}}^{u-} du \leq \delta \, \mathbf{E} \sup_{s \in [0,t]} \int_{\tau_{n_s}}^s \sup_{w \in (\tau_{n_s},u]} I_{\alpha-}^2[g]_{\tau_{n_s}}^{w-} du$$

$$\leq \delta \, \mathbf{E} \int_0^t \sup_{w \in (\tau_{n_u},u]} I_{\alpha-}^2[g]_{\tau_{n_u}}^{w-} du = \delta \int_0^t \mathbf{E} \sup_{w \in (\tau_{n_u},u]} I_{\alpha-}^2[g]_{\tau_{n_u}}^{w-} du$$

$$\leq \delta \int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha-)} \cdot$$

$$\cdot \int_{\tau_{n_u}}^u \mathbf{E} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} g^2(w, x_1, x_2, ... x_{k(\alpha)}) \nu(dx_{k(\alpha)}) \, \nu(dx_{k(\alpha)-1}) \, ... \, \nu(dx_1) dw \, du$$

$$\leq \delta \, \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha-)} \int_0^t \int_{\tau_{n_u}}^u G_{\tau_{n_u},w}^\alpha[g] \, dw \, du$$

$$\leq \delta^2 \, \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha-)} \int_0^t G_{\tau_{n_u},u}^\alpha[g] \, du$$

$$= \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \int_0^t G_{0,u}^\alpha[g] \, du.$$

Finally we obtain:

$$F_t^\alpha[g] \leq 2 \cdot 4^{w(\alpha)+\tilde{n}(\alpha)} \left\{ 2(4 + \delta\nu(B')) \right\}^{n(\alpha)} \delta^{l(\alpha)+s(\alpha)-1} \int_0^t G_{0,u}^\alpha[g] \, du.$$

b2) $\{\, w(\alpha) > 0$ or $\tilde{n}(\alpha) > 0\}$ and $\alpha_{l(\alpha)} = 1$

By Doob's inequality, the isometric formula for Wiener integrals and Lemma 4.3 we obtain:

$$F_t^\alpha[g] = \mathbf{E}\sup_{s \le t}\left(\int_0^s I_{\alpha-}[g]_{\tau_{n_u}}^{u-}\, dW_u\right)^2 \le 4\sup_{s \le t}\mathbf{E}\left(\int_0^s I_{\alpha-}[g]_{\tau_{n_u}}^{u-}\, dW_u\right)^2$$

$$= 4\sup_{s \le t}\mathbf{E}\int_0^s I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}\, du = 4\int_0^t \mathbf{E}\left(I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}\right) du \le 4\int_0^t \mathbf{E}\sup_{w \in (\tau_{n_u}, u]} I_{\alpha-}^2[g]_{\tau_{n_u}}^{w-}\, du$$

$$\le 4\int_0^t \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha-)}\int_{\tau_{n_u}}^u G_{\tau_{n_u},s}^{\alpha-}[g]\, ds\, du$$

$$\le 4\delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha-)}\delta\int_0^t G_{\tau_{n_u},u}^{\alpha-}[g]\, du$$

$$= \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha)}\int_0^t G_{0,u}^{\alpha}[g]\, du.$$

b3) $\{\, w(\alpha) > 0$ or $\tilde{n}(\alpha) > 0\}$ and $\alpha_{l(\alpha)} = 2$

By Doob's inequality, the isometric formula for integrals with respect to the compensated Poisson measure and Lemma 4.3 we obtain:

$$F_t^\alpha[g] = \mathbf{E}\sup_{s \le t}\left(\int_0^s\int_B I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\tilde{N}(du, dx_1)\right)^2 \le 4\sup_{s \le t}\mathbf{E}\left(\int_0^s\int_B I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\tilde{N}(du, dx_1)\right)^2$$

$$= 4\sup_{s \le t}\mathbf{E}\left(\int_0^s\int_B I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right) = 4\mathbf{E}\left(\int_0^t\int_B I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right)$$

$$\le 4\left(\int_0^t\int_B \mathbf{E}\sup_{w \in (\tau_{n_u}, u]} I_{\alpha-}^2[g]_{\tau_{n_u}}^{w-}(x_1)\nu(dx_1)du\right)$$

$$\le 4\int_0^t\int_B \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha-)}.$$

$$\cdot\int_{\tau_{n_u}}^u \mathbf{E}\int_{B_1^{\alpha-}}\int_{B_2^{\alpha-}}\ldots\int_{B_{k(\alpha-)}^{\alpha-}} g^2(w, x_1, x_2, \ldots x_{k(\alpha)})\nu(dx_{k(\alpha)})\,\nu(dx_{k(\alpha)-1})\ldots\nu(dx_2)dw\,\nu(dx_1)du$$

$$\le 4\delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)}\left\{2(4+\delta\nu(B'))\right\}^{n(\alpha-)}.$$

$$\cdot\int_0^t \mathbf{E}\int_{\tau_{n_u}}^u\int_{B_1^{\alpha}}\int_{B_2^{\alpha}}\ldots\int_{B_{k(\alpha)}^{\alpha}} g^2(w, x_1, x_2, \ldots x_{k(\alpha)})\nu(dx_{k(\alpha)})\,\nu(dx_{k(\alpha)-1})\ldots\nu(dx_2)\nu(dx_1)dw\,du$$

$$\leq 4\delta^{l(\alpha-)+s(\alpha-)-1}4^{w(\alpha-)+\tilde{n}(\alpha-)}\Big\{2(4+\delta\nu(B^{'}))\Big\}^{n(\alpha-)}.$$

$$\cdot\int_0^t\mathbf{E}\int_{\tau_{n_u}}^u\sup_{s\in[\tau_{n_u},w]}\int_{B_1^\alpha}\int_{B_2^\alpha}...\int_{B_{k(\alpha)}^\alpha}g^2(s,x_1,...x_{k(\alpha)})\nu(dx_{k(\alpha)})\,\nu(dx_{k(\alpha)-1})\,...\,\nu(dx_2)\nu(dx_1)dwdu$$

$$\leq 4\delta^{l(\alpha-)+s(\alpha-)-1}4^{w(\alpha-)+\tilde{n}(\alpha-)}\Big\{2(4+\delta\nu(B^{'}))\Big\}^{n(\alpha-)}.$$

$$\cdot\int_0^t\mathbf{E}\delta\sup_{s\in[\tau_{n_u},u]}\int_{B_1^\alpha}\int_{B_2^\alpha}...\int_{B_{k(\alpha)}^\alpha}g^2(s,x_1,x_2,...x_{k(\alpha)})\nu(dx_{k(\alpha)})\,\nu(dx_{k(\alpha)-1})\,...\,\nu(dx_2)\nu(dx_1)du$$

$$= 4\delta\,\delta^{l(\alpha-)+s(\alpha-)-1}4^{w(\alpha-)+\tilde{n}(\alpha-)}\Big\{2(4+\delta\nu(B^{'}))\Big\}^{n(\alpha-)}\int_0^t G_{\tau_{n_u},u}^\alpha[g]du$$

$$\leq \delta^{l(\alpha)+s(\alpha)-1}4^{w(\alpha)+\tilde{n}(\alpha)}\Big\{2(4+\delta\nu(B^{'}))\Big\}^{n(\alpha)}\int_0^t G_{0,u}^\alpha[g]du.$$

b4) $\{\,w(\alpha)>0$ or $\tilde{n}(\alpha)>0\}$ and $\alpha_{l(\alpha)}=3$
We have the following inequality:

$$F_t^\alpha[g] = \mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)N(du,dx_1)\right)^2$$

$$= \mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\tilde{N}(du,dx_1)+\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right)^2$$

$$\leq 2\,\mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\tilde{N}(du,dx_1)\right)^2+2\,\mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right)^2.$$

The first term is bounded as in the case (b3). For the second term we have the following inequalities:

$$\mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}I_{\alpha-}[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right)^2$$

$$\leq \mathbf{E}\sup_{s\leq t}\left(\int_0^s\int_{B^{'}}\mathbf{1}\,\nu(dx_1)du\cdot\int_0^s\int_{B^{'}}I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}(x_1)\nu(dx_1)du\right)$$

$$\leq \delta\nu(B^{'})\int_0^t\int_{B^{'}}\mathbf{E}\left(I_{\alpha-}^2[g]_{\tau_{n_u}}^{u-}(x_1)\right)\nu(dx_1)du \leq \delta\nu(B^{'})\int_0^t\int_{B^{'}}\mathbf{E}\sup_{w\in(\tau_{n_u},u]}I_{\alpha-}^2[g]_{\tau_{n_u}}^{w-}(x_1)\nu(dx_1)du$$

$$\leq \delta\nu(B') \int_0^t \int_{B'} \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha-)} \cdot$$

$$\cdot \int_{\tau_{n_u}}^{u} \mathbf{E} \int_{B_1^{\alpha-}} \int_{B_2^{\alpha-}} \dots \int_{B_{k(\alpha-)}^{\alpha-}} g^2(w, x_1, x_2, \dots x_{k(\alpha)}) \nu(dx_{k(\alpha)}) \, \nu(dx_{k(\alpha)-1}) \dots \nu(dx_2) dw \, \nu(dx_1) du$$

$$= \delta\nu(B') \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha-)} \cdot$$

$$\cdot \int_0^t \mathbf{E} \int_{\tau_{n_u}}^{u} \int_{B_1^{\alpha}} \int_{B_2^{\alpha}} \dots \int_{B_{k(\alpha)}^{\alpha}} g^2(w, x_1, x_2, \dots x_{k(\alpha)}) \nu(dx_{k(\alpha)}) \, \nu(dx_{k(\alpha)-1}) \dots \nu(dx_1) dw du \dots$$

and omitting identical operations as in (b3) we obtain:

$$\dots \leq \delta^2 \nu(B') \delta^{l(\alpha-)+s(\alpha-)-1} 4^{w(\alpha-)+\tilde{n}(\alpha-)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha-)} \int_0^t G_{0,u}^\alpha[g] du$$

$$= \delta^{l(\alpha)+s(\alpha)-1} \delta\nu(B') 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha)-1} \int_0^t G_{0,u}^\alpha[g] du.$$

Finally, for this case we have:

$$F_t^\alpha[g] \leq 2 \, \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha)} \int_0^t G_{0,u}^\alpha[g] du$$

$$+ 2 \, \delta^{l(\alpha)+s(\alpha)-1} \delta\nu(B') 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha)-1} \int_0^t G_{0,u}^\alpha[g] du$$

$$\leq 2 \, \delta^{l(\alpha)+s(\alpha)-1} 4^{w(\alpha)+\tilde{n}(\alpha)} \Big\{ 2(4 + \delta\nu(B')) \Big\}^{n(\alpha)-1} \Big\{ 8 + 3\delta\nu(B') \Big\} \int_0^t G_{0,u}^\alpha[g] du.$$

c) $n(\alpha) > 0$ and $w(\alpha) = \tilde{n}(\alpha) = 0$

In this case the multiindex α consists of 0 and 3 only. If $\alpha_{l(\alpha)} = 3$ then the desired inequality follows from (b4). In opposite case let us denote $r(\alpha) := \max\{i : \alpha_i = 3\}$. For simplicity of exposition we show the case when $r(\alpha) = l(\alpha) - 1$. The idea for other cases is exactly the same. We have the following inequality:

$$F_t^\alpha[g] \leq 2 \, \mathbf{E} \sup_{s \leq t} \left(\int_0^s \int_{\tau_{n_u}}^{u-} \int_{B'} I_{\alpha--}[g]_{\tau_{n_u}}^{w-}(x_1) \tilde{N}(dx_1, dw) \, du \right)^2$$

$$+ 2 \, \mathbf{E} \sup_{s \leq t} \left(\int_0^s \int_{\tau_{n_u}}^{u-} \int_{B'} I_{\alpha--}[g]_{\tau_{n_u}}^{w-}(x_1) \nu(dx_1) dw \, du \right)^2.$$

Calculations for the first term in the sum above are covered by (b1). Applying the Schwartz inequality and Lemma 4.3 for the second term we obtain:

$$\mathbf{E} \sup_{s \leq t} \left(\int_0^s \int_{\tau_{n_u}}^{u-} \int_{B'} I_{\alpha--}[g]_{\tau_{n_u}}^{w-}(x_1) \nu(dx_1) dw \, du \right)^2 \leq t \mathbf{E} \int_0^t \left(\int_{\tau_{n_u}}^{u-} \int_{B'} I_{\alpha--}[g]_{\tau_{n_u}}^{w-}(x_1) \nu(dx_1) dw \right)^2 du$$

$$\leq t \int_0^t \mathbf{E} \sup_{s \in (\tau_{n_u}, u]} \left(\int_{\tau_{n_u}}^{s^-} \int_{B'} I_{\alpha--}[g]_{\tau_{n_u}}^{w-}(x_1) \nu(dx_1) dw \right)^2 du$$

$$\leq t \delta \nu(B') \int_0^t \int_{\tau_{n_u}}^u \int_{B'} \mathbf{E} I_{\alpha--}^2[g]_{\tau_{n_u}}^{w-}(x_1) \nu(dx_1) dw \, du$$

$$\leq t \delta \nu(B') \int_0^t \int_{\tau_{n_u}}^u \int_{B'} \mathbf{E} \sup_{s \in (\tau_{n_u}, w]} I_{\alpha--}^2[g]_{\tau_{n_u}}^{s-}(x_1) \nu(dx_1) dw \, du$$

$$\leq t \delta \nu(B') \delta^{l(\alpha--)+s(\alpha--)-1} \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha--)}.$$

$$\cdot \int_0^t \int_{\tau_{n_u}}^u \int_{B'} \int_{\tau_{n_u}}^w \mathbf{E} \int_{B_1^{\alpha--}} ... \int_{B_{k(\alpha--)}^{\alpha--}} g^2(s, x_1, ..., x_{k(\alpha)}) \nu(dx_{k(\alpha)}) ... \nu(dx_2) ds \, \nu(dx_1) \, dw \, du$$

$$\leq t \delta^{l(\alpha--)+s(\alpha--)} \nu(B') \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha--)}.$$

$$\int_0^t \int_{\tau_{n_u}}^u \int_{\tau_{n_u}}^w \mathbf{E} \int_{B_1^\alpha} ... \int_{B_{k(\alpha)}^\alpha} g^2(s, x_1, ..., x_{k(\alpha)}) \nu(dx_{k(\alpha)}) ... \nu(dx_1) \, ds \, dw \, du$$

$$\leq t \delta^{l(\alpha--)+s(\alpha--)} \nu(B') \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha--)} \int_0^t \int_{\tau_{n_u}}^u \delta G_{\tau_{n_u}, w}^\alpha \, dw \, du$$

$$\leq t \delta^{l(\alpha--)+s(\alpha--)} \nu(B') \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha--)} \int_0^t \delta^2 G_{\tau_{n_u}, u}^\alpha du$$

$$\leq t \delta^{l(\alpha)+s(\alpha)-1} \nu(B') \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha)-1} \int_0^t G_{0,u}^\alpha du.$$

Finally we have:

$$F_t^\alpha[g] \leq \delta^{l(\alpha)+s(\alpha)-1} \left\{ 2(4 + \delta \nu(B')) \right\}^{n(\alpha)-1} \left\{ 8 + 2t\nu(B') \right\} \int_0^t G_{0,u}^\alpha[g] du.$$

$\square$

Proposition 4.5 *Let $\mathcal{A}$ be any hierarchical set. If for each $\alpha \in \mathcal{A}$ the condition:*

$$\int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} \mid f_\alpha(y, x_1, x_2, ..., x_{k(\alpha)}) \mid^2 \nu(dx_{k(\alpha)}) ... \nu(dx_1) \leq L_\alpha(1 + y^2) \quad (4.22)$$

holds, then the approximation Y^δ given by (3.7) satisfies:

$$\mathbf{E} \sup_{0 \leq s \leq T} |Y_s^\delta|^2 \leq C_3(1 + |Y_0^\delta|^2) \quad \forall \, t \in [0, T],$$

where $C_3 \geq 0$.

Proof: Due to (3.7) we write the approximation in the following form

$$Y_s^\delta = Y_0^\delta + \sum_{\alpha \in \mathcal{A} \backslash \{v\}} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\tau_{n_s}}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right).$$

By Lemma 4.4 and assumption (4.22) we have the following inequalities:

$$\mathbf{E}\sup_{s \leq t} |Y_s^\delta|^2 \leq \sharp(\mathcal{A}) \bigg\{ |Y_0^\delta|^2 + \sum_{\alpha \in \mathcal{A}} \mathbf{E} \sup_{0 \leq s \leq T} \bigg(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}}$$

$$+ I_\alpha[f_\alpha(Y_{\tau_{n_s}}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \bigg)^2 \bigg\}$$

$$\leq \sharp(\mathcal{A}) \bigg\{ |Y_0^\delta|^2 + \sum_{\alpha \in \mathcal{A}} \int_0^t \mathbf{E} \sup_{u \in [0,s]} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} f_\alpha^2(Y_{\tau_{n_u}}^\delta, x_1, ..., x_{k(\alpha)}) \nu(dx_{k(\alpha)})...\nu(dx_1) \, ds \bigg\}$$

$$\leq \sharp(\mathcal{A}) \bigg\{ |Y_0^\delta|^2 + \sum_{\alpha \in \mathcal{A}} \int_0^t \mathbf{E} \sup_{u \in [0,s]} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} f_\alpha^2(Y_u^\delta, x_1, ..., x_{k(\alpha)}) \nu(dx_{k(\alpha)})...\nu(dx_1) \, ds \bigg\}$$

$$\leq \sharp(\mathcal{A}) \bigg\{ |Y_0^\delta|^2 + \sum_{\alpha \in \mathcal{A}} \int_0^t \mathbf{E} \sup_{u \in [0,s]} L_\alpha(1 + |Y_u^\delta|^2) \, ds \bigg\}$$

$$\leq \sharp(\mathcal{A}) \bigg\{ |Y_0^\delta|^2 + T \sum_{\alpha \in \mathcal{A}} L_\alpha + \sum_{\alpha \in \mathcal{A}} L_\alpha \int_0^t \mathbf{E} \sup_{u \leq s} |Y_u^\delta|^2 ds \bigg\}.$$

By applying the Gronwall lemma 2.7 we obtain the required result. $\qquad \square$

Now we are ready to present the main result's proof.

Proof of Theorem 4.1: We write the solution Y of (2.2) and its approximation Y^δ in the forms:

$$Y_s = Y_0 + \sum_{\alpha \in \mathcal{A}_\gamma \backslash \{v\}} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right)$$

(4.23)

$$+ \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right),$$

$$Y_s^\delta = Y_0^\delta + \sum_{\alpha \in \mathcal{A}_\gamma \backslash \{v\}} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\tau_{n_s}}^\delta, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right).$$

Due to Proposition 2.4 and Proposition 4.5 the error of the approximation
$Z_t := \mathbf{E}\sup_{s \leq t} | Y_s - Y_s^\delta |^2$ is finite and satisfies the inequality:

$$Z_t \leq D_1(\gamma) \left(| Y_0 - Y_0^\delta |^2 + \sum_{\alpha \in \mathcal{A}_\gamma \backslash \{v\}} R_t^\alpha + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} U_t^\alpha \right), \qquad (4.24)$$

21

where

$$D_1(\gamma) = \sharp\{\mathcal{A}_\gamma \cup \mathcal{B}(\mathcal{A}_\gamma)\},$$

$$R_t^\alpha = \mathbf{E} \sup_{s \le t} \Bigg(\sum_{i=0}^{n_s-1} I_\alpha \Big[f_\alpha(Y_{\tau_i}, x_1, ..., x_{k(\alpha)}) - f_\alpha(Y_{\tau_i}^\delta, x_1, ..., x_{k(\alpha)}) \Big]_{\tau_i}^{\tau_{i+1}} \tag{4.25}$$

$$+ I_\alpha \Big[f_\alpha(Y_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)}) - f_\alpha(Y_{\tau_{n_s}}^\delta, x_1, ..., x_{k(\alpha)}) \Big]_{\tau_{n_s}}^{s} \Bigg)^2,$$

$$U_t^\alpha = \mathbf{E} \sup_{s \le t} \Bigg(\sum_{i=0}^{n_s-1} I_\alpha [f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha [f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^{s} \Bigg)^2.$$
$$\tag{4.26}$$

Let us denote $D(\alpha, T) := \sup_{t \in [0,T]} \max\{t, C(\alpha, t)\}$ where $C(\alpha, t)$ is a constant from Lemma (4.4). Since $\delta^{l(\alpha)+s(\alpha)-1} < \delta^{l(\alpha)+s(\alpha)-2} < 1$, by Lemma 4.4 and assumption **(A3)** we have the following inequality for any $\alpha \in \mathcal{A}_\gamma \backslash \{v\}$:

$$R_t^\alpha \le D(\alpha, T) \int_0^t \mathbf{E} \sup_{s \le u} \Big[f_\alpha(Y_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)}) - f_\alpha(Y_{\tau_{n_s}}^\delta, x_1, ..., x_{k(\alpha)}) \Big] du$$

$$\le D(\alpha, T) \int_0^t \mathbf{E} \sup_{s \le u} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} \Big[f_\alpha(Y_s, x_1, ..., x_{k(\alpha)})$$

$$- f_\alpha(Y_s^\delta, x_1, ..., x_{k(\alpha)}) \Big]^2 \nu(dx_{k(\alpha)}) ... \nu(dx_1) \, du$$

$$\le D(\alpha, T) K_\alpha \int_0^t \mathbf{E} \sup_{s \le u} | Y_s - Y_s^\delta |^2 \, du = D(\alpha, T) K_\alpha \int_0^t Z_u \, du.$$

For any $\alpha \in \mathcal{B}(\mathcal{A}_\gamma)$ inequality: $l(\alpha) + s(\alpha) - 1 > l(\alpha) + s(\alpha) - 2 \ge 2\gamma$ is satisfied. Due to this fact, assumption **(A4)**, Proposition 2.4 and Lemma 2.8 we have the following inequalities:

$$U_t^\alpha \le D(\alpha, T) \delta^{2\gamma} \int_0^t G_{0,u}^\alpha \Big[f_\alpha(Y_{\bullet-}, x_1, ..., x_{k(\alpha)}) \Big] du$$

$$\le D(\alpha, T) \delta^{2\gamma} \int_0^t \mathbf{E} \sup_{s \le u} \int_{B_1^\alpha} \int_{B_2^\alpha} ... \int_{B_{k(\alpha)}^\alpha} \Big[f_\alpha(Y_{s-}, x_1, ..., x_{k(\alpha)}) \Big]^2 \nu(dx_{k(\alpha)}) ... \nu(dx_1) \, du$$

$$\le D(\alpha, T) \delta^{2\gamma} L_\alpha \int_0^t \mathbf{E} \sup_{s \le u} | 1 + Y_{s_-}^2 | \, du \le D(\alpha, t) \delta^{2\gamma} L_\alpha \int_0^t (1 + C_2(1 + Y_0^2)) \, du$$

$$\le \delta^{2\gamma} D(\alpha, T) L_\alpha T (1 + C_2(1 + Y_0^2)).$$

Finally, denoting shorter relevant constants we have:

$$R_t^\alpha \le D_2(\alpha, T) \int_0^t Z_u \, du, \qquad U_t^\alpha \le D_3(\alpha, T, Y_0) \delta^{2\gamma}.$$

Coming back to (4.24) we obtain

$$Z_t \le D_1(\gamma) | Y_0 - Y_0^\delta |^2 + \tilde{D}_2(\gamma, T) \int_0^t Z_u du + \tilde{D}_3(\gamma, T, Y_0) \delta^{2\gamma}, \tag{4.27}$$

where $\tilde{D}_2(\gamma, T) := D_1(\gamma) \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} D_2(\alpha, T)$ and
$\tilde{D}_3(\gamma, T, Y_0) := D_1(\gamma) \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} D_3(\alpha, T, Y_0)$. Applying the Gronwall lemma 2.7 to (4.27) we obtain:

$$Z_t \le E_1(\gamma, T) \mid Y_0 - Y_0^\delta \mid^2 + E_2(\gamma, T, Y_0) \delta^{2\gamma},$$

where:

$$E_1(\gamma, T) = D_1(\gamma) e^{\tilde{D}_2(\gamma, T)T}, \quad E_2(\gamma, T, Y_0) = \tilde{D}_3(\gamma, Y_0, T) e^{\tilde{D}_2(\gamma, T)T}.$$

$\square$

5 Infinite Lévy measure

The strong approximation described by Theorem 4.1 can not always be easily constructed in practice even for low order of convergence. In case when $\nu(B) = \infty$ the integrals with respect to the compensated Poisson measure are difficult to obtain even for simple integrands. In this section we formulate alternative theorem which describes approximation with the use of integrals which can be practically derived.

For a fixed $\varepsilon \in (0, 1)$ we split the unit ball B into the ball B_ε with radius ε and the disc $D_\varepsilon = B \setminus B_\varepsilon$. Our idea is to modify the approximation given by Theorem 4.1 by exchanging all the integrals on unit balls with respect to the compensated Poisson measure for integrals on discs D_ε.

For the use of this section we extend the inductive definition of multiple stochastic integral introduced in Section 2. To this end for any multiindex α let us define a set of subscripts $\Pi(\alpha)$ consisting of vectors $\pi(\alpha) = (\pi_1(\alpha), \pi_2(\alpha), ..., \pi_{\tilde{n}(\alpha)}(\alpha))$ of length $\tilde{n}(\alpha)$ with coordinates equal to 0 or 1, i.e.

$$\pi(\alpha) \in \Pi(\alpha) \iff \begin{cases} \pi_i(\alpha) = 0 \text{ or } \pi_i(\alpha) = 1 \text{ for } i = 1, 2, ..., \tilde{n}(\alpha) & \text{if } \tilde{n}(\alpha) > 0 \\ v & \text{if } \tilde{n}(\alpha) = 0. \end{cases}$$

The empty subscript v, i.e. the subscript of length zero is introduced for technical reasons. The subscripts for the multiindices α and $\alpha-$ are related in the following way:

$$\pi(\alpha-) = \begin{cases} \pi(\alpha) & \text{if } \alpha_{l(\alpha)} = 0, 1, 3 \\ (\pi_1(\alpha), \pi_2(\alpha), ..., \pi_{\tilde{n}(\alpha)-1}(\alpha)) & \text{if } \alpha_{l(\alpha)} = 2. \end{cases}$$

For a process $g = g(s, x_1, ..., x_l)$, a multiindex α s.t. $k(\alpha) \le l$ and a subscript $\pi(\alpha) \in \Pi(\alpha)$ we define the multiple integral by the induction procedure.
If $\tilde{n}(\alpha) = 0$ then

$$I^\varepsilon_{\alpha_v}[g]^\tau_\rho(x_1, ..., x_l) = I_\alpha[g]^\tau_\rho(x_1, ..., x_l).$$

Assume that $I^\varepsilon_{\alpha-\pi(\alpha-)}[g]^\tau_\rho(x_1, x_2, ..., x_k)$ depends on k parameters, where $k \le l$. Then:

1) if $\alpha_{l(\alpha)} = 0$ then

$$I^\varepsilon_{\alpha_{\pi(\alpha)}}[g]^\tau_\rho(x_1, ..., x_k) = \int_\rho^\tau I_{\alpha-\pi(\alpha-)}[g]^{s-}_\rho(x_1, ..., x_k) ds,$$

2) if $\alpha_{l(\alpha)} = 1$ then

$$I^\varepsilon_{\alpha_{\pi(\alpha)}}[g]^\tau_\rho(x_1, ..., x_k) = \int_\rho^\tau I_{\alpha-\pi(\alpha-)}[g]^{s-}_\rho(x_1, ..., x_k) dW_s,$$

3) if $\alpha_{l(\alpha)} = 2$ and $\pi_{\tilde{n}(\alpha)}(\alpha) = 0$ and $k \geq 1$ then

$$I^{\varepsilon}_{\alpha_{\pi(\alpha)}}[g]^{\tau}_{\rho}(x_1, ..., x_{k-1}) = \int_{\rho}^{\tau} \int_{B_{\varepsilon}} I_{\alpha - \pi(\alpha-)}[g]^{s-}_{\rho}(x_1, ..., x_k) \tilde{N}(ds, dx_k),$$

4) if $\alpha_{l(\alpha)} = 2$ and $\pi_{\tilde{n}(\alpha)}(\alpha) = 1$ and $k \geq 1$ then

$$I^{\varepsilon}_{\alpha_{\pi(\alpha)}}[g]^{\tau}_{\rho}(x_1, ..., x_{k-1}) = \int_{\rho}^{\tau} \int_{D_{\varepsilon}} I_{\alpha - \pi(\alpha-)}[g]^{s-}_{\rho}(x_1, ..., x_k) \tilde{N}(ds, dx_k),$$

5) if $\alpha_{l(\alpha)} = 3$ and $k \geq 1$ then

$$I^{\varepsilon}_{\alpha_{\pi(\alpha)}}[g]^{\tau}_{\rho}(x_1, ..., x_{k-1}) = \int_{\rho}^{\tau} \int_{B'} I_{\alpha - \pi(\alpha-)}[g]^{s-}_{\rho}(x_1, ..., x_{l-k}) N(ds, dx_k).$$

Example Assume that g is of the form $g(s, x_1, x_2,)$. Then:

$$I_{(212)_{(1,0)}}[g]^t_0 = \int_0^t \int_{B_{\varepsilon}} \int_0^{s_1-} \int_0^{s_2-} \int_{D_{\varepsilon}} g(s_3-, x_1, x_2) \, \tilde{N}(ds_3, dx_2) \, dW_{s_2} \, \tilde{N}(ds_1, dx_1).$$

For any hierarchical set $\mathcal{A}$ let us denote by $\mathcal{A}^2$ a subset of multiindices containing at least one element equal to 2, i.e. $\alpha \in \mathcal{A}^2$ iff $\alpha \in \mathcal{A}$ and $\tilde{n}(\alpha) > 0$.

Remark 5.1 *Let $\varepsilon > 0$. For any $\alpha \in \mathcal{A}^2$ and a process $g = g(s, x_1, x_2, ..., x_{k(\alpha)})$ the following equality holds:*

$$I_{\alpha}[g]^{\tau}_{\rho} = \sum_{\pi \in \Pi(\alpha)} I^{\varepsilon}_{\alpha_{\pi(\alpha)}}[g]^{\tau}_{\rho}.$$

Remark 5.2 *If we replace in the formulas (4.20),(4.21) the unit balls in integrals with respect to the compensated Poisson measure by ε-balls or ε-discs, then Lemma 4.4 remains true. As a consequence, for a process:*

$$Y^{\delta, \varepsilon}_s = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{A}^2} I_{\alpha}[f_{\alpha}(Y^{\delta, \varepsilon}_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)})]^s_{\tau_{n_s}} + \sum_{\alpha \in \mathcal{A}^2} I^{\varepsilon}_{\alpha_{(1,1,...,1)}}[f_{\alpha}(Y^{\delta, \varepsilon}_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)})]^s_{\tau_{n_s}}$$

we obtain analogous estimation as in Proposition 4.5, i.e.

$$\mathbf{E} \sup_{s \leq t} |Y^{\delta, \varepsilon}_s|^2 \leq C_4(1 + |Y^{\delta, \varepsilon}_0|^2) \quad \forall \, t \in [0, T],$$

where $C_4 > 0$, assuming that (4.22) is satisfied.

Theorem 5.3 *Assume that coefficients in equation (2.2) satisfy conditions **(A1)**,**(A2)**. Let $\mathcal{A}_{\gamma}$ be a hierarchical set given by (4.9) and assume that **(A3)**,**(A4)** hold. Assume that for any $\alpha \in \mathcal{A}^2_{\gamma}$ there exists a constant L^{ε}_{α} such that for every i s.t. $\alpha_i = 2$ holds:*

$$\int_{B^{\alpha}_1} \int_{B^{\alpha}_2} ... \int_{B_{\varepsilon}} ... \int_{B^{\alpha}_{k(\alpha)}} |f_{\alpha}(y, x_1, x_2, ..., x_{k(\alpha)})|^2 \nu(dx_{k(\alpha)})...\nu(dx_1) \leq L^{\varepsilon}_{\alpha}(1 + y^2), \quad (5.28)$$

where B_{ε} is on the position $k(\alpha) - i + 1$ and $L^{\varepsilon}_{\alpha} \xrightarrow[\varepsilon \longrightarrow 0]{} 0$.
Then the approximation defined by the formula:

$$Y^{\delta, \varepsilon}_s = \sum_{\alpha \in \mathcal{A}_{\gamma} \setminus \mathcal{A}^2_{\gamma}} I_{\alpha}[f_{\alpha}(Y^{\delta, \varepsilon}_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)})]^s_{\tau_{n_s}} + \sum_{\alpha \in \mathcal{A}^2_{\gamma}} I^{\varepsilon}_{\alpha_{(1,1,...,1)}}[f_{\alpha}(Y^{\delta, \varepsilon}_{\tau_{n_s}}, x_1, ..., x_{k(\alpha)})]^s_{\tau_{n_s}}$$

satisfies:

$$\mathbf{E} \sup_{s \in [0,T]} | Y_s - Y_s^{\delta,\varepsilon} |^2 \leq N_1(\gamma, T) \mid Y_0 - Y_0^{\delta,\varepsilon} \mid^2 + N_2(\gamma, T, Y_0) \delta^{2\gamma} + N_3(\gamma, T, Y_0^{\delta,\varepsilon}, \varepsilon),$$

where $N_3(\gamma, T, Y_0^{\delta,\varepsilon}, \varepsilon) \xrightarrow[\varepsilon \longrightarrow 0]{} 0.$

Proof: We write the approximation in the form:

$$Y_s^{\delta,\varepsilon} = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s - \sum_{\alpha \in \mathcal{A}_\gamma^2} I_\alpha[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s$$

$$+ \sum_{\alpha \in \mathcal{A}_\gamma^2} I_{\alpha_{(1,1,...,1)}}^\varepsilon [f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{\tilde{n}(\alpha)+n(\alpha)})]_{\tau_{n_s}}^s$$

$$= Y_0^{\delta,\varepsilon} + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right)$$

$$- \sum_{\alpha \in \mathcal{A}_\gamma^2} \left(\sum_{i=0}^{n_s-1} I_\alpha[f_\alpha(Y_{\tau_i}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_\alpha[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right)$$

$$+ \sum_{\alpha \in \mathcal{A}_\gamma^2} \left(\sum_{i=0}^{n_s-1} I_{\alpha_{(1,1,...,1)}}^\varepsilon [f_\alpha(Y_{\tau_i}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} + I_{\alpha_{(1,1,...,1)}}^\varepsilon [f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right).$$

By Remark 5.2 and Proposition 2.4 the error $Z_t := \mathbf{E} \sup_{s \leq t} |Y_s - Y_s^{\delta,\varepsilon}|^2$ is finite. Taking into account (4.23) we have:

$$Z_t \leq M_1(\gamma) \left(| Y_0 - Y_0^{\delta,\varepsilon} |^2 + \sum_{\alpha \in \mathcal{A}_\gamma \setminus \{v\}} R_t^\alpha + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_\gamma)} U_t^\alpha + \sum_{\alpha \in \mathcal{A}_\gamma^2} S_t^\alpha \right), \qquad (5.29)$$

where

$$M_1(\gamma) = \sharp\{\mathcal{A}_\gamma\} + \sharp\{\mathcal{B}(\mathcal{A}_\gamma)\} + \sharp\{\mathcal{A}_\gamma^2\},$$

R_t^α is defined by (4.25) with Y^δ replaced by $Y^{\delta,\varepsilon}$ and U_t^α by (4.26) and

$$S_t^\alpha = \mathbf{E} \sup_{s \leq t} \left(\sum_{i=0}^{n_s-1} \left(I_\alpha[f_\alpha(Y_{\tau_i}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} - I_{\alpha_{(1,1,...,1)}}^\varepsilon [f_\alpha(Y_{\tau_i}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}} \right) \right.$$

$$\left. + I_\alpha[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s - I_{\alpha_{(1,1,...,1)}}^\varepsilon [f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon}, x_1, ..., x_{k(\alpha)})]_{\tau_{n_s}}^s \right)^2.$$

Due to Remark 5.1 we have:

$$S_t^\alpha = \mathbf{E}\sup_{s\le t}\left(\sum_{i=0}^{n_s-1}\left(\sum_{\pi(\alpha)\in\Pi(\alpha),\pi(\alpha)\ne(1,1,...,1)}I_{\alpha_{\pi(\alpha)}}^\varepsilon[f_\alpha(Y_{\tau_i}^{\delta,\varepsilon},x_1,...,x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}}\right)\right.$$

$$\left.+\left(\sum_{\pi(\alpha)\in\Pi(\alpha),\pi(\alpha)\ne(1,1,...,1)}I_{\alpha_{\pi(\alpha)}}^\varepsilon[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon},x_1,...,x_{k(\alpha)})]_{\tau_{n_s}}^{s}\right)\right)^2$$

$$\le\left(\sharp\{\Pi(\alpha)\}-1\right)\sum_{\pi\in\Pi(\alpha),\pi\ne(1,1,...,1)}\mathbf{E}\sup_{s\le t}\left(\sum_{i=0}^{n_s-1}I_{\alpha_{\pi(\alpha)}}^\varepsilon[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon},x_1,...,x_{k(\alpha)})]_{\tau_i}^{\tau_{i+1}}\right.$$

$$\left.+I_{\alpha_{\pi(\alpha)}}^\varepsilon[f_\alpha(Y_{\tau_{n_s}}^{\delta,\varepsilon},x_1,...,x_{k(\alpha)})]_{\tau_{n_s}}^{s}\right)^2.$$

In the sum above each integral contains at least one integral on ε-ball. Using assumption (5.28) and Remark 5.2 we obtain:

$$S_t^\alpha \le \left(\sharp\{\Pi(\alpha)\}-1\right)\sum_{\pi(\alpha)\in\Pi(\alpha),\pi\ne(1,1,...,1)}C(\alpha,t)L_\alpha^\varepsilon\int_0^t\mathbf{E}\sup_{s\le u}(1+|Y_s^{\delta,\varepsilon}|^2)du$$

$$\le\left((\sharp\{\Pi(\alpha)\}-1)\right)^2 D(\alpha,T)L_\alpha^\varepsilon T(1+C_4(1+|Y_0^{\delta,\varepsilon}|^2))=:L_\alpha^\varepsilon\cdot D_4(\alpha,T,Y_0^{\delta,\varepsilon}),$$

Coming back to (5.29) and using notation of constants from the proof of Theorem 4.1 we obtain:

$$Z_t\le M_1(\gamma)\mid Y_0-Y_0^{\delta,\varepsilon}\mid^2 +M_2(\gamma,T)\int_0^t Z_u du + M_3(\gamma,T,Y_0)\delta^{2\gamma}+M_4(\gamma,T,Y_0^{\delta,\varepsilon},\varepsilon),$$

where $M_2(\gamma,T)=M_1(\gamma)\sum_{\alpha\in\mathcal{A}_\gamma\setminus v}D_2(\alpha,T)$, $M_3(\gamma,T,Y_0)=M_1(\gamma)\sum_{\alpha\in\mathcal{B}(\mathcal{A}_\gamma)}D_3(\alpha,T,Y_0)$ and $M_4(\gamma,T,Y_0^{\delta,\varepsilon},\varepsilon)=M_1(\gamma)\sum_{\alpha\in\mathcal{A}_\gamma^2}L_\alpha^\varepsilon\cdot D_4(\alpha,t,Y_0^{\delta,\varepsilon})$. Finally, applying the Gronwall lemma 2.7 we obtain:

$$Z_t\le N_1(\gamma,T)\mid Y_0-Y_0^{\delta,\varepsilon}\mid^2 +N_2(\gamma,T,Y_0)\delta^{2\gamma}+N_3(\gamma,T,Y_0^{\delta,\varepsilon},\varepsilon),$$

where $N_1(\gamma,T)=M_1(\gamma)e^{M_2(\gamma,T)T}$; $N_2(\gamma,T,Y_0)=M_3(\gamma,T,Y_0)e^{M_2(\gamma,T)T}$; $N_3(\gamma,T,Y_0^{\delta,\varepsilon},\varepsilon)=M_4(\gamma,T,Y_0^{\delta,\varepsilon},\varepsilon)e^{M_2(\gamma,T)T}=e^{M_2(\gamma,T)T}M_1(\gamma)\sum_{\alpha\in\mathcal{A}_\gamma^2}L_\alpha^\varepsilon\cdot D_4(\alpha,t,Y_0^{\delta,\varepsilon})$.

$\square$

6 Examples

We present the Euler ($\gamma=\frac{1}{2}$) and Milstein ($\gamma=1$) schemes in for linear coefficients, i.e.

$$b(y)=by,\ \sigma(y)=\sigma y,\ F(y,x)=Fyp(x),\ G(y,x)=Gyq(x)$$

where σ,b,F,G are constants and functions $p(\cdot),q(\cdot)$ satisfy integrability conditions: $\int_B p^2(x)\nu(dx)<\infty$, $\int_{B'}q^2(x)\nu(dx)<\infty$. Then assumptions **(A1)**,**(A2)** are satisfied.

For finding integrals with respect to the Poisson random measure we use the representation of

random measures given by Theorem 7.2 in [4] applied for a set E s.t. $\nu(E) < \infty$. It states that the random measure $N(\cdot, \cdot)$ can be represented as

$$N(t, E) = \sum_{n>0} \mathbf{1}_{[0,t] \times E}(\eta_n, \xi_n),$$

where $\eta_n = r_1 + r_2 + ... + r_n$ and $\{\xi_n\}, \{r_n\}$ are mutually independent random variables with distributions:

$$P(\xi_n \in A) = \frac{\nu(A \cap E)}{\nu(E)}, \ \forall A \in \mathcal{B}(\mathbb{R}), \qquad P(r_n > s) = e^{-\nu(E)s}, \ s \geq 0.$$

In the following constructions we assume that $\nu(B) < \infty$ and as a consequence that $N((\tau_i, \tau_{i+1}], B \cup B') =: K(i) < \infty$. Then all the moments of jumps generated by the Poisson random measure N in the interval $(\tau_i, \tau_{i+1}]$ form a sequence: $\eta_1 < \eta_2 < ... < \eta_{K(i)}$. We omit the dependence of this sequence on i to simplify notation. For the sake of clarity we use the following notation:

$$\bar{\eta}_n = \min\{\eta_k : \eta_k > \eta_n \text{ and } \xi_k \in B'\} \wedge \tau_{i+1},$$

$$\underline{\eta}_n = \min\{\eta_k : \eta_k > \eta_n \text{ and } \xi_k \in B\} \wedge \tau_{i+1}.$$

Condition $\nu(B) < \infty$ guaranties that all the formulas below can be practically derived. If it is not satisfied, then we apply Theorem 5.3 by replacing all unit balls in the approximation by ε- discs. In this case $K(i)$ and $\underline{\eta}_n$ are defined with the use of D_ε instead of B. Since $N((\tau_i, \tau_{i+1}], D_\varepsilon \cup B') < \infty$ the modified approximation can be calculated. We also find the dependence of the approximation error on ε.

Notational remark: if the range of indices in the sums below is empty, then the sum is assumed to be zero.

The Euler scheme

The hierarchical set and the remainder sets are of the form $\mathcal{A}_{\frac{1}{2}} = \{v, 0, 1, 2, 3\}$, $\mathcal{B}(\mathcal{A}_{\frac{1}{2}}) = \{00, 10, 20, 30, 01, 11, 21, 31, 02, 12, 22, 32, 03, 13, 23, 33\}$. It can be easily checked that conditions **(A3)**, **(A4)** are also satisfied. The approximation has the following form:

$$Y_{\tau_{i+1}}^\delta = Y_{\tau_i}^\delta + I_0[f_0(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_1[f_1(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_2[f_2(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}} + I_3[f_3(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}$$

where:

$$I_0[f_0(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} b Y_{\tau_i}^\delta ds = b Y_{\tau_i}^\delta \triangle_i,$$

$$I_1[f_1(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} \sigma Y_{\tau_i}^\delta dW_s = \sigma Y_{\tau_i}^\delta \triangle_i^W,$$

$$I_2[f_2(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} \int_B F \cdot Y_{\tau_i}^\delta p(x) \tilde{N}(ds, dx),$$

27

$$= FY_{\tau_i}^\delta \left(\sum_{n=1}^{K(i)} \mathbf{1}_B(\xi_n) p(\xi_n) - \triangle_i \int_B p(x)\nu(dx) \right),$$

$$I_3[f_3(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}} = \int_{\tau_i}^{\tau_{i+1}} \int_{B'} G \cdot Y_{\tau_i}^\delta q(x) N(ds, dx) = GY_{\tau_i}^\delta \left(\sum_{n=1}^{K(i)} \mathbf{1}_{B'}(\xi_n) q(\xi_n) \right),$$

$$\triangle_i = \tau_{i+1} - \tau_i, \qquad \triangle_i^W = W_{\tau_{i+1}} - W_{\tau_i}.$$

If $\nu(B) = \infty$ we apply Theorem 5.3. Notice that condition (5.28) is satisfied since

$$\int_{B_\varepsilon} |\, f_2(y, x)\,|^2 \,\nu(dx) = F^2 y^2 \int_{B_\varepsilon} p^2(x)\nu(dx) \xrightarrow[\varepsilon \to 0]{} 0,$$

so $L_\alpha^\varepsilon = \int_{B_\varepsilon} p^2(x)\nu(dx)$. It follows from the proof of Theorem 5.3 that:

$$N_3\left(\frac{1}{2}, T, Y_0^{\delta,\varepsilon}, \varepsilon\right) = K(T, Y_0^{\delta,\varepsilon}) \int_{B_\varepsilon} p^2(x)\nu(dx),$$

where $K(T, Y_0^{\delta,\varepsilon}) > 0$.

The Milstein scheme

The hierarchical and remainder sets are of the form:

$$\mathcal{A}_1 = \{v, 0, 1, 2, 3, 11, 21, 31, 12, 22, 32, 13, 23, 33\},$$

$$\mathcal{B}(\mathcal{A}_1) = \{00, 10, 20, 30, 01, 02, 03, 011, 111, 211, 311, 021, 121, 221, 321, 031, 131, 231, 331,$$
$$012, 112, 212, 312, 022, 122, 222, 322, 032, 132, 232, 332, 013, 113, 213, 313,$$
$$023, 123, 223, 323, 033, 133, 233, 333, \}.$$

Assumptions $(\mathbf{A3})$, $(\mathbf{A4})$ are satisfied. The approximation is of the following form:

$$Y_{\tau_{i+1}}^\delta = Y_{\tau_i}^\delta + I_0[f_0(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_1[f_1(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_2[f_2(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}} + I_3[f_3(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}$$

$$+ I_{11}[f_{11}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{21}[f_{21}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{31}[f_{31}(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}$$

$$+ I_{12}[f_{12}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{22}[f_{22}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{32}[f_{32}(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}$$

$$+ I_{13}[f_{13}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{23}[f_{23}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} + I_{33}[f_{33}(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}},$$

where $I_0[f_0(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}}, I_1[f_1(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}}, I_2[f_2(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}, I_3[f_3(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}}$ are like in the Euler scheme and

$$I_{11}[f_{11}(Y_{\tau_i}^\delta)]_{\tau_i}^{\tau_{i+1}} = \frac{1}{2}\sigma^2 Y_{\tau_i}^\delta \left((\triangle_i^W)^2 - \triangle_i \right),$$

$$I_{13}[f_{13}(Y_{\tau_i}^\delta, x)]_{\tau_i}^{\tau_{i+1}} = G\sigma Y_{\tau_i} \int_{\tau_i}^{\tau_{i+1}} \int_{B'} q(x)(W_s - W_{\tau_i}) N(ds, dx)$$

$$= G\sigma Y_{\tau_i} \sum_{n=1}^{K(i)} q(\xi_n)(W_{\eta_n} - W_{\tau_i}) \mathbf{1}_{B'}(\xi_n),$$

$$I_{12}[f_{12}(Y_{\tau_i}^{\delta}, x)]_{\tau_i}^{\tau_{i+1}} = F\sigma Y_{\tau_i}\left(\int_{\tau_i}^{\tau_{i+1}}\int_B p(x)(W_s - W_{\tau_i})N(ds, dx)\right.$$

$$\left. -\int_{\tau_i}^{\tau_{i+1}}\int_B p(x)(W_s - W_{\tau_i})\nu(dx)ds\right)$$

$$= F\sigma Y_{\tau_i}\left(\sum_{n=1}^{K(i)} p(\xi_n)(W_{\eta_n} - W_{\tau_i})\mathbf{1}_B(\xi_n) - \int_B p(x)\nu(dx)\cdot\triangle_i^Z\right),$$

where $\triangle_i^Z = \int_{\tau_i}^{\tau_{i+1}}(W_s - W_{\tau_i})ds$ is a random variable with distribution $N(0, \frac{1}{3}\triangle_i^3)$, correlated with $\triangle_i^W$, i.e. $\mathbf{E}(\triangle_i^W\triangle_i^Z) = \frac{1}{2}\triangle_i^2$. The pair $\triangle_i^W, \triangle_i^Z$ can be generated by transformation of two independent random variables U_1, U_2 with distributions $N(0, 1)$ in the following way: $\triangle_i^W = U_1\sqrt{\triangle_i}$, $\triangle_i^Z = \frac{1}{2}\triangle_i^{\frac{3}{2}}(U_1 + \frac{1}{\sqrt{3}}U_2)$, for more details see [3].

$$I_{31}[f_{31}(Y_{\tau_i}^{\delta}, x)]_{\tau_i}^{\tau_{i+1}} = G\sigma Y_{\tau_i}\int_{\tau_i}^{\tau_{i+1}}\int_{\tau_i}^{s-}\int_{B'} q(x)N(du, dx)dW_s$$

$$= G\sigma Y_{\tau_i}\sum_{n=1}^{K(i)}\left\{\sum_{0<k\leq n} q(\xi_k)\mathbf{1}_{B'}(\xi_k)\right\}(W_{\bar{\eta}_n} - W_{\eta_n})$$

$$I_{21}[f_{21}(Y_{\tau_i}^{\delta}, x)]_{\tau_i}^{\tau_{i+1}} = F\sigma Y_{\tau_i}\left(\int_{\tau_i}^{\tau_{i+1}}\int_{\tau_i}^{s-}\int_B p(x)N(du, dx)dW_s\right.$$

$$\left. -\int_{\tau_i}^{\tau_{i+1}}\int_{\tau_i}^{s-}\int_B p(x)\nu(dx)du\ dW_s\right)$$

$$= F\sigma Y_{\tau_i}\left(\sum_{n=1}^{K(i)}\left\{\sum_{0<k\leq n} p(\xi_k)\mathbf{1}_B(\xi_k)\right\}(W_{\underline{\eta}_n} - W_{\eta_n})\right.$$

$$\left. -\int_B p(x)\nu(dx)\cdot(\triangle_i^W\triangle_i - \triangle_i^Z)\right)$$

$$I_{33}[f_{33}(Y_{\tau_i}, x_1, x_2)]_{\tau_i}^{\tau_{i+1}} = G^2 Y_{\tau_i}\int_{\tau_i}^{\tau_{i+1}}\int_{B'}\int_{\tau_i}^{s-}\int_{B'} q(x_1)q(x_2)N(du, dx_2)\,N(ds, dx_1)$$

$$= G^2 Y_{\tau_i}\sum_{n=1}^{K(i)}\left\{\sum_{0<k<n} q(\xi_k)\mathbf{1}_{B'}(\xi_k)\right\}q(\xi_n)\mathbf{1}_{B'}(\xi_n)$$

$$I_{32}[f_{32}(Y_{\tau_i}, x_1, x_2)]_{\tau_i}^{\tau_{i+1}} = FGY_{\tau_i}\Big(\int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B'} p(x_1)q(x_2)N(du, dx_2)N(ds, dx_1)$$

$$- \int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B'} p(x_1)q(x_2)N(du, dx_2)\nu(dx_1)ds \Big)$$

$$= FGY_{\tau_i}\Big(\sum_{n=1}^{K(i)} \Big\{ \sum_{0<k<n} q(\xi_k)\mathbf{1}_{B'}(\xi_k) \Big\} p(\xi_n)\mathbf{1}_B(\xi_n)$$

$$- \int\limits_{B} p(x_1)\nu(dx_1) \cdot \sum_{n=1}^{K(i)} \Big\{ \sum_{0<k\leq n} q(\xi_k)\mathbf{1}_B(\xi_k) \Big\}(\underline{\eta}_n - \eta_n) \Big)$$

$$I_{23}[f_{23}(Y_{\tau_i}, x_1, x_2)]_{\tau_i}^{\tau_{i+1}} = FGY_{\tau_i}\Big(\int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B'} \int\limits_{\tau_i}^{s-} \int\limits_{B} q(x_1)p(x_2)N(du, dx_2)N(ds, dx_1)$$

$$- \int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B'} \int\limits_{\tau_i}^{s-} \int\limits_{B} q(x_1)p(x_2)du\,\nu(dx_2)\,N(ds, dx_1) \Big)$$

$$= FGY_{\tau_i}\Big(\sum_{n=1}^{K(i)} \Big\{ \sum_{0<k<n} q(\xi_k)\mathbf{1}_B(\xi_k) \Big\} p(\xi_n)\mathbf{1}_{B'}(\xi_n)$$

$$- \int_B p(x_2)\nu(dx_2) \cdot \sum_{n=1}^{K(i)} (\eta_n - \tau_i)q(\xi_n)\mathbf{1}_{B'}(\xi_n) \Big)$$

$$I_{22}[f_{22}(Y_{\tau_i}, x_1, x_2)]_{\tau_i}^{\tau_{i+1}} = F^2 Y_{\tau_i}\Big(\int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B} p(x_1)p(x_2)N(du, dx_2)N(ds, dx_1)$$

$$- \int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B} p(x_1)p(x_2)\nu(dx_2)du\,N(ds, dx_1)$$

$$- \int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B} p(x_1)p(x_2)\,N(du, dx_2)\,\nu(dx_1)ds$$

$$+ \int\limits_{\tau_i}^{\tau_{i+1}} \int\limits_{B} \int\limits_{\tau_i}^{s-} \int\limits_{B} p(x_1)p(x_2)\,\nu(dx_2)du\,\nu(dx_1)ds \Big)$$

$$= F^2 Y_{\tau_i} \left(\sum_{n=1}^{K(i)} \left\{ \sum_{0 < k < n} p(\xi_k) \mathbf{1}_B(\xi_k) \right\} p(\xi_n) \mathbf{1}_B(\xi_n) \right.$$

$$- \int_B p(x_2) \nu(dx_2) \cdot \sum_{n=1}^{K(i)} (\eta_n - \tau_i) p(\xi_n) \mathbf{1}_B(\xi_n)$$

$$- \int_B p(x_1) \nu(dx_1) \sum_{n=1}^{K(i)} \left\{ \sum_{0 < k \leq n} p(\xi_k) \mathbf{1}_B(\xi_k) \right\} (\underline{\eta}_n - \eta_n)$$

$$\left. + \frac{1}{2} \int_B p(x_1) \nu(dx_1) \int_B p(x_2) \nu(dx_2) \triangle_i^2 \right)$$

It is easy to check that all of the integrals below:

$$\int_{B_\varepsilon} |f_2(y,x)|^2 \nu(dx), \int_{B_\varepsilon} |f_{12}(y,x)|^2 \nu(dx), \int_{B_\varepsilon} |f_{21}(y,x)|^2 \nu(dx)$$

$$\int_{B_\varepsilon} \int_B |f_{22}(y,x_1,x_2)|^2 \nu(dx_2)\nu(dx_1), \int_B \int_{B_\varepsilon} |f_{22}(y,x_1,x_2)|^2 \nu(dx_2)\nu(dx_1)$$

$$\int_B \int_{B_\varepsilon} |f_{23}(y,x_1,x_2)|^2 \nu(dx_2)\nu(dx_1), \int_{B_\varepsilon} \int_B |f_{32}(y,x_1,x_2)|^2 \nu(dx_2)\nu(dx_1),$$

are bounded above by $Ky^2 \int_{B_\varepsilon} p^2(x)\nu(dx)$ where K is some constant, so we can assume that $L_\alpha^\varepsilon = L^\varepsilon = \int_{B_\varepsilon} p^2(x)\nu(dx)$ for all $\alpha \in \mathcal{A}_1^2$. The part of the error of the modified approximation connected with the procedure of ε-balls cutting satisfies:

$$N_3(1,T,Y_0^{\delta,\varepsilon},\varepsilon) \leq K(T,Y_0^{\delta,\varepsilon}) \int_{B_\varepsilon} p^2(x)\nu(dx),$$

where $K(T,Y_0^{\delta,\varepsilon}) > 0$.

References

[1] Applebaum D. (2004) *Lévy Processes and Stochastic Calculus*, Cambridge University Press

[2] Gardoń A. (2004) *The Order of Approximations for Solutions of Ito-type Stochastic Differential Equations with Jumps*, Stochastic Analysis and Applications, Vol.22 No.3

[3] Kloeden P., Platen E. (1995) *Numerical Solutions of Stochastic Differential Equations*, Springer-Verlag

[4] Peszat S., Zabczyk J. (forthcoming) *Stochastic Partial Differential Equations Driven by Lévy Processes*

[5] Rubenthaler S. (2003) *Numerical simulation of the solution of a stochastic differential equation driven by a Lévy process* Stochastic Processes and their Applications, Vol. 103,Is. 2, p.311-349.

[6] Cohen S., Rosiński J. *Gaussian approximation of multivariate Lévy processes with application to simulation of tempered stable processes*, Preprint

[7] Protter P., Talay D. (1997) *The Euler scheme for Lévy driven stochastic differential equation*, The Annals of Probability, Vol. 25, No. 1, 393-423