INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

IM PAN Preprint 701 (2009)

Jan Kisyński

Cauchy's problem for system of PDE with constant coefficients and semigroups of operators

Published as manuscript

Received 13 January 2009

Cauchy's problem for systems of PDE with constant coefficients and semigroups of operators

Jan Kisyński

Institute of Mathematics, Polish Academy of Sciences,
Śniadeckich 8, 00-956 Warszawa, Poland

Abstract

The paper deals with Cauchy's problem $\frac{\partial}{\partial t}u(t,x) = P(D)u(t,x)$, $u(0,x) = u_0(x)$, $t \geq 0$, $x \in \mathbb{R}^n$, for $\mathbb{C}^m$-valued u and $P(D) = \sum_{|\alpha| \leq p} A_\alpha i^{-|\alpha|}(\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ where A_α are $m \times m$ matrices with constant complex entries. Let $\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\sum_{|\alpha|\leq p} \xi^\alpha A_\alpha),\ \xi \in \mathbb{R}^n\}$ where σ stands for the spectrum. Let E denote any of the three l.c.v.s.: (i) the T. Ushijima space $\{u \in L^2(\mathbb{R}^n; \mathbb{C}^m) : P(D)^k u \in L^2(\mathbb{R}^n; \mathbb{C}^m)$ for every $k \in \mathbb{N}\}$, (ii) the space of $\mathbb{C}^m$-valued rapidly decreasing C^∞-functions on $\mathbb{R}^n$, (iii) the space of $\mathbb{C}^m$-valued tempered distributions on $\mathbb{R}^n$. It is proved that the operator $P(D)|_E$ is the infinitesimal generator of a (C_0)-semigroup $(S_t)_{t\geq 0} \subset L(E)$ if and only if $\omega_0 < \infty$, and then $\omega_0 = \inf\{\omega \in \mathbb{R}$: the semigroup $(e^{-\omega t}S_t)_{t \geq 0} \subset L(E)$ is equicontinuous$\}$.

MSC: Primary 35E15, 47D06; Secondary 46F05, 15A42

Key words: Cauchy's problem; Petrovskiĭ correct system; (C_0)-semigroup

1. Introduction

1.1. The ACP perspective. $D(A^\infty)$-well posed operators A of T. Ushijima

Let X be a complex Banach space, A a closed linear operator from X into X, $D(A^n)$ the domain of the n-th power of A and

$$D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n).$$

If $n = 1, 2, \ldots$, then $D(A^n)$ equipped with the norm

Email address: `jan.kisynski@gmail.com` (Jan Kisyński)

$$\|x\|_n = \|x\|_X + \|Ax\|_X + \cdots + \|A^n x\|_X, \qquad x \in D(A^n),$$

is a Banach space continuously imbedded in X. $D(A^\infty)$ equipped with the topology determined by the system of norms $\| \ \|_n$, $n = 1, 2, \ldots$, is a Fréchet space continuously imbedded in X.

Let $\mathbb{R}^+ = [0, \infty[$. Consider the abstract Cauchy problem (ACP)

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) \quad \text{for } t \in \mathbb{R}^+, \\ u(0) &= u_0. \end{aligned} \tag{C}$$

For every $n \in \mathbb{N} \cup \{\infty\}$ put

$$C_n(A) = \{u_0 \in D(A) : C^n(\mathbb{R}^+; X) \cap C^{n-1}(\mathbb{R}^+; D(A)) \text{ contains exactly one solution of (C)}\},$$

and for every $u_0 \in C_n(A)$ let $\mathbb{R} \ni t \mapsto u_n(t; u_0) \in D(A)$ be the unique solution of (C) belonging to $C^n(\mathbb{R}^+; X) \cap C^{n-1}(\mathbb{R}^+; D(A))$. Closedness of A implies that

$$C_n(A) \subset D(A^n) \quad \text{for every } n \in \mathbb{N} \cup \{\infty\}.$$

$C_n(A)$ carries the natural topology determined by the countable system of seminorms $p_{k,l,m,n}$, $0 \le k < n$, $0 \le l < n - 1$, $m = 1, 2, \ldots$, defined by the formula

$$p_{k,l,m,n}(u_0) = \sup \left\{ \left\| \frac{d^k}{dt^k} u_n(t; u_0) \right\|_X, \left\| \frac{d^l}{dt^l} A u_n(t; u_0) \right\|_X : t \in [0, m] \right\}.$$

If $\varrho(A) \neq \emptyset$ and the resolvent of A satisfies the growth condition from Yu. I. Lyubich's uniqueness theorem ([Lyu], Theorems 9.2–9.4; [P], p. 101, Theorem 1.2), then $C_n(A)$ with the above topology is complete, and hence it is a Fréchet space. The uniqueness condition in the definition of $C_n(A)$ implies that

$$u_n(t; u_n(s; u_0)) = u_n(t + s; u_0) \quad \text{for every } s, t \in \mathbb{R}^+ \text{ and } u_0 \in C_n(A).$$

Consequently, the formula

$$S_n(t) u_0 = u_n(t; u_0), \qquad t \in \mathbb{R}^+, \ u_0 \in C_n(A),$$

defines a semigroup $(S_n(t))_{t \ge 0}$ of continuous linear operators from $C_n(A)$ into $C_n(A)$.

If $n \in \mathbb{N} \cup \{\infty\}$ and $C_n(A) = D(A^n)$, then $(S_n(t))_{t\geq 0} \subset L(D(A^n))$ is a (C_0)-semigroup with infinitesimal generator equal to $A|_{D(A^{n+1})}$. In the case of $n = \infty$ the generator $A|_{D(A^\infty)}$ is a closed operator defined on the whole Fréchet space $D(A^\infty)$, so that it is a continuous operator from $D(A^\infty)$ into $D(A^\infty)$, by the closed graph theorem.

T. Ushijima [U], p. 74, defines a closed operator A from X into X to be $D(A^\infty)$-*well posed* if $D(A^\infty)$ is dense in X and $A|_{D(A^\infty)}$ is the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t\geq 0} \subset L(D(A^\infty))$. Thus a closed operator A from X into X is $D(A^\infty)$-well posed if and only if $D(A^\infty)$ is dense in X and $D(A^\infty) = C_\infty(A)$.

The paper [U] of T. Ushijima is devoted to $D(A^\infty)$-well posed operators A from a complex Banach space into itself, and to corresponding semigroups of operators acting in the Fréchet space $D(A^\infty)$. Except in Section 4 of Chapter I, it is not assumed in [U] that $\varrho(A) \neq \emptyset$, where $\varrho(A)$ denotes the resolvent set of A treated as an operator from X into X. In Section 10 of Chapter II of [U] T. Ushijima proves $D(A^\infty)$-well posedness of an operator A related to a Petrovskiĭ correct system of PDE with constant coefficients. The proof involves the spectral theory of matrices and depends on E. A. Gorin's Lemma 3 from [G1] asserting that the coefficients of an interpolation polynomial for a given holomorphic function are linear combinations of some complex contour integrals involving that function.

1.2. The subject of the present paper

We simplify the proof of Ushijima's theorem by avoiding the theory of interpolation polynomials, but still using contour integrals of Gorin's type. A refined formulation of Ushijima's theorem is given in Section 1.4. Earlier, in Section 1.3, in order to elucidate the position of $D(A^\infty)$-well posedness in the theory of one-parameter semigroups and distribution semigroups of linear operators, we quote some theorems of E. Hille, D. Fujiwara and T. Ushijima. Chapter 4 is devoted to some other results in the theory of Petrovskiĭ correct systems. Section 4.2 emphasises the role played in [P] by the space $\mathcal{O}_M$ of slowly increasing C^∞-functions. In Section 4.3 the bounded subsets of $\mathcal{O}_M$ are characterized as equicontinuous sets of multipliers on the space $\mathcal{S}$ of rapidly decreasing C^∞-functions. In Section 4.4 the Petrovskiĭ correctness is expressed in terms of one-parameter (C_0)-semigroups of operators in the spaces $\mathcal{S}$ and $\mathcal{S}'$.

1.3. The case of non-empty resolvent set

Lemma ([W], Corollary 3.3). *If the resolvent set $\varrho(A)$ of A is non-empty and $D(A)$ is dense in X, then $D(A^\infty)$ is dense in X, and for every $n = 1, 2, \ldots$, $D(A^\infty)$ is dense in the Banach space $D(A^n)$.*

The role of the equality $C_n(A) = D(A^n)$ in semigroup theory is elucidated by the following two theorems.

Theorem 1. *Let A be a closed densely defined linear operator from a complex Banach space X into X such that $\varrho(A) \neq \emptyset$. Fix $n \in \mathbb{N}$. Then $C_n(A) = D(A^n)$ if and only if A is the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t\geq 0} \subset L(X)$.*

Theorem 2. *Let A be a closed densely defined linear operator from a complex Banach space X into X such that $\varrho(A) \neq \emptyset$. Then A is $D(A^\infty)$-well posed if and only if A is the generator of an $L(X)$-valued L. Schwartz distribution semigroup. Furthermore, if $(S(t))_{t\geq 0} \subset L(D(A^\infty))$ is the semigroup with infinitesimal generator $A|_{D(A^\infty)}$ and S is the distribution semigroup with generator A, then for every $\kappa \in \mathbb{R}$ the following conditions are equivalent:*

(a_κ) *the semigroup $(e^{-\kappa t}S(t))_{t\geq 0} \subset L(D(A^\infty))$ is equicontinuous,*
(b_κ) *$e_{-\kappa}S$ is an $L(X)$-valued tempered distribution.*

In (b_κ), $e_{-\kappa}(t) = e^{-\kappa t}$ for $t \in \mathbb{R}$, and "tempered distribution" means a member of the L. Schwartz space $\mathcal{S}'(L(X))$. Theorem 1 (for $n = 1$) goes back to E. Hille [H]. See also [H-P], p. 622, Theorem 28.8.3. A proof of this theorem is also presented in [Pa], pp. 102–104. Theorem 2 follows from Theorem 4.1, p. 92, of T. Ushijima [U] and Theorems 2 and 3 of D. Fujiwara [Fu] (see also [U], p. 94, Theorem 4.2). The distribution semigroups for which (b_κ) is satisfied for some $\kappa \in \mathbb{R}$ are called *exponential*, after J.-L. Lions [L]. It follows from results of [L] and J. Chazarain [C] that not all distribution semigroups of L. Schwartz are exponential. Hence, in Theorem 2, there may be no κ for which (b_κ) holds, and then there is no κ for which (a_κ) holds.

There are closed densely defined operators A from X into X with non-empty resolvent set for which $C_\infty(A)$ is a Fréchet space densely and continuously imbedded in X and

$$C_\infty(A) \subsetneq D(A^\infty).$$

An example of such an operator may be constructed as follows. Take a non-negative continuous function ω on $\mathbb{R}$ such that $\omega(0) = 0$, $\omega(-x) \equiv \omega(x)$, $\omega|_{\mathbb{R}^+}$ is concave, $\int_1^\infty x^{-2}\omega(x)\,dx < \infty$ and $\lim_{x\to\infty} \omega(x)/\ln x = \infty$. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z < \omega(\operatorname{Im} z)\}$, and let $X = L^2(\Omega)$. Define

$$D(A) = \{f \in L^2(\Omega) : \text{the function } \Omega \ni z \mapsto zf(z) \in \mathbb{C} \text{ belongs to } L^2(\Omega)\},$$
$$Af(z) = zf(z) \quad \text{for every } f \in D(A) \text{ and almost every } z \in \Omega.$$

Then A is closed, $D(A)$ is dense in X, $\varrho(A) = \mathbb{C} \setminus \overline{\Omega}$ and

$$\sup_{\lambda \in (\mathbb{C}\setminus\Omega)+1} \|(\lambda - A)^{-1}\|_{L(X)} < \infty, \tag{$*$}$$

so that A is the generator of a $\mathcal{D}_\omega$-distribution semigroup S. See [K2], Sections 1.4 and 2.7. Furthermore, $C_\infty(A)$ coincides with the space of infinitely differentiable vectors of S (the latter being defined similarly to [K1]), and hence (by an argument similar to one in the proof of Proposition 4.6 in [C-Z], pp. 157–158) the estimate $(*)$ implies that $C_\infty(A)$ is dense in X. Finally, one has $C_\infty(A) \subsetneq D(A^\infty)$ because, by Theorem 2, the equality would imply that S is a distribution semigroup of L. Schwartz. But then, by Theorem 5.1, p. 403, of J. Chazarain [C] (and by inequalities in Sec. 9 of [K1]) one would have
$$\mathbb{C} \setminus \overline{\Omega} = \varrho(A) \supset \{z : \operatorname{Re} z \geq a \ln(1 + |\operatorname{Im} z|) + b\}$$
for some constants $a \geq 0$ and $b \in \mathbb{R}$. However, such an inclusion is impossible because $\lim_{x\to\infty} \omega(x)/\ln x = \infty$.

1.4. Theorem of T. Ushijima concerning Petrovskiĭ correct systems of linear partial differential equations with constant coefficients

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$, and let $m, n \in \mathbb{N}$ be fixed. Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{R}^n$ and for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ let
$$D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Suppose that $p \in \mathbb{N}$ and that for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ of length $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq p$ there is given an $m \times m$ matrix A_α with complex entries. Consider the differential operator

$$P(D) = \sum_{|\alpha| \leq p} A_\alpha D^\alpha$$

and the corresponding polynomial matrix

$$A(\xi) = \sum_{|\alpha| \leq p} \xi^\alpha A_\alpha$$

where $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Denote by $\sigma(A(\xi))$ the spectrum of $A(\xi)$. Define

$$X = L^2(\mathbb{R}^n; \mathbb{C}^m), \quad D(A) = \{u \in X : P(D)u \in X\},$$
$$Au = P(D)u \quad \text{for } u \in D(A),$$

where $P(D)u$ is meant in the sense of distributions. It is easy to see that A is a closed operator from X into X, and that $D(A^\infty)$ is dense in X. Endowed with the topology determined by the sequence of norms $\|u\|_j = (\|u\|_X^2 + \|Au\|_X^2 + \cdots + \|A^j u\|_X^2)^{1/2}$, $j = 0, 1, \dots$, $D(A^\infty)$ is a Fréchet space continuously imbedded in X.

Theorem 3. *The following conditions are equivalent:*

(a) $A|_{D(A^\infty)}$ *is the infinitesimal generator of a* (C_0)*-semigroup* $(S_t)_{t \geq 0} \subset L(D(A^\infty))$,

(b) $\omega_0 := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A(\xi)), \xi \in \mathbb{R}^n\}$ *is finite.*

Furthermore, if these equivalent conditions are satisfied, then

$$\omega_0 = \omega_1$$

where

$$\omega_1 = \inf\{\omega \in \mathbb{R} : \textit{the semigroup } (e^{-\omega t} S_t)_{t \geq 0} \subset L(D(A^\infty)) \textit{ is equicontinuous}\}.$$

The theory of semigroups of operators in locally convex spaces is presented in Chapter IX of the monograph of K. Yosida [Y]. The equivalence (a)⇔(b) was proved by T. Ushijima [U], Theorem 10.1, p. 118. If $p = 1$, then condition (b) is equivalent to hyperbolicity of the polynomial $\det(\zeta_0 \mathbb{1} - P(\zeta_1, \dots, \zeta_n))$ of the variables $\zeta_0, \zeta_1, \dots, \zeta_n \in \mathbb{C}$ with respect to the real vector $N = (1, 0, \dots, 0) \in \mathbb{R}^{1+n}$. See [H3], Definition 12.3.3. In the terminology of [C-P], p. 346, condition (b) means that the matricial differential operator $\mathbb{1}\frac{\partial}{\partial t} - P(D)$ is Petrovskiĭ correct in the direction $(1, 0, \dots, 0)$. An inspection of the operators $P(D) = \sum_{|\alpha| \leq p} A_\alpha D^\alpha$ with subdiagonal matrices A_α shows that (b) does not imply that A treated as an operator from X into X has non-empty resolvent set.

2. Functions of matrices as polynomials with coefficients expressed by complex contour integrals

Fix $m \in \mathbb{N}$ and let

$$\tau_1(x_1, \dots, x_m) = x_1 + \cdots + x_m,$$

$$\tau_k(x_1,\dots,x_m)=\sum_{1\le i_1<\dots<i_k\le m} x_{i_1}\cdots x_{i_k}\quad \text{for } k=2,\dots,m$$

be elementary symmetric polynomials of m variables $x_1,\dots,x_m$. Let A be a complex $m\times m$ matrix, and let $\lambda_1,\dots,\lambda_m$ be a sequence of eigenvalues of A in which the number of occurrences of any eigenvalue is equal to its spectral multiplicity. Let $P(z)=\det(z\mathbb{1}-A)$ be the characteristic polynomial of A. The spectrum of A, equal to the set $\{\lambda_1,\dots,\lambda_m\}$, is denoted by $\sigma(A)$.

Lemma 1. *For every $z\in\mathbb{C}\setminus\sigma(A)$ one has*

$$(z\mathbb{1}-A)^{-1}=\sum_{k=0}^{m-1} r_k(A,z)A^k$$

where

$$r_k(A,z)=\sum_{l=0}^{m-1-k}\binom{k+l}{k}(-z)^l\tau_{k+l+1}\left(\frac{1}{z-\lambda_1},\dots,\frac{1}{z-\lambda_m}\right).$$

Furthermore,

$$\tau_\mu\left(\frac{1}{z-\lambda_1},\dots,\frac{1}{z-\lambda_m}\right)=\frac{1}{\mu!}\,\frac{P^{(\mu)}(z)}{P(z)}\qquad \text{for } \mu=1,\dots,m,$$

so that

$$\tau_\mu\left(\frac{1}{z-\lambda_1},\dots,\frac{1}{z-\lambda_m}\right),\qquad \mu=1,\dots,m,$$

are rational functions of z and of the coefficients of the characteristic polynomial $P(z)$.

Proof. Lemma 1 is related to the solution of Problem 124 in [G-L]. We present an independent proof. By Taylor's formula and the Cayley–Hamilton theorem,

$$P(z)\mathbb{1}+\sum_{\mu=1}^{m}\frac{1}{\mu!}P^{(\mu)}(z)(A-z\mathbb{1})^\mu=P(A)=0,$$

whence

$$(z\mathbb{1}-A)^{-1}=\sum_{\mu=1}^{m}\frac{1}{\mu!}\,\frac{P^{(\mu)}(z)}{P(z)}(A-z\mathbb{1})^{\mu-1}\qquad \text{for } z\in\mathbb{C}\setminus\sigma(A).$$

Since $\frac{d}{dz}\tau_\mu(z-\lambda_1,\dots,z-\lambda_m)=(m-\mu+1)\tau_{\mu-1}(z-\lambda_1,\dots,z-\lambda_m)$ for $\mu=2,\dots,m$, it follows that

$$\begin{aligned} P^{(\mu)}(z) &= \left(\frac{d}{dz}\right)^{\mu} \tau_m(z-\lambda_1,\ldots,z-\lambda_m) \\ &= \left(\frac{d}{dz}\right)^{\mu-1} \tau_{m-1}(z-\lambda_1,\ldots,z-\lambda_m) \\ &= 2\left(\frac{d}{dz}\right)^{\mu-2} \tau_{m-2}(z-\lambda_1,\ldots,z-\lambda_m) = \cdots \\ &= \mu!\tau_{m-\mu}(z-\lambda_1,\ldots,z-\lambda_m) \end{aligned}$$

for $\mu = 1, \ldots, m-1$. Consequently,

$$\frac{1}{\mu!}\,\frac{P^{(\mu)}(z)}{P(z)} = \frac{\tau_{m-\mu}(z-\lambda_1,\ldots,z-\lambda_m)}{(z-\lambda_1)\cdots(z-\lambda_m)} = \tau_\mu\left(\frac{1}{z-\lambda_1},\ldots,\frac{1}{z-\lambda_m}\right)$$

for $\mu = 1, \ldots, m-1$. Furthermore,

$$\frac{1}{m!}\,\frac{P^{(m)}(z)}{P(z)} = \frac{1}{(z-\lambda_1)\cdots(z-\lambda_m)} = \tau_m\left(\frac{1}{z-\lambda_1},\ldots,\frac{1}{z-\lambda_m}\right).$$

Therefore

$$(z\mathbb{1}-A)^{-1} = \sum_{\mu=1}^{m} \tau_\mu\left(\frac{1}{z-\lambda_1},\ldots,\frac{1}{z-\lambda_m}\right)(A-z\mathbb{1})^{\mu-1} \qquad \text{for } z \in \mathbb{C}\setminus\sigma(A),$$

whence the expressions for the coefficients $r_k(A,z)$ follow by Newton's binomial formula.

Corollary 1. *Suppose that f is a function holomorphic in an open neighbourhood U of the spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_m\}$ of A. Let C be a system of closed rectifiable curves contained in $U \setminus \sigma(A)$ such that the whole C winds once about $\sigma(A)$. Then*

$$\frac{1}{2\pi i}\int_C f(z)(z\mathbb{1}-A)^{-1}\,dz = \sum_{k=0}^{m-1} a_k A^k \tag{2.1}$$

where

$$a_k = \sum_{l=0}^{m-1-k} \binom{k+l}{k} I^l_{k+l+1}(f;\lambda_1,\ldots,\lambda_m)$$

for $k = 0, \ldots, m-1$ and

$$I^l_\mu(f;\lambda_1,\ldots,\lambda_m) = \frac{1}{2\pi i}\int_C f(z)\left[(-z)^l \tau_\mu\left(\frac{1}{z-\lambda_1},\cdots,\frac{1}{z-\lambda_m}\right)\right]dz$$

for $\mu = 1, \ldots, m$ and $l = 0, \ldots, \mu-1$.

The integral $\frac{1}{2\pi i}\int_C f(z)(z\mathbb{1}-A^{-1})\,dz$ can be used as definition of the $m\times m$ matrix $f(A)$ when f is a function holomorphic in a neighbourhood of $\sigma(A)$. In another definition $f(A)$ is expressed as a polynomial of A of order no greater than $m-1$. The coefficients $a_0, a_1, \ldots, a_{m-1}$ of that polynomial (i.e. the coefficients for which (2.1) holds if f is holomorphic in a neighbourhood of $\sigma(A)$) are uniquely determined by the values of $f^{(k)}(\lambda)$ for $\lambda \in \sigma(A)$ and $k = 0, 1, \ldots, \mu(\lambda) - 1$ where $\mu(\lambda)$ is the spectral multiplicity of λ as a root of the characteristic equation $\det(\lambda\mathbb{1} - A) = 0$. See [D-S], Chap. VII, Sec. 1; [Hig], Sec. 1. The fact that if f is holomorphic in a neighbourhood of $\sigma(A)$, then the coefficients $a_0, a_1, \ldots, a_{m-1}$ are linear combinations of the integrals

$$I^l_{i_1,\ldots,i_k} = \frac{1}{2\pi i}\int_C f(z)\frac{(-z)^l}{(z-\lambda_{i_1})\cdots(z-\lambda_{i_k})}\,dz, \qquad 1 \le i_1 < \cdots < i_k \le m,$$

was discovered and exploited by E. A. Gorin in [G1]. This fact was also used by T. Ushijima in Sec. 10 of [U].

Remark. It should be noted that in [G1] the proof that $a_k \in \operatorname{lin}\{I^l_{i_1,\ldots,i_k}\}$ is presented only for simple characteristic roots $\lambda_1, \ldots, \lambda_m$, and without computing the coefficients of linear combinations. Passage to multiple roots then causes difficulties because the integrals $I^l_{i_1,\ldots,i_k}$ depend on the numbering of roots.

Lemma 2. *Let A be a complex $m \times m$ matrix, and let $z_0 \in \mathbb{C} \setminus \sigma(A)$. Then*

$$(A - z_0\mathbb{1})^{-m-1}\exp(tA) = \frac{1}{2\pi i}\int_C (z-z_0)^{-m-1}e^{tz}(z\mathbb{1}-A)^{-1}\,dz$$

for every $t \in \mathbb{R}$ and every rectifiable closed path C contained in $\mathbb{C} \setminus \{z_0\}$, winding once about $\sigma(A)$ and not winding about z_0.

Proof. For any $R > \|A\|$,

$$\begin{aligned}\frac{1}{2\pi i}\int_C e^{tz}(z\mathbb{1}-A)^{-1}\,dz &= \sum_{n=0}^{\infty}\frac{t^n}{2\pi i n!}\int_{|z|=R} z^n(z\mathbb{1}-A)^{-1}\,dz \\ &= \sum_{n=0}^{\infty}\frac{t^n}{2\pi i n!}\int_{|z|=R} z^n(z^{-1}\mathbb{1}+z^{-2}A+\cdots)\,dz\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{2\pi i n!} \int_{|z|=R} z^{-1} A^n \, dz$$
$$= \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = \exp(tA).$$

Furthermore,

$$\frac{1}{2\pi i} \int_C (z - z_0)^{-1} (z\mathbb{1} - A)^{-1} \, dz$$
$$= \left[\frac{1}{2\pi i} \int_C (z - z_0)^{-1} \, dz - \frac{1}{2\pi i} \int_C (z\mathbb{1} - A)^{-1} \, dz \right] (z_0 \mathbb{1} - A)^{-1}$$
$$= [0 - \mathbb{1}](z_0\mathbb{1} - A)^{-1} = (A - z_0\mathbb{1})^{-1},$$

by the resolvent equation. These equalities imply the lemma, by Theorem 10 in Sec. 3 of Chap. VII of [D-S] or the Theorem in Chap. VIII, Sec. 7 of [Y], or Fact 3 in [Hig].

Lemma 3. *Let A be a C^∞-map of $\mathbb{R}$ into the set of complex $m \times m$ matrices. Suppose that*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A(\xi)), \, \xi \in \mathbb{R}^n\} = \omega_0 < \infty.$$

Then there are functions $a_k \in C^\infty(\mathbb{R}^{1+n}; \mathbb{C})$, $k = 0, \ldots, 2m$, such that

$$\exp(tA(\xi)) = \sum_{k=0}^{2m} a_k(t, \xi) A(\xi)^k \quad \textit{for every } (t, \xi) \in \mathbb{R}^{1+n}$$

and

$$\sup\{e^{-(\omega_0+\epsilon)t} |a_k(t, \xi)| : k = 0, \ldots, 2m, \, t \in [0, \infty[, \, \xi \in \mathbb{R}^n\} < \infty$$

for every $\epsilon > 0$.

Proof. Fix $z_0 \in \mathbb{C}$ such that $\operatorname{Re} z_0 > \omega_0$. It is sufficient to show that there are complex-valued functions b_k, $k = 0, \ldots, m-1$, defined on $\mathbb{R}^{1+n}$ and having the following three properties:

$$(A(\xi) - z_0\mathbb{1})^{-m-1} \exp(tA(\xi)) = \sum_{k=0}^{m-1} b_k(t, \xi) A(\xi)^k, \quad (t, \xi) \in \mathbb{R}^{1+n}, \tag{2.2}$$

$$b_k \in C^\infty(\mathbb{R}^{1+n}; \mathbb{C}), \quad k = 0, \ldots, m-1, \tag{2.3}$$

$$\sup\{e^{-(\omega_0+\epsilon)t} |b_k(t, \xi)| : k = 0, \ldots, m-1, \, t \in [0, \infty[, \, \xi \in \mathbb{R}^n\} < \infty \tag{2.4}$$

for every $\epsilon > 0$. By Corollary 1 and Lemma 2, the functions b_k, $k = 0, \dots, m-1$, satisfying (2.2) are uniquely determined on $\mathbb{R}^{1+n}$ and may be represented in the form

$$b_k(t,\xi) = \sum_{l=0}^{m-1-k} \binom{k+l}{k} I^l_{k+l+1}(t,\xi)$$

where

$$I^l_\mu(t,\xi) = \frac{1}{2\pi i} \int_{C_\xi} (z-z_0)^{-m-1} e^{tz} (-z)^l \tau_\mu\Big(\frac{1}{z-\lambda_1(\xi)}, \dots, \frac{1}{z-\lambda_m(\xi)}\Big)\, dz$$

for $\mu = 1, \dots, m$ and $l = 0, \dots, \mu - 1$. In the last formula $\lambda_1(\xi), \dots, \lambda_m(\xi)$ is any sequence of eigenvalues of $A(\xi)$ in which the number of occurrences of any eigenvalue is equal to its spectral multiplicity, and C_ξ is a rectifiable closed path contained in $\{z \in \mathbb{C} : \operatorname{Re} z < \operatorname{Re} z_0\} \setminus \sigma(A(\xi))$ and winding once about $\sigma(A(\xi))$.

Every $\xi_0 \in \mathbb{R}^n$ has an open neighbourhood U such that $C_{\xi_0} \subset \mathbb{C} \setminus \sigma(A(\xi))$ and C_{ξ_0} winds once about $\sigma(A(\xi))$ for every $\xi \in U$. This follows from Theorem 9.17.4 in [D]. Consequently, for every $\xi \in U$ one can replace C_ξ by C_{ξ_0} without changing the values of the integrals $I^l_\mu(t,\xi)$. Since, by Lemma 1, each $\tau_\mu\big(\frac{1}{z-\lambda_1(\xi)}, \dots, \frac{1}{z-\lambda_m(\xi)}\big)$ is a C^∞ function on $\{(z,\xi) \in \mathbb{C} \times \mathbb{R}^n : z \notin \sigma(A(\xi))\}$, it follows that $I^l_\mu \in C^\infty(\mathbb{R}^{1+n}; \mathbb{C})$, so that (2.3) holds.

It remains to prove (2.4). To this end, fix $\epsilon > 0$ and take $\delta \in \,]0, \epsilon]$ such that $\omega_0 + \delta < \operatorname{Re} z_0$. Let $\xi \in \mathbb{R}^n$. Since $\sigma(A(\xi)) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le \omega_0\}$, without changing the values of the integrals $I^l_\mu(t,\xi)$ one can choose a closed rectifiable path C_ξ winding once about $\sigma(A(\xi))$ such that

$$C_\xi \subset D_\xi := \{z \in \mathbb{C} : \operatorname{Re} z - \omega_0 \le \delta \le \operatorname{dist}(z, \sigma(A(\xi)))\}.$$

For every $\xi \in \mathbb{R}^n$ the straight line

$$\mathbf{L} = \{z \in \mathbb{C} : \operatorname{Re} z = \omega_0 + \delta\}$$

is contained in D_ξ. Furthermore, for every $t \in [0, \infty[$, $\xi \in \mathbb{R}^n$, $z \in D_\xi$, $\mu = 1, \dots, m$ and $l = 0, \dots, \mu - 1$, one has

$$\Big| (z-z_0)^{-m-1} e^{tz} (-z)^l \tau_\mu\Big(\frac{1}{z-\lambda_1(\xi)}, \dots, \frac{1}{z-\lambda_m(\xi)}\Big) \Big| \le C |z-z_0|^{-2} e^{(\omega_0+\delta)t} \quad (2.5)$$

with some finite constant C depending only on δ. Therefore, by Cauchy's integral theorem, in the definition of $I^l_\mu(t,\xi)$ one can replace integration along the closed path C_ξ by integration along $\mathbf{L}$. From (2.5) it follows that

$$|I^l_\mu(t,\xi)| \le \frac{C}{2\pi}\int_{\mathbf{L}} |z-z_0|^{-2}\,dz \cdot e^{(\omega_0+\delta)t}$$

for every $\mu = 1,\dots,m$, $l = 0,\dots,\mu-1$, $t \in [0,\infty[$, and $\xi \in \mathbb{R}^n$, whence (2.4) follows because $\delta \in\,]0,\epsilon]$.

3. Proof of Theorem 3

Theorem 3 is a conjunction of three implications: (a)$\Rightarrow$(b), (b) $\Rightarrow$ (a) $\wedge$ $(\omega_1 \le \omega_0)$ and (a) $\wedge$ $(\omega_1 < \infty) \Rightarrow (\omega_0 \le \omega_1)$.

Proof of (a)$\Rightarrow$(b). Suppose that (a) holds. Then $S_1 \in L(D(A^\infty))$ and hence there are $C \in\,]0,\infty[$ and $j \in \mathbb{N}$ such that $\|S_1 u\|_X \le C(\sum_{0\le i\le j}\|A^i u\|^2_X)^{1/2}$ for every $u \in D(A^\infty)$. Consequently, by Plancherel's theorem, there are $K \in\,]0,\infty[$ and $k \in \mathbb{N}$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \|(\exp A(\eta))\varphi(\eta)\|^2\,d\eta\right)^{1/2} &\le C\sum_{0\le i\le j}\left(\int_{\mathbb{R}^n}\|A(\eta)\|^{2i}\|\varphi(\eta)\|^2\,d\eta\right)^{1/2}\\ &\le K\left(\int_{\mathbb{R}^n}(1+|\eta|)^{2k}\|\varphi(\eta)\|^2\,d\eta\right)^{1/2} \qquad (3.1)\end{aligned}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$. For any $\xi \in \mathbb{R}^n$ take $z(\xi) \in \mathbb{C}^m$ such that $\|z(\xi)\|_{\mathbb{C}^m} = 1$ and $\|(\exp A(\xi))z(\xi)\|_{\mathbb{C}^m} = \|\exp A(\xi)\|_{L(\mathbb{C}^m)}$. Let $(\phi_\nu)_{\nu=1,2,\dots} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence of non-negative functions such that the support of ϕ_ν is contained in the ball with center at ξ and radius $1/\nu$, and $\int_{\mathbb{R}^n}\phi_\nu(\eta)^2\,d\eta = 1$. Applying (3.1) to $\varphi(\eta) = \phi_\nu(\eta)z(\xi)$, one concludes that

$$\begin{aligned}\|\exp A(\xi)\| &= \|(\exp A(\xi))z(\xi)\|\\ &= \lim_{\nu\to\infty}\left(\int_{\mathbb{R}^n}\|(\exp A(\eta))z(\xi)\|^2_{\mathbb{C}^m}\phi_\nu(\eta)^2\,d\eta\right)^{1/2}\\ &\le K\lim_{\nu\to\infty}\left(\int_{\mathbb{R}^n}(1+|\eta|)^{2k}\phi_\nu(\eta)^2\,d\eta\right)^{1/2} = K(1+|\xi|)^k. \qquad (3.2)\end{aligned}$$

Let ρ stand for the spectral radius. By Corollary 2.4 on p. 252 of [E-N] and by (3.2), for every $\xi \in \mathbb{R}^n$ one has

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi))\} = \log \rho(\exp A(\xi)) \\ \leq \log \|\exp A(\xi)\| \leq \log K + k \log(1 + |\xi|). \quad (3.3)$$

By a theorem of Hurwitz ([S-Z], Sec. III.11), or by Theorem 9.17.4 in [D], for every $r \in [0, \infty[$ the set $\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n,\ |\xi| \leq r\}$ is compact and

$$\Lambda(r) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n,\ |\xi| \leq r\} \quad (3.4)$$

is a continuous function of r. From (3.3) it follows that

$$\Lambda(r) \leq \log K + k \log(1 + r) \quad \text{for every } r \in [0, \infty[. \quad (3.5)$$

In order to prove (b) it remains to recall that (3.5) implies a seemingly stronger condition

$$\sup_{r \in [0,\infty[} \Lambda(r) < \infty. \quad (3.6)$$

Proof of the implication (3.5)⇒(3.6). Validity of the implication (3.5)⇒ (3.6) was conjectured by I. G. Petrovskiĭ [P], footnote on p. 24. L. Gårding [G], pp. 11–14, proposed a method of proving this conjecture by an argument that consists in

(A) constructing a polynomial $P(z, w)$ of two variables such that $P(r, \Lambda(r)) = 0$ for every $r \in [0, \infty[$, and

(B) applying Puiseux series of algebraic functions $\mathcal{R}$ of one complex variable z satisfying the equation $P(z, \mathcal{R}(z)) = 0$.

L. Hörmander [H1], proof of Lemma 3.9, noticed that stage (A) may be realized by an application of A. Seidenberg's theorem (also called the Tarski–Seidenberg theorem) asserting that the projection onto $\mathbb{R}^d$ of a semi-algebraic subset of $\mathbb{R}^{d+k}$ is a semi-algebraic subset of $\mathbb{R}^d$. This projection theorem is a particular case of Seidenberg's decision theorem [Se] (belonging to mathematical logic). Detailed presentations of Seidenberg's proof in the case of the projection theorem are given in [G2] and [F]. An argument from P. Cohen's proof of a decision theorem [Co1,2] is used in the proof of the projection theorem in the Appendix to [H3].

Let us present a proof of the implication (3.5)⇒(3.6) consisting of the stages (A) and (B). At stage (A) we describe a standard application of the Tarski–Seidenberg theorem. At stage (B) we give detailed references to algebraic functions of one complex variable.

(A) Let R and S be a real polynomials on $\mathbb{R}^{2+n}$ such that

$$(R+iS)(\sigma,\tau,\xi) = \det((\sigma+i\tau)\mathbb{1} - A(\xi)),$$

and let

$$E = \{(r,\sigma) \in \mathbb{R}^2 : \exists_{(\tau,\xi)\in\mathbb{R}^{1+n}}\ (r,\sigma,\tau,\xi) \in F\}$$

where

$$F = \{(r,\sigma,\tau,\xi) \in \mathbb{R}^{3+n} : r \geq 0,\ \xi_1^2 + \cdots + \xi_n^2 \leq r^2,\ R(\sigma,\tau,\xi) = 0,\ S(\sigma,\tau,\xi) = 0\}.$$

Then F is equal to a finite union of finite intersections of subsets of $\mathbb{R}^d$, $d = 3+n$, each defined by a real polynomial equality or strict inequality. In other words, in the terminology of the Appendices in [Tr] and [H3], F is a semi-algebraic subset of $\mathbb{R}^d$. The set E is the projection of F onto $\mathbb{R}^2$, and hence, by the Tarski–Seidenberg theorem, E is a semi-algebraic subset of $\mathbb{R}^2$. Consequently,

$$E = \bigcup_{i=1}^{k} F_i \cap G_i$$

where $F_i = \{(r,\sigma) \in \mathbb{R}^2 : P_i(r,\sigma) = 0\}$ and $G_i = \{(r,\sigma) \in \mathbb{R}^2 : Q_{ij}(r,\sigma) > 0$ for $j = 1,\ldots,j(i)\}$, P_i and Q_{ij} being real polynomials on $\mathbb{R}^2$. Some P_i may vanish identically on $\mathbb{R}^2$, and some Q_{ij} may be strictly positive on $\mathbb{R}^2$. However, since the sets G_i are open and the sets

$$E_r := \{\sigma \in \mathbb{R} : (r,\sigma) \in E\} = \{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n,\ |\xi| \leq r\}, \quad r \in [0,\infty[,$$

are compact, it follows that

$$J = \{i = 1,\ldots,k : P_i \text{ is not identically zero}\} \neq \emptyset.$$

Pick any $r \in [0,\infty[$. Since $\Lambda(r) = \max E_r \in E_r$ and all the sets G_i are open, it follows that

$$\begin{aligned}\Lambda(r) &= \max\{\sigma \in \mathbb{R} : (r,\sigma) \in F_i\} \\ &= \max\{\sigma \in \mathbb{R} : P_i(r,\sigma) = 0\} \quad \text{for some } i = i(r) \in J. \end{aligned} \tag{3.7}$$

Consequently, $P_i(r,\Lambda(r)) = 0$ for $i = i(r)$, and if $P(r,\sigma) = \prod_{i\in J} P_i(r,\sigma)$, then P is not identically zero and

$$P(r,\Lambda(r)) = 0 \quad \text{for every } r \in [0,\infty[. \tag{3.8}$$

(B) Since the function $\Lambda(\cdot)$ is continuous on $[0,\infty[$, its boundedness on $[0,\infty[$ follows at once from (3.5) and (3.8) by virtue of the Proposition below. Let $Q[w]$ be the ring of polynomials of one variable w with coefficients in the field Q of rational functions of one complex variable. Any polynomial $P \in Q[w]$ of the form $P(w) = \sum_{k=0}^{n} A_k w^k$ where $A_0, \ldots, A_n \in Q$ and $A_n \neq 0$ may be treated as a complex-valued function $P(z,w) = \sum_{k=0}^{n} A_k(z) w^k$ of two complex variables z and w defined for $(z,w) \in (\mathbb{C} \setminus S) \times \mathbb{C}$ where $S = \{z \in \mathbb{C} : \text{either } A_n(z) = 0 \text{ or } z \text{ is a pole of } A_k \text{ for some } k = 0, \ldots, n\}$.

Proposition. *Let $P \in Q[w]$ and let Λ be a real function defined on $[0,\infty[$ such that*

$$\limsup_{x\to\infty} x^{-\alpha}\Lambda(x) \leq 0 \quad \textit{for every } \alpha > 0. \tag{3.9}$$

Suppose that the set

$$Z = \{x \in [0,\infty[\, : x \notin S,\ P(x, \Lambda(x)) = 0\}$$

is unbounded. Then

$$\limsup_{Z \ni x\to\infty} \Lambda(x) < \infty. \tag{3.10}$$

Proof of the Proposition. P may be represented as a product $P = P_1 \ldots P_s$ of irreducible elements of $Q[w]$. Let $Z_j = \{x \in [0,\infty[\, : x \notin S_j, P_j(x,\Lambda(x)) = 0\}$ for $j = 1, \ldots, s$, and let $J = \{j = 1, \ldots, s : Z_j \text{ is unbounded}\}$. Then $[a,\infty[\, \cap Z \subset \bigcup_{j\in J} Z_j$ for sufficiently large $a \in [0,\infty[$, so that (3.10) will follow once it is shown that $\limsup_{Z_j \ni x\to\infty} \Lambda(x) < \infty$ for every $j \in J$. Hence it is sufficient to prove the Proposition under the additional assumption that $P \in Q[w]$ is irreducible. So, suppose that $P = \sum_{k=0}^{n} A_k w^k \in Q[w]$ is irreducible and $A_n \neq 0$. Then, by Theorem VI.13.7 of [S-Z] there is a finite set $F \subset \mathbb{C} \setminus S$ such that for every $z_0 \in \mathbb{C} \setminus (S \cup F)$ the polynomial $P(z_0, w) \in \mathbb{C}[w]$ of degree n has n distinct simple roots belonging to $\mathbb{C}$. By Theorems VI.14.2 and VI.14.3 of [S-Z] *there is a multivalued analytic function $\mathcal{R}$ defined on $\mathbb{C} \setminus (S \cup F)$ such that for every $z_0 \in \mathbb{C} \setminus (S \cup F)$ the set of values of $\mathcal{R}$ at z_0 coincides with the set of roots of $P(z_0, w)$.* (Notice that in [S-Z] an analytic function is, by definition, holomorphic on a *connected* analytic space.) If $R \in [0,\infty[$ is so large that $S \cup F \subset \{z \in \mathbb{C} : |z| \leq R\}$, then, by Theorem VI.9.3 of [S-Z] there is a function Φ holomorphic in $0 < |z| < R^{-1/n}$ such that

$$\mathcal{R}(z) = \{\Phi(\zeta) : \zeta \in \mathbb{C},\ \zeta^n = z^{-1}\} \quad \text{whenever } R < |z| < \infty. \tag{3.11}$$

Furthermore, an argument presented at the end of the proof of Theorem VI.14.2 of [S-Z], based on the Casorati–Weierstrass theorem, shows that Φ has at $z = 0$ either a removable singularity or a pole. It follows that Φ has in $0 < |z| < R^{-1/n}$ the Laurent expansion $\Phi(z) = \sum_{k=m}^{\infty} a_k z^k$, $m \in \mathbb{Z}$, $a_m \neq 0$, where the series is absolutely convergent, uniformly on $0 < |z| \leq R^{-1/n} - \epsilon$ for every $\epsilon \in]0, R^{-1/n}[$. Consequently, if $x \in]R, \infty[\cap Z$, then by (3.10) and (3.11) one has

$$\Lambda(x) \in \mathcal{R}(x) = \Big\{ \sum_{k=m}^{\infty} a_k (x^{-1/n} z)^k : z \in U \Big\} \tag{3.12}$$

where $x^{-1/n}$ is real and strictly positive, and U is the set of n-th roots of unity. If $m \geq 0$, then (3.10) holds because $\Lambda(\cdot)$ is bounded on $[R+1, \infty[\cap Z$, by (3.12). If $m < 0$, then (3.12) implies that

$$\lim_{Z \ni x \to \infty} \operatorname{dist}(x^{m/n} \Lambda(x), a_m U) = 0. \tag{3.13}$$

From (3.9) and (3.13) it follows that $-|a_m| \in a_m U$ and

$$\lim_{Z \ni x \to \infty} x^{m/n} \Lambda(x) = -|a_m|.$$

Since $-|a_m| < 0$, one concludes that $\lim_{Z \ni x \to \infty} \Lambda(x) = -\infty$, so that (3.10) holds.

Proof of (b) $\Rightarrow$ (a) $\vee$ $(\omega_1 \leq \omega_0)$**.** Suppose that (b) is satisfied. By Lemma 3 for every $\epsilon > 0$ there is $C_\epsilon \in]0, \infty[$ such that if $u \in D(A^\infty)$, $j \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $\widehat{u} = \mathcal{F} u$, then

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} \| A(\xi)^j (\exp(tA(\xi))) \widehat{u}(\xi) \|_{\mathbb{C}^m}^2 \, d\xi \Big)^{1/2} \\
&\qquad = \Big(\int_{\mathbb{R}^n} \Big\| \sum_{k=0}^{2m} a_k(t, \xi) A(\xi)^{k+j} \widehat{u}(\xi) \|_{\mathbb{C}^m}^2 \, d\xi \Big)^{1/2} \\
&\qquad \leq C_\epsilon e^{(\omega_0 + \epsilon)t} \sum_{k=0}^{2m} \Big(\int_{\mathbb{R}^n} \| A(\xi)^{k+j} \widehat{u}(\xi) \|_{\mathbb{C}^m}^2 \, d\xi \Big)^{1/2}.
\end{aligned}$$

By Plancherel's theorem, the last estimate implies that the operators $S_t = \mathcal{F}^{-1} \exp(tA(\cdot)) \mathcal{F}$, $t \in [0, \infty[$, constitute a one-parameter semigroup $(S_t)_{t \geq 0} \subset L(D(A^\infty))$ such that

$$\|S_t u\|_j \leq C_\epsilon e^{(\omega_0+\epsilon)t} \|u\|_{j+2m}$$

for every $j \in \mathbb{N}$, $t \in \mathbb{R}^+$ and $u \in D(A^\infty)$. Consequently, for every $\epsilon > 0$ the semigroup $(e^{-(\omega_0+\epsilon)t} S_t)_{t\geq 0} \subset L(D(A^\infty))$ is equicontinuous, whence $\omega_1 \leq \omega_0$. It remains to prove that $(S_t)_{t\geq 0} \subset L(D(A^\infty))$ is a (C_0)-semigroup whose infinitesimal generator is equal to the operator $A|_{D(A^\infty)} = P(D)|_{D(A^\infty)} \in L(D(A^\infty))$. To this end, it is sufficient to observe that if $u \in D(A^\infty)$, then

$$\begin{aligned}
\|S_t u - S_\tau u\| &= \|\mathcal{F}^{-1}(\exp(tA(\cdot)) - \exp(\tau A(\cdot)))\mathcal{F}u\|_j \\
&\leq |t-\tau| \sup_{(\sigma-\tau)(\sigma-t)\leq 0} \|\mathcal{F}^{-1}A(\cdot)\exp(\sigma A(\cdot))\mathcal{F}u\|_j \\
&= |t-\tau| \sup_{(\sigma-\tau)(\sigma-t)\leq 0} \|S_\sigma A u\|_j \\
&\leq |t-\tau| C_\epsilon \sup_{(\sigma-\tau)(\sigma-t)\leq 0} e^{(\omega_0+\epsilon)\sigma} \|u\|_{j+2m+1}
\end{aligned}$$

for $t, \tau \in [0,\infty[$, and

$$\begin{aligned}
\left\| \frac{1}{t}(S_t u - u) - Au \right\|_j &= \left\| \mathcal{F}^{-1} \frac{1}{t}[\exp(tA(\cdot)) - 1 - tA(\cdot)]\mathcal{F}u \right\|_j \\
&= \left\| \mathcal{F}^{-1} \frac{1}{t} \int_0^t (t-\tau)A(\cdot)^2 \exp(\tau A(\cdot))\, d\tau\, \mathcal{F}u \right\|_j \\
&= \left\| \frac{1}{t} \int_0^t (t-\tau) S_\tau A^2 u\, d\tau \right\|_j \\
&\leq \frac{1}{2}\, t \max_{0\leq\tau\leq t} \|S_\tau u\|_{j+2} \\
&\leq \frac{1}{2}\, t C_\epsilon \max_{0\leq\tau\leq t} e^{(\omega_0+\epsilon)\tau} \|u\|_{j+2m+2}
\end{aligned}$$

for every $t \in]0,\infty[$.

Proof of (a) $\wedge$ $(\omega_1 < \infty) \Rightarrow (\omega_0 \leq \omega_1)$**.** The proof of this implication is similar to that of (a)$\Rightarrow$(b), but does not employ anything similar to the implication (3.5)$\Rightarrow$(3.6). Suppose that (a) holds and $\omega_1 < \infty$. Pick an arbitrary $\omega \in]\omega_1, \infty[$. Then the semigroup $(e^{-\omega t} S_t)_{t\geq 0} \subset L(D(A^\infty))$ is equicontinuous, and hence there are $C \in]0,\infty[$ and $j \in \mathbb{N}$ such that

$$\|S_t u\|_X \leq e^{\omega t} C \|u\|_j \quad \text{for every } t \in \mathbb{R}^+ \text{ and } D(A^\infty).$$

Consequently, by Plancherel's theorem, there are $K \in]0,\infty[$ and $k \in \mathbb{N}$ such that whenever $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^m)$, then

$$\left(\int_{\mathbb{R}^n} \|\exp(tA(\eta))\varphi(\eta)\|^2\,d\eta\right)^{1/2} \le e^{\omega t} C \sum_{0\le i\le j} \left(\int_{\mathbb{R}^n} \|A(\eta)^i \varphi(\eta)\|^2\,d\eta\right)^{1/2}$$
$$\le e^{\omega t} K \left(\int_{\mathbb{R}^n} (1+|\eta|)^{2k} \|\varphi(\eta)\|^2\,d\eta\right)^{1/2}. \quad (3.14)$$

For any $(t,\xi) \in \mathbb{R}^{n+1}$ choose $z(t,\xi) \in \mathbb{C}^m$ such that $\|z(t,\xi)\|_{\mathbb{C}^m} = 1$ and $\|\exp(tA(\xi))z(t,\xi)\|_{\mathbb{C}^m} = \|\exp(tA(\xi))\|_{L(\mathbb{C}^m)}$. Let $(\phi_\nu)_{\nu=1,2,\dots} \subset C_c(\mathbb{R}^n)$ be a sequence of non-negative functions such that $\int_{\mathbb{R}^n} \phi_\nu(\eta)^2\,d\eta = 1$ and ϕ_ν vanishes outside the ball with center at ξ and radius $1/\nu$. Applying (3.14) to $\varphi(\eta) = \phi_\nu(\eta) z(t,\xi)$, one concludes that

$$\begin{aligned}\|\exp(tA(\xi))\|_{L(\mathbb{C}^m)} &= \|\exp(tA(\xi))z(t,\xi)\|_{\mathbb{C}^m} \\ &= \lim_{\nu\to\infty} \left(\int_{\mathbb{R}^n} \|\exp(tA(\eta))\phi_\nu(\eta)z(t,\xi)\|^2\,d\eta\right)^{1/2} \\ &\le \lim_{\nu\to\infty} e^{\omega t} K \left(\int_{\mathbb{R}^n} (1+|\eta|)^{2k} \phi_\nu(\eta)^2\,d\eta\right)^{1/2} = e^{\omega t} K (1+|\xi|)^k\end{aligned}$$

for every $(t,\xi) \in \mathbb{R}^{n+1}$. Hence, by Proposition 2.2, p. 251, and Corollary 2.4, p. 252, in [E-N], for every $\xi \in \mathbb{R}^n$ one has

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi))\} = \lim_{t\to\infty} \frac{1}{t} \log \|\exp(tA(\xi))\|_{L(\mathbb{C}^m)} \le \omega.$$

Since ω is an arbitrary number in $]\omega_1,\infty[$, it follows that $\omega_0 \le \omega_1$. □

4. Remarks on Petrovskiĭ correct systems of partial differential equations with constant coefficients

4.1. The one-parameter group of operators $G_t = \exp(tP(D))$, $-\infty < t < \infty$, in the space Z' dual to $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$

Let $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ be the space of C^∞ maps of $\mathbb{R}^n$ into $\mathbb{C}^m$ with compact support. $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ is endowed with the topology of the inductive limit of the Fréchet spaces $\mathcal{D}_K(\mathbb{R}^n;\mathbb{C}^m) = \{\varphi \in C^\infty(\mathbb{R}^n;\mathbb{C}^m) : \operatorname{supp}\varphi \subset K\}$ for K running through the family of compact subsets of $\mathbb{R}^n$. Let $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ be the space of $\mathbb{C}^m$-valued distributions of L. Schwartz on $\mathbb{R}^n$. $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$

is endowed with the topology of uniform convergence on bounded subsets of $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$. The above topologies on $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ are compatible with the duality determined by the bilinear form $(\varphi, T) \to \sum_{k=1}^m T_k(\varphi_k)$, $\varphi = (\varphi_1, \ldots, \varphi_m) \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$, $T = (T_1, \ldots, T_m) \in \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$. The spaces $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ are barrelled, reflexive with respect to the above duality form, and complete. Furthermore, the space $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ is bornological. See [E], Sec. 5.3; [Y], Sec. I.7-8 and Appendix to Chapter V; [S], Sec. III.2, Theorem VIII. The space $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ of $\mathbb{C}^m$-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ and the space $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ of $\mathbb{C}^m$-valued tempered distributions on $\mathbb{R}^n$ constitute another dual pair with analogous properties. Reflexivity of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, and bornologicity of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ are essential for the proof of the Corollary in Section 4.4.

The inverse Fourier transformation

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \varphi(\xi)\, d\xi, \qquad \varphi \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^m),$$

is an isomorphism of $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$ onto the space $Z(\mathbb{C}^n;\mathbb{C}^m)$ of $\mathbb{C}^m$-valued functions holomorphic on $\mathbb{C}^n$, satisfying suitable growth-decay conditions. The topology of $Z(\mathbb{C}^n;\mathbb{C}^m)$ is transported by $\mathcal{F}^{-1}$ from $\mathcal{D}(\mathbb{R}^n;\mathbb{C}^m)$. See [G-S2], Chap. III. Let Z' be the space dual to $Z(\mathbb{C}^n;\mathbb{C}^m)$ endowed with topology of uniform convergence on bounded subsets of $Z(\mathbb{C}^n;\mathbb{C}^m)$. Similarly to $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$, also Z' is a complete l.c.v.s. The above definifions imply that for every $S \in Z'$ there is a unique $T \in \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ such that

$$S(\mathcal{F}\phi) = (2\pi)^n S(\mathcal{F}^{-1}\phi^\vee) = T(\phi) \quad \text{ for every } \phi \in \mathcal{D}(\mathbb{R}^n;\mathbb{C}^m).$$

In view of the Parseval equality ([Y], p. 148, formula (11)) one can say that S is equal to the Fourier transform of T.

As in Section 1.3, define

$$P(D) = \sum_{|\alpha|\le p} A_\alpha D^\alpha, \qquad D^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

and

$$A(\xi) = \sum_{|\alpha|\le p} \xi^\alpha A_\alpha, \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

The map $\mathbb{R}^n \ni \xi \mapsto A(\xi) \in L(\mathbb{C}^m)$ is infinitely differentiable, and the map

$$\mathbb{R}^{1+n} \ni (t,\xi) \mapsto \phi(t,\xi) = \exp(tA(\xi)) \in L(\mathbb{C}^m) \tag{4.1}$$

satisfies the differential equation $\frac{d}{dt}\phi(t,\xi) = A(\xi)\phi(t,\xi)$. Therefore the theorem on differentiation of a solution of an ordinary differential equation with respect to a parameter ([Ha], Chap. V, Sec. 4, Theorem 4.1) implies that the map (4.1) is infinitely differentiable. Consequently, the formula

$$\widehat{G}_t T = (\exp tA(\cdot))T, \qquad t \in \mathbb{R},\, T \in \mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m),$$

defines a one-parameter (C_0)-group $(\widehat{G}_t)_{t\in\mathbb{R}} \subset L(\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m))$ with infinitely differentiable trajectories. See [S], Chap. III, Theorem XI. Since $\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m)$ is a barrelled space, by the Banach–Steinhaus theorem, the group $(\widehat{G}_t)_{t\in\mathbb{R}} \subset L(\mathcal{D}'(\mathbb{R}^n;\mathbb{C}^m))$ is locally equicontinuous. It follows that the operators

$$G_t = \mathcal{F}^{-1}\widehat{G}_t\mathcal{F}, \qquad t \in \mathbb{R}, \tag{4.2}$$

constitute a one-parameter locally equicontinuous (C_0)-group $(G_t)_{t\in\mathbb{R}} \subset L(Z')$ with infinitely differentiable trajectories. Local equicontinuity implies that the map

$$\mathbb{R} \times Z' \ni (t,U) \mapsto G_t U \in Z' \tag{4.3}$$

is continuous. The infinitesimal generator of the one-parameter group (4.2) is the operator $P(D) = \sum_{|\alpha|\le p} A_\alpha D^\alpha$ defined on the whole Z' and belonging to $L(Z')$.

Let $t_0 \in \,]0,\infty]$ and $u_0 \in Z'$. For I equal to either $[0,t_0[$ or $]-t_0,0]$ the Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t) &= P(D)u(t) \qquad \text{for } t \in I, \\ u(0) &= u_0, \end{aligned} \tag{4.4}$$

has in the class $C^1(I;Z')$ a unique solution $u(\cdot)$, and this unique solution is given by

$$u(t) = G_t u_0 \qquad \text{for } t \in I.$$

We will prove the above for $I = [0,t_0[$, the proof for $I = \,]-t_0,0]$ being similar. Fix any $t \in \,]0,t_0[$ and let $\tau \in [0,t]$. Then

$$\lim_{h\to 0} G_{t-\tau}\frac{1}{h}[G_{-h}u(\tau) - u(\tau)] = -G_{t-\tau}P(D)u(\tau)$$

and, by continuity of the map (4.3),

$$\lim_{[-\tau,t-\tau]\ni h\to 0} G_{t-\tau-h}\frac{1}{h}[u(\tau+h) - u(\tau)] = -G_{t-\tau}P(D)u(\tau),$$

so that

$$\lim_{[-\tau,t-\tau]\ni h\to 0}\frac{1}{h}[G_{t-\tau-h}u(\tau+h)-G_{t-\tau}u(\tau)]=0.$$

This shows that for every $t\in\,]0,t_0[$ the function $[0,t]\ni\tau\mapsto G(t-\tau)u(\tau)\in Z'$ has derivative vanishing everywhere on $[0,t]$ (the derivative at the ends of $[0,t]$ being one-sided). Consequently, $\frac{d}{d\tau}[G_{t-\tau}u(\tau)](\varphi)=0$ for every $\tau\in[0,t]$ and $\varphi\in Z(\mathbb{R}^n;\mathbb{C}^m)$, whence

$$[u(t)-G_tu_0](\varphi)=[G_{t-\tau}u(\tau)](\varphi)|_{\tau=0}^{\tau=t}=0,$$

and so $G_tu_0=u(t)$. Notice that the above argument resembles one used in the proof of E. R. van Kampen's uniqueness theorem for solutions of ordinary differential equations. See [K] and [Ha], Chap. III, Sec. 7.

An important consequence of the uniqueness of solutions of (4.4) is the following. Suppose that E is a function space continuously imbedded in Z' and that the operator $P(D)$ restricted to the domain $\{u\in E: P(D)u\in E\}$ is the infinitesimal generator of a (C_0)-semigroup $(S_t)_{t\geq 0}\subset L(E)$. Then

$$G_tE\subset E\quad\text{and}\quad S_t=G_t|_E\qquad\text{for every } t\in[0,\infty[.$$

We will show that if the Petrovskiĭ correctness condition

$$\sup\{\operatorname{Re}\lambda:\lambda\in\sigma(A(\xi)),\ \xi\in\mathbb{R}^n\}<\infty$$

is satisfied, then there are various function spaces E with the above properties. One of them is $E=D(A^\infty)$ from Theorem 3 in Section 1.4.

4.2. Conditions on $\sigma(A(\xi))$ and $\exp(tA(\xi))$ equivalent to the Petrovskiĭ correctness

For any $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ let $|\xi|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$. For any $\omega\in\mathbb{R}$ consider the conditions:

$\sup\{\operatorname{Re}\lambda:\lambda\in\sigma(A(\xi)),\ \xi\in\mathbb{R}^n\}\leq\omega$ (the Petrovskiĭ correctness); (4.5)

there is $k\in\mathbb{N}$ such that $\sup\{e^{-(\omega+\epsilon)t}(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{L(\mathbb{C}^m)}:$ $0\leq t<\infty,\ \xi\in\mathbb{R}^n\}<\infty$ for every $\epsilon>0$; (4.6)

for every multiindex $\alpha\in\mathbb{N}_0^n$ there is $k_\alpha\in\mathbb{N}$ such that for every $\epsilon>0$, $\sup\{e^{-(\omega+\epsilon)t}(1+|\xi|)^{-k_\alpha}\|(\partial/\partial\xi)^\alpha\exp(tA(\xi))\|_{L(\mathbb{C}^m)}:$ $0\leq t<\infty,\ \xi\in\mathbb{R}^n\}<\infty.$ (4.7)

Then (4.6) implies (4.5) because

$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi))\} = t^{-1}\log \rho(\exp(tA(\xi))) \le t^{-1}\log \|\exp(tA(\xi))\|_{L(\mathbb{C}^m)}$$

where ρ denotes the spectral radius. See [E-N], p. 252. The converse implication is a consequence of the estimate

$$\begin{aligned}\|\exp(tA(\xi))\| &\le e^{\omega t}(1 + 2t\|A(\xi)\| + \cdots + (2t\|A(\xi)\|)^{m-1}) \\ &\le e^{\omega t}(1 + (2t)^2 + \cdots + (2t)^{2(m-1)})^{1/2} \\ &\quad \times (1 + \|A(\xi)\|^2 + \cdots + \|A(\xi)\|^{2(m-1)})^{1/2} \qquad (4.8)\end{aligned}$$

for every $t \in [0,\infty[$ and $\xi \in \mathbb{R}^n$, where ω is defined by (4.5). Inequality (4.8) is stated in [G-S2] in Section 6 of Chapter II, and is also an immediate consequence of Theorem 2 in Section 2 of Chapter 7 of [F]. Obviously (4.7) implies (4.6), and the proof of the converse implication will be given shortly. Therefore for any fixed $\omega \in \mathbb{R}$ the conditions (4.5), (4.6) and (4.7) are equivalent.

I. G. Petrovskiĭ considered in [P] the following conditions which are similar to (4.5)–(4.7), but are not uniform with respect to t on the whole $[0,\infty[$:

$$\sup\{(1+\log(1+|\xi|))^{-1}\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n\} < \infty, \qquad (4.9)$$

for every $T \in \,]0,\infty[$ there is $k \in \mathbb{N}$ such that
$\sup\{(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{L(\mathbb{C}^n)} : 0 \le t \le T,\ \xi \in \mathbb{R}\} < \infty,$ (4.10)

for every multiindex $\alpha \in \mathbb{N}_0^n$ and every $T \in \,]0,\infty[$ there is
$k_{\alpha,T} \in \mathbb{N}$ such that $\sup\{(1+|\xi|)^{-k_{\alpha,T}}\|(\partial/\partial\xi)^\alpha \exp(tA(\xi))\|_{L(\mathbb{C}^m)} :$
$0 \le t \le T,\ \xi \in \mathbb{R}^n\} < \infty.$ (4.11)

Each of the three conditions (4.9)–(4.11) is equivalent to every of the other two, and each is equivalent to the existence of an $\omega \in \mathbb{R}$ for which the conditions (4.5)–(4.7) are satisfied . This follows from the implication (3.5)⇒(3.6) and arguments similar to those proving the mutual equivalence of (4.5), (4.6) and (4.7).

Proof of the implication (4.6)⇒(4.7). For every $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, $\xi \in \mathbb{R}^n$ and $t \in [0,\infty[$ put

$$A_\alpha = \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n} A(\xi),$$
$$U_\alpha(t,\xi) = \left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n} \exp(tA(\xi)).$$

If $\alpha, \beta \in \mathbb{N}_0^n$, then let $\beta \leq \alpha$ mean that $\beta_\nu \leq \alpha_\nu$ for every $\nu = 1, \dots, n$. If $\beta \leq \alpha$, then $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$ where $\binom{\alpha_\nu}{\beta_\nu} = \frac{\alpha_\nu!}{\beta_\nu!(\alpha_\nu - \beta_\nu)!}$. Condition (4.7) means that whenever $\alpha \in \mathbb{N}_0^n$, then

there is $k \in \mathbb{N}$ such that $\sup\{e^{-(\omega+\epsilon)t}(1+|\xi|)^{-k}\|U_\alpha(t,\xi)\| :$
$0 \leq t \leq \infty,\, \xi \in \mathbb{R}^n\} < \infty$ for every $\epsilon > 0$. $\qquad (4.12)_\alpha$

Condition (4.6) is identical with $(4.12)_0$. Hence the implication (4.6)$\Rightarrow$(4.7) will follow once we prove that if $l \in \mathbb{N}_0$ and $(4.12)_\beta$ holds for every $\beta \in \mathbb{N}_0^n$ such that $|\beta| = \beta_1 + \cdots + \beta_n \leq l$, then $(4.12)_\alpha$ holds for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = l+1$. So, pick any α such that $|\alpha| = l+1$. Then

$$\frac{d}{dt}U_\alpha(t,\xi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} A_{\alpha-\beta}(\xi) U_\beta(t,\xi) = A(\xi)U_\alpha(t,\xi) + V_\alpha(t,\xi) \qquad (4.13)$$

where

$$V_\alpha(t,\xi) = \sum_{\beta\leq\alpha,\, |\beta|\leq l} \binom{\alpha}{\beta} A_{\alpha-\beta}(\xi) U_\beta(t,\xi).$$

Since $(4.12)_\beta$ holds whenever $|\beta| \leq l$, it follows that

there is $k \in \mathbb{N}$ such that $\sup\{e^{-(\omega+\epsilon)t}(1+|\xi|)^{-k}\|V_\alpha(t,\xi)\| :$
$0 \leq t \leq \infty,\, \xi \in \mathbb{R}^n\} < \infty$ for every $\epsilon > 0$. $\qquad (4.14)$

By (4.13) one has

$$U_\alpha(t,\xi) = \int_0^t U_0(t-\tau,\xi) V_\alpha(\tau,\xi)\, d\tau, \qquad t \in [0,\infty[,\ \xi \in \mathbb{R}^n. \qquad (4.15)$$

Conditions $(4.12)_0$ and (4.14) imply $(4.12)_\alpha$, by (4.15).

Remark. Notice that the above proof is similar to the proof of Lemma 2 in Sec. 2 of Chap. 1 of [P]. Furthermore, (4.5) implies condition (4.6) with $k = p(m-1)$, and this last implies condition (4.7) with $k_\alpha = p(m-1)(|\alpha|+1)$.

4.3. The space $\mathcal{O}_M$

A continuous function ϕ defined on $\mathbb{R}^n$ is called *slowly increasing* if there is $k \in \mathbb{N}_0$ such that $\sup\{(1+|\xi|)^{-k}|\phi(\xi)| : \xi \in \mathbb{R}^n\} < \infty$. The space $\mathcal{O}_M = \mathcal{O}_M(\mathbb{R}^n; \mathbb{C})$ of $\mathbb{C}$-valued slowly increasing infinitely differentiable functions on $\mathbb{R}^n$ consists of $\mathbb{C}$-valued C^∞-functions ϕ on $\mathbb{R}^n$ such that ϕ and all its partial derivatives are slowly increasing. We will say that a subset B of

O_M is bounded if for every multiindex $\alpha \in \mathbb{N}_0^n$ there is $k_\alpha \in \mathbb{N}_0$ such that $\sup\{(1+|\xi|)^{-k_\alpha}|(\partial/\partial\xi)^\alpha \phi(\xi)| : \phi \in B, \xi \in \mathbb{R}^n\} < \infty$. Lemma 4 will give a topological justification of this terminology. See [S], Chap. VII, Sec. 5, pp. 243–244. Things are similar for $L(\mathbb{C}^m)$-valued functions ϕ. Condition (4.7) may be formulated in the equivalent form

for every $\epsilon \in]0,\infty[$ the set $\{e^{-(\omega+\epsilon)t}\exp(tA(\cdot)) : 0 \le t < \infty\}$ is a bounded subset of $\mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$. $(4.7)_{\mathcal{O}_M}$

The condition (4.11) may also be formulated in terms of $\mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$.

Let $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ be the space of $\mathbb{C}^m$-valued infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$.

Lemma 4. *For every $L(\mathbb{C}^m)$-valued function ϕ defined on $\mathbb{R}^n$ the following two conditions are equivalent:*

$$\phi \in \mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m)), \tag{4.16}$$

ϕ is a multiplier for $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, i.e. $\phi\cdot\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$. (4.17)

Furthermore,

a subset B of $\mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$ is bounded if and only if the family of multiplication operators $\{\phi\cdot : \phi \in B\} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. (4.18)

Remark. From (4.18) and bornologicity of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, by an argument similar to that in the proof of Theorem 3 in Sec. I.7 of [Y], it follows that

a subset B of $\mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$ is bounded if and only if the subset $B\cdot C$ of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is bounded for every bounded subset C of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$.

Proof of Lemma 4. It is obvious that (4.16) implies (4.17) and if $B \subset \mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$ is bounded, then $B\cdot \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. Equivalence of (4.16) to an analogue of (4.17) for the space of tempered distributions is stated without proof on p. 246 of Chapter VII of [S].

Suppose that ϕ is a multiplier for $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$. Then $\phi \in C^\infty(\mathbb{R}^n; L(\mathbb{C}^m))$ and the operator $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m) \ni \varphi \mapsto \phi\cdot\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is closed. Hence, by the closed graph theorem, $\phi\cdot \in L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ From properties of the Fourier transformation it follows that $\mathcal{F}^{-1}(\phi\cdot)\mathcal{F} \in L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ and $\mathcal{F}^{-1}(\phi\cdot)\mathcal{F}$

commutes with translations. Therefore, by a theorem of L. Schwartz, there is a unique distribution $T \in \mathcal{S}'(\mathbb{R}^n; L(\mathbb{C}^m))$ such that $\mathcal{F}^{-1}(\phi \cdot \mathcal{F}\varphi) = T * \varphi$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$, so that, if the $L(\mathbb{C}^m)$-valued function ϕ is treated as a distribution, then $\phi = \mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n; L(\mathbb{C}^m))$. Let J be a set of indices such that

all $\phi_\iota \in C^\infty(\mathbb{R}; L(\mathbb{C}^m))$, $\iota \in J$, *are multipliers for* $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ *and the family of operators* $\{\phi_\iota \cdot : \iota \in J\} \subset L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ *is equicontinuous.* (4.19)

If $\phi_\iota^{(\alpha)} \cdot = (D^\alpha \phi_\iota)\cdot$ are considered as operators defined on $S(\mathbb{R}^n; \mathbb{C}^m)$, then

$$\phi_\iota^{(\alpha)} \cdot = D^\alpha(\phi_\iota \cdot) - \sum_{\beta \le \alpha,\, |\beta| < |\alpha|} \binom{\alpha}{\beta} (\phi_\iota^{(\beta)} \cdot) D^{\alpha - \beta} \quad \text{for every } \alpha \in \mathbb{N}_0^n$$

where $D^\alpha, D^{\alpha-\beta} \in L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$. Consequently, induction on $|\alpha|$ shows that if (4.19) holds, then for every $\alpha \in \mathbb{N}_0^n$ all $\phi_\iota^{(\alpha)}$, $\iota \in J$, are multipliers for $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ and the family of operators $\{\phi_\iota^{(\alpha)} \cdot : \iota \in J\}$ is contained in $L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ and is equicontinuous. This reduces the proofs of the implication (4.17)⇒(4.16) and of (4.18) to showing that if (4.19) holds, then

there is $k \in \mathbb{N}_0$ such that
$\sup\{(1 + |\xi|)^{-k} \|\phi_\iota(\xi)\|_{L(\mathbb{C}^m)} : \iota \in J,\, \xi \in \mathbb{R}^n\} < \infty.$ (4.20)

So, suppose that (4.19) holds. Let $T_\iota \in \mathcal{S}'(\mathbb{R}^n; L(\mathbb{C}^m))$ be the inverse Fourier image of ϕ_ι. Then (4.20) will follow once we prove that there are $f_\iota, g_\iota \in L^1(\mathbb{R}^n; L(\mathbb{C}^m))$ and a polynomial P such that

$$\sup\{\|f_\iota\|_{L^1(\mathbb{R}^n; L(\mathbb{C}^m))}, \|g_\iota\|_{L^1(\mathbb{R}^n; L(\mathbb{C}^m))} : \iota \in J\} < \infty \tag{4.21}$$

and

$$T_\iota = P(D) f_\iota + g_\iota \quad \text{for every } \iota \in J \tag{4.22}$$

where $P(D)$ acts on f_ι in the sense of distributions. Indeed, if (4.21) and (4.22) hold, then $\phi_\iota(\xi) = P(\xi)\widehat{f_\iota}(\varphi) + \widehat{g_\iota}(\xi)$ where $\widehat{f_\iota}$, $\widehat{g_\iota}$ are continuous and bounded on $\mathbb{R}^n$, and $\sup\{\|\widehat{f_\iota}(\xi)\|_{L(\mathbb{C}^m)}, \|\widehat{g_\iota}(\xi)\|_{L(\mathbb{C}^m)} : \iota \in J,\, \xi \in \mathbb{R}^n\} < \infty$, so that (4.20) is satisfied. In this way we are reduced to proving an analogue of Theorem 3.10 of [Ch], p. 82, and Theorem XXV of Sec. VI.8 of [S], p. 201.

We will construct $P(D)$, f_ι and g_ι in the form $P(D) = \Delta^k$, $f_\iota = T_\iota * u$, $g_\iota = T_\iota * \nu$ where $u \in C^l_K(\mathbb{R}^n;\mathbb{C})$, $\nu \in C^\infty_K(\mathbb{R}^n;\mathbb{C})$ are independent of ι, $K = \{x \in \mathbb{R}^n : |x| = (x_1^2 + \cdots + x_n^2)^{1/2} \le 1\}$, $k, l \in \mathbb{N}$, $2k \ge l + n + 2$, and l is sufficiently large. Since $T_\iota * \varphi = (2n)^{-n}\mathcal{F}^{-1}(\phi_\iota \cdot \mathcal{F}\varphi)$ for every $\iota \in J$ and $\varphi \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$, from (4.19) it follows that the family of convolution operators $\{T_\iota * : \iota \in J\} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. Consequently, if the convolution is understood as a bilinear map of $\mathcal{S}'(\mathbb{R}^n;L(\mathbb{C}^m))\times C^\infty_K(\mathbb{R}^n;\mathbb{C})$ into $C^\infty(\mathbb{R}^n;L(\mathbb{C}^m))$, then the range of every operator $T_\iota *|_{C^\infty_K(\mathbb{R}^n;\mathbb{C})}$, $\iota \in J$, is contained in $\mathcal{S}(\mathbb{R}^n;L(\mathbb{C}^m)) \subset L^1(\mathbb{R}^n;L(\mathbb{C}^m))$, and the family of operators

$$\{T_\iota *|_{C^\infty_K(\mathbb{R}^n;\mathbb{C})} : \iota \in J\} \subset L(C^\infty_K(\mathbb{R}^n;\mathbb{C}); L^1(\mathbb{R}^n;L(\mathbb{C}^m)))$$

is equicontinuous. Therefore there are $l \in \mathbb{N}_0$ and $C \in\,]0,\infty[$ such that $\|T_\iota * \varphi\|_{L^1(\mathbb{R}^n;L(\mathbb{C}^m))} \le C\|\varphi\|_{C^l_K(\mathbb{R}^n;\mathbb{C})}$ for every $\iota \in J$ and $\varphi \in C^\infty_K(\mathbb{R}^n;\mathbb{C})$. Since $C^\infty_K(\mathbb{R}^n;\mathbb{C})$ is dense in $C^l_K(\mathbb{R}^n;\mathbb{C})$, it follows that whenever $\iota \in J$ and $\varphi \in C^l_K(\mathbb{R}^n;\mathbb{C})$, then the convolution $T_\iota * \varphi$ of the vector-valued distribution $T_\iota \in \mathcal{S}'(\mathbb{R}^n;L(\mathbb{C}^m))$ with the scalar distribution $\varphi \in C^l_K(\mathbb{R}^n;\mathbb{C})$ is represented by a function belonging to $L^1(\mathbb{R}^n;L(\mathbb{C}^m))$ such that

$$\|T_\iota * \varphi\|_{L^1(\mathbb{R}^n;L(\mathbb{C}^m))} \le C\|\varphi\|_{C^l_K(\mathbb{R}^n;\mathbb{C})} \tag{4.23}$$

for every $\iota \in J$ and $\varphi \in C^l_K(\mathbb{R}^n;\mathbb{C})$.

Now we are ready to write down and explain the formulas for $P(D) = \Delta^k$, $f_\iota = T_\iota * u$ and $g_\iota = T_\iota * \nu$. To this end we will use the radial (i.e. depending only on $|x|$) functions E which are fundamental solutions for Δ^k (i.e. satisfy $\Delta^k E = \delta$, in the sense of distributions). For every $n = 1, 3, 5, \ldots$ and every $k \in \mathbb{N}$ such that $2k \ge n$ there is $A_{n,k} \in\,]0,\infty[$ such that $E(x) = A_{n,k}|x|^{2k-n}$, $x \in \mathbb{R}^n$, is a fundamental solution for Δ^k. For every $n = 2, 4, \ldots$ and $k \in \mathbb{N}$ such that $2k \ge n + 1$ there are $B_{n,k}, C_{n,k} \in\,]0,\infty[$ such that $E(x) = (B_{n,k}\log|x| + C_{n,k})|x|^{2k-n}$, $x \in \mathbb{R}^n$, is a fundamental solution for Δ^k. If $2k \ge l + n + 1$, then $E \in C^l(\mathbb{R}^n)$. See [Ch], Theorem 5.1, p. 99; [G-S1], Chap. III, Example at the end of Sec. 2.1. Fix a function $\gamma \in C^\infty_K(\mathbb{R}^n;\mathbb{C})$ equal to one in some neighbourhood of 0, and fix $k \in \mathbb{N}$ such that $2k \ge l+n+1$ where $l \in \mathbb{N}_0$ is the number occurring in (4.23). For every $\iota \in J$ define

$$f_\iota = T_\iota * \gamma E, \qquad g_\iota = T_\iota * \Delta^k((1-\gamma)E.$$

Then $\gamma E \in C^l_K(\mathbb{R}^n;\mathbb{C})$ and $\Delta^k((1-\gamma)E) \in C^\infty_K(\mathbb{R}^n;\mathbb{C})$, so that, by (4.23), the condition (4.21) is satisfied. Furthermore, $T_\iota = T_\iota * \delta = T_\iota * \Delta^k E = \Delta^k(T_\iota * \gamma E) + T_\iota * \Delta^k((1-\gamma)E) = \Delta^k f_\iota + g_\iota$, so that the condition (4.22) is satisfied for $P(D) = \Delta^k$.

4.4. Operator semigroups generated by $P(D)$ in the L. Schwartz spaces $\mathcal{S}$ and $\mathcal{S}'$

Let A_α, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le p$, be complex $m \times m$ matrices. Consider the matricial differential operator $P(D) = \sum_{|\alpha|\le p} A_\alpha D^\alpha$ and the corresponding $m \times m$ matrices $A(\xi) = \sum_{|\alpha|\le p} \xi^\alpha A_\alpha$, $\xi \in \mathbb{R}^n$.

Theorem 4. *The following two conditions are equivalent:*

(i) $P(D)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$ *is the infinitesimal generator of a* (C_0)*-semigroup* $(U_t)_{t\ge 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$,

(ii) $\omega_0 := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n\}$ *is finite.*

Furthermore, if these equivalent conditions are satisfied, then

$$\omega_0 = \omega_2$$

where

$$\omega_2 := \inf\{\omega \in \mathbb{R} : \textit{the semigroup } (e^{-\omega t}U_t)_{t\ge 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)) \textit{ is equicontinuous}\}.$$

For a single PDE of higher order a result analogous to Theorem 4 may be found in Sec. 3.10 of the book of J. Rauch [R]. Theorem 4 resembles Theorem 3 from Section 1.4. Since $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m) \subset D(A^\infty)$, from remarks at the end of Section 4.1 it follows that $U_t = S_t|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$ for every $t \in [0,\infty[$ where $(S_t)_{t\ge 0} \subset L(D(A^\infty))$ is the semigroup from Theorem 3.

Theorem 4 is a conjunction of three implications: (i)$\Rightarrow$(ii), (ii) $\Rightarrow$ (i)$\wedge(\omega_2 \le \omega_0)$ and (i) $\wedge$ $(\omega_2 < \infty) \Rightarrow (\omega_0 \le \omega_2)$.

Proof of (i)$\Rightarrow$(ii). Suppose that (i) holds. Since $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is invariant with respect to the Fourier transformation, it follows that for every $t \in [0,\infty[$ the multiplication operator $\exp(tA(\cdot))\cdot = \mathcal{F}U_t\mathcal{F}^{-1}$ maps $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ into itself. Hence, by Lemma 4, the function $\xi \mapsto \exp A(\xi)$ belongs to $\mathcal{O}_M(\mathbb{R}^n; L(\mathbb{C}^m))$, so that

$$\sup\{(1+|\xi|)^{-k}\|\exp A(\xi)\|_{L(\mathbb{C}^m)} : \xi \in \mathbb{R}^n\} < \infty \quad \text{for some } k \in \mathbb{N}_0.$$

The last condition implies (ii), by an argument identical with that used in the proof of (a)$\Rightarrow$(b) in Chapter 3.

Proof of (ii) $\Rightarrow$ (i) $\wedge$ $(\omega_2 \le \omega_0)$. Suppose that (ii) holds. Then, by the equivalence (4.5)$\Leftrightarrow$(4.7)$_{\mathcal{O}_M}$ and Lemma 4, for every $\epsilon > 0$ the family of multiplication operators $\{e^{-(\omega_0+\epsilon)t}\exp(tA(\cdot))\cdot : 0 \le t < \infty\} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. By invariance of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ with respect to the Fourier transformation, it follows that the operators $U_t = \mathcal{F}^{-1}[\exp(tA(\cdot))\cdot]\mathcal{F}|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$, $0 \leq t < \infty$, constitute a semigroup $(U_t)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ such that the semigroup $(e^{-(\omega_0+\epsilon)t}U_t)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous. Consequently, $\omega_2 \leq \omega_0$. Finally, estimations similar to those in the proof (b) $\Rightarrow$ (a) $\wedge$ $(\omega_1 \leq \omega_0)$ in Chapter 3 show that $(U_t)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ is a (C_0)-semigroup with the infinitesimal generator equal to $P(D)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$.

Remark. In contrast to the proof of (b) $\Rightarrow$ (a) $\wedge$ $(\omega_1 \leq \omega_0)$ in Chapter 3, the above proof of (ii) $\Rightarrow$ (i) $\wedge$ $(\omega_2 \leq \omega_0)$ is independent of Chapter 2. The role analogous to that of Lemma 3 from Chapter 2 is now played by the estimate (4.8).

Proof of (i) $\wedge$ $(\omega_2 < \infty) \Rightarrow (\omega_0 \leq \omega_2)$. Suppose that (i) holds and ω_2 is finite. Then for every $\epsilon > 0$ the family of multiplication operators

$$\{e^{-(\omega_2+\epsilon)t}\exp(tA(\cdot))\cdot|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)} : 0 \leq t < \infty\}$$
$$= \{\mathcal{F}e^{-(\omega_2+\epsilon)t}U_t\mathcal{F}^{-1}|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)} : 0 \leq t < \infty\} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$$

is equicontinuous. Hence, by Lemma 4, the condition $(4.7)_{\mathcal{O}_M}$ is satisfied for $\omega = \omega_2$. It follows that also for $\omega = \omega_2$ the equivalent condition (4.5) is satisfied. This last means that $\omega_0 \leq \omega_2$.

Corollary. *Let $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ be endowed with the topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$. The matricial differential operator $P(D)$ is Petrovskiĭ correct if and only if $P(D)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)}$ is the infinitesimal generator of a (C_0)-semigroup $(V_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$. Furthermore, $\omega_0 = \omega_3$ where $\omega_0 = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n\}$ and $\omega_3 = \inf\{\omega \in \mathbb{R}$: the semigroup $(e^{-\omega t}V_t)_{t\geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous$\}$.*

Sketch of the proof. Let $Q(D) = \sum_{|\alpha|\leq p} B_\alpha D^\alpha$ where $B_\alpha = (-1)^{|\alpha|}A_\alpha^\dagger$, the superscript $\dagger$ denoting transposition. Then $B(\xi) = \sum_{|\alpha|\leq p}\xi^\alpha B_\alpha = A(-\xi)^\dagger$ for every $\xi \in \mathbb{R}^n$. Consequently, the operator $P(D)$ is Petrovskiĭ correct if and only if the same is true for $Q(D)$, and hence if and only if the operator $Q(D)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)}$ is the infinitesimal generator of a (C_0)-semigroup $(W_t)_{t\geq 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ with properties as in Theorem 4. The spaces $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ are reflexive with respect to the duality form

$$\langle\varphi, T\rangle = \sum_{\mu=1}^{m} T_\mu(\varphi_\mu), \quad \varphi = (\varphi_1, \ldots, \varphi_m) \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m),$$
$$T = (T_1, \ldots, T_m) \in \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m).$$

Moreover, $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is bornological. Therefore the proof of the Corollary may be based on the equality $\langle V_t T, \varphi\rangle = \langle T, W_t\varphi\rangle$.

4.5. Examples of function spaces E invariant with respect to the semigroup $(V_t)_{t\geq 0}$

In the whole present subsection we assume that the $m \times m$ matricial differential operator $P(D) = \sum_{|\alpha|\leq p} A_\alpha D^\alpha$ described in Section 1.4 satisfies the Petrovskiĭ correctness condition

$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n\} = \omega_0 < \infty$$

where $A(\xi) = \sum_{|\alpha|\leq p} \xi^\alpha A_\alpha$. Under this assumption there are remarkable function spaces E densely continuously imbedded in $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ such that

$V_t E \subset E$ for every $t \in [0,\infty[$ and the operators $S_t = V_t|_E$ constitute a (C_0)-semigroup $(S_t)_{t\geq 0} \subset L(E)$ with the infinitesimal generator G defined by the conditions $D(G) = \{u \in E : P(D)u \in E\}$, $Gu = P(D)u$ for $u \in D(G)$. (4.24)

We already know two examples of such function spaces E:

Example 1. $E = D(A^\infty)$ from Theorem 3 of Section 1.4, where

$$D(A) = \{u \in L^2(\mathbb{R}^n;\mathbb{C}^m) : P(D)u \in L^2(\mathbb{R}^n;\mathbb{C}^m)\}$$

and

$$Au = P(D)u \quad \text{for } u \in D(A).$$

Example 2. $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ considered in Theorem 4 of Section 4.4.

In the first example the definition of $E = D(A^\infty)$ involves a possibly limited number of derivatives. In the second example $E = \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ is a standard function space independent of $P(D)$. Let us mention further examples.

Example 3. $E = C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$.The spaces $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ and $C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$ are both endowed with the topology determined by the sequence of norms

$$\|u\|_j = \sup\{\|D^\alpha u(x)\|_{\mathbb{C}^m} : \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq j,\ x \in \mathbb{R}^n\}, \qquad j \in \mathbb{N}_0.$$

Properties (4.24) of $E = C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ follow from some variants of estimates going back to I. G. Petrovskiĭ [P]. In contrast to original estimates, these variants are uniform with respect to t on the whole $[0,\infty[$. Let us present the modified estimates. One has

$$\min(1,a^{-k}) \le 2^k(1+a)^{-k} \quad \text{for every } a \in\,]0,\infty[\text{ and } k \in \mathbb{N}. \tag{4.25}$$

Let $(U_t)_{t\ge 0} \subset L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ be the (C_0)-semigroup from Theorem 4. If $u \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^m)$, $x \in \mathbb{R}^n$ and $x_\nu \neq 0$ for $\nu = 1,\dots,n$, then

$$\begin{aligned}(U_t u)(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\exp(tA(\xi))\widehat{u}(\xi)\,d\xi \\ &= (-2\pi)^{-n}(x_1\cdots x_n)^{-2}\int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\left(\frac{\partial^n}{\partial\xi_1\cdots\partial\xi_n}\right)^2(\exp(tA(\xi))\widehat{u}(\xi))\,d\xi,\end{aligned}$$

so that, from (4.25) and (4.7) with $k_\alpha = p(m-1)(|\alpha|+1)$, it follows that for every $\epsilon > 0$ there is $K_\epsilon \in\,]0,\infty[$ such that

$$\begin{aligned}\|U_t u(x)\|_{\mathbb{C}^m} \le K_\epsilon e^{(\omega_0+\epsilon)t}\prod_{\nu=1}^n(1+|x_\nu|)^{-2} \\ \times\int_{\mathbb{R}^n}\prod_{\nu=1}^n(1+|\xi_\nu|)^{p(m-1)(2n+1)}\sup_{|\alpha|\le 2n}\left\|\left(\frac{\partial}{\partial\xi}\right)^\alpha\widehat{u}(\xi)\right\|_{\mathbb{C}^m}d\xi\end{aligned} \tag{4.26}$$

for every $z \in \mathbb{R}^n$ and $t \in [0,\infty[$.

If $u \in C^\infty_{[-1/2,1/2]^n}(\mathbb{R}^n;\mathbb{C}^m)$, then

$$\begin{aligned}\left(\frac{\partial}{\partial\xi}\right)^\alpha\widehat{u}(\xi) &= (-i)^{|\alpha|}\int e^{-i\langle x,\xi\rangle}x^\alpha u(x)\,dx \\ &= (-i)^{|\alpha|}(\xi^\beta)^{-1}\int e^{-i\langle x,\xi\rangle}D^\beta(x^\alpha u(x))\,dx\end{aligned}$$

for every $\alpha,\beta \in \mathbb{N}_0^n$ and $\xi \in \mathbb{R}^n$ such that $\xi_\nu \neq 0$ for $\nu = 1,\dots,n$. Consequently, from (4.25) it follows that for every $l \in \mathbb{N}$ there is $C_l \in\,]0,\infty[$ such that

$$\begin{aligned}&\left\|\left(\frac{\partial}{\partial\xi}\right)^\alpha\widehat{u}(\xi)\right\|_{\mathbb{C}^m} \\ &\le C_l\prod_{\nu=1}^n(1+|\xi_\nu|)^{-l}\sup\{\|D^\beta u(x)\|_{\mathbb{C}^m} : |\beta| \le ln,\ x \in [-1/2,1/2]^n\}\end{aligned} \tag{4.27}$$

for every $u \in C^\infty_{[-1/2,1/2]^n}(\mathbb{R}^n;\mathbb{C}^m)$, $\xi \in \mathbb{R}^n$, and $\alpha \in \mathbb{N}_0^n$. The estimates (4.26) and (4.27) imply that

for every $\epsilon > 0$ there is $M_\epsilon \in [0,\infty[$ such that whenever $u \in C^\infty_{[-1/2,1/2]^n}(\mathbb{R}^n;\mathbb{C}^m)$, $x \in \mathbb{R}^n$ and $t \in [0,\infty[$, then

$$\|(U_t u)(x)\|_{\mathbb{C}^m} \le M_\epsilon \|u\|_k e^{(\omega_0+\epsilon)t} \prod_{\nu=1}^n (1+|x_\nu|)^{-2}$$

where $k = n(p(m-1)(2n+1)+2)$. (4.28)

Let $\mathbb{Z}$ be the set of integers and for any $z = (z_1,\ldots,z_n) \in \mathbb{Z}^n$ denote by τ_z the operator of translation by $\frac{1}{2}z$: $(\tau_z f)(x) = f(x_1 + \frac{1}{2}z_1,\ldots,x_n + \frac{1}{2}z_n)$ for every function f defined on $\mathbb{R}^n$ and every $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Following [P], fix a function $\nu \in C_{[-1/2,1/2]^n}(\mathbb{R}^n)$ with values in $[0,1]$ such that $\sum_{z\in\mathbb{Z}^n} \tau_z \nu \equiv 1$ on $\mathbb{R}^n$. Since the operators U_t, τ_z and D^α commute, one has $D^\alpha U_t(u\tau_z\nu) = \tau_z U_t(D^\alpha(\nu\tau_{-z}u))$. Therefore from (4.28) it follows that whenever $u \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$, $\alpha \in \mathbb{N}_0^n$, $z \in \mathbb{Z}^n$, $t \in [0,\infty[$, $x \in \mathbb{R}^n$ and $\epsilon > 0$, then

$$\|(D^\alpha U_t(u\tau_z\nu))(x)\|_{\mathbb{C}^m} \le M_{|\alpha|,\epsilon} \|u\|_{k+|\alpha|} e^{(\omega_0+\epsilon)t} \prod_{\nu=1}^n (1+|x_\nu + \tfrac{1}{2}z_\nu|)^{-2} \quad (4.29)$$

where $k = n(p(m-1)(2n+1)+2)$ and $M_{|\alpha|,\epsilon}$ depends only on $|\alpha|$ and ϵ.

Again following [P], consider the series

$$\sum_{z\in\mathbb{Z}^n} \prod_{\nu=1}^n (1+|x_\nu + \tfrac{1}{2}z_\nu|)^{-2}. \quad (4.30)$$

The terms of this series are functions of x continuous on $\mathbb{R}^n$, the series is uniformly convergent on every bounded subset of $\mathbb{R}^n$, and its sum $s(x)$ is periodic ($s(x+\frac{1}{2}z) = s(x)$ for every $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$). Therefore $s \in C_b(\mathbb{R}^n)$. In particular, $K = \sup_{x\in\mathbb{R}^n} s(x)$ is finite. From (4.29), the theorem on term by term differentiation and properties of the series (4.30) it follows that whenever $u \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$, then $\sum_{z\in\mathbb{Z}^n} U_t(u\tau_z\nu) \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$, and for every $\epsilon > 0$ and $j \in \mathbb{N}_0$ one has

$$\Big\| \sum_{z\in\mathbb{Z}^n} U_t(u\tau_z\nu) \Big\|_j \le K M_{j,\epsilon} e^{(\omega_0+\epsilon)t} \|u\|_{j+k} \quad (4.31)$$

where again $k = n(p(m-1)(2n+1)+2)$. Furthermore, if $u \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$, then $u = \sum_{z\in\mathbb{Z}^n} u\tau_z\nu$ in the sense of the topology of $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, so that

$$\sum_{z\in\mathbb{Z}^n} U_t(u\tau_z\nu) = \sum_{z\in\mathbb{Z}^n} V_t(u\tau_z\nu) = V_t \sum_{z\in\mathbb{Z}^n} u\tau_z\nu = V_t u \tag{4.32}$$

for every $t \in [0,\infty[$. From (4.31) and (4.32) it follows that *the formula*

$$S_t u := \sum_{z\in\mathbb{Z}^n} U_t(u\tau_z\nu) = V_t(u), \quad t \in [0,\infty[,\, u \in C_b^\infty(\mathbb{R}^n;\mathbb{C}^m), \tag{4.33}$$

defines a semigroup $(S_t)_{t\geq 0} \subset L(C_b^\infty(\mathbb{R}^n;\mathbb{C}^m))$ *such that for every* $\epsilon > 0$ *the semigroup* $(e^{(\omega_0+\epsilon)t}S_t)_{t\geq 0} \subset L(C_b^\infty(\mathbb{R}^n;\mathbb{C}^m))$ *is equicontinuous. Moreover,* $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ *is a closed subspace of* $C_b^\infty(\mathbb{R}^n;\mathbb{C}^m)$ *invariant with respect to the semigroup* (4.33). This last follows from the observation that $C_c^\infty(\mathbb{R}^n;\mathbb{C}^m)$ is dense in $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$, and if $u \in C_c^\infty(\mathbb{R}^n;\mathbb{C}^m)$ then only finitely many functions $u\tau_z\nu, z \in \mathbb{Z}^n$, are different from zero, so that

$$S_t u = \sum_{z\in\mathbb{Z}^n} U_t(u\tau_z\nu) = U_t\Big(\sum_{z\in\mathbb{Z}^n} u\tau_z\nu\Big) = U_t u \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m) \subset C_0^\infty(\mathbb{R}^n;\mathbb{C}^m).$$

It remains to prove that $(S_t|_{C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)})_{t\leq 0} \subset L(C_0^\infty(\mathbb{R}^n;\mathbb{C}^m))$ is a (C_0)-semigroup and that its infinitesimal generator is equal to $P(D)|_{C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)}$. To this end, pick any $u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$. Then there is a sequence $(u_k)_{k=1,2,\ldots} \subset C_c^\infty(\mathbb{R}^n;\mathbb{C}^m)$ such that $\lim_{k\to\infty} u_k = u_0$ and hence also $\lim_{k\to\infty} P(D)u_k = P(D)u_0$, both in the sense of the topology of $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$. Since the semigroup $(e^{-(\omega_0+1)t}S_t)_{t\geq 0} \subset L(C_b^\infty(\mathbb{R}^n;\mathbb{C}^m))$ is equicontinuous, it follows that

1° $\lim_{k\to\infty} S_t u_k = S_t u_0$ and $\lim_{k\to\infty} S_t P(D)u_k = S_t P(D)u_0$ in the sense of the topology of $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$, uniformly with respect to t on every bounded interval $[0,T]$.

Furthermore,

2° $\frac{d}{dt}S_t u_k = \frac{d}{dt}U_t u_k = U_t P(D)u_k = S_t P(D)u_k$ for every $t \in [0,\infty[$ and $k = 1,2,\ldots$, the derivative being computed in the sense of the topology of $C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$.

By the theorem on term by term differentiation, from 1° and 2° it follows that the maps $[0,\infty[\ni t \mapsto S_t u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ and $[0,\infty[\ni t \mapsto S_t P(D)u_0 \in C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$ are continuous, and $\frac{d}{dt}S_t u_0 = S_t P(D)u_0$ for every $[0,\infty[$, the derivative being computed in the sense of the topology of

$C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)$. Consequently, u_0 belongs to the domain $D(G)$ of the infinitesimal generator G of the semigroup $(S_t|_{C_0^\infty(\mathbb{R}^n;\mathbb{C}^m)})_{t\geq 0} \subset L(C_0^\infty(\mathbb{R}^n;\mathbb{C}^m))$, and $Gu_0 = \frac{d}{dt}|_{t=0} S_t u_0 = S_0 P(D) u_0 = P(D) u_0$.

Example 4. $E = \mathcal{B}_{\mathcal{N},2}$ where the Hilbert space $\mathcal{B}_{\mathcal{N},2}$ of G. Birkhoff is equal to the completion of the prehilbert space $(Z(\mathbb{R}^n;\mathbb{C}^m), \|\ \|_{\mathcal{N}})$. The norm $\|\ \|_{\mathcal{N}}$ is defined on $Z(\mathbb{R}^n;\mathbb{C}^m)$ as follows:

$$\|u\|_{\mathcal{N}} = \left(\int_{\operatorname{supp} \widehat{u}} \|\mathcal{N}(\xi)\widehat{u}(\xi)\|^2 \, d\xi \right)^{1/2}, \qquad u \in Z(\mathbb{R}^n;\mathbb{C}^m),$$

where $\mathbb{R}^n \ni \xi \mapsto \mathcal{N}(\xi) \in L(\mathbb{C}^m)$ is a Lebesgue measurable map such that for every $\xi \in \mathbb{R}^n$ the matrix $\mathcal{N}(\xi)$ has two properties:

(I) $\mathcal{N}(\xi)$ is invertible and $\|\mathcal{N}(\xi)^{-1}\|_{L(\mathbb{C}^m)} \leq 1$,
(II) $\mathcal{N}(\xi)A(\xi)\mathcal{N}(\xi)^{-1}$ is a superdiagonal Jordan matrix.

The existence of such a measurable reduction of $A(\xi)$ to the canonical Jordan form was proved by K. Baker in [Ba]. A matrix-valued function $\mathcal{N}$ is not unique: for instance $\mathcal{N}$ may be replaced by $f\mathcal{N}$ where $f \geq 1$ is any real Lebesgue measurable function on $\mathbb{R}^n$. Thanks to condition (I) for every $u \in Z(\mathbb{R}^n;\mathbb{C}^m)$ and $\xi \in \mathbb{R}^n$ one has

$$\begin{aligned}\|\widehat{u}(\xi)\|_{\mathbb{C}^m} &= \|\mathcal{N}(\xi)^{-1}\mathcal{N}(\xi)\widehat{u}(\xi)\|_{\mathbb{C}^m} \leq \|\mathcal{N}(\xi)^{-1}\|_{L(\mathbb{C}^m)}\|\mathcal{N}(\xi)\widehat{u}(\xi)\|_{\mathbb{C}^m} \\ &\leq \|\mathcal{N}(\xi)\widehat{u}(\xi)\|_{\mathbb{C}^m},\end{aligned}$$

whence $\mathcal{F}^{-1}\mathcal{B}_{\mathcal{N},2} \subset L^2(\mathbb{R}^n;\mathbb{C}^m)$, and so $\mathcal{B}_{\mathcal{N},2} \subset L^2(\mathbb{R}^n;\mathbb{C}^m) \subset \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$. The results of G. Birkhoff's paper [B] show that $E = \mathcal{B}_{\mathcal{N},2}$ satisfies (4.24), and for the semigroup $(S_t)_{t\geq 0} = (V_t|_{\mathcal{B}_{\mathcal{N},2}})_{t\geq 0} \subset L(\mathcal{B}_{\mathcal{N},2})$ one has

$$\inf_{t>0} \frac{1}{t} \log \|S_t\|_{L(\mathcal{B}_{\mathcal{N},2})} = \lim_{t\to\infty} \frac{1}{t} \log \|S_t\|_{L(\mathcal{B}_{\mathcal{N},2})} = \omega_0.$$

Example 5. $E = \mathcal{L}_B$ where $\mathcal{L}_B$ is the Hilbert space of C^m-valued functions on $\mathbb{R}^n$ with "differentiable norm" of S. D. Eidelman and S. G. Krein. Construction of the scalar product in $\mathcal{L}_B$ is presented in Section 8 of Chapter I of S. G. Krein's monograph [Kr].

In Examples 1, 2 and 4, $\omega_0 = \omega_E := \inf\{\omega \in \mathbb{R} : \text{the semigroup} (e^{-\omega t}V_t|_E)_{t\geq 0} \subset L(E) \text{ is equicontinuous}\}$. In Example 3, $\omega_0 \geq \omega_E$. In Example 5 no relation between ω_0 and ω_E is proved.

References

[Ba] K. Baker, Borel functions for transformation group orbits, J. Math. Anal. Appl. 11 (1965) 217–225.

[B] G. Birkhoff, Well-set Cauchy problems and C_0 -semigroups, J. Math. Anal. Appl. 8 (1964) 303–324.

[C] J. Chazarain, Problèmes de Cauchy abstraits et applications à quelques problèmes mixtes, J. Funct. Anal. 7 (1971) 386–446.

[C-P] J. Chazarain and A. Piriou, Introduction to the Theory of Linear Partial Differential Equations, North-Holland, 1982.

[Ch] C. Chevalley, Theory of Distributions, Lectures given at Columbia University, 1950–1951. Notes prepared by K. Nomizu (mimeographed).

[C-Z] I. Cioranescu and L. Zsidó, ω-Ultradistributions and their application to operator theory, in: Spectral Theory, Banach Center Publ. 8, PWN, Warszawa, 1982, 77–220.

[Co1] P. J. Cohen, A simple proof of Tarski's theorem on elementary algebra, mimeographed manuscript, Stanford University, 1967.

[Co2] P. J. Cohen, Decision procedures for real and p-adic fields, Comm. Pure Appl. Math. 22 (1969) 131–151.

[D] J. Dieudonné, Foundations of Modern Analysis, Academic Press, 1960.

[D-S] N. Dunford and J. T. Schwartz, Linear Operators. Part I: General Theory, Interscience Publishers, 1958.

[E] R. E. Edwards, Functional Analysis, Theory and Applications, Holt, Reinehart and Winston, 1965.

[E-N] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 2000.

[F] A. Friedman, Generalized Functions and Partial Differential Equations, Prentice-Hall, 1963.

[Fu] D. Fujiwara, A chacterization of exponential distribution semi-groups, J. Math. Soc. Japan 18 (1966) 267–274.

[G] L. Gårding, Linear hyperbolic partial differential equations with constant coefficients, Acta Math. 85 (1951) 1–62.

[G-L] I. M. Glazman and Yu. I. Lyubich, Finite-Dimensional Linear Analysis: A Systematic Presentation in Problem Form, The MIT Press, 1974 (translation); Russian original: Finite-Dimensional Linear Analysis in Problems, "Nauka", Moscow, 1969.

[G-S1] I. M. Gelfand and G. E. Shilov, Generalized Functions, Vol. 1, Fizmatgiz, Moscow, 1958 (in Russian); English transl.: Academic Press, 1964.

[G-S2] I. M. Gelfand and G. E. Shilov, Generalized Functions, Vol. 2, Spaces of Fundamental Functions and Generalized Functions, Fizmatgiz, Moscow, 1958 (in Russian); English transl.: Academic Press, 1968.

[G1] E. A. Gorin, On quadratic summability of solutions of partial differential equations with constant coefficients, Sibirsk. Mat. Zh. 2 (1961) 221–232 (in Russian).

[G2] E. A. Gorin, Asymptotic properties of polynomials and algebraic functions of several variables, Uspekhi Mat. Nauk 16 (1) (1961) 91–118 (in Russian).

[Ha] P. Hartman, Ordinary Differential Equations, Wiley, 1964.

[H] E. Hille, Une généralisation du problème de Cauchy, Ann. Inst. Fourier (Grenoble) 4 (1952) 31–48.

[H-P] E. Hille and R. S. Phillips, Functional Analysis and Semi-groups, Amer. Math. Soc., 1957.

[Hig] N. J. Higham, Functions of Matrices, Chapter 11 in: Handbook of Linear Algebra, L. Hogben, R. A. Brualdi, A. Greenbaum and R. Mathias (eds.), Chapman and Hall/CRC, 2006.

[H1] L. Hörmander, On the theory of general partial differential operators, Acta Math. 94 (1955) 161–248.

[H2] L. Hörmander, Linear Partial Differential Operators, Springer, 1963.

[H3] L. Hörmander, The Analysis of Linear Partial Differential Operators II. Operators with Constant Coefficients, Springer, 1983.

[K] E. R. van Kampen, Remarks on systems of ordinary differential equations. Amer. J. Math. 59 (1937) 144–152.

[K1] J. Kisyński, Distribution semigroups and one parameter semigroups, Bull. Polish Acad. Sci. Math. 50 (2002) 189–216.

[K2] J. Kisyński, On Fourier transforms of distribution semigroups, J. Funct. Anal. 242 (2007) 400–441.

[Kr] S. G. Krein, Linear Differential Equations in Banach Space, Transl. Math. Monogr. 29, Amer. Math. Soc., 1971 (Russian original, Moscow, 1967).

[L] J.-L. Lions, Les semi groupes distributions, Portugal. Math. 19 (1960) 141–164.

[Lyu] Yu. I. Lyubich, The classical and local Laplace transformation in an abstract Cauchy problem, Uspekhi Mat. Nauk 21 (3) (1966) 3–51 (in Russian); English transl.: Russian Math. Surveys 21 (3) (1966) 1–52.

[Pa] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, 2nd printing, Springer, 1983.

[P] I. G. Petrovskiĭ, Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nichtanalytischen Funktionen, Bulletin de l'Université d'État de Moscou 1 (7) (1938) 1–74.

[R] J. Rauch, Partial Differential Equations, Springer, 1991.

[S] L. Schwartz, Théorie des Distributions, nouvelle éd., Hermann, Paris, 1966.

[Se] A. Seidenberg, A new decision method for elementary algebra, Ann. of Math. 60 (1954) 365–374.

[S-Z] S. Saks and A. Zygmund, Analytic Functions, 3rd ed., PWN, Warszawa, 1959 (in Polish); English transl.: PWN, 1965; French transl.: Masson, 1970.

[Tr] F. Trèves, Lectures on Partial Differential Equations with Constant Coefficients, Notas de Mat. 7, Rio de Janeiro, 1961.

[U] T. Ushijima, On the generation and smoothness of semi-groups of linear operators, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 19 (1972) 65–127; Correction, ibid. 20 (1973) 187–189.

[W] V. Wrobel, Spectral properties of operators generating Fréchet–Montel spaces, Math. Nachr. 129 (1986) 9–20.

[Y] K. Yosida, Functional Analysis, 6th ed., Springer, 1980.