INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

IM PAN Preprint 690 (2008)

Michał Baran
Jacek Jakubowski
Jerzy Zabczyk

On incompleteness of bond markets with infinite number of random factors

Published as manuscript

Received 15 September 2008

On incompleteness of bond markets with infinite number of random factors *

Michał Baran

Mathematics Department of Cardinal Stefan Wyszyński University in Warsaw, Poland

Jacek Jakubowski

Institute of Mathematics, University of Warsaw, Warsaw, Poland

and

Jerzy Zabczyk

Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

Abstract

The completeness of a bond market model with infinite number of sources of randomness on a finite time interval in the Heath-Jarrow-Morton framework is studied. It is proved that the market in the case of trading strategies is not complete. An explicit construction of a bounded contingent claim, which can not be replicated, is provided. Moreover, a new concept of generalized strategies is introduced and sufficient conditions for the market completeness with such strategies are given. An example of a complete model is provided.

Key words: bond market, completeness, infinite dimensional model

AMS Subject Classification: 91B28, 91B70, 91B24.

JEL Classification Numbers: G10,G11

1 Introduction

We investigate the completeness of a continuous time zero-coupon bond market model in the Heath-Jarrow-Morton framework. Let $P(t,T)$, $0 \le t, T \le \bar{S} < \infty$ be the price at time t of a bond paying 1 at maturity T and $\hat{P}(t,T)$ be the discounted price. $\bar{S}$ denotes a finite time horizon of the model. The paper is devoted to the problem of completeness of the market. We

*Research supported by Polish KBN Grant P03A 034 29 „Stochastic evolution equations driven by Lévy noise"

say that a market is complete if an arbitrary bounded random variable can be replicated by an admissible strategy. Thus we are concerned with a problem of replicating for, each $S \leq \bar{S}$, contingent claims depending on the information available up to time S. If $S = \bar{S}$ we have a situation which occurs in practice, when a trader hedges a payoff which depends on the bond's prices up to time $\bar{S}$, uses bonds with maturities not exceeding $\bar{S}$. Strategies of this type will be called *natural*.

The bond market we study is infinite in the sense that the price process is a function-valued process. The concept of the portfolio and the strategy may be formalized in many possible ways. We base our approach on the stochastic integration theory with respect to the Hilbert space-valued martingales, as presented in Métivier [15] and Da Prato, Zabczyk [5].

The problem of completeness has been investigated by several authors. Björk et. al [2], [3] regarded the price process in the space of continuous functions and obtained conditions for approximate completeness. Cylindrical integration theory, due to Mikulevicius, Rozovski [16], [17], and signed measures as portfolios were used in the paper of De Donno, Pratelli [6]. The problem of replicating with the help of natural strategies was discussed and partially solved in Carmona, Tehranchi [4]. It was shown in [4], with the use of the Malliavin calculus, that each contingent claim of a special form can be replicated. The problem was also considered in Aihara, Bagchi [1]. The lack of completeness was shown in Taflin [19] however, for a non-standard definition of the set of contingent claims, namely for $D_0 := \bigcap_{p>1} L^p(\Omega)$.

In the present paper the problem of completeness in the class of all bounded random variables is studied. We consider two cases, the first one when strategies take values in the space G^* - the dual of the Sobolev space $G = H^1[0, \bar{S}]$ and the second one when strategies are only $(\hat{P}_t)$ integrable processes. The strategies in the latter case are called generalized strategies. We prove that in the first case, under some natural conditions, the bond market model is not complete. The main contribution of this paper is the result on incompleteness for bounded contingent claims. Moreover, we provide a construction of a bounded random variable which can not be replicated. As a corollary we obtain incompleteness when the strategies take values in $L^2[0, \bar{S}]$, compare [1]. Our results seem to be in a contrast with Theorem 4.1 in [1] which states that the bond market is complete (see our Remark 3.8).

For the class of generalized strategies we establish a sufficient condition for completeness. Although these strategies have a clear mathematical meaning, portfolios used can be even non-continuous functionals on G. We provide an example of a Gaussian bond market which is complete.

2 The model

2.1 Process of bond prices

We consider a bond market with a finite time horizon $\bar{S}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \bar{S}]}, \mathbb{P})$ satisfying usual conditions. The filtration is generated by a sequence of independent standard Wiener processes W^i, $i = 1, 2, \ldots$, i.e. $\mathcal{F}_t$ is generated by $W^i(s)$, where $i = 1, 2, \ldots$, $s \leq t$. We interpret $W_t = (W^1(t), W^2(t), \ldots)$ as a cylindrical Wiener process W in l^2. The dynamics of the forward rate curve is given by the equation:

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{\infty} \sigma^i(t,T)dW^i(t), \qquad t \in [0,\bar{S}],\ T \in [0,\bar{S}], \tag{2.1}$$

where we put

$$\alpha(t,T) = \sigma^i(t,T) = 0 \qquad \text{for } t \geq T \text{ and } i = 1, 2, \ldots. \tag{2.2}$$

The formula (2.1) defines a family of equations parametrized by a continuous parameter $T \in [0,\bar{S}]$. For each $T \in [0,\bar{S}]$ the equation describes the evolution of the forward rate on the interval $[0,\bar{S}]$. Denote by σ_t a linear operator from l^2 into $H := L^2[0,\bar{S}]$ given by the formula

$$(\sigma_t u)(T) := \sum_{i=1}^{\infty} \sigma^i(t,T)u^i, \quad u = (u^i)_{i=1}^{\infty} \in l^2, \quad t, T \in [0,\bar{S}],$$

and $\alpha_t(T) = \alpha(t,T)$. In this notation the formula (2.1) has the following form:

$$df_t = \alpha_t dt + \sigma_t dW_t, \tag{2.3}$$

or equivalently, using a stochastic integral:

$$f_t = f_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s, \qquad t \in [0,\bar{S}]. \tag{2.4}$$

For the concept of the stochastic integral in Hilbert spaces see [15], [5].
The drift coefficient α is assumed to be a process taking values in the space H, with the norm denoted by $|\cdot|_H$, satisfying Bochner integrability condition:

$$\int_0^{\bar{S}} |\ \alpha_t\ |_H\ dt = \int_0^{\bar{S}} \left(\int_0^{\bar{S}} \alpha^2(t,T)dT \right)^{\frac{1}{2}} dt < \infty \qquad \mathbb{P} - a.s.. \tag{2.5}$$

Processes (σ_t) is assumed to be a predictable process taking values in the space of Hilbert-Schmidt operators $L_{HS}(l^2, H)$:

$$\|\sigma_t\|^2_{L_{HS}(l^2,H)} = \sum_{i=1}^{\infty} |\ \sigma_t^i\ |_H^2 = \sum_{i=1}^{\infty} \left(\int_0^{\bar{S}} \sigma^i(t,T)^2 dT \right) < \infty \qquad \mathbb{P} - a.s., \tag{2.6}$$

and satisfied the integrability condition:

$$\int_0^{\bar{S}} \|\sigma_t\|^2_{L_{HS}(l^2,H)} dt = \sum_{i=1}^{\infty} \int_0^{\bar{S}} \left(\int_0^{\bar{S}} \sigma^i(t,T)^2 dT \right) dt < \infty \qquad \mathbb{P} - a.s.. \tag{2.7}$$

Conditions (2.5) and (2.7) are necessary and sufficient for the forward rate curve process (f_t) to be continuous in H.
The short rate process r is defined by $r(t) := f(t,t)$ and the evolution of the money held in the savings account is given by the equation:

$$dB(t) = r(t)B(t)dt.$$

The bond price P is a process defined by the following formula:

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \tag{2.8}$$

and the discounted bond price $\hat{P}$, due to (2.2), satisfies:

$$\hat{P}(t,T) := B^{-1}(t)P(t,T) = e^{-\int_0^T f(t,u)du}. \tag{2.9}$$

Let $G := H^1[0,\bar{S}]$ be the Hilbert space of absolutely continuous functions with square integrable first derivative equipped with the norm:

$$\mid g \mid_G^2 := \mid g(0) \mid^2 + \int_0^{\bar{S}} \Big(\frac{dg}{ds}(s)\Big)^2 ds, \qquad g \in G.$$

Note that the process $\hat{P}$ takes values in G. In fact, since:

$$\frac{d}{dT}\,\hat{P}(t,T) = -\hat{P}(t,T)f(t,T),$$

we have:

$$\begin{aligned}\mid \hat{P}_t \mid_G^2 &= \hat{P}(t,0)^2 + \int_0^{\bar{S}} \Big(\hat{P}(t,T)f(t,T)\Big)^2 dT \\ &\leq \hat{P}(t,0)^2 + C(t)\int_0^{\bar{S}} f(t,T)^2 dT < \infty,\end{aligned}$$

where $C(t) := \sup_{T\in[0,\bar{S}]} \hat{P}^2(t,T)$ is finite because $\hat{P}(t,T)$ is a continuous function of T. We will also assume that the model is arbitrage-free, in the sense that the process $\hat{P}(\cdot,T)$ is a local martingale for every $T \in [0,\bar{S}]$, see Delbaen, Schachermayer [7], [8]. This postulate is satisfied if and only if the following HJM-condition holds, see Jakubowski, Zabczyk [10]:

$$\int_t^T \alpha(t,u)du = \frac{1}{2}\Big|\int_t^T \sigma(t,u)du\Big|_{l^2}^2 \quad \forall\, t,T \in [0,\bar{S}]. \tag{2.10}$$

Differentiating (2.10) with respect to T gives the following formula:

$$\alpha(t,T) = \sum_{i=1}^{\infty} \sigma^i(t,T)\int_t^T \sigma^i(t,u)du \quad \forall\, t,T \in [0,\bar{S}]. \tag{2.11}$$

As a consequence of (2.11) we can write the following expression for the process $\hat{P}$, see [10]:

$$d\hat{P}(t,T) = \hat{P}(t,T)\Big(\sum_{i=1}^{\infty} b^i(t,T)dW^i(t)\Big), \tag{2.12}$$

where

$$b^i(t,T) := -\int_0^T \sigma^i(t,u)du, \qquad t,T \in [0,\bar{S}]. \tag{2.13}$$

Let (Γ_t) be a stochastic process with values in the space of linear operators from l^2 into G given by the formula:

$$(\Gamma_t u)(T) := \hat{P}(t,T)\sum_{i=1}^{\infty} b^i(t,T)u^i, \quad u \in l^2, \quad t,T \in [0,\bar{S}]. \tag{2.14}$$

Proposition 2.1 *For every $t \in [0,\bar{S}]$, Γ_t given by* (2.14) *is a Hilbert-Schmidt operator form l^2 to G. Moreover with probability one*

$$\int_0^{\bar{S}} \|\Gamma_t\|^2_{L_{HS}(l^2,G)}dt < \infty. \tag{2.15}$$

Proof: At the beginning we show an auxiliary estimation. By (2.9) we have:

$$\mid \hat{P}(t,T) \mid \leq e^{\int_0^{\bar{S}}|f(t,u)|du} \leq e^{(\int_0^{\bar{S}}|f(t,u)|^2du)^{\frac{1}{2}}\ (\bar{S})^{\frac{1}{2}}} = e^{|f_t|_H\ (\bar{S})^{\frac{1}{2}}}. \tag{2.16}$$

Since (f_t) is a continuous process in H, we conclude that the function $t \longrightarrow \mid f_t \mid_H$ is bounded as a continuous function on $[0,\bar{S}]$, i.e. there exists a constant $A > 0$ such that:

$$\sup_{t\in[0,\bar{S}]} \mid f_t \mid_H \leq A. \tag{2.17}$$

Thus for some $B > 0$ we have:

$$\mid \hat{P}(t,T) \mid \leq B \quad \forall\, t,T \in [0,\bar{S}]. \tag{2.18}$$

For any $i = 1,2,...$, we have:

$$\begin{aligned}\frac{d}{dT}\Big(\hat{P}(t,T)b^i(t,T)\Big) &= \frac{d}{dT}\hat{P}(t,T)\, b^i(t,T) + \hat{P}(t,T)\frac{d}{dT}b^i(t,T)\\ &= f(t,T)\hat{P}(t,T)\int_0^T \sigma^i(t,u)du - \hat{P}(t,T)\sigma^i(t,T).\end{aligned} \tag{2.19}$$

Using (2.17), (2.18), (2.19) we can estimate the G-norm of $\hat{P}(t,\cdot)b^i(t,\cdot)$ by the H-norm of σ^i_t:

$$
\begin{aligned}
\mid \hat{P}(t,\cdot)b^i(t,\cdot)\mid^2_G &= \int_0^{\bar{S}} \hat{P}(t,T)^2 \left(f(t,T)\int_0^T \sigma^i(t,u)du - \sigma^i(t,T)\right)^2 dT \\
&\le B^2 \int_0^{\bar{S}} 2\left(\left(f(t,T)\int_0^T \sigma^i(t,u)du\right)^2 + \sigma^i(t,T)^2\right) dT \\
&\le B^2 \left(2\int_0^{\bar{S}} f(t,T)^2 \left(T\int_0^T \sigma^i(t,u)^2 du\right) dT + 2 \mid \sigma^i_t \mid^2_H\right) \\
&\le B^2 \left(2\int_0^{\bar{S}} f(t,T)^2 \bar{S} \mid \sigma^i_t \mid^2_H dT + 2 \mid \sigma^i_t \mid^2_H\right) \\
&\le 2B^2 \mid \sigma^i_t \mid^2_H \left(\bar{S}\int_0^{\bar{S}} f(t,T)^2 dT + 1\right) = 2B^2 \mid \sigma^i_t \mid^2_H \left(\bar{S} \mid f_t \mid^2_H + 1\right) \\
&\le 2B^2(\bar{S}A+1) \mid \sigma^i_t \mid^2_H . \qquad (2.20)
\end{aligned}
$$

Let $(e^i)_{i=1}^\infty$ be a standard basis in l^2. In virtue of (2.7) and (2.20) we obtain the desired estimation:

$$
\begin{aligned}
\int_0^{\bar{S}} \|\Gamma_t\|^2_{L_{HS}(l^2,G)} dt &= \int_0^{\bar{S}} \sum_{i=1}^\infty \mid \Gamma_t e^i \mid^2_G dt = \int_0^{\bar{S}} \sum_{i=1}^\infty \mid \hat{P}(t,\cdot)b^i(t,\cdot) \mid^2_G dt \\
&\le 2B^2(\bar{S}A+1) \int_0^{\bar{S}} \sum_{i=1}^\infty \mid \sigma^i_t \mid^2_H dt \le 2B^2(\bar{S}A+1)\int_0^{\bar{S}} \|\sigma_t\|^2_{L_{HS}(l^2,H)} dt < \infty.
\end{aligned}
$$

■

As an immediate consequence we obtain:

Corollary 2.2 *The process $\hat{P}$ of discounted bond prices is a G-valued local martingale.*

Since (2.15), hence the equation (2.12) can be written in the form, see [15], [5]:

$$
\hat{P}_t = \hat{P}_0 + \int_0^t \Gamma_s dW_s, \quad t \in [0,\bar{S}]. \qquad (2.21)
$$

2.2 Portfolios and strategies

2.2.1 Trading strategies

In general portfolios φ are identified with linear functionals acting on a space in which the price process lives. For the bond market the following classes of portfolios are considered in literature.

A) Portfolios consisting of finite or infinite number of bonds:

$$\varphi = \sum_{i=1}^{m} \alpha_i \delta_{\{T_i\}}, \quad m \in \mathbb{N} \cup \infty, \ \alpha_i, T_i \in [0, \bar{S}], \ i = 1, 2, ..., m, \ \sum_{i=1}^{\infty} | \alpha_i | < \infty.$$

B) φ are finite signed measures on the interval $[0, \bar{S}]$.

C) φ are bounded functionals on the space $G = H^1[0, \bar{S}]$, shortly $\varphi \in G^*$, where G^* denotes the dual space.

The class A has an obvious interpretation. Some justification for using portfolios as finite signed measures or as elements of G^* can be found in [6] or in [2] and [3]. Let us recall that the space G^* contains finite signed measures on the interval $[0, \bar{S}]$.

Definition 2.3 *A trading strategy is any predictable process with values in some fixed class in A - C.*

Definition 2.4 *The (discounted) wealth process $\hat{V}^\varphi$ corresponding to φ is given by:*

$$\hat{V}_t^\varphi = \hat{V}_0^\varphi + \int_0^t < \varphi_s, d\hat{P}_s >_{G^*, G} , \qquad t \in [0, \bar{S}],$$

The concept of stochastic integral will be discussed now.

Since $\hat{P}$ is a G-valued local martingale of the form (2.21), the class of integrands, see [15], consists of all G^*-valued predictable processes φ satisfying:

$$\int_0^{\bar{S}} | \varphi_t(Q_t^{\frac{1}{2}}) |_{G^*}^2 \, dt < \infty, \tag{2.22}$$

where

$$Q_t = \Gamma_t \Gamma_t^{'}, \qquad t \in [0, \bar{S}]. \tag{2.23}$$

In (2.23) $\Gamma_t^{'}$ is the conjugate of Γ_t, i.e. for all $a \in G$ and $b \in l^2$ we have $< \Gamma_t^{'} a, b >_{l^2} = < a, \Gamma_t b >_G$. We say that a predictable process φ is $\hat{P}$ integrable if φ satisfies (2.22). The construction of the stochastic integral in [15] is developed for a square integrable martingale, but it can be extended to local martingales by the localization procedure. Moreover, identifying Hilbert space G with its dual, with $\tilde{\varphi} \in G^*$ corresponding to $\varphi \in G$, we have

$$| \varphi(Q_t^{\frac{1}{2}}) |_{G^*}^2 = < (\Gamma_t \Gamma_t^{'})^{\frac{1}{2}} \tilde{\varphi}, (\Gamma_t \Gamma_t^{'})^{\frac{1}{2}} \tilde{\varphi} >_G = | \Gamma_t^{'} \tilde{\varphi} |_G^2 = | \Gamma_t^* \tilde{\varphi} |_{G^*}^2 .$$

So, the condition (2.22) can be reformulated as:

$$\int_0^{\bar{S}} | \Gamma_s^* \varphi_s |_{l^2}^2 \, ds < \infty. \tag{2.24}$$

Therefore, φ is $(\hat{P}_t)$ integrable if (2.24) holds and then

$$\int_0^t < \varphi_s, d\hat{P}_s >_{G^*,G} = \int_0^t < \Gamma_s^* \varphi_s, dW_s >_{l^2,l^2} . \tag{2.25}$$

We define a class of admissible strategies in a standard way, see, for instance, Karatzas, Shreve [13], Hunt, Kennedy [9], Jarrow, Madan [11].

Definition 2.5 *A strategy φ is admissible if it is $(\hat{P}_t)$ integrable and if the process*

$$\int_0^t < \varphi_s, d\hat{P}_s >_{G^*,G}, \qquad t \in [0, \bar{S}]$$

is a martingale. The class of all admissible strategies will be denoted by $\mathcal{A}$.

Example 2.6 Assume that φ is a G^*-valued predictable process.

1) If φ satisfies the integrability condition:

$$\mathbf{E}\left(\int_0^{\bar{S}} \mid \Gamma_s^* \varphi_s \mid_{l^2}^2 ds\right)^{\frac{1}{2}} < \infty,$$

then $\varphi \in \mathcal{A}$. Indeed, the Burkholder-Davies-Gundy inequality implies that the integral $\int_0^{\cdot} < \varphi_s, d\hat{P}_s >_{G^*,G}$ is a martingale.

2) If there exists a constant $K > 0$ such that the following condition holds:

$$\left| \int_0^t < \varphi_s, d\hat{P}_s >_{G^*,G} \right| < K, \qquad \forall\, t \in [0, \bar{S}],$$

then $\varphi \in \mathcal{A}$, because a bounded local martingale is a martingale. □

2.2.2 Generalized strategies

A class of generalized strategies consists of predictable processes for which the integral with respect to the process $\hat{P}$ is well defined. This class contains G^*-valued strategies but also strategies which take values in the space of unbounded functionals on G. The integral with respect to the process $\hat{P}$ given by (2.21) is defined in [15], see also Peszat, Zabczyk [18], where the space of integands is explicitly described. The class of integrands consists of all predictable processes φ taking values in the space of linear but not necessarily continuous functionals on G, satisfying the following conditions:

$$\mathrm{Im} Q_t^{\frac{1}{2}} \subseteq \mathrm{Dom}\varphi_t, \quad \varphi_t(Q_t^{\frac{1}{2}}) \in G^*, \quad \forall t \in [0, S], \tag{2.26}$$

$$\int_0^{\bar{S}} \mid \varphi_t(Q_t^{\frac{1}{2}}) \mid_{G^*}^2 dt < \infty. \tag{2.27}$$

Definition 2.7 *An admissible generalized strategy is a predictable process φ taking values in the space of linear functionals on G, satisfying* (2.26) *and the following condition:*

$$\mathbf{E}\left(\int_0^{\bar{S}} \mid \varphi_t(Q_t^{\frac{1}{2}}) \mid_{G^*}^2 dt\right) < \infty.$$

The stochastic integral of φ with respect to $(\hat{P}_t)$ is denoted by $\int(\varphi_s, d\hat{P}_s)$. Notice that

$$\int_0^t (\varphi_s, d\hat{P}_s) = \int_0^t < \Gamma_s^* \circ \varphi_s, dW_s >_{l^2,l^2}, \qquad t \in [0, \bar{S}], \tag{2.28}$$

where by $\Gamma_s^* \circ \varphi_s$ is given by the formula $\Gamma_s^* \circ \varphi_s(u) = \varphi_s(\Gamma_s u), u \in l^2$.

3 Completeness

3.1 Case of trading strategies

We define the completeness of the market in a usual way.

Definition 3.1 *Let $S \leq \bar{S}$. The bond market is complete on $[0, S]$ if for any $\mathcal{F}_S$-measurable, bounded random variable ξ there exists an admissible strategy φ and a constant c such that*

$$\xi = c + \int_0^S < \varphi_t, d\hat{P}_t >_{G^*,G} . \tag{3.29}$$

Theorem 3.2 *The bond market is not complete on $[0, S]$ for any $S \leq \bar{S}$.*

For the proof we will need the following lemmas.

Lemma 3.3 *(Appendix B in [5]) Let X, Y, Z be three Hilbert spaces and $A : X \longrightarrow Z$, $B : Y \longrightarrow Z$ two linear bounded operators. Then $ImA \subseteq ImB$ if and only if there exists constant $c > 0$ such that $\|A^* f\| \leq c\|B^* f\|$ for all $f \in Z^*$.*

The next lemma states a uniqueness of random variables representation in a class of admissible strategies. This property is crucial in a method which is used in the proof of the main result.

Lemma 3.4 *Let $x, y \in \mathbb{R}$. Assume that ψ is integrable with respect to W and such that $\int_0^t \psi_s dW_s,\ t \in [0, S]$ is a martingale. If for $\varphi \in \mathcal{A}$ the following condition*

$$x + \int_0^S < \varphi_s, d\hat{P}_s >_{G^*,G} = y + \int_0^S < \psi_s, dW_s >_{l^2,l^2}, \tag{3.30}$$

is satisfied, then $x = y$ and $\Gamma_s^* \varphi_s = \psi_s$ a.s. wrt. $P \otimes \lambda$ on $\Omega \times [0, S]$, where λ denotes a Lebesgue measure.

Proof: Taking expectations of both sides in (3.30) we immediately obtain that $x = y$. Thus the following condition is satisfied:

$$\int_0^S < \Gamma_s^* \varphi_s - \psi_s, dW_s >_{l^2,l^2} = 0.$$

By Lemma 10.15 in [5], there exists a standard, one dimensional Wiener process B for which we have:

$$\int_0^t < \Gamma_s^* \varphi_s - \psi_s, dW_s >_{l^2,l^2} = \int_0^t \mid \Gamma_s^* \varphi_s - \psi_s \mid_{l^2} dB(s) \qquad \forall\, t \in [0, S].$$

Thus the process $\int_0^t \mid \Gamma_s^* \varphi_s - \psi_s \mid_{l^2} dB(s)$ is a martingale which is equal to zero at time S. So, it is equal to zero for every $t \in [0, S]$. By the uniqueness of the martingale representation, see Theorem 18.10 in Kallenberg [12], we conclude that the integrand must be zero which implies: $\Gamma_s^* \varphi_s = \psi_s$, $P \otimes \lambda$ a.s.

■

3.1.1 Proof of Theorem 3.2

Due to the definition of the integral with respect to $\hat{P}$, the set of all final portfolio values starting from zero initial endowment has the following structure:

$$\left\{ \int_0^S < \varphi_t, d\hat{P}_t >_{G^*,G} : \ \varphi \in \mathcal{A} \right\} = \left\{ \int_0^S < \Gamma_t^* \varphi_t, dW_t >_{l^2,l^2} : \ \varphi \in \mathcal{A} \right\}.$$

We will show that the operators $\Gamma_t^*,\ t \in [0, S]$, are not surjective and we construct process ψ taking values in the space l^2 satisfying conditions:

1) ψ is integrable wrt. W and such that $\int_0^t < \psi_t, dW_t >_{l^2,l^2},\ t \in [0, S]$, is a martingale,

2) $\psi_t \notin Im(\Gamma_t^*)$ with positive $\mathbb{P} \otimes \lambda$ measure,

3) the random variable $\int_0^S < \psi_t, dW_t >_{l^2,l^2}$ is bounded.

Then, in virtue of Lemma 3.4, for any $c \in \mathbb{R}$ and $\varphi \in \mathcal{A}$ we have:

$$\int_0^S < \psi_t, dW_t >_{l^2,l^2} \neq c + \int_0^S < \Gamma_t^* \varphi_t, dW_t >_{l^2,l^2} .$$

Thus the integral $\int_0^S < \psi_t, dW_t >_{l^2,l^2}$ is a bounded random variable which can not be replicated.

By Proposition 2.1 the operator Γ_t is a Hilbert-Schmidt operator for any $t \in [0, S]$. Thus Γ_t is compact, so is Γ_t^*. As a compact operator with values in infinite dimensional Hilbert space, Γ_t^* is not surjective.

Let us consider the self-adjoint operator $(\Gamma_t^*\Gamma_t)^{\frac{1}{2}} : l^2 \longrightarrow l^2$ which is also compact. For any $u \in l^2$ we have:

$$\mid (\Gamma_t^*\Gamma_t)^{\frac{1}{2}} u \mid_{l^2}^2 = < (\Gamma_t^*\Gamma_t)^{\frac{1}{2}} u, (\Gamma_t^*\Gamma_t)^{\frac{1}{2}} u >_{l^2} = < \Gamma_t^*\Gamma_t u, u >_{l^2} = < \Gamma_t u, \Gamma_t u >_G = \mid \Gamma_t u \mid_G^2 ,$$

so by Lemma 3.3 it follows that $Im(\Gamma_t^*) = Im((\Gamma_t^*\Gamma_t)^{\frac{1}{2}})$.
By Proposition 1.8 in [5] the operator $(\Gamma_t^*\Gamma_t)^{\frac{1}{2}}$ can be represented by the formula:

$$(\Gamma_t^*\Gamma_t)^{\frac{1}{2}} = \sum_{i=1}^{\infty} \lambda_i(t) \;\; g_t^i \otimes g_t^i,$$

where $\lambda_i(t)$ is a random variable and g_t^i is an l^2- valued random variable for i=1,2,... . Here "$\otimes$" denotes the linear operation: $(a \otimes b)h = a < b, h >$ for $a, b, h \in l^2$. Moreover, λ_i and g^i are predictable as functions of (ω, t) and $\lambda_i(t) \longrightarrow_{i\to\infty} 0$ by compactness of $(\Gamma_t^*\Gamma_t)^{\frac{1}{2}}$.
Our aim now is to construct the process $\tilde{\psi}$ where $\tilde{\psi}_t = (\tilde{\psi}^1(t), \tilde{\psi}^2(t), ...) \in l^2$ such that it is not of the form $\sum_{i=1}^{\infty} \lambda_i(t) g_t^i < g_t^i, u >_{l^2}$ for any $u \in l^2$. This process process must thus satisfy:

$$\sum_{i=1}^{\infty} \left(\frac{\tilde{\psi}^i(t)}{\lambda_i(t)} \right)^2 = \infty,$$

$$\sum_{i=1}^{\infty} (\tilde{\psi}^i(t))^2 < \infty.$$

Let us define the sequence $(i_k(t))_{k=1,2,...}$ in the following way:

$$i_1(t) := \inf \left\{ i : \frac{1}{\lambda_i(t)} \geq 1 \right\},$$

$$i_{k+1}(t) := \inf \left\{ i > i_k : \frac{1}{\lambda_i(t)} \geq k \right\}$$

and put

$$\tilde{\psi}^i(t) = \begin{cases} 0 & \text{if } i \neq i_k(t) \\ \frac{1}{k} & \text{if } i = i_k(t). \end{cases}$$

Then we have $\sum_{i=1}^{\infty} \left(\frac{\tilde{\psi}^i(t)}{\lambda_i(t)} \right)^2 \geq \sum_{k=1}^{\infty} \frac{1}{k^2} \, k^2 = \infty$ and $\sum_{i=1}^{\infty} \tilde{\psi}^i(t)^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so the process is bounded in l^2. It is also predictable since it is obtained by measurable operations on predictable elements. Thus the process $\tilde{\psi}$ is integrable with respect to W and $\int_0^{\cdot} < \tilde{\psi}, dW >_{l^2, l^2}$ is a martingale.
Now let us define a stopping time τ as:

$$\tau := \inf\{t > 0 : \left| \int_0^t < \tilde{\psi}_t, dW_t >_{l^2, l^2} \right| \geq 1\} \wedge S.$$

Finally, we define the required process as:

$$\psi_t := \tilde{\psi}_t \mathbf{1}_{[0,\tau)}(t).$$

■

Remark 3.5 *From Theorem 3.2 with $S = \bar{S}$ it follows that the bond market is not complete on $[0, \bar{S}]$ if traders can use natural strategies only.*

3.1.2 Incompleteness in $L^2[0, \bar{S}]$

Proposition 3.6 *There exists a model of the bond market in which all $L^2[0, \bar{S}]$-valued processes φ such that*

$$\mathbf{E}\left(\int_0^{\bar{S}} \mid \varphi_t \mid^2_{L^2[0,\bar{S}]} dt\right) < \infty \tag{3.31}$$

are admissible strategies (absolutely continuous measures are identified with their densities).

Proof: We will construct the model by defining the volatility coefficient. Let us assume that $\tilde{\sigma}$ satisfies the following conditions:

$$0 \leq \tilde{\sigma}^i(t,T) \leq K \quad i = 1, 2, \ldots, \ (t,T) \in [0, \bar{S}] \times [0, \bar{S}] \text{ for some } \ K > 0, \tag{3.32}$$

$$\mid \tilde{\sigma}^i_t \mid^2_H \leq \frac{1}{i^2} \quad i = 1, 2, \ldots, t \in [0, \bar{S}], \tag{3.33}$$

and define a new operator σ as

$$\sigma^i(t,T) = \begin{cases} \tilde{\sigma}^i(t,T) & \text{if} \quad \sum_{i=1}^{\infty} \int_0^t \tilde{\sigma}^i(s,T) dW^i(s) \geq 0, \\ 0 & \text{if} \quad \sum_{i=1}^{\infty} \int_0^t \tilde{\sigma}^i(s,T) dW^i(s) < 0. \end{cases} \tag{3.34}$$

Let the coefficient α be given by the HJM condition (2.11):

$$\alpha(t,T) = \sum_{i=1}^{\infty} \sigma^i(t,T) \int_t^T \sigma^i(t,s) ds. \tag{3.35}$$

It follows from (3.32), (3.33), (3.34) and (3.35) that coefficients α and σ satisfy (2.5) and (2.7). Assume that the initial forward rate curve is nonnegative: $f(0,T) \geq 0$ for $T \in [0, \bar{S}]$. Then

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T) ds + \sum_{i=1}^{\infty} \int_0^t \sigma^i(s,T) dW^i(s) \geq 0, \quad (t,T) \in [0, \bar{S}] \times [0, \bar{S}]$$

and thus $\hat{P}(t,T) = e^{-\int_0^T f(t,u)du} \leq 1$. It follows from the condition (3.33) that:

$$\sum_{i=1}^{\infty}\int_0^{\bar{S}} b^i(t,T)^2 dT = \sum_{i=1}^{\infty}\int_0^{\bar{S}} \left(\int_0^T \sigma^i(t,u)du\right)^2 dT$$
$$\leq \sum_{i=1}^{\infty}\int_0^{\bar{S}} \left(T\int_0^T \sigma^i(t,u)^2 du\right) dT \leq \bar{S}^2\sum_{i=1}^{\infty}\frac{1}{i^2} \qquad \forall t\in[0,\bar{S}].$$

As a consequence we obtain the following inequalities:

$$\mathbf{E}\left(\int_0^{\bar{S}} \mid \Gamma_t^*\varphi_t \mid_{l^2}^2 dt\right) = \mathbf{E}\left(\int_0^{\bar{S}}\sum_{i=1}^{\infty}\Big(\int_0^{\bar{S}} \varphi_t(T)\hat{P}(t,T)b^i(t,T)dT\Big)^2 dt)\right)$$
$$\leq \mathbf{E}\left(\int_0^{\bar{S}}\Big(\int_0^{\bar{S}} \varphi_t(T)^2 dT\sum_{i=1}^{\infty}\int_0^{\bar{S}} b^i(t,T)^2 dT\Big)dt\right)$$
$$\leq \bar{S}^2\sum_{i=1}^{\infty}\frac{1}{i^2}\mathbf{E}\left(\int_0^{\bar{S}}\int_0^{\bar{S}} \varphi_t(T)^2 dT dt\right) < \infty. \tag{3.36}$$

In view of (3.36) we conclude that for each φ satisfying (3.31) the process $\int_0^{\cdot} < \varphi_t, d\hat{P}_t >_{G^*,G}$ is a martingale and thus φ is admissible. ■

Corollary 3.7 *There exists an incomplete bond market for which all $L^2[0,\bar{S}]$-valued processes satisfying* (3.31) *are admissible. To see this is true it is enough to remark that the model presented in Proposition 3.6 is incomplete.*

Remark 3.8 *Due to relation $L^2[0,\bar{S}] = H \supseteq G = H^1[0,\bar{S}]$ we can treat the process $(\hat{P}_t)$ as taking values in the space H. The inclusion $H^* \subseteq G^*$ expresses the fact, that if we admit H as a state space for the discounted bonds prices, then the investor can use smaller class of strategies and as a consequence, in this case, the market is also incomplete. It follows from Corollary 3.7 that Theorem 4.1. in [1] is false.*

Remark 3.9 *It was pointed out by one of the reviewers that the market considered in [1] is approximately complete but the limit passage, performed in [1] to get completeness, is not correct.*

3.1.3 Comments on admissibility

Notice that Lemma 3.4 can be reformulated in the following way. If for the W integrable processes γ, ψ, such that the integrals $\int < \gamma_s, dW_s >_{l^2,l^2}$ and $\int < \psi_s, dW_s >_{l^2,l^2}$ are martingales, we have:

$$x + \int_0^S < \gamma_s, dW_s >_{l^2,l^2} = y + \int_0^S < \psi_s, dW_s >_{l^2,l^2}$$

for some $x, y \in \mathbb{R}$, then $x = y$ and $\gamma = \psi$. It turns out that this assertion is not true if we assume only the existence of the integrals or if we additionally assume that the integrals are bounded from below. A counterexample in a one dimensional case which we show is based on Example 8 p. 259 in Lipcer, Shiryaev [14].

Example 3.10 Let us consider a one dimensional Wiener process B on the interval $[0, 1]$. The following stopping time:

$$\tau := \inf\{t \in [0, 1] : B^2(t) + t = 1\}$$

satisfies $P(0 < \tau < 1) = 1$. The process

$$X(t) := -\frac{2B(t)}{(1 - t^2)}\mathbf{1}_{\{t \le \tau\}}$$

is integrable with respect to B because the following estimation holds:

$$\int_0^1 X(s)^2 ds = 4\int_0^{\tau} \frac{B(s)^2}{(1 - s)^4} ds < \infty.$$

Applying the Itô formula to the process $\frac{B(t)^2}{(1-t)^2}$ we obtain:

$$\int_0^1 X(s)dB(s) - \frac{1}{2}X(s)^2 ds = -1 - 2\int_0^{\tau} B(s)^2 \left(\frac{1}{(1 - s)^4} - \frac{1}{(1 - s)^3}\right) ds < -1.$$

As a consequence, the Doléans-Dade exponent $M = \mathcal{E}(X)$, which is a local martingale, is not a martingale because

$$\mathbf{E}(M(1)) = \mathbf{E}\left(e^{\int_0^1 X(s)dB(s) - \frac{1}{2}X(s)^2 ds}\right) < e^{-1} < M(0) = 1.$$

The random variable $M(1)$ satisfies estimation $0 < M(1) < e^{-1}$ and thus the application of the martingale representation theorem to the square integrable martingale $\mathbf{E}[M(1) \mid \mathcal{F}_t]$ provides:

$$M(1) = \mathbf{E}[M(1)] + \int_0^1 \gamma(s)dB(s),$$

where $\mathbf{E}\int_0^1 \gamma^2(s)ds < \infty$. On the other hand the application of the martingale representation theorem to the local martingale M provides, see Theorem 18.10 in [12]:

$$M(1) = M(0) + \int_0^1 \psi(s)dB(s) = 1 + \int_0^1 \psi(s)dB(s),$$

where $P(\int_0^1 \psi^2(s)ds < \infty) = 1$. Moreover, ψ satisfies condition $\mathbf{E}\int_0^1 \psi^2(s)ds = \infty$ because M is not a martingale.

Summarizing, we have two different representations of the same bounded random variable $M(1)$, i.e.

$$\mathbf{E}[M(1)] = x \neq y = 1; \qquad \gamma \neq \psi.$$

Moreover, both representations are bounded from below by zero. $\square$

3.2 Completeness with generalized strategies

In this subsection we formulate a sufficient condition for completeness in the class of generalized strategies. Let $S = \bar{S}$. We start from the auxiliary lemma.

Lemma 3.11 *Let ξ be a square integrable, $\mathcal{F}_S$ measurable random variable such that $\mathbf{E}\xi = 0$. Then ξ can be represented in the following form:*

$$\xi = \int_0^S < \psi_s, dW_s >_{l^2,l^2},$$

where ψ is a predictable, l^2-valued process satisfying condition $\mathbf{E}\int_0^S \mid \psi_s \mid_{l^2}^2 < \infty$. Moreover, $\mathbf{E}\xi^2 = \mathbf{E}\int_0^S \mid \psi_s \mid_{l^2}^2$.

Proof: This result is true when W is finite dimensional, see Lemma 18.11 in [12]. The generalization can be obtained by the following arguments. Let $\mathcal{G}_n \subseteq \mathcal{F}$ be a σ-field generated by

$$W_n(s) = (W^1(s), W^2(s), ..., W^n(s)), \quad s \leq S,$$

and let $\xi_n := \mathbf{E}(\xi \mid \mathcal{G}_n)$. Since $\xi \in L^2(\Omega)$ and $\mathcal{G}_n \uparrow \mathcal{F} = \bigcup_{n\geq 1} \mathcal{G}_n$, so by the classical convergence theorem for martingales, see Corollary 7.22 and Theorem.7.23 in [12], $\xi_n \longrightarrow \xi$ in $L^2(\Omega)$. On the other hand we have:

$$\xi_n = \int_0^S \varphi_n(s) dW_n(s).$$

Setting $\tilde{\varphi}_s^n := (\varphi_n(s), 0, ...) \in l^2$ we see that

$$\xi_n = \int_0^S < \tilde{\varphi}_s^n, dW_s >_{l^2,l^2} .$$

Hence $\mathbf{E}(\xi_m - \xi_n)^2 = \mathbf{E}\int_0^S (\tilde{\varphi}_s^n - \tilde{\varphi}_s^m)^2 ds \longrightarrow 0$ with $m, n \longrightarrow \infty$ and therefore $\{\tilde{\varphi}^n\}$ is a Cauchy sequence in $F = L^2(\Omega \times [0, S], \mathcal{F}_S \otimes \mathcal{B}[0, S], \mathbb{P} \times \lambda; l^2)$. By the completeness of F there exists a predictable limit $\varphi \in F$ of the sequence $(\tilde{\varphi}_n)$. Due to the fact that

$$\int_0^S < \tilde{\varphi}_s^n, dW_s >_{l^2,l^2} \longrightarrow \int_0^S < \varphi_s, dW_s >_{l^2,l^2} \text{ in } L^2(\Omega),$$

we have the required representation: $\xi = \int_0^S < \varphi_s, dW_s >_{l^2,l^2}$. ∎

Theorem 3.12 *If the operator Γ is injective $\mathbb{P} \otimes \lambda$ a.s. then the bond market with admissible generalized strategies is complete. Moreover, any square integrable random variable can be replicated.*

Proof: Let $\xi \in L^2(\Omega)$. Using Lemma 3.11 we can represent ξ in the following form

$$\xi = x + \int_0^S < \psi_s, dW_s >_{l^2,l^2},$$

where $x = \mathbf{E}(\xi)$ and ψ is a predictable, l^2-valued process satisfying condition $\mathbf{E} \int_0^S \mid \psi_s \mid_{l^2}^2 \; ds < \; \infty$. We will find an admissible generalized strategy φ such that

$$\int_0^S < \psi_s, dW_s >_{l^2,l^2} = \int_0^S (\varphi_s, d\hat{P}_s). \tag{3.37}$$

Since Γ_t is injective, so

$$\varphi_t(v) := < \psi_t, \Gamma_t^{-1} v >_{l^2}, \qquad \forall v \in \mathrm{Im}\Gamma_t,$$

is a well defined linear functional. The process $(\varphi_t(\Gamma_t))$ is predictable and for any $u \in l^2$ we have:

$$< \psi_t, u >_{l^2} = < \psi_t, \Gamma_t^{-1} \Gamma_t u >_{l^2} = \varphi_t(\Gamma_t u).$$

Therefore

$$\psi_t = \varphi_t(\Gamma_t), \qquad \forall t \in [0, S], \tag{3.38}$$

so by (2.28) the formula (3.37) is satisfied. ∎

Now, we give an example of a bond market with deterministic volatility (Gaussian HJM-model) which is complete.

Example 3.13 Let $\sigma^j, j = 1, 2, ...$ be given by the formula:

$$\sigma^j(t, T) := \gamma_j \sin\left(j\pi \Big(\frac{T - t}{\bar{S} - t} \Big) \vee 0 \right), \quad 0 \leq t, T \leq \bar{S},$$

where $\gamma_j > 0$, and $\sum_{j=1}^{\infty} \gamma_i^2 < \infty$. Notice that for any $t \in [0, \bar{S}]$ the sequence $(\sigma^j(t, \cdot))_j$ is an orthogonal system in $L^2[t, \bar{S}]$ and

$$\mid \sigma^j(t, \cdot) \mid_{L^2[0,\bar{S}]} \leq \frac{\bar{S}}{2} \gamma_j^2.$$

Hence the process (σ_t) satisfies (2.6). So the deterministic process (Γ_t) satisfies (2.15). Therefore the corresponding process $(\hat{P}_t)$ is a G-valued martingale. To prove that in this case the bond market with generalized strategies is complete, it is enough, by Theorem 3.12, to show that Γ_t is injective for all $t \in [0, \bar{S}]$. To this end we prove that if $\Gamma_t u = 0$ for some $u \in l^2$, then u=0. Differentiating

$$\Gamma_t u(T) = - \sum_{j=1}^{\infty} \Big(\int_0^T \sigma^j(t, s) ds \Big) u^j$$

with respect to T, we see that

$$\sum_{j=1}^{\infty} \sigma^j(t,s)u^j = 0$$

in the sense of $L^2[t,\bar{S}]$. By orthogonality of the sequence $(\sigma^j(t,\cdot))_j$ we obtain that $u^j = 0$ for $j = 1, 2, ...$, hence $u = 0$. So, Γ is injective. $\square$

Remark 3.14 *In general if (Γ_t) is not injective $\mathbb{P} \otimes \lambda$ a.s., then in a similar way as in the proof of Theorem 3.2 one can show incompleteness even with the use of admissible generalized strategies.*

Acknowledgement
The authors express thanks to the anonymous reviewers for their valuable comments which were helpful while improving the early version of the paper.

References

[1] Aihara, S.I., A. Bagchi (2005): "Stochastic hyperbolic dynamics for infinite-dimensional forward rates and option pricing", *Mathematical Finance*, Vol.15, 27-47.

[2] Björk, Th., G. Di Masi, Y. Kabanov and W. Runggaldier (1997): "Towards a general theory of bond markets", *Finance and Stochastic*, 1, 141-174.

[3] Björk, Th., Y. Kabanov and W. Runggaldier (1997): "Bond market structure in the presence of marked point processes", *Mathematical Finance*, 7, 211-239.

[4] Carmona, R., M. Tehranchi (2004): "A characterization of hedging portfolios for interest rate contingent claims", *Annals of Applied Probability*, 14, 1267-1294.

[5] Da Prato, G., J. Zabczyk (1992): *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

[6] De Donno, M., M.Pratelli (2004): "On the use of measured-valued strategies in bond markets", *Finance and Stochastics*, 8, 87-109.

[7] Delbaen, F., W. Schachermayer (1994): "A general version of the fundamental theorem of asset pricing", *Mathematische Annalen* 300, 463-520.

[8] Delbaen, F., W. Schachermayer (1998): "The fundamental Theorem of Asset Pricing for unbounded stochastic processes", *Mathematische Annalen* 312, 215-250.

[9] Hunt, P.J., J.E. Kennedy (2005): "Financial Derivatives in Theory and Practice", John Wiley & Sons, Ltd.

[10] Jakubowski, J., J. Zabczyk (2007): "Exponential moments for HJM models with jumps", *Finance and Stochastics*, 11, 429-445.

[11] Jarrow, R.A, D.B. Madan (1991): "A characterization of complete security markets on a Brownian filtration", *Mathematical Finance*, Vol.1, 31-43.

[12] Kallenberg, O. (2001): "Foundations of Modern Probability", 2nd ed. Springer-Verlag.

[13] Karatzas, I., S.E. Shreve (1998): "Methods of mathematical finance", New York Springer.

[14] Lipcer, R.S., A.N. Shiryaev (1974): "Statistika slucajnych processov", Nauka, Moskwa.

[15] Métivier M. (1982): "Semimartingales: a course on stochastic processes", Berlin, New York: Walter de Gruyter.

[16] Mikulevicius, R., B.L. Rozovskii (1998): "Normalized stochastic integrals in topological vector spaces", *Séminaire de probabilités XXXII (Lecture notes in Mathematics)*, Berlin, Heidelberg, New York Springer.

[17] Mikulevicius, R., B.L. Rozovskii (1999): "Martingales problems for stochastic PDE's", In Carmona, R., Rozovski, B. (eds) Stochastic partial differential equations: Six perspectives. (Math. Surveys and Monographs, 64) Amer. Math. Soc., 243-326.

[18] Peszat, Sz., J. Zabczyk (2007): "Stochastic partial differential equations with Lévy noise", Cambridge University Press.

[19] Taflin, E. (2005): "Bond market completeness and attainable contingent claims", *Finance and Stochastics*, 9, 429 - 452.