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**Finite Closed Coverings
of Compact Quantum Spaces**

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FINITE CLOSED COVERINGS OF COMPACT QUANTUM SPACES

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ABSTRACT. We show that a projective space $\mathbb{P}^\infty(\mathbb{Z}/2)$ endowed with the Alexandrov topology is a classifying space for finite closed coverings of compact quantum spaces in the sense that any such a covering is functorially equivalent to a sheaf over this projective space. In technical terms, we prove that the category of finitely supported flabby sheaves of algebras is equivalent to the category of algebras with a finite set of ideals that intersect to zero and generate a distributive lattice. In particular, the Gelfand transform allows us to view finite closed coverings of compact Hausdorff spaces as flabby sheaves of commutative C^* -algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$. As a noncommutative example, we construct from Toeplitz cubes a quantum projective space whose defining covering lattice is free.

Dedicated to Henri Moscovici on the occasion of his 65th birthday.

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INTRODUCTION

Motivation. In the day-to-day practice of the mathematical art, one can see a recurrent theme of reducing a complicated mathematical construct into its simpler constituents, and then putting these constituents together using gluing data that prescribes how these pieces fit together consistently. The (now) classical manifestation of such gluing arguments in various flavours of geometry is the concept of a sheaf on a topological space, or more generally on a topos. Another manifestation of such gluing arguments appeared in noncommutative geometry as the description of a noncommutative space via a finite closed covering. Here a covering is defined as a distinguished finite set of ideals that intersect to zero and generate a distributive lattice [19]. This can be considered as a noncommutative analogue of [45, Prop. 1.10].

Manifolds without boundary fit particularly well this piecewise approach because they are defined as spaces that are locally diffeomorphic to \mathbb{R}^n . Thus a manifold appears assembled from standard pieces by the gluing data. The standard pieces are contractible — they are homeomorphic to a ball. They encode only the dimension of a manifold. All the rest, topological properties of the manifold included, are described by the gluing data. The aim of this article is to explore the method of constructing noncommutative deformations of manifolds by deforming the standard pieces. This method is an alternative to the global deformation methods. Thus it is expected to yield new examples or provide a new perspective on already known cases.

Main results. Following [19], we express the gluing data of a compact Hausdorff space as a sheaf of algebras over a certain universal topological space, and extend it to the noncommutative setting. This universal topological space is explicitly constructed as the infinite $\mathbb{Z}/2$ -projective space $\mathbb{P}^\infty(\mathbb{Z}/2)$ endowed with the Alexandrov topology. The advantages of this new theorem over its predecessor [19, Cor. 4.3] are twofold. First, it considers coverings rather than topologically unnatural ordered coverings. To this end, we need to construct more refined morphisms between sheaves than natural transformations. Next, as $\mathbb{P}^\infty(\mathbb{Z}/2) := \operatorname{colimit}_{N \geq 0} \mathbb{P}^N(\mathbb{Z}/2)$, it takes care of all finite coverings at once.

Theorem 2.13. *The category of finite coverings of algebras is equivalent to the category of finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$ whose morphisms are obtained by taking a certain quotient of the usual class of morphisms enlarged by the actions of a specific family of endofunctors.*

Our second main result concerns a new noncommutative deformation of complex projective space and the lattice generated its covering. The guiding principle of our deformation is to preserve the gluing data of this manifold while deforming the standard pieces. We refine the affine covering of a complex projective space to the Cartesian powers of unit discs, and replace the algebra of continuous functions on the disc by the Toeplitz algebra commonly regarded as the algebra of a quantum disc [24]. The main point here is that we preserve the freeness property enjoyed by the lattice generated by the affine covering of a complex projective space:

Theorem 3.7. *Let $C(\mathbb{P}^N(\mathcal{T})) \subset \prod_{i=0}^N \mathcal{T}^{\otimes N}$ be the C^* -algebra of the Toeplitz quantum projective space, and let $\pi_i: C(\mathbb{P}^N(\mathcal{T})) \rightarrow \mathcal{T}^{\otimes N}$, $i \in \{0, \dots, N\}$, be the family of restrictions of the*

canonical projections onto the components. Then the family of ideals $\{\ker \pi_i \mid 0 \leq i \leq N\}$ generates a free distributive lattice.

Sheaves, patterns, and P -diagrams. The idea of using lattices to study closed coverings of noncommutative spaces has already been widely studied (see [26]). The coverings are closed to afford the C^* -algebraic description. Therefore, a natural framework for coverings uses sheaf-like objects defined on the lattice of closed subsets of a topological space, or more generally, topoi modelled upon finite closed coverings of topological spaces. Interestingly, the original definition of sheaves by Leray was given in terms of the lattice of closed subspaces of a topological space [27, p. 303]. This definition changed in the subsequent years into the nowadays standard open-set formulation for various reasons.

Recently, however, a closed-set approach re-appeared in the form sheaf-like objects called *patterns* [30]. We show in Proposition 1.19 that for our combinatorial models based on finite Alexandrov spaces, the distinction between sheaves and patterns is immaterial. Another reformulation of sheaves over Alexandrov spaces is given by the concept of a P -*diagram*. It is widely known among commutative algebraists (e.g., see [10, Prop. 6.6] and [45, p. 174]) that any sheaf on an Alexandrov space P can be recovered from its P -diagram (cf. Theorem 1.21). See also [17] for a different approach.

Noncommutative projective spaces as homogeneous spaces over quantum groups. Complex projective spaces are fundamental examples of compact manifolds without boundary. They can be viewed as the quotient spaces of odd-dimensional spheres divided by an action of the group $U(1)$ of unitary complex numbers. This presentation allows for a noncommutative deformation coming from the world of compact quantum groups via Soibelman-Vaksman spheres. This construction has been widely studied, and recently entered the very heart of noncommutative geometry via the study of Dirac operators on the thus constructed quantum projective spaces [1].

Recall that the C^* -algebra $C(\mathbb{C}P_q^N)$ of functions on a quantum projective space, as defined by Soibelman and Vaksman [44], is the invariant subalgebra for an action of $U(1)$ on the C^* -algebra of the odd-dimensional quantum sphere $C(S_q^{2N+1})$ (cf. [31]). By analyzing the space of characters, we want to show that this C^* -algebra is not isomorphic to the C^* -algebra $C(\mathbb{P}^N(\mathcal{T}))$ of the Toeplitz quantum projective space proposed in this paper, unless $N = 0$. To this end, we observe first that one can easily see from Definition 3.2 that the space of characters on $C(\mathbb{P}^N(\mathcal{T}))$ contains the N -torus. On the other hand, since $C(\mathbb{C}P_q^N)$ is a graph C^* -algebra [21], its space of characters is at most a circle. Hence these C^* -algebras can coincide only for $N = 0, 1$. For $N = 0$, they both degenerate to \mathbb{C} , and for $N = 1$, they are known to be the standard Podleś and mirror quantum spheres, respectively. The latter are non-isomorphic, so that the claim follows.

Better still, one can easily show that the space of characters of the C^* -algebras of the quantum-group projective spaces admit only one character. Indeed, these C^* -algebras are obtained by iterated extensions by the ideal of compact operators, i.e., for any N , there is the short exact sequence of C^* -algebras [21, eq. 4.11]:

$$(1) \quad 0 \longrightarrow \mathcal{K} \longrightarrow C(\mathbb{C}P_q^N) \longrightarrow C(\mathbb{C}P_q^{N-1}) \longrightarrow 0.$$

On the other hand, any character on a C^* -algebra containing the ideal \mathcal{K} of compact operators must evaluate to 0 on \mathcal{K} , as otherwise it would define a proper ideal in \mathcal{K} , which is impossible. Therefore, not only any character on $C(\mathbb{C}P_q^{N-1})$ naturally extends to a character on $C(\mathbb{C}P_q^N)$, but also any character on $C(\mathbb{C}P_q^N)$ naturally descends to a character on $C(\mathbb{C}P_q^{N-1})$. Hence the space of characters on $C(\mathbb{C}P_q^N)$ coincides with the space of characters on $C(\mathbb{C}P_q^{N-1})$. Remembering that $C(\mathbb{C}P_q^0) = \mathbb{C}$, we conclude the claim.

Outline. Sections 2 and 3 are devoted to the main results of this paper. Section 1 is of preliminary nature. It is focused on explaining the emergence of the projective space $\mathbb{P}^\infty(\mathbb{Z}/2)$ as the classifying space of finite coverings. We show how finite closed coverings of compact Hausdorff spaces naturally yield finite partition spaces with Alexandrov topology. Then we interpret them as projective spaces $\mathbb{P}^N(\mathbb{Z}/2)$ and take the colimit with $N \rightarrow \infty$. We continue with analyzing in detail the topological properties of $\mathbb{P}^\infty(\mathbb{Z}/2)$ to be ready for studying sheaves of algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$ — the main objects of Section 2.

In Section 1, we also present the classical complex projective space $\mathbb{P}^N(\mathbb{C})$ as the colimit of a certain diagram, so that, via the Gelfand transform, we can dualize it to the limit of the dual diagram of C^* -algebras. It is this construction that we deform (quantize) in Section 3 to obtain the C^* -algebra of our noncommutative (quantum) complex projective space as a limit (multirestricted pullback) of C^* -algebras. Since a key tool to study lattice properties of this C^* -algebra is the Birkhoff Representation Theorem, we explain it at the very beginning of Section 1.

Notation and conventions. Throughout the article we fix a ground field k of an arbitrary characteristic. We assume that all algebras are over k and are associative and unital but not necessarily commutative. We will use \mathbb{N} and \mathbb{Z} to denote the set of natural numbers and the ring of integers, respectively. We will assume $0 \in \mathbb{N}$. The finite set $\{0, \dots, N\}$ will be denoted by \underline{N} for any natural number N . However, the finite set $\{0, 1\}$ when viewed as the finite field of 2 elements will be denoted by $\mathbb{Z}/2$. We will use 2^X to denote the set of all subsets of an arbitrary set X . If \underline{x} is a sequence of elements from a set X , we will use $\kappa(\underline{x})$ to denote the underlying set of elements of \underline{x} . The symbol $|X|$ will stand for the cardinality of the set X .

1. PRIMER ON LATTICES AND ALEXANDROV TOPOLOGY

We first recall definitions and simple facts about ordered sets and lattices to fix notation. Our main references on the subject are [9, 11, 40].

A set P together with a binary relation \leq is called a *partially ordered set*, or a *poset* in short, if the relation \leq is (i) reflexive, i.e., $p \leq p$ for any $p \in P$, (ii) transitive, i.e., $p \leq q$ and $q \leq r$ implies $p \leq r$ for any $p, q, r \in P$, and (iii) anti-symmetric, i.e., $p \leq q$ and $q \leq p$ implies $p = q$ for any $p, q \in P$. If only the conditions (i)-(ii) are satisfied we call \leq a *preorder*. For every preordered set (P, \leq) there is an opposite preordered set $(P, \leq)^{\text{op}}$ given by $P = P^{\text{op}}$ and $p \leq^{\text{op}} q$ if and only if $q \leq p$ for any $p, q \in P$.

A poset (P, \leq) is called a *semi-lattice* if for every $p, q \in P$ there exists an element $p \vee q$ such that (i) $p \leq p \vee q$, (ii) $q \leq p \vee q$, and (iii) if $r \in P$ is an element which satisfies $p \leq r$ and $q \leq r$

then $p \vee q \leq r$. The binary operation \vee is called *the join*. A poset is called a *lattice* if both (P, \leq) and $(P, \leq)^{\text{op}}$ are semi-lattices. The join operation in P^{op} is called *the meet*, and traditionally denoted by \wedge . One can equivalently define a lattice P as a set with two binary associative commutative and idempotent operations \vee and \wedge . These operations satisfy two absorption laws: $p = p \vee (p \wedge q)$ and $p = p \wedge (p \vee q)$ for any $p, q \in P$. A lattice (P, \vee, \wedge) is called *distributive* if one has $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ for any $p, q, r \in P$. Note that one can prove that the distributivity of meet over join we have here is equivalent to the distributivity of join over meet.

Let (P, \leq) be an ordered set, and let $\uparrow p = \{q \in P \mid p \leq q\}$ for any $p \in P$. As a natural extension of notation, we define $\uparrow U := \bigcup_{p \in U} \uparrow p$ for all $U \subseteq P$. The sets $U \subseteq P$ that satisfy $U = \uparrow U$ are called *upper sets* or *dual order ideals*. The topological space we obtain from an ordered set using the upper sets as open sets is called an *Alexandrov space*. Note that a set U is open in the Alexandrov topology if and only if for any $u \in U$ one has $\uparrow u \subseteq U$.

Next, let Λ be any lattice. Element $c \in \Lambda$ is called *meet irreducible* if

- (1) $c = a \wedge b \Rightarrow (c = a \text{ or } c = b)$,
- (2) $\exists \lambda \in \Lambda : \lambda \not\leq c$.

The set of meet irreducible elements of the lattice Λ is denoted $\mathcal{M}(\Lambda)$. The *join irreducibles* $\mathcal{J}(\Lambda)$ are defined dually. Celebrated *Birkhoff's Representation Theorem* [8] states that, if Λ is a *finite distributive lattice*, then the map

$$(2) \quad \Lambda \ni a \longmapsto \{x \in \mathcal{M}(\Lambda) \mid x \geq a\} = \mathcal{M}(\Lambda) \cap \uparrow a \in \text{Up}(\mathcal{M}(\Lambda))$$

assigning to a the set of all meet irreducible elements greater or equal then a is a lattice isomorphism between Λ and the lattice $\text{Up}(\mathcal{M}(\Lambda))$ of upper sets of meet-irreducible elements of Λ with the set intersection and union as its join and meet, respectively. We refer to this isomorphism as the Birkhoff transform. Let us observe that it is analogous to the Gelfand transform: every finite distributive lattice is the lattice of uppersets of a certain poset just as every unital commutative C^* -algebra is the algebra of continuous functions on a certain compact Hausdorff space.

1.1. Projective spaces over $\mathbb{Z}/2$ as partition spaces.

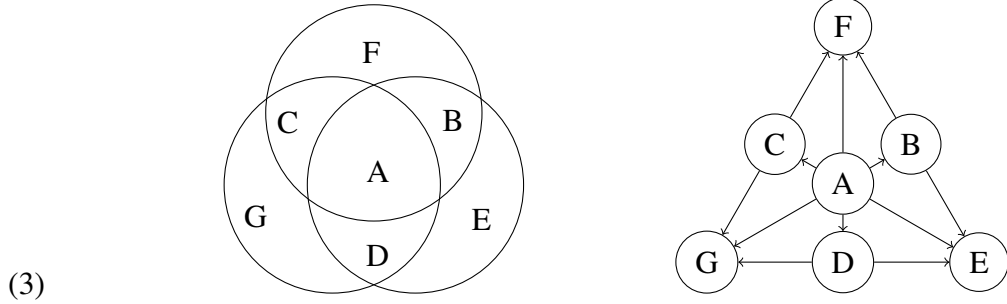
In [39], Sorkin defined and investigated the order structure on the spaces here we call *partition spaces*. For the lattice of subsets covering a space, the partition spaces play a role analogous to the set of meet-irreducible elements of an arbitrary finite distributive lattice, i.e., they are much smaller than lattices themselves while encoding important lattice properties. Sorkin's primary objective was to develop finite approximations for topological spaces via their finite open coverings (see also [5, 14]). We will use the dual approach: we will investigate spaces with finite closed coverings. See also [45, 46] for a more algebraic approach. We begin by analyzing properties of partition spaces.

Definition 1.1. Let X be a set and let $\mathcal{C} = \{C_0, \dots, C_N\}$ be a finite covering of X , i.e., let $\bigcup \mathcal{C} := \bigcup_i C_i = X$. For any $x \in X$, we define its support $\text{supp}_{\mathcal{C}}(x) = \{C \in \mathcal{C} \mid x \in C\}$. A preorder $\prec_{\mathcal{C}}$ on X is defined by $x \prec_{\mathcal{C}} y$ if and only if $\text{supp}_{\mathcal{C}}(x) \supseteq \text{supp}_{\mathcal{C}}(y)$. We also define an

equivalence relation $\sim_{\mathcal{C}}$ by letting $x \sim_{\mathcal{C}} y$ if and only if $\text{supp}_{\mathcal{C}}(x) = \text{supp}_{\mathcal{C}}(y)$. Note that $\prec_{\mathcal{C}}$ induces on $X/\sim_{\mathcal{C}}$ a partial order. This space is called the partition space associated with the finite covering \mathcal{C} of X .

Definition 1.2. Let X and \mathcal{C} be as before. We use $(X, \prec_{\mathcal{C}})$ to denote the set X with its Alexandrov topology induced from the preorder relation $\prec_{\mathcal{C}}$ defined above.

Example 1.3. Consider a region on the 2-dimensional Euclidean plane covered by three disks in generic position, and the corresponding poset, as described below:



Here an arrow \rightarrow indicates the existence of an order relation between the source and the target.

Definition 1.4. Let X be a set and $\mathcal{C} = \{C_0, \dots, C_N\}$ be a finite covering of X . The covering \mathcal{C} viewed as a subbasis for closed sets induces a topology on X . The space X together with the topology induced from \mathcal{C} is denoted by (X, \mathcal{C}) .

Proposition 1.5. Let X be a set and let \mathcal{C} be a finite covering. The Alexandrov topology defined by the preorder $\prec_{\mathcal{C}}$ coincides with the topology in Definition 1.4.

Proof. We need to prove that a subset L is closed in (X, \mathcal{C}) if and only if it is closed in $(X, \prec_{\mathcal{C}})$. By Lemma 1.18 and the definition of Alexandrov topology, we see that sets of the form $L(x) = \{x' \in X \mid x' \prec_{\mathcal{C}} x\}$ form a basis for the set of closed sets in $(X, \prec_{\mathcal{C}})$. So, it is enough to prove the same statement for $L(x)$ for any $x \in X$. Let

$$(4) \quad C_x := \bigcap \text{supp}_{\mathcal{C}}(x) := \bigcap_{C \in \text{supp}_{\mathcal{C}}(x)} C$$

We have $x' \prec_{\mathcal{C}} x$ in X if and only if x' is covered by the same sets from \mathcal{C} , or more. In other words $x' \prec_{\mathcal{C}} x$ if and only if $x' \in C_x$, i.e. $C_x = L(x)$. So, both topologies share the same basis for the closed sets. The result follows. \square

Corollary 1.6. The canonical quotient map $\pi: (X, \mathcal{C}) \rightarrow (X/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})$ is a continuous map which is both open and closed.

Proof. The canonical quotient map $\pi: (X, \prec_{\mathcal{C}}) \rightarrow (X/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})$ is continuous, open and closed since π is an epimorphism of posets, and the preorder relation $\prec_{\mathcal{C}}$ on the quotient $X/\sim_{\mathcal{C}}$ is induced by the preorder relation $\prec_{\mathcal{C}}$ on X , and therefore upper and lower sets (basic open and closed sets respectively) in $(X, \prec_{\mathcal{C}})$ are sent to corresponding upper and lower sets in $(X/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})$.

The result follows immediately since (X, \mathcal{C}) and the Alexandrov space $(X, \prec_{\mathcal{C}})$ are homeomorphic. \square

Lemma 1.7. *Let \mathcal{C} be a finite covering of a set X . Let $X/\sim_{\mathcal{C}}$ be the partition space associated with the covering \mathcal{C} and $\pi: X \rightarrow X/\sim_{\mathcal{C}}$ be the canonical surjection on the quotient. Denote by $\Lambda_{\mathcal{C}}$ the lattice of subsets of X generated by the covering \mathcal{C} and by $\Lambda_{\pi(\mathcal{C})}$ the lattice of subsets of $X/\sim_{\mathcal{C}}$ generated by $\pi(\mathcal{C}) := \{\pi(C) \mid C \in \mathcal{C}\}$. The two lattices are isomorphic via the induced morphism of lattices*

$$\hat{\pi}: \Lambda_{\mathcal{C}} \longrightarrow \Lambda_{\pi(\mathcal{C})}, \quad \lambda \longmapsto \pi(\lambda),$$

whose inverse is given by

$$\hat{\pi}^{-1}: \Lambda_{\pi(\mathcal{C})} \longrightarrow \Lambda_{\mathcal{C}}, \quad \lambda \longmapsto \pi^{-1}(\lambda).$$

Proof. Inverse images preserve set unions and intersections. Hence $\hat{\pi}^{-1}$ is a lattice morphism. On the other hand, though in general images preserve only unions, here we have

$$(5) \quad \pi(x) \in \pi(C_i) \quad \Leftrightarrow \quad x \in C_i$$

for any i . It follows that

$$(6) \quad \begin{aligned} \pi(x) \in \pi(C_{i_1}) \cap \cdots \cap \pi(C_{i_k}) &\Leftrightarrow x \in C_{i_1} \cap \cdots \cap C_{i_k} \\ &\Rightarrow \pi(x) \in \pi(C_{i_1} \cap \cdots \cap C_{i_k}). \end{aligned}$$

In other words, $\pi(C_{i_1}) \cap \cdots \cap \pi(C_{i_k})$ is a subset of $\pi(C_{i_1} \cap \cdots \cap C_{i_k})$. As the containment in the other direction always holds, it follows that $\hat{\pi}$ is also a lattice morphism. Since $\pi^{-1}(\pi(C_i)) = C_i$ for all i , one also sees immediately that $\hat{\pi}^{-1}$ is the inverse of $\hat{\pi}$. \square

Let \underline{N} be the set $\{0, \dots, N\}$ for any $N \in \mathbb{N}$. The projective space over a field \mathbb{k} is denoted by $\mathbb{P}^N(\mathbb{k})$. It is defined as the space \mathbb{k}^{N+1} divided by the diagonal action of the non-zero scalars $\mathbb{k}^{\times} := \mathbb{k} \setminus \{0\}$. For $\mathbb{k} = \mathbb{Z}/2$, we obtain

$$(7) \quad \mathbb{P}^N(\mathbb{Z}/2) := \{(z_0, \dots, z_N) \in (\mathbb{Z}/2)^{N+1} \mid \exists i \in \underline{N}, z_i = 1\}.$$

The projective space $\mathbb{P}^N(\mathbb{Z}/2)$ has a natural poset structure: for any $a = (a_i)_{i \in \underline{N}}$ and $b = (b_i)_{i \in \underline{N}}$ in $\mathbb{P}^N(\mathbb{Z}/2)$ we write $a \leq b$ if and only if $a_i \leq b_i$ for any $i \in \underline{N}$. We are ready now to compare partition spaces with $\mathbb{Z}/2$ -projective spaces with Alexandrov topology. The following theorem is a direct generalization of [19, Prop. 4.1]:

Theorem 1.8. *Let $\underline{\mathcal{C}} = (C_0, \dots, C_N)$ be a finite covering of X with a fixed ordering on the elements of the covering. Let χ_a be the characteristic function of a subset $a \subseteq \underline{N}$. Then the map $\xi: X \rightarrow \mathbb{P}^N(\mathbb{Z}/2)$ defined by*

$$\xi(x) = \chi_{s(x)}, \quad s(x) = \{i \in \underline{N} \mid x \in C_i\},$$

yields a morphism of posets $\xi: (X, \prec_{\mathcal{C}})^{op} \rightarrow (\mathbb{P}^N(\mathbb{Z}/2), \leq)$ and, consequently, a continuous map between Alexandrov spaces. Moreover, ξ is both open and closed, and it factors through the quotient $(X/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})^{op}$ as $\xi = \hat{\xi} \circ \pi$, where $\hat{\xi}: (X/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})^{op} \rightarrow (\mathbb{P}^N(\mathbb{Z}/2), \leq)$ is an embedding of Alexandrov topological spaces.

1.2. Topological properties of partition spaces.

Let $\mathbf{2}^N$ denote the set of all subsets of $\underline{N} = \{0, \dots, N\}$ and $\mathbf{2}^N \setminus \{\emptyset\}$ denote the set of all non-empty subsets of \underline{N} . Both $\mathbf{2}^N$ and $\mathbf{2}^N \setminus \{\emptyset\}$ are posets with respect to the inclusion relation \subseteq . For any non-empty subset $a \subseteq \underline{N}$, one has a sequence (a_0, \dots, a_N) where

$$(8) \quad a_i = \begin{cases} 1 & \text{if } i \in a, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the sequence (a_0, \dots, a_N) is the characteristic function χ_a of the subset $a \subseteq \underline{N}$. The assignment $a \mapsto \chi_a$ determines a bijection between the set of non-empty subsets of \underline{N} and the projective space $\mathbb{P}^N(\mathbb{Z}/2)$. Its inverse is defined as

$$(9) \quad \nu(z) := \{i \in \underline{N} \mid z_i = 1\}, \quad z = (z_i)_{i \in \underline{N}} \in (\mathbb{Z}/2)^{N+1}.$$

With this bijection, one has $(a_i)_{i \in \underline{N}} \leq (b_i)_{i \in \underline{N}}$ if and only if $\nu((a_i)_{i \in \underline{N}}) \subseteq \nu((b_i)_{i \in \underline{N}})$. In other words, we have the following:

Proposition 1.9. *The map $\nu: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbf{2}^N \setminus \{\emptyset\}$ is an isomorphism of posets, and thus a homeomorphism of Alexandrov spaces.*

Definition 1.10. *For any $i \in \underline{N}$ and any non-empty subset $a \subseteq \underline{N}$, we define open sets*

$$\mathbb{A}_i^N = \{(z_0, \dots, z_N) \in (\mathbb{Z}/2)^{N+1} \mid z_i = 1\} = \uparrow\chi_{\{i\}} \quad \text{and} \quad \mathbb{A}_a^N := \bigcap_{i \in a} \mathbb{A}_i^N = \uparrow\chi_a.$$

For brevity, when there is no risk of confusion we omit the superscripts and write \mathbb{A}_i and \mathbb{A}_a instead of \mathbb{A}_i^N and \mathbb{A}_a^N .

Lemma 1.11. *For all $N \geq 0$, the map $\phi_N: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^{N+1}(\mathbb{Z}/2)$ defined by*

$$(10) \quad \phi_N(z_0, \dots, z_N) := (z_0, \dots, z_N, 0)$$

is an embedding of topological spaces.

Proof. The fact that the maps ϕ_N are injective is obvious. They are also continuous since we have

$$(11) \quad \phi_N^{-1}(\mathbb{A}_i^{N+1}) = \begin{cases} \mathbb{A}_i^N & \text{if } i \leq N, \\ \emptyset & \text{if } i = N + 1, \end{cases}$$

and

$$(12) \quad \phi_N(\mathbb{P}^N(\mathbb{Z}/2)) \cap \mathbb{A}_i^{N+1} = \begin{cases} \phi_N(\mathbb{A}_i^N) & \text{if } i \in \underline{N}, \\ \emptyset & \text{otherwise,} \end{cases}$$

for the open subsets in the subbasis of the Alexandrov topology. □

The maps $\phi_N: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^{N+1}(\mathbb{Z}/2)$ form an injective system of maps of Alexandrov spaces. Hence we can define:

Definition 1.12. $\mathbb{P}^\infty(\mathbb{Z}/2) := \text{colimit}_{N \geq 0} \mathbb{P}^N(\mathbb{Z}/2)$.

We can represent the points of $\mathbb{P}^\infty(\mathbb{Z}/2)$ as infinite sequences $\{(z_i)_{i \in \mathbb{N}} \mid z_i \in \mathbb{Z}/2\}$ where the number of non-zero terms is finite and greater than zero. The canonical morphisms of the colimit $i_N: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^\infty(\mathbb{Z}/2)$ send a finite sequence (z_0, \dots, z_N) to the infinite sequence $(z_0, \dots, z_N, 0, 0, \dots)$ obtained from the finite sequence by padding it with countably many 0's. The topology on the colimit is the topology induced by the maps $\{i_N\}_{N \in \mathbb{N}}$.

Let Fin denote the set of all finite subsets of \mathbb{N} . By convention 0 is an element of \mathbb{N} . One can extend the bijection $\nu: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbf{2}^N \setminus \{\emptyset\}$ to a bijection of the form $\nu: \mathbb{P}^\infty(\mathbb{Z}/2) \rightarrow \text{Fin} \setminus \{\emptyset\}$. The inverse of ν is given by the assignment $a \mapsto \chi_a := (a_i)_{i \in \mathbb{N}}$ which is defined as

$$(13) \quad a_i = \begin{cases} 1 & \text{if } i \in a, \\ 0 & \text{otherwise,} \end{cases}$$

for any $a \in \text{Fin}$. The map $\nu: \mathbb{P}^\infty(\mathbb{Z}/2) \rightarrow \text{Fin} \setminus \{\emptyset\}$ is an isomorphism of posets, and therefore the Alexandrov spaces $\mathbb{P}^\infty(\mathbb{Z}/2)$ and $\text{Fin} \setminus \{\emptyset\}$ are homeomorphic.

We also have a natural poset structure on $\mathbb{P}^\infty(\mathbb{Z}/2)$ where $(a_i)_{i \in \mathbb{N}} \leq (b_i)_{i \in \mathbb{N}}$ if and only if $a_i \leq b_i$ for any $i \in \mathbb{N}$. Therefore, we have two possibly different topologies on $\mathbb{P}^\infty(\mathbb{Z}/2)$: one coming from the preorder structure, and the other coming from the colimit.

Theorem 1.13. *The following statements hold:*

- (1) *The Alexandrov topology and the colimit topology on $\mathbb{P}^\infty(\mathbb{Z}/2)$ are the same.*
- (2) *The spaces $\mathbb{P}^N(\mathbb{Z}/2)$ are T_0 but not T_1 for any $N = 1, \dots, \infty$.*
- (3) *$\mathbb{P}^N(\mathbb{Z}/2)$ is a connected topological space for any $N = 0, 1, \dots, \infty$.*
- (4) *The topology on $\mathbb{P}^\infty(\mathbb{Z}/2)$ is compactly generated.*

Proof. For any $i \in \mathbb{N}$ and $a \in \text{Fin} \setminus \{\emptyset\}$, we define

$$(14) \quad \mathbb{A}_i^\infty := \uparrow \chi_{\{i\}} \quad \text{and} \quad \mathbb{A}_a^\infty := \bigcap_{i \in a} \mathbb{A}_i^\infty = \uparrow \chi_a$$

which are open in the Alexandrov topology.

Proof of (1): Let $i_N: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^\infty(\mathbb{Z}/2)$ be the structure maps of the colimit. We need to prove that an open set in one topology is open in the other, and vice versa. The set $\{\mathbb{A}_a^\infty \mid a \in \text{Fin} \setminus \{\emptyset\}\}$ is a basis for the Alexandrov topology since each \mathbb{A}_a^∞ is an upper set. Then

$$(15) \quad i_N^{-1}(\mathbb{A}_a^\infty) = \begin{cases} \mathbb{A}_a^N & \text{if } a \subseteq N, \\ \emptyset & \text{if } a \not\subseteq N \end{cases}$$

is an open set in $\mathbb{P}^N(\mathbb{Z}/2)$ for any $N \geq 0$ and $a \in \mathbf{2}^N \setminus \{\emptyset\}$. Therefore, every open set in Alexandrov topology is open in the colimit topology. Now, assume $U \subseteq \mathbb{P}^\infty(\mathbb{Z}/2)$ is open in the colimit topology. We can assume every sequence in $\mathbb{P}^\infty(\mathbb{Z}/2)$ is of the form χ_a for a unique $a \in \text{Fin} \setminus \{\emptyset\}$ since $z = \chi_{\nu(z)}$ for any $z \in \mathbb{P}^\infty(\mathbb{Z}/2)$. Now assume $\chi_a \in U$ and we have $\chi_a \leq \chi_b$ for some $\chi_b \in \mathbb{P}^\infty(\mathbb{Z}/2)$. We need to show that $\chi_b \in U$. Since $b \in \text{Fin} \setminus \{\emptyset\}$, we have a natural number $N = \max(b) \geq 1$. Moreover, we have an inequality

$$(16) \quad i_N^{-1}(\chi_a) = \chi_a \leq \chi_b = i_N^{-1}(\chi_b)$$

in $\mathbb{P}^N(\mathbb{Z}/2)$. Since $i_N^{-1}(U)$ is open in the Alexandrov topology of $\mathbb{P}^N(\mathbb{Z}/2)$, we must have $\chi_b \in i_N^{-1}(U)$ in $\mathbb{P}^N(\mathbb{Z}/2)$, which in turn implies $\chi_b \in U$.

Proof of (2): Let $p, q \in \mathbb{P}^N(\mathbb{Z}/2)$, $p \neq q$. Then $\nu(p) \neq \nu(q)$. Let us suppose without loss of generality that $i \in \nu(p)$ and $i \notin \nu(q)$. Then $q \notin \uparrow p$ which proves that $\mathbb{P}^N(\mathbb{Z}/2)$ is T_0 . On the other hand if $p < q$ then for any open set $U \subseteq \mathbb{P}^N(\mathbb{Z}/2)$ such that $p \in U$ also $q \in U$ (as U is an upper set). It follows that $\mathbb{P}^N(\mathbb{Z}/2)$ is not T_1 .

Proof of (3): Suppose there exists a non-empty subset $V \subsetneq \mathbb{P}^N(\mathbb{Z}/2)$ that is both open and closed. Let $\chi_a \in \mathbb{P}^N(\mathbb{Z}/2)$ and $\chi_b \in \mathbb{P}^N(\mathbb{Z}/2) \setminus V$. Then, because V and $\mathbb{P}^N(\mathbb{Z}/2) \setminus V$ are open, we have $\chi_{a \cup b} \in V$ and $\chi_{a \cup b} \in \mathbb{P}^N(\mathbb{Z}/2) \setminus V$, which is a contradiction.

Proof of (4): In order to prove our assertion, we need to show that for any $a \in \text{Fin} \setminus \{\emptyset\}$ the set \mathbb{A}_a^∞ is compact. Let $a \in \text{Fin} \setminus \{\emptyset\}$ and suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of \mathbb{A}_a^∞ . Since a is finite we have $\chi_a \in \mathbb{A}_a^\infty$ and since \mathcal{U} is a covering, there exists $j \in I$ such that $\chi_a \in U_j$. Since U_j is open in the Alexandrov topology we obtain $\uparrow \chi_a = \mathbb{A}_a^\infty \subseteq U_j$. Consequently, for any finite subset α of $\text{Fin} \setminus \{\emptyset\}$, the set $\bigcup_{a \in \alpha} \mathbb{A}_a^\infty$ is compact. The result follows. \square

1.3. Continuous maps between partition spaces.

In the following, unless explicitly stated otherwise, N will be a finite natural number or ∞ . Accordingly, the set $\{0, \dots, N\}$ will be a finite set or will be \mathbb{N} if $N = \infty$. For example, a permutation $\sigma : \{0, \dots, N\} \rightarrow \{0, \dots, N\}$ is either a finite permutation or an arbitrary bijection $\mathbb{N} \rightarrow \mathbb{N}$.

Let $\text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2))$ be the lattice of open subsets of $\mathbb{P}^\infty(\mathbb{Z}/2)$. It is obvious that any continuous map $f : \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^M(\mathbb{Z}/2)$ defines a morphism between lattices of open sets of the form $\mathfrak{X}_f : \text{Op}(\mathbb{P}^M(\mathbb{Z}/2)) \rightarrow \text{Op}(\mathbb{P}^N(\mathbb{Z}/2))$, where

$$(17) \quad \mathfrak{X}_f(U) := f^{-1}(U).$$

Conversely, we have the following:

Proposition 1.14. *Let M and N be finite natural numbers or ∞ . Let $\mathfrak{X} : \text{Op}(\mathbb{P}^M(\mathbb{Z}/2)) \rightarrow \text{Op}(\mathbb{P}^N(\mathbb{Z}/2))$ be a lattice morphism with the property that*

$$(18a) \quad \bigcup_{i \in \{0, \dots, M\}} \mathfrak{X}(\mathbb{A}_i^M) = \mathbb{P}^N(\mathbb{Z}/2),$$

$$(18b) \quad \bigcap_{i \in a} \mathfrak{X}(\mathbb{A}_i^M) = \emptyset, \quad \text{for all infinite } a \subseteq \{0, \dots, M\}.$$

Then there exists a unique continuous function $f_{\mathfrak{X}} : \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^M(\mathbb{Z}/2)$ such that, for all open subsets $U \subseteq \mathbb{P}^M(\mathbb{Z}/2)$, we have $\mathfrak{X}(U) = f_{\mathfrak{X}}^{-1}(U)$.

Proof. We define a map $f_{\mathfrak{X}} : \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^M(\mathbb{Z}/2)$ as

$$(19) \quad f_{\mathfrak{X}} : z \mapsto \chi_a, \quad \text{where } a := \{i \in \{0, \dots, M\} \mid z \in \mathfrak{X}(\mathbb{A}_i^M)\}.$$

We observe that a is non-empty due to the condition (18a), and finite due to the condition (18b). By definition, $z \in f_{\mathfrak{X}}^{-1}(\mathbb{A}_i^M) \Leftrightarrow f_{\mathfrak{X}}(z) \in \mathbb{A}_i^M \Leftrightarrow i \in \nu(f_{\mathfrak{X}}(z)) \Leftrightarrow z \in \mathfrak{X}(\mathbb{A}_i^M)$. This proves both the uniqueness and continuity of $f_{\mathfrak{X}}$. \square

Note that the conditions (18a) and (18b) are satisfied for \mathfrak{X}_f for any continuous f because $\bigcap_{i \in a} \mathbb{A}_i = \emptyset$ for any infinite a , and f^{-1} preserves infinite unions and intersections.

Finally, in order to characterize the continuous maps of the projective spaces $\mathbb{P}^N(\mathbb{Z}/2)$, we will need the following technical lemma.

Lemma 1.15. *Let N and M be finite natural numbers or ∞ . Let $f: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^M(\mathbb{Z}/2)$ be a continuous map of Alexandrov spaces.*

- (1) *If f is injective then $|\nu(z)| \leq |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$.*
- (2) *If f is surjective then $|\nu(z)| \geq |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$.*

Therefore, if f is bijective then $|\nu(z)| = |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$.

Proof. Observe that for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$, one can compute $|\nu(z)|$ as

$$(20) \quad |\nu(z)| = \max\{n \in \underline{N} \mid a_1 < \cdots < a_n < z, a_i \in \mathbb{P}^N(\mathbb{Z}/2)\}$$

If f is injective, for any proper chain of elements $a_1 < \cdots < a_n < z$ in $\mathbb{P}^N(\mathbb{Z}/2)$ the elements $f(a_i)$ form a proper chain of elements $f(a_1) < \cdots < f(a_n) < f(z)$ in $\mathbb{P}^M(\mathbb{Z}/2)$. Thus $|\nu(z)| \leq |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$. If f is surjective then for any proper chain of elements $b_1 < \cdots < b_m < f(z)$ in $\mathbb{P}^M(\mathbb{Z}/2)$ we have a proper chain of elements $a_1 < \cdots < a_m < z$ in $\mathbb{P}^N(\mathbb{Z}/2)$ such that $f(a_i) = b_i$ for $i = 1, \dots, m$. This means $|\nu(z)| \geq |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$. \square

Theorem 1.16. *Let N and M be finite natural numbers or ∞ . A map $f: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^M(\mathbb{Z}/2)$ is continuous if and only if f is a morphism of posets. Furthermore, a map $f: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^N(\mathbb{Z}/2)$ is a homeomorphism if and only if there exists a bijection $\sigma: \underline{N} \rightarrow \underline{N}$ such that $f(\chi_a) = \chi_{\sigma^{-1}(a)}$, for any subset a .*

Proof. We will prove the first statement for $N = M = \infty$. Assume f is a continuous map. Then by definition $f^{-1}(U)$ is open for any $U \subseteq \mathbb{P}^\infty(\mathbb{Z}/2)$. We would like to show that f is a morphism of posets. Assume $t \leq z$ in $\mathbb{P}^\infty(\mathbb{Z}/2)$. We would like to compare $f(t)$ and $f(z)$ which will be equivalent to comparing the upper sets $\uparrow f(t)$ and $\uparrow f(z)$. Since f is continuous, the set $f^{-1}(\uparrow f(t))$ is open. Moreover, since $t \in f^{-1}(\uparrow f(t))$ we have $z \in f^{-1}(\uparrow f(t))$ because $t \leq z$ and $f^{-1}(\uparrow f(t))$ is open. Then $f(z) \in (f \circ f^{-1})(\uparrow f(t)) \subseteq \uparrow f(t)$, or equivalently $f(t) \leq f(z)$. Now assume f is a morphism of posets. In order to prove continuity, it is enough to show that $f^{-1}(\mathbb{A}_a)$ is open for any $a \in \text{Fin} \setminus \{\emptyset\}$. So, fix $a \in \text{Fin} \setminus \{\emptyset\}$ and consider $t \in f^{-1}(\mathbb{A}_a)$ which means $\chi_a \leq f(t)$. Let $z \in \mathbb{P}^\infty(\mathbb{Z}/2)$ such that $t \leq z$. Since f is a morphism of posets we have $\chi_a \leq f(t) \leq f(z)$, i.e. $z \in f^{-1}(\mathbb{A}_a)$ as we wanted to show.

For the second statement, we consider a bijection $\sigma: \underline{N} \rightarrow \underline{N}$. It induces a bijection of the form $f_\sigma: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^N(\mathbb{Z}/2)$ with the inverse $(f_\sigma)^{-1} = f_{\sigma^{-1}}$. Since $f_\sigma^{-1}(\mathbb{A}_i) = \mathbb{A}_{\sigma(i)}$ for

all i , f_σ is a homeomorphism. Conversely, assume we have a homeomorphism $f: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^N(\mathbb{Z}/2)$. Consider $\ell \subseteq \underline{N}$ and $\chi_\ell \in \mathbb{P}^N(\mathbb{Z}/2)$. By Lemma 1.15 the function f satisfies $|\nu(z)| = |\nu(f(z))|$ for any $z \in \mathbb{P}^N(\mathbb{Z}/2)$. This means f determines a unique permutation σ of \underline{N} such that $f(\chi_{\{i\}}) = \chi_{\{\sigma^{-1}(i)\}}$. Suppose that we have already proven that $f(\chi_\ell) = \chi_{\sigma^{-1}(\ell)}$ for all ℓ such that $0 < |\ell| \leq n$. Pick $\ell \subseteq \underline{N}$, with $|\ell| = n$, and $j \in \underline{N} \setminus \ell$. By the induction hypothesis we know that $\chi_{\sigma^{-1}(\ell)} = f(\chi_\ell)$ and therefore $\chi_{\sigma^{-1}(\ell)} \leq f(\chi_{\ell \cup \{j\}})$ in $\mathbb{P}^N(\mathbb{Z}/2)$. Then by Lemma 1.15 we see that $|\nu(f(\chi_{\ell \cup \{j\}}))| = n + 1$, hence $f(\chi_{\ell \cup \{j\}}) = \chi_{\sigma^{-1}(\ell) \cup \{k\}}$ for some $k \notin \sigma^{-1}(\ell)$. It remains to prove that $k = \sigma^{-1}(j)$. By definition $\chi_{\sigma^{-1}(\ell) \cup \{k\}} \in \mathbb{A}_k$. Hence $\chi_{\ell \cup \{j\}} \in f^{-1}(\mathbb{A}_k) = \mathbb{A}_{\sigma(k)}$ and therefore $\sigma(k) \in \ell \cup \{j\}$. But as $\sigma(k) \notin \ell$ we must have $\sigma(k) = j$. \square

We end this subsection by introducing a monoid that acts on $\mathbb{P}^\infty(\mathbb{Z}/2)$ by continuous maps and is pivotal on our classification theorem. It is a monoid that labels all finite sequences that can be formed from a given finite set.

Definition 1.17. A surjection $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ is called tame if

- (1) $\alpha^{-1}(i)$ is finite for any $i \in \mathbb{N}$,
- (2) $|\alpha^{-1}(i)| > 1$ for finitely many $i \in \mathbb{N}$.

We denote the monoid of all such tame surjections by \mathcal{M} .

It is clear that the composition of any two tame surjections is again a tame surjection, and that the monoid is generated by bijections and the following tame surjection:

$$(21) \quad \partial(i) = \begin{cases} i & \text{if } i = 0, \\ i - 1 & \text{if } i > 0. \end{cases}$$

We can view elements of $\mathbb{P}^\infty(\mathbb{Z}/2)$ as maps from \mathbb{N} to $\mathbb{Z}/2$. Then the monoid \mathcal{M} acts on $\mathbb{P}^\infty(\mathbb{Z}/2)$ by pullbacks. Here the tameness property ensures that such pullbacks preserve $\mathbb{P}^\infty(\mathbb{Z}/2)$, and the following definition

$$(22) \quad f_\alpha(\chi_a) := \alpha^*(\chi_a) = \chi_{\alpha^{-1}(a)} \quad \text{for all } a \in \text{Fin} \setminus \emptyset,$$

guarantees that they are morphisms of posets, whence continuous in the Alexandrov topology. Observe that this pullback representation of the monoid \mathcal{M} is faithful.

1.4. Sheaves and patterns on Alexandrov spaces.

In [30], Maszczyk defined the topological dual of a sheaf, called a *pattern*, akin to Leray's original definition of sheaves [27, pg. 303] using closed sets instead of open sets. A pattern is a sheaf-like object defined on the category of closed subsets $\text{Cl}(X)$ of a topological space X with inclusions. Explicitly, a pattern of sets on a topological space X is a covariant functor $F: \text{Cl}(X)^{\text{op}} \rightarrow \text{Set}$ satisfying the property that, for any given finite closed covering $\{C_\lambda\}_\lambda$ of X , the canonical diagram

$$(23) \quad F(X) \rightarrow \prod_{\lambda} F(C_\lambda) \rightrightarrows \prod_{\lambda, \mu} F(C_\lambda \cap C_\mu)$$

is an equalizer diagram. A pattern F on a topological space is called *global* if for any inclusion of closed sets $C' \subseteq C$ the restriction morphism $F(C) \rightarrow F(C')$ is an epimorphism.

We would like to note that for compact Hausdorff spaces Leray's definition of *faisceau continu* is equivalent to the definition of a sheaf. However, in this paper we only consider sheaves over Alexandrov spaces which are of completely different nature, and thus we cannot exchange these two concepts. On the other hand, for any finite Alexandrov space, we show below that the category of global patterns and the category of flabby sheaves are equivalent up to a natural duality.

Lemma 1.18. *Let (P, \leq) be a preordered set. A subset $C \subseteq P$ is closed in the Alexandrov topology of P if and only if C is open in the Alexandrov topology of the opposite preordered set $(P, \leq)^{op}$.*

Proof. Since $(P, \leq) = ((P, \leq)^{op})^{op}$ and the statement is symmetric, we need to prove only one implication. Assume C is closed and let $x \in C$. In order to prove that C is open in the opposite Alexandrov topology, we need to show that $y \in C$ for any $y \leq x$. Suppose the contrary that $y \leq x$ and $y \in C^c := P \setminus C$. Since C^c is open in the Alexandrov topology of (P, \leq) and $y \leq x$, we must have $x \in C^c$, which is a contradiction. \square

It follows that the lattice of open sets of an Alexandrov space (P, \leq) is isomorphic to the lattice of closed sets of the dual Alexandrov space $(P, \leq)^{op}$. Hence:

Proposition 1.19. *Let (P, \leq) be a finite preordered set. The category of (flabby) sheaves on an Alexandrov space (P, \leq) is isomorphic to the category of (global) patterns on the opposite Alexandrov space $(P, \leq)^{op}$.*

Proof. Since the lattice of closed subsets of $(P, \leq)^{op}$ is isomorphic to the lattice of open subsets of (P, \leq) , we conclude that any (flabby) sheaf on (P, \leq) is a (global) pattern on $(P, \leq)^{op}$ regardless of P being finite. Conversely, assume F is a (global) pattern on $(P, \leq)^{op}$, and let \mathcal{U} be an open cover of (P, \leq) . Since P is finite, the number of open and closed subsets of P is finite as well. Thus \mathcal{U} is a finite collection closed sets in $(P, \leq)^{op}$ sets covering P . Since F is a (global) pattern on $(P, \leq)^{op}$,

$$(24) \quad F(P) \rightarrow \prod_{U \in \mathcal{U}} F(U) \rightrightarrows \prod_{U, U' \in \mathcal{U}} F(U \cap U')$$

is an equalizer diagram. Then we conclude immediately that F is a sheaf. \square

The restriction that P is finite comes from the definition of a pattern. A pattern is a sheaf-like object where Diagram (24) is an equalizer for only *finite* closed coverings, as opposed to a sheaf where Diagram (24) is an equalizer for every (finite or infinite) open covering.

Next, we consider a poset (P, \leq) as a category by letting

$$(25) \quad Ob(P) = P \quad \text{and} \quad Hom_P(p, q) = \begin{cases} \{p \rightarrow q\} & \text{if } p \leq q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then a functor $X: P \rightarrow k\text{-Mod}$ is just a collection of k -modules $\{X_p\}_{p \in P}$ together with morphisms of k -modules $T_{qp}: X_p \rightarrow X_q$ such that (i) $T_{pp} = \text{id}_{X_p}$ and (ii) $T_{rq} \circ T_{qp} = T_{rp}$. Any such object will be called a *right P -module*. The category of right P -modules and their morphisms will be denoted by $P\text{-Mod}$. We will call a P -module flabby if each T_{pq} is an epimorphism. If $X: P \rightarrow \mathbf{Alg}_k$ is a functor into the category of k -algebras, then it will be referred as a right P -algebra. The category of P -algebras and their morphisms will be denoted by \mathbf{Alg}_P .

For a topological space X and a covering \mathcal{O} of X , we say that \mathcal{O} is stable under finite intersections if for any finite collection O_1, \dots, O_n of sets from \mathcal{O} there exists a subset $\mathcal{O}' \subseteq \mathcal{O}$ such that

$$(26) \quad \bigcap_{i=1}^n O_i = \bigcup_{O' \in \mathcal{O}'} O'.$$

Lemma 1.20. *Let F be a sheaf of algebras on a topological space X . Then for any open subset $U \subseteq X$ and any open covering \mathcal{U} of U that is stable under finite intersections, the canonical morphism $F(U) \rightarrow \lim_{V \in \mathcal{U}} F(V)$ is an isomorphism.*

Proof. First, we recall that F is a sheaf of algebras on a topological space X if and only if given an open subset U and a covering \mathcal{U} of U we have:

- (1) for any $s \in F(U)$, if $\text{Res}_V^U(s) = 0$ for all $V \in \mathcal{U}$, then $s = 0$,
- (2) for a collection of elements $\{s_V \in F(V)\}_{V \in \mathcal{U}}$ indexed by \mathcal{U} and satisfying $\text{Res}_{V \cap W}^V(s_V) = \text{Res}_{V \cap W}^W(s_W)$, there exists $s \in F(U)$ with $\text{Res}_V^U(s) = s_V$ for any $V \in \mathcal{U}$.

Now assume that F is a sheaf and \mathcal{U} is an open covering of an open subset U that is stable under finite intersections. Recall that

$$(27) \quad \lim_{V \in \mathcal{U}} F(V) = \{(f_V)_{V \in \mathcal{U}} \mid f_V \in F(V) \text{ and } f_W = \text{Res}_W^V(f_V) \text{ for any } V \supseteq W \in \mathcal{U}\}.$$

The canonical morphism $F(U) \rightarrow \lim_{V \in \mathcal{U}} F(V)$ sends each element $s \in F(U)$ to the sequence $(\text{Res}_V^U(s))_{V \in \mathcal{U}}$. The condition (1) implies that the canonical morphism is injective. Every element $(f_V)_{V \in \mathcal{U}}$ of $\lim_{V \in \mathcal{U}} F(V)$ satisfies $\text{Res}_{V \cap W}^V(f_V) = \text{Res}_{V \cap W}^W(f_W)$ because of the fact that \mathcal{U} is stable under finite intersections, and F is a sheaf. Then the condition (2) implies that the canonical morphism is an epimorphism. \square

The following result is well-known for sheaves of modules. See [10, Prop. 6.6] for a proof. Here we prove an analogous result for sheaves of algebras.

Theorem 1.21. *Let (P, \leq) be a poset. Then the category of sheaves of k -algebras on the Alexandrov space (P, \leq) is equivalent to the category P -algebras.*

Proof. Consider an arbitrary sheaf of algebras $F \in \mathbf{Sh}(P)$. Define a collection of k -algebras $\{F_p\}_{p \in P}$ indexed by elements of P by letting $F_p := F(\uparrow p)$ for any $p \in P$. Then $\uparrow p \supseteq \uparrow q$ for any $p \leq q$. Therefore, since F is a sheaf, we have well-defined morphisms of k -modules $T_{qp}^F: F_p \rightarrow F_q$ for any $p \leq q$ that satisfy (i) $T_{pp}^F = \text{id}_{F_p}$ for any $p \in P$, and (ii) $T_{rq}^F \circ T_{qp}^F = T_{rp}^F$ for

any $p \leq q \leq r$ in P . In other words, $\{F_p\}_{p \in P}$ is a right P -algebra. Also, given any morphism of sheaves $f: F \rightarrow G$, we have well-defined morphisms of algebras $\{f_p\}_{p \in P}$ that fit into a commutative diagram of the form:

$$(28) \quad \begin{array}{ccc} F_p & \xrightarrow{T_{qp}^F} & F_q \\ f_p \downarrow & & \downarrow f_q \\ G_p & \xrightarrow{T_{qp}^G} & G_q. \end{array}$$

This means that we have a functor of the form $\Phi: \mathbf{Sh}(P) \rightarrow \mathbf{Alg}_P$.

Conversely, assume that we have such a collection of algebras $\mathcal{F} = \{F_p\}_{p \in P}$ with structure morphisms $T_{qp}: F_p \rightarrow F_q$ for any $p \leq q$ satisfying the conditions (i) and (ii) described above. We let $\Upsilon(\mathcal{F})(V) = \lim_{v \in V} F_v$ viewing P as a category as in (25). Now, for any inclusion of open sets $V \subseteq W$, we have a morphism of algebras $\text{Res}_V^W(\Upsilon(\mathcal{F})) : \Upsilon(\mathcal{F})(W) \rightarrow \Upsilon(\mathcal{F})(V)$. By definition, it is the canonical morphism of limits $\lim_{w \in W} F_w \rightarrow \lim_{v \in V} F_v$. Thus we see that $\Upsilon(\mathcal{F})$ is a pre-sheaf.

To show that $\Upsilon(\mathcal{F})$ is a sheaf, we fix an open subset $U \subseteq P$ and an open covering \mathcal{U} of U . Using the description analogous to Equation (27), one can see that for any $(f_u)_{u \in U} \in \Upsilon(\mathcal{F})(U) = \lim_{u \in U} F_u$ we have $\text{Res}_V^U(f) = 0$ for any $V \in \mathcal{U}$ if and only if $f_u = 0$ for any $u \in U$. Moreover, assume that we have a collection of elements $f^V := (f_v^V)_{v \in V} \in \Upsilon(\mathcal{F})(V)$ indexed by $V \in \mathcal{U}$ satisfying $\text{Res}_{V \cap W}^V \Upsilon(\mathcal{F})(f^V) = \text{Res}_{V \cap W}^W \Upsilon(\mathcal{F})(f^W)$ for any $V, W \in \mathcal{U}$. This means $f_z^V = f_z^W$ for any $z \in V \cap W$. Notice that $\text{Res}_Z^V(f_v^V)_{v \in V} = (f_z^V)_{z \in Z} \in \Upsilon(\mathcal{F})(Z)$ for any open subset Z of V . Therefore, one can patch $\{(f_v^V)_{v \in V}\}_{V \in \mathcal{U}}$ by letting $f = (f_u)_{u \in U}$ by forgetting the superscripts indicating which open subset we consider. Hence we can conclude that $\Upsilon(\mathcal{F})$ indeed is a sheaf.

Next, to show that Υ is compatible with morphisms, for an arbitrary morphism $f: \{F_p\}_{p \in P} \rightarrow \{G_p\}_{p \in P}$ of right P -algebras and for any open subset $V \subseteq P$, we define:

$$(29) \quad \Upsilon(f)(V) := \lim_{v \in V} f_v : \Upsilon(\{F_p\}_{p \in P})(V) \longrightarrow \Upsilon(\{G_p\}_{p \in P})(V).$$

Thus we obtain a functor of the from the category of P -algebras into the category of sheaves of k -algebras on (P, \leq) . One can see that $\Upsilon(\Phi(F))(V) = \lim_{v \in V} F(\uparrow v)$. Since $\{\uparrow v \mid v \in V\}$ is an open cover of V that is stable under finite intersections and F is a sheaf, it follows from Lemma 1.20 (cf. [22, Sect. 2.2, pg.85]) that $F(V) \cong \lim_{v \in V} F(\uparrow v)$. Hence we conclude that the endofunctors id and $\Upsilon \circ \Phi$ are isomorphic. It is easy to see that $\Phi \circ \Upsilon$ is the identity functor since any $p \in P$ is the unique minimal element of the open set $\uparrow p$. The result follows. \square

We end this subsection with the following direct generalization of [19, Subsect. 2.2]. It is needed to upgrade the flabby-sheaf classification of ordered N -coverings [19, Cor. 4.3] to a classification of arbitrary finite ordered coverings we develop in Lemma 2.9.

Lemma 1.22. *Let $(\text{Lat}(A), \cap, +)$ denote the lattice of all ideals in an algebra A . Assume that $(I_i)_{i \in \mathbb{N}}$ is a sequence of ideals such that only finitely many of them are different from A . Then, for any open subset $U \subseteq \mathbb{P}^\infty(\mathbb{Z}/2)$, the map given by*

$$(30) \quad R^{(I_i)_i}(U) := \bigcap_{a \in \nu(U)} \sum_{i \in a} I_i$$

defines a morphism of lattices $R^{(I_i)_i} : \text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow \text{Lat}(A)$.

Proof. By Proposition 1.9, the map $\nu : \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbf{2}^N \setminus \{\emptyset\}$ given by (9) is an isomorphism of posets. For an open subset $U \subseteq \mathbb{P}^N(\mathbb{Z}/2)$, we let $\nu(U) = \{\nu(z) \mid z \in U\}$. Since ν is a bijection we have

$$(31) \quad \nu(U_1 \cap U_2) = \nu(U_1) \cap \nu(U_2) \quad \text{and} \quad \nu(U_1 \cup U_2) = \nu(U_1) \cup \nu(U_2)$$

for any $U_1, U_2 \in \text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2))$. In order to prove that $R^{(I_i)_i}$ is a morphism of lattices, we need to show that

$$(32) \quad R^{(I_i)_i}(U_1 \cap U_2) = R^{(I_i)_i}(U_1) + R^{(I_i)_i}(U_2), \quad R^{(I_i)_i}(U_1 \cup U_2) = R^{(I_i)_i}(U_1) \cap R^{(I_i)_i}(U_2),$$

for all $U_1, U_2 \in \text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2))$. First note that although the intersection in formula (30) is potentially infinite, the number of intersecting ideals different from A is always finite. It is also trivial to see that the second identity in (32) is satisfied.

To prove the first identity, we see that for all upper sets $\alpha_1, \alpha_2 \subseteq \text{Fin}$ we have

$$(33) \quad \alpha_1 \cap \alpha_2 = \{a_1 \cup a_2 \mid a_1 \in \alpha_1, a_2 \in \alpha_2\}.$$

Since $a_1 \subseteq a_1 \cup a_2$ and $a_2 \subseteq a_1 \cup a_2$, we see that the left hand side contains the right hand side. The other inclusion follows from the fact that $\alpha_1 \cap \alpha_2 \subseteq \alpha_1$ and $\alpha_1 \cap \alpha_2 \subseteq \alpha_2$ and $a = a \cup a$. From the distributivity of the lattice generated by ideals I_i and the fact that ν is a bijection, we obtain:

$$(34) \quad \begin{aligned} \bigcap_{a \in \nu(U_1 \cap U_2)} \sum_{i \in a} I_i &= \bigcap_{a \in \nu(U_1) \cap \nu(U_2)} \sum_{i \in a} I_i \\ &= \bigcap_{a \in \nu(U_1)} \bigcap_{b \in \nu(U_2)} \sum_{i \in a \cup b} I_i \\ &= \bigcap_{a \in \nu(U_1)} \sum_{i \in a} I_i + \bigcap_{b \in \nu(U_2)} \sum_{i \in b} I_i. \end{aligned}$$

The result follows. □

1.5. Closed covering of $\mathbb{P}^N(\mathbb{C})$ as an example of a free lattice.

In [19], a closed refinement of the affine covering of $\mathbb{P}^N(\mathbb{C})$ was constructed as an example of a finite closed covering of a compact Hausdorff space. Let us recall this construction. The elements of this covering are given by:

$$(35) \quad V_i := \{[x_0 : \dots : x_N] \mid |x_i| = \max\{|x_0|, \dots, |x_N|\}\}, \quad i \in \underline{N}.$$

It is easy to see that the family $\{V_i\}_{i \in \underline{N}}$ of closed subsets of $\mathbb{P}^N(\mathbb{C})$ is a covering of $\mathbb{P}^N(\mathbb{C})$, i.e., $\bigcup_i V_i = \mathbb{P}^N(\mathbb{C})$. This covering is interesting because of its following property:

Proposition 1.23. *The distributive lattice generated by the subsets $V_i \subset \mathbb{P}^N(\mathbb{C})$, $i \in \underline{N}$, is free.*

Proof. Sets $\{\mathbb{A}_i\}_i$ covering $\mathbb{P}^N(\mathbb{Z}/2)$ generate the free distributive lattice on $N + 1$ generators. One can demonstrate this by showing that for any distributive lattice Λ generated by $N + 1$ elements there exists a lattice morphism from the lattice generated by \mathbb{A}_i 's into Λ . (See, e.g., [19, Sect. 2.2] and Proposition 1.9, cf. Lemma 1.22.) We prove the proposition by showing that this free lattice is isomorphic with the lattice generated by V_i 's.

Let $\mathcal{C} = \{V_0, \dots, V_N\}$, and $\sim_{\mathcal{C}}$ be the equivalence relation from Definition 1.1. The partition space $(\mathbb{P}^N(\mathbb{C})/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})^{\text{op}}$ associated with the covering \mathcal{C} is homeomorphic with $\mathbb{P}^N(\mathbb{Z}/2)$ ([19, Ex. 4.2]). The \mathbb{A}_i 's subgenerate the topology of $\mathbb{P}^N(\mathbb{Z}/2)$ as open subsets, and the homeomorphism between $\mathbb{P}^N(\mathbb{Z}/2)$ and $(\mathbb{P}^N(\mathbb{C})/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})^{\text{op}}$ defines an isomorphism between the respective lattices of nonempty open sets. Hence, by Lemma 1.7, the canonical surjection $\pi : \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{C})/\sim_{\mathcal{C}}$ provides an isomorphism between the lattice of nonempty open sets of $(\mathbb{P}^N(\mathbb{C})/\sim_{\mathcal{C}}, \prec_{\mathcal{C}})^{\text{op}}$ and the lattice generated by V_i 's. \square

Now we use the covering $\{V_i\}_{i \in \underline{N}}$ to present $\mathbb{P}^N(\mathbb{C})$ as a multipushout, and, consequently, its C^* -algebra $C(\mathbb{P}^N(\mathbb{C}))$ as a multipullback. To this end, we first define a family of homeomorphisms:

$$(36) \quad \begin{aligned} \psi_i : V_i &\longrightarrow D^{\times N} := \underbrace{D \times \dots \times D}_{N \text{ times}}, \\ [x_0 : \dots : x_N] &\longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_N}{x_i} \right), \end{aligned}$$

for all $i \in \underline{N}$, from V_i onto the Cartesian product of N -copies of 1-disk. Inverses of ψ_i 's are given explicitly by

$$(37) \quad \psi_i^{-1} : D^{\times N} \ni (d_1, \dots, d_N) \longmapsto [d_1 : \dots : d_i : 1 : d_{i+1} : \dots : d_N] \in \mathbb{P}^N(\mathbb{C}).$$

Pick indices $0 \leq i < j \leq N$ and consider the following commutative diagram:

$$(38) \quad \begin{array}{ccccc} & & \mathbb{P}^N(\mathbb{C}) & & \\ & \swarrow \text{---} & \uparrow & \nwarrow \text{---} & \\ & D^{\times N} & \leftarrow \psi_i & V_i & \rightarrow \psi_j & V_j & \rightarrow D^{\times N} \\ & \uparrow & & \swarrow & \searrow & \uparrow & \\ D^{\times j-1} \times S \times D^{\times N-j} & \xleftarrow{\psi_{ij}} & V_i \cap V_j & \xrightarrow{\psi_{ji}} & D^{\times i} \times S \times D^{\times N-i-1}. \end{array}$$

Here, for

$$(39) \quad k = \begin{cases} n & \text{if } m < n, \\ n + 1 & \text{if } m > n, \end{cases}$$

we have

$$(40) \quad \psi_{mn} := \psi_m|_{V_m \cap V_n} : V_m \cap V_n \longrightarrow D^{\times k-1} \times S \times D^{\times N-k}.$$

In other words, counting from 1, the 1-circle S appears on the k 'th position among disks. It follows immediately from the definition of ψ_i 's that the maps

$$(41) \quad \Upsilon_{ij} := \psi_{ji} \circ \psi_{ij}^{-1} : D^{\times j-1} \times S \times D^{\times N-j} \longrightarrow D^{\times i} \times S \times D^{\times N-i-1}, \quad i < j,$$

can be explicitly written as

$$(42) \quad \Upsilon_{ij}(d_1, \dots, d_{j-1}, s, d_{j+1}, \dots, d_N) = (s^{-1}d_1, \dots, s^{-1}d_i, s^{-1}, s^{-1}d_{i+1}, \dots, s^{-1}d_{j-1}, s^{-1}d_{j+1}, \dots, s^{-1}d_N).$$

One sees from Diagram (38) that $\mathbb{P}^N(\mathbb{C})$ is homeomorphic to the disjoint union $\bigsqcup_{i=0}^N D_i^{\times N}$ of $N + 1$ -copies of $D^{\times N}$ divided by the identifications prescribed by the the following diagrams indexed by $i, j \in \underline{N}$, $i < j$,

$$(43) \quad \begin{array}{ccc} D_i^{\times N} & & D_j^{\times N} \\ \uparrow \wr & & \uparrow \wr \\ D^{\times j-1} \times S \times D^{\times N-j} & \xrightarrow{\Upsilon_{ij}} & D^{\times i} \times S \times D^{\times N-i-1}. \end{array}$$

Consequently, one sees that the C*-algebra $C(\mathbb{P}^N(\mathbb{C}))$ of continuous functions on $\mathbb{P}^N(\mathbb{C})$ is isomorphic with the limit of the dual diagram to Diagram (43):

$$(44) \quad \begin{array}{ccc} C(D)_i^{\otimes N} & & C(D)_j^{\otimes N} \\ \downarrow & & \downarrow \\ C(D)^{\otimes j-1} \otimes C(S) \otimes C(D)^{\otimes N-j} & \xleftarrow{\Upsilon_{ij}^*} & C(D)^{\otimes i} \otimes C(S) \otimes C(D)^{\otimes N-i-1}. \end{array}$$

Here the tensor product is the completed tensor product. Note that, by construction, $C(\mathbb{P}^N(\mathbb{C})) \subseteq \prod_{i=0}^N C(D)_i^{\otimes N}$.

Finally, observe that the covering of $\mathbb{P}^N(\mathbb{C})$ discussed above naturally gives rise to a flabby sheaf F over $\mathbb{P}^N(\mathbb{Z}/2)$ of unital commutative C*-algebras. Explicitly, for any open $U \subseteq \mathbb{P}^N(\mathbb{Z}/2)$,

$$(45) \quad F(U) := C(\xi^{-1}(U)),$$

where $\xi : \mathbb{P}^N(\mathbb{C}) \rightarrow \mathbb{P}^N(\mathbb{Z}/2)$ is the map defined in Theorem 1.8. Restriction morphisms are defined by

$$(46) \quad \text{Res}_V^U : F(U) \longrightarrow F(V), \quad f \longmapsto f|_{\xi^{-1}(V)}.$$

In particular, $F(\mathbb{A}_i)$ is isomorphic to $C(D)^{\otimes N}$ for all i .

2. CLASSIFICATION OF FINITE COVERINGS VIA THE UNIVERSAL PARTITION SPACE $\mathbb{P}^\infty(\mathbb{Z}/2)$

The aim of this section is to establish an equivalence between the category of finite coverings of algebras and an appropriate category of finitely-supported flabby sheaves of algebras. To this end, we first define a number of different categories of coverings and sheaves. Then we explore their interrelations to assemble a path of functors yielding the desired equivalence of categories.

2.1. Categories of coverings.

Let X be a topological space and \mathcal{C} be a collection of subsets of X that cover X , i.e., $\bigcup_{U \in \mathcal{C}} U = X$. We allow $\emptyset \in \mathcal{C}$. Recall that such a set is called a *covering* of X . A covering \mathcal{C} is called finite if the set \mathcal{C} is finite. A covering \mathcal{C} of a topological space X is called *closed* (resp. *open*) if \mathcal{C} consists of closed (resp. open) subsets of X . Now we consider the category of pairs of the form (X, \mathcal{C}) where X is a topological space and \mathcal{C} is a closed (or open) covering. A morphism $f: (X, \mathcal{C}) \rightarrow (X', \mathcal{C}')$ is a continuous map of topological spaces $f: X \rightarrow X'$ such that for any $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}'$ with the property that $C \subseteq f^{-1}(C')$. In the spirit of the Gelfand transform, we are going to dualize this category to the category of algebras.

Let $\Pi = \{\pi_i: A \rightarrow A_i\}_i$ be a finite set of epimorphisms of algebras. We allow the case $A_i = \mathbf{0}$ for some i . Denote by Λ the lattice of ideals generated by $\ker \pi_i$, where \cap and $+$ denote the join and meet operations, respectively. Recall from [19] that the set Π is called a *covering* if the lattice Λ is distributive and $\bigcap_i \ker(\pi_i) = \mathbf{0}$. Finally, an ordered family $\underline{\Pi} = (\pi_i: A \rightarrow A_i)_i$ is called an *ordered covering* if the set $\kappa(\underline{\Pi}) := \{\pi_i: A \rightarrow A_i\}_i$ is a covering. In such an ordered sequence $(\pi_i: A \rightarrow A_i)_i$ we allow repetitions.

In [19], for each natural number $N \geq 1$, the authors defined a category \mathbf{C}_N whose objects are pairs $(A; \pi_1, \dots, \pi_N)$ where A is a unital algebra, and the ordered sequence (π_1, \dots, π_N) is an ordered covering of A . A morphism between two objects $f: (A; \pi_1, \dots, \pi_N) \rightarrow (A'; \pi'_1, \dots, \pi'_N)$ is a morphism of algebras $f: A \rightarrow A'$ such that $f(\ker(\pi_i)) \subseteq \ker(\pi'_i)$, or equivalently that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_i))$ for any $i = 1, \dots, N$. This category is called *the category of ordered N -coverings of algebras*.

For any natural number $N \geq 0$, there is a functor $e_N: \mathbf{C}_N \rightarrow \mathbf{C}_{N+1}$ which is defined as $e_N(A; \pi_1, \dots, \pi_N) := (A; \pi_1, \dots, \pi_N, A \rightarrow \mathbf{0})$ on the set of objects for any $(A; \pi_1, \dots, \pi_N) \in \text{Ob}(\mathbf{C}_N)$. The functor is defined to be identity on the set of morphisms. Moreover, observe that for any $(A, \underline{\Pi})$ and $(A', \underline{\Pi}')$ in $\text{Ob}(\mathbf{C}_N)$, $N \geq 0$, we have

$$(47) \quad \text{Hom}_{\mathbf{C}_{N+1}}(e_N(A, \underline{\Pi}), e_N(A', \underline{\Pi}')) = \text{Hom}_{\mathbf{C}_N}((A, \underline{\Pi}), (A', \underline{\Pi}')).$$

Therefore, e_N is both full and faithful. Now, we define

Definition 2.1. $\mathcal{OCov}_{\text{fin}} := \text{colim}_N \mathbf{C}_N$. *The category $\mathcal{OCov}_{\text{fin}}$ is called the category of finite ordered coverings of algebras.*

One can think of $\mathcal{OCov}_{\text{fin}}$ as the category of pairs of the form $(A, \underline{\Pi})$, where A is again a unital algebra. This time $\underline{\Pi}$ is an infinite sequence of epimorphisms $\pi_i: A \rightarrow A_i$ indexed

by $i \in \mathbb{N}$ with the property that (i) all but finitely many of these epimorphisms have zero codomain, and (ii) the underlying set $\kappa(\underline{\Pi})$ of epimorphisms is a covering of A . A morphism $f: (A; \pi_0, \pi_1, \dots) \rightarrow (A'; \pi'_0, \pi'_1, \dots)$ is a morphism of algebras $f: A \rightarrow A'$ with the property that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_i))$ for any $i \in \mathbb{N}$.

Next, recall from the beginning of this section that, in the category of topological spaces together with a prescribed finite covering, a covering is a collection of sets devoid of an ordering on the covering sets. Thus, it is necessary for us to replace the ordered sequences of epimorphisms in the objects of the category $\mathcal{OCov}_{\text{fin}}$, and work with *finite sets* of epimorphisms of algebras.

Definition 2.2. Let $\mathcal{Cov}_{\text{fin}}$ be a category whose objects are pairs (A, Π) , where A is a unital algebra and Π is a finite set of unital algebra epimorphisms that is a covering of the algebra A . A morphism $f: (A, \Pi) \rightarrow (A', \Pi')$ in this category is a morphism of algebras $f: A \rightarrow A'$ satisfying the condition that for any epimorphism $\pi'_i: A' \rightarrow A'_i$ in the covering Π' there exists an epimorphism $\pi_j: A \rightarrow A_j$ in the covering Π such that $\ker(\pi_j) \subseteq f^{-1}(\ker(\pi'_i))$. This category will be called the category of finite coverings of algebras.

If $f: (A, \Pi) \rightarrow (A', \Pi')$ is a morphism in $\mathcal{Cov}_{\text{fin}}$, we will say that f is implemented by the morphism of algebras $f: A \rightarrow A'$. Note that the matching of the epimorphisms, or rather the kernels, is not part of the datum defining a morphism. We also need the following auxiliary category.

Definition 2.3. Category \mathcal{Aux} is a category whose objects are the same as the objects of $\mathcal{OCov}_{\text{fin}}$. A morphism $f: (A, \underline{\Pi}) \rightarrow (A', \underline{\Pi}')$ in \mathcal{Aux} is a morphism of algebras $f: A \rightarrow A'$ satisfying the property that for every π'_j appearing in the sequence $\underline{\Pi}'$ there exists an epimorphism π_i appearing in the ordered sequence $\underline{\Pi}$ such that $\ker(\pi_i) \subseteq f^{-1}(\ker(\pi'_j))$.

As before, the matching of the epimorphisms is not part of the datum defining a morphism.

Now we want to prove that the categories \mathcal{Aux} and $\mathcal{Cov}_{\text{fin}}$ are equivalent. Recall first that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called *essentially surjective* if for every $X \in \text{Ob}(\mathcal{D})$ there exists an object $C_X \in \text{Ob}(\mathcal{C})$ and an isomorphism $\omega_X: F(C_X) \rightarrow X$ in \mathcal{D} .

Theorem 2.4. [29, IV. 4 Thm.1] Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor that is fully faithful and essentially surjective. Then F is an equivalence of categories.

Lemma 2.5. Consider the assignment

$$\mathfrak{Z}(A; \pi_0, \pi_1, \dots) := (A; \{\pi_i \mid i \in \mathbb{N}\}) \quad \text{and} \quad \mathfrak{Z}(f) := f$$

for every object $(A; \pi_0, \pi_1, \dots)$ and morphism $f: (A, \underline{\Pi}) \rightarrow (A', \underline{\Pi}')$ in the category \mathcal{Aux} . Then \mathfrak{Z} defines an equivalence of categories of the form $\mathfrak{Z}: \mathcal{Aux} \rightarrow \mathcal{Cov}_{\text{fin}}$.

Proof. One can see that

$$(48) \quad \text{Hom}_{\mathcal{Aux}}((A, \underline{\Pi}), (B, \underline{\Theta})) = \text{Hom}_{\mathcal{Cov}_{\text{fin}}}((A, \kappa(\underline{\Pi})), (B, \kappa(\underline{\Theta}))).$$

This implies that \mathfrak{J} is fully faithful, and that it makes sense for the functor \mathfrak{J} to act as identity on the set of morphisms. Given an object (A, Π) in $\mathcal{C}ov_{\text{fin}}$, one can choose an ordering on the finite set Π and obtain an ordered sequence of epimorphisms

$$(49) \quad (\pi_0: A \rightarrow A_0, \pi_1: A \rightarrow A_1, \dots, \pi_N: A \rightarrow A_N),$$

where $N = |\Pi|$. We can pad this sequence with $A \rightarrow \mathbf{0}$ to get an infinite sequence $\underline{\Pi}$ of epimorphisms where only finitely many epimorphisms are non-trivial. This infinite sequence has the property that the corresponding finite set $\kappa(\underline{\Pi})$ of epimorphisms is the set $\Pi \cup \{A \rightarrow \mathbf{0}\}$. Since the identity morphism $\text{id}_A: A \rightarrow A$ implements an isomorphism

$$(50) \quad (A, \Pi \cup \{A \rightarrow \mathbf{0}\}) \longrightarrow (A, \Pi)$$

in $\mathcal{C}ov_{\text{fin}}$, we conclude that \mathfrak{J} is essentially surjective. Now the result follows from Theorem 2.4. \square

The category $\mathcal{A}ux$ sits in between the category $\mathcal{O}\mathcal{C}ov_{\text{fin}}$ of ordered coverings and the category $\mathcal{C}ov_{\text{fin}}$ of coverings:

$$(51) \quad \mathcal{O}\mathcal{C}ov_{\text{fin}} \hookrightarrow \mathcal{A}ux \xrightarrow{\simeq} \mathcal{C}ov_{\text{fin}}.$$

The definitions of morphisms in the categories $\mathcal{A}ux$ and $\mathcal{C}ov_{\text{fin}}$ coincide even though the classes of objects are different. Even though the categories $\mathcal{O}\mathcal{C}ov_{\text{fin}}$ and $\mathcal{A}ux$ share the same objects, there are more morphisms in $\mathcal{A}ux$ than in $\mathcal{O}\mathcal{C}ov_{\text{fin}}$:

$$(52) \quad \text{Hom}_{\mathcal{O}\mathcal{C}ov_{\text{fin}}}((A, \underline{\Pi}), (B, \underline{\Pi}')) \subseteq \text{Hom}_{\mathcal{A}ux}((A, \underline{\Pi}), (B, \underline{\Pi}')).$$

Explicitly, one can describe $\text{Hom}_{\mathcal{A}ux}((A, \underline{\Pi}), (B, \underline{\Pi}'))$ as the set of morphisms of algebras $f: A \rightarrow B$ for which there exists a sequence of epimorphisms $\underline{\Pi}''$ obtained from $\underline{\Pi}$ by permutations and insertions of already existing epimorphisms, such that f is a morphism in $\text{Hom}_{\mathcal{O}\mathcal{C}ov_{\text{fin}}}((A, \underline{\Pi}''), (B, \underline{\Pi}'))$. This can be expressed elegantly by introducing another auxiliary category $\mathcal{A}ux$ where $\mathcal{A}ux$ comes out as the quotient of $\mathcal{A}ux$ by an equivalence relation on the morphisms (c.f. Definition 2.6 and Lemma 2.7 below).

The reason why we prefer working with ordered sequences of epimorphisms in $\mathcal{A}ux$ rather than the sets of epimorphisms in $\mathcal{C}ov_{\text{fin}}$ is that we want to interpret coverings in the language of sheaves. Working with sheaves inevitably introduces order on the set of epimorphisms because of the particular nature of morphism in the category of sheaves (c.f. Lemma 2.9). Fortunately, by Lemma 2.5, our auxiliary category $\mathcal{A}ux$, where the objects are based on ordered sequences, is equivalent to $\mathcal{C}ov_{\text{fin}}$, the category of finite coverings of algebras where the objects are based on finite sets of epimorphisms.

Let $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ be a tame surjection from the monoid \mathcal{M} (Definition 1.17). Any such α gives rise to an endofunctor $\check{\alpha}: \mathcal{O}\mathcal{C}ov_{\text{fin}} \rightarrow \mathcal{O}\mathcal{C}ov_{\text{fin}}$ defined on objects by

$$(53) \quad \check{\alpha}(A, (\pi_i)_i) := (A, (\pi_{\alpha(i)})_i),$$

and by identity on the morphisms.

Definition 2.6. *Category $\widetilde{\mathcal{A}ux}$ is a category whose objects are the same as in $\mathcal{OCov}_{\text{fin}}$ and $\mathcal{A}ux$. Morphisms in $\widetilde{\mathcal{A}ux}$ are pairs of the form $(f, \alpha) : (A, \underline{\Pi}) \rightarrow (A', \underline{\Pi}')$, where $\alpha \in \mathcal{M}$ and*

$$f : \check{\alpha}(A, \underline{\Pi}) \longrightarrow (A', \underline{\Pi}')$$

is a morphism in $\mathcal{OCov}_{\text{fin}}$. The identity morphisms are simply $(\text{id}_A, \text{id}_{\mathbb{N}})$, and the composition of morphisms is defined as

$$(g, \beta) \circ (f, \alpha) = (g \circ (\check{\beta}f), \alpha \circ \beta).$$

Note that we have $(\beta \circ \alpha)^{\check{}} = \check{\alpha}\check{\beta}$.

We define an equivalence relation on $\widetilde{\mathcal{A}ux}$ as follows. We say that two morphisms (f, α) , (f', α') in $\text{Hom}_{\widetilde{\mathcal{A}ux}}((A, \underline{\Pi}), (A', \underline{\Pi}'))$ are equivalent (here denoted by $(f, \alpha) \sim (f', \alpha')$) if $f = f'$ as morphisms of algebras. By [29, Proposition II.8.1], we know the quotient category $\widetilde{\mathcal{A}ux}/\sim$ exists. Moreover, it is easy to see that the relation \sim preserves the compositions of morphisms. Hence, by the proof of [29, Proposition II.8.1], we do not need to extend the relation \sim to form a quotient category. We are now ready for:

Lemma 2.7. *The category $\mathcal{A}ux$ and the quotient category $\widetilde{\mathcal{A}ux}/\sim$ are isomorphic.*

Proof. We implement the isomorphism with two functors

$$(54) \quad F : \widetilde{\mathcal{A}ux}/\sim \longrightarrow \mathcal{A}ux, \quad G : \mathcal{A}ux \longrightarrow \widetilde{\mathcal{A}ux}/\sim,$$

defined as identities on objects. For any equivalence class $[f, \alpha]_{\sim}$ of morphisms in $\widetilde{\mathcal{A}ux}/\sim$, we define $F([f, \alpha]_{\sim}) := f$. On the other hand, for any morphism $f : (A, (\pi_i)_{i \in \mathbb{N}}) \rightarrow (A', (\pi'_i)_{i \in \mathbb{N}})$ in $\mathcal{A}ux$, we set $G(f) := [f, \alpha]_{\sim}$, where α is any element of \mathcal{M} satisfying:

$$(55) \quad \alpha(i) = \begin{cases} i - N & \text{for } i > N, \\ j, \text{ where } j \text{ is such that } \ker \pi_j \subseteq f^{-1}(\ker \pi'_i), & \text{for } i \leq N. \end{cases}$$

Here $N \in \mathbb{N}$ is a number such that for any $i > N$ we have $\pi'_i := A' \rightarrow 0$. It is obvious that $F \circ G = \text{id}_{\mathcal{A}ux}$ and $G \circ F = \text{id}_{\widetilde{\mathcal{A}ux}/\sim}$. One can easily see that F and G are functorial — it is enough to note that $\check{\alpha}f = f$ as morphisms of algebras. \square

2.2. The sheaf picture for coverings.

Let $\text{Sh}(\mathbb{P}^{\infty}(\mathbb{Z}/2))$ be the category of flabby sheaves of algebras over $\mathbb{P}^{\infty}(\mathbb{Z}/2)$. A morphism $f : F \rightarrow G$ in $\text{Sh}(\mathbb{P}^{\infty}(\mathbb{Z}/2))$ is a collection $\{f_U : F(U) \rightarrow G(U)\}_{U \in \text{Op}(\mathbb{P}^{\infty}(\mathbb{Z}/2))}$ of morphisms of algebras (indexed by the open subsets of $\mathbb{P}^{\infty}(\mathbb{Z}/2)$) that fit into the following diagram

$$(56) \quad \begin{array}{ccc} F(U) & \xrightarrow{f_U} & G(U) \\ \text{Res}_V^U(F) \downarrow & & \downarrow \text{Res}_V^U(G) \\ F(V) & \xrightarrow{f_V} & G(V) \end{array}$$

for any chain $V \subseteq U$ of open subsets of $\mathbb{P}^{\infty}(\mathbb{Z}/2)$.

Definition 2.8. A flabby sheaf $F \in \text{Ob}(\text{Sh}(\mathbb{P}^\infty(\mathbb{Z}/2)))$ is said to have finite support if there exists $N \in \mathbb{N}$ such that $F(\mathbb{A}_n) = 0$ for any $n > N$. The full subcategory of flabby sheaves with finite support will be denoted by $\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$.

Here is an alternative way of seeing sheaves with finite support on $\mathbb{P}^\infty(\mathbb{Z}/2)$. Any sheaf of algebras on $\mathbb{P}^N(\mathbb{Z}/2)$ can be extended to a sheaf of algebras on $\mathbb{P}^{N+1}(\mathbb{Z}/2)$ by the direct image functor

$$(57) \quad \text{Sh}(\mathbb{P}^N(\mathbb{Z}/2)) \ni F \longmapsto (\phi_N)_*(F) \in \text{Sh}(\mathbb{P}^{N+1}(\mathbb{Z}/2))$$

with respect to the canonical embedding $\phi_N: \mathbb{P}^N(\mathbb{Z}/2) \rightarrow \mathbb{P}^{N+1}(\mathbb{Z}/2)$ defined in Lemma 1.11. Then we obtain an injective system of small categories $(\text{Sh}(\mathbb{P}^N(\mathbb{Z}/2)), j_N)$, and we observe that $\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ is $\text{colim}_N \text{Sh}(\mathbb{P}^N(\mathbb{Z}/2))$.

For a flabby sheaf F in $\text{Ob}(\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)))$, we will use $\text{Res}_i(F)$ to denote the canonical restriction epimorphism $F(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow F(\mathbb{A}_i)$ for any $i \in \mathbb{N}$. Note that, since F is a sheaf with finite support, all but finitely many morphisms $\text{Res}_i(F)$ are of the form $F(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow \mathbf{0}$. The following Lemma is a reformulation of [19, Cor. 4.3] in a new setting. The proof is essentially the same as in [19, Prop. 2.2] using Lemma 1.22. Note that we can apply the generalized Chinese Remainder Theorem (e.g., see [35, Thm. 18 on p. 280] and [30]) as there is always only a finite number of non-trivial congruences.

Lemma 2.9. For any $(A, \underline{\Pi}) \in \text{Ob}(\mathcal{OCov}_{\text{fin}})$ and $F \in \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$, the following assignments

$$(58) \quad \Psi(A, \underline{\Pi}) := \{U \mapsto A/R^{\underline{\Pi}}(U)\}_{U \in \text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2))} \in \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)),$$

$$(59) \quad \Phi(F) := (F(\mathbb{P}^\infty(\mathbb{Z}/2)); \text{Res}_0(F), \text{Res}_1(F), \dots, \text{Res}_n(F), \dots) \in \mathcal{OCov}_{\text{fin}},$$

are functors establishing an equivalence between the category $\mathcal{OCov}_{\text{fin}}$ of ordered coverings and the category $\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ of finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$.

We would like to extend the equivalence we constructed in Lemma 2.9 to an equivalence of categories between \mathcal{Aux} (and therefore $\mathcal{Cov}_{\text{fin}}$) and a suitable category of sheaves filling the following diagram:

$$(60) \quad \begin{array}{ccccc} \mathcal{OCov}_{\text{fin}} & \xrightarrow{\quad} & \mathcal{Aux} & \xrightarrow[\simeq]{3} & \mathcal{Cov}_{\text{fin}} \\ \Psi \downarrow \simeq & & \downarrow \simeq & & \\ \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) & \cdots \rightarrow & \text{Sh}_{\text{fin}}^{???}(\mathbb{P}^\infty(\mathbb{Z}/2)) & & \end{array}$$

As \mathcal{Aux} is isomorphic to a quotient category, we expect $\text{Sh}_{\text{fin}}^{???}(\mathbb{P}^\infty(\mathbb{Z}/2))$ to be a quotient of the following category of sheaves with extended morphisms:

Definition 2.10. The objects of $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ are finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$. A morphism $[\tilde{f}, \alpha^*]: P \rightarrow Q$ in $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ is a pair consisting of a

continuous map (see (22))

$$\alpha^* : \mathbb{P}^\infty(\mathbb{Z}/2) \longrightarrow \mathbb{P}^\infty(\mathbb{Z}/2), \quad \chi_a \longmapsto \chi_{\alpha^{-1}(a)},$$

where $\mathcal{M} \ni \alpha : \mathbb{N} \rightarrow \mathbb{N}$ is a tame surjection (Definition 1.17), and a morphism of sheaves

$$\tilde{f} : \alpha_*^* P \rightarrow Q.$$

Composition of morphisms is given by

$$[\tilde{g}, \beta^*] \circ [\tilde{f}, \alpha^*] := [\tilde{g} \circ (\beta_*^* \tilde{f}), \beta^* \circ \alpha^*].$$

Lemma 2.11. *Let $\Psi : \mathcal{OCov}_{\text{fin}} \rightarrow \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ and $\Phi : \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow \mathcal{OCov}_{\text{fin}}$ be functors defined in Lemma 2.9. Then the functors*

$$\tilde{\Psi} : \widetilde{\mathcal{A}ux} \longrightarrow \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)), \quad \tilde{\Phi} : \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) \longrightarrow \widetilde{\mathcal{A}ux},$$

defined on objects by

$$\tilde{\Psi}(A, \underline{\Pi}) = \Psi(A, \underline{\Pi}), \quad \tilde{\Phi}(P) = \Phi(P),$$

and on morphisms by

$$\tilde{\Psi}(f, \alpha) = [\Psi f, \alpha^*], \quad \tilde{\Phi}[\tilde{f}, \alpha^*] = (\Phi \tilde{f}, \alpha),$$

establish an equivalence of categories between $\widetilde{\mathcal{A}ux}$ and $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$.

Proof. We divide the proof into several steps.

(1) $(\alpha^*)^{-1}(\mathbb{A}_i) = \mathbb{A}_{\alpha(i)}$ for all $i \in \mathbb{N}$. Indeed,

$$\begin{aligned} (\alpha^*)^{-1}(\mathbb{A}_i) &= (\alpha^*)^{-1}(\{\chi_a \mid i \in a \subset \mathbb{N}\}) \\ &= \{\chi_b \mid \alpha^*(\chi_b) = \chi_a \text{ and } i \in a \subset \mathbb{N}\} \\ &= \{\chi_b \mid \chi_{\alpha^{-1}(b)} = \chi_a \text{ and } i \in a \subset \mathbb{N}\} \\ &= \{\chi_b \mid i \in \alpha^{-1}(b)\} \\ &= \{\chi_b \mid \alpha(i) \in b \subset \mathbb{N}\} \\ &= \mathbb{A}_{\alpha(i)}. \end{aligned}$$

(2) As α is tame by assumption, $\alpha^{-1}(a)$ is finite for any finite $a \subseteq \mathbb{N}$. Hence α^* is well defined.

(3) Equality $\alpha^* = \beta^*$ implies $\alpha = \beta$ for any surjective maps $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{N}$. Hence the functor $\tilde{\Phi}$ is well defined.

(4) $\alpha_*^* \Psi = \Psi \tilde{\alpha}$. Indeed, for any $(A, (\pi_i)_i) \in \widetilde{\mathcal{A}ux}$, we see that

$$\begin{aligned} (\alpha_*^* \Psi)((A, (\pi_i)_i)) &= \alpha_*^*(U \mapsto A/R^{(\pi_i)_i}(U)) \\ &= U \mapsto A/R^{(\pi_i)_i}((\alpha^*)^{-1}(U)), \\ (\Psi \tilde{\alpha})((A, (\pi_i)_i)) &= \tilde{\Psi}((A, (\pi_{\alpha(i)})_i)) \\ &= U \mapsto A/R^{(\pi_{\alpha(i)})_i}(U). \end{aligned}$$

On the other hand, the observation that for any open $U \subseteq \mathbb{P}^\infty(\mathbb{Z}/2)$ we have $U = \bigcup_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_i$, and the result from Step (1), yield:

$$\begin{aligned}
R^{(\pi_i)_i}((\alpha^*)^{-1}(U)) &= R^{(\pi_i)_i}((\alpha^*)^{-1}(\bigcup_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_i)) \\
&= R^{(\pi_i)_i}(\bigcup_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \bigcap_{i \in a} (\alpha^*)^{-1}(\mathbb{A}_i)) \\
&= R^{(\pi_i)_i}(\bigcup_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_{\alpha(i)}) \\
&= \bigcap_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \left(\sum_{i \in a} \ker \pi_{\alpha(i)} \right) \\
&= R^{(\pi_{\alpha(i)})_i}(\bigcup_{\substack{a \text{ s.t.} \\ \chi_a \in U}} \bigcap_{i \in a} \mathbb{A}_i) \\
&= R^{(\pi_{\alpha(i)})_i}(U).
\end{aligned}$$

(5) Let $\alpha, \beta : \mathbb{N} \rightarrow \mathbb{N}$ be maps from \mathcal{M} . Then $(\alpha \circ \beta)^* = \beta^* \circ \alpha^*$. Indeed, for any $\chi_a \in \mathbb{P}^\infty(\mathbb{Z}/2)$, we obtain:

$$(\beta^* \circ \alpha^*)(\chi_a) = \beta^*(\chi_{\alpha^{-1}(a)}) = \chi_{(\beta^{-1} \circ \alpha^{-1})(a)} = \chi_{(\alpha \circ \beta)^{-1}(a)} = (\alpha \circ \beta)^*(\chi_a).$$

(6) $\tilde{\Psi}$ is functorial. Indeed, take any composable morphisms (f, α) and (g, β) in $\widetilde{\mathcal{A}ux}$. Then the previous two steps and the functoriality of Ψ yield

$$\begin{aligned}
\tilde{\Psi}((g, \beta) \circ (f, \alpha)) &= \tilde{\Psi}((g \circ (\check{\beta}f), \alpha \circ \beta)) \\
&= (\Psi(g \circ (\check{\beta}f)), (\alpha \circ \beta)^*) \\
&= (\Psi(g) \circ \Psi(\check{\beta}f)), \beta^* \circ \alpha^* \\
&= ((\Psi g) \circ (\beta_*^* \Psi f)), \beta^* \circ \alpha^* \\
&= [\Psi g, \beta^*] \circ [\Psi f, \alpha^*] \\
&= \tilde{\Psi}((g, \beta)) \circ \tilde{\Psi}((f, \alpha)).
\end{aligned}$$

(7) $\Phi \alpha_*^* = \check{\alpha} \Phi$. Indeed, take any $P \in \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$. Using the result of Step (1), we obtain:

$$\begin{aligned}
(\Phi \alpha_*^*)(P) &= \Phi(U \mapsto P(\alpha^{-1}(U))) \\
&= (P(\mathbb{P}^\infty(\mathbb{Z}/2)), (P(\mathbb{P}^\infty(\mathbb{Z}/2)) \mapsto P((\alpha^*)^{-1}(\mathbb{A}_i)))_i \\
&= (P(\mathbb{P}^\infty(\mathbb{Z}/2)), (P(\mathbb{P}^\infty(\mathbb{Z}/2)) \mapsto P(\mathbb{A}_{\alpha(i)}))_i \\
&= \check{\alpha}((P(\mathbb{P}^\infty(\mathbb{Z}/2)), (P(\mathbb{P}^\infty(\mathbb{Z}/2)) \mapsto P(\mathbb{A}_i))_i)) \\
&= (\check{\alpha} \Phi)(P).
\end{aligned}$$

- (8) $\tilde{\Phi}$ is functorial. The proof uses the result from the previous step, and is analogous to the proof of Step (6).
- (9) The natural isomorphism $\eta: \Psi\Phi \rightarrow \text{id}_{\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))}$ comes from a family of isomorphisms of sheaves $\eta_P: \Psi\Phi P \rightarrow P$. The latter are given by the canonical isomorphisms between the image of an epimorphism and the quotient of its domain by its kernel (c.f. [19, Prop. 2.2]):

$$\eta_{P,U}: P(\mathbb{P}^\infty(\mathbb{Z}/2))/\ker(P(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow P(U)) \longrightarrow P(U).$$

To see that, for any sheaf P ,

$$\alpha_*^* \eta_P = \eta_{\alpha_*^* P},$$

note that $\alpha_*^* \eta_P: \alpha_*^* \Psi\Phi P = \Psi\Phi \alpha_*^* P \longrightarrow \alpha_*^* P$, and

$$\begin{aligned} & (\alpha_*^* \eta_P)_U \\ & := \eta_{P,(\alpha^*)^{-1}(U)} \\ & = P(\mathbb{P}^\infty(\mathbb{Z}/2))/\ker(P(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow P((\alpha^*)^{-1}(U))) \longrightarrow P((\alpha^*)^{-1}(U)) \\ & = \eta_{\alpha_*^* P,U}. \end{aligned}$$

Here the first equality is just the definition of action of direct image functor on morphisms.

- (10) The family of maps

$$\tilde{\eta}_P := [\eta_P, \text{id}_{\mathbb{N}}^*]: \tilde{\Psi}\tilde{\Phi}P \longrightarrow P$$

establishes a natural isomorphism between $\tilde{\Psi}\tilde{\Phi}$ and $\text{id}_{\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))}$. It is clear that $\tilde{\eta}_P$'s are isomorphisms. We know that η is a natural isomorphism. In particular, for any $\alpha \in \mathcal{M}$ and any morphism $\tilde{f}: \alpha_*^* P \rightarrow Q$ in $\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$, the following diagram is commutative:

$$\begin{array}{ccc} \Psi\Phi \alpha_*^* P & \xrightarrow{\eta_{\alpha_*^* P}} & \alpha_*^* P \\ \Psi\Phi \tilde{f} \downarrow & & \downarrow \tilde{f} \\ \Psi\Phi Q & \xrightarrow{\eta_Q} & Q. \end{array}$$

On the other hand, we need to establish the commutativity of the diagrams

$$\begin{array}{ccc} \tilde{\Psi}\tilde{\Phi}P & \xrightarrow{\tilde{\eta}_P} & P \\ \tilde{\Psi}\tilde{\Phi}[\tilde{f}, \alpha^*] \downarrow & & \downarrow [\tilde{f}, \alpha^*] \\ \tilde{\Psi}\tilde{\Phi}Q & \xrightarrow{\tilde{\eta}_Q} & Q. \end{array}$$

Using Equation (10) and the results of Steps (4),(7) and (9), we obtain the desired:

$$\begin{aligned}
\tilde{\eta}_Q \circ (\tilde{\Psi}\tilde{\Phi}[\tilde{f}, \alpha^*]) &= [\eta_Q, \text{id}_{\mathbb{N}}^*] \circ [\Psi\Phi\tilde{f}, \alpha^*] \\
&= [\eta_Q \circ (\Psi\Phi\tilde{f}), \alpha^*] \\
&= [\tilde{f} \circ \eta_{\alpha^*P}, \alpha^*] \\
&= [\tilde{f} \circ (\alpha^*\eta_P), \alpha^*] \\
&= [\tilde{f}, \alpha^*] \circ [\eta_P, \text{id}_{\mathbb{N}}^*] \\
&= [\tilde{f}, \alpha^*] \circ \tilde{\eta}_P.
\end{aligned}$$

(11) By [19, Prop. 2.2]), we have $\Phi\Psi = \text{id}_{\mathcal{O}Cov_{\text{fin}}}$. Hence, it is easy to see that the family of identity morphisms $(\text{id}_A, \text{id}_{\mathbb{N}})$ in $\widetilde{\mathcal{A}ux}$ establishes a natural isomorphism between $\tilde{\Phi}\tilde{\Psi}$ and $\text{id}_{\widetilde{\mathcal{A}ux}}$. □

Our next step is to define an equivalence relation on $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$. Let $[\tilde{f}, \alpha^*], [\tilde{g}, \beta^*] : P \rightarrow Q$ be morphisms in $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$. We say that they are equivalent ($[\tilde{f}, \alpha^*] \sim [\tilde{g}, \beta^*]$) if $\tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)} = \tilde{g}_{\mathbb{P}^\infty(\mathbb{Z}/2)}$ as morphisms of algebras (c.f. the equivalence relation on $\mathcal{A}ux$, Lemma 2.7). By [29, Proposition II.8.1], we know that the quotient category $\widetilde{\mathcal{A}ux}/\sim$ exists. Moreover, it is easy to see that the relation \sim preserves the compositions of morphisms. Hence, by the proof of [29, Proposition II.8.1], we do not need to extend the relation \sim to form a quotient category. Note that that the equivalence class of the morphism $[\tilde{f}, \alpha^*]$ in $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ can be represented by $\tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)}$. Therefore, the quotient functor $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim$ is defined on morphisms as

$$(61) \quad [\tilde{f}, \alpha^*] \longmapsto \tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)}.$$

In other words,

$$(62) \quad [\tilde{f}, \alpha^*]_{\sim} := \tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)}.$$

The final step to arrive our classification of finite coverings by finitely-supported flabby sheaves is as follows:

Lemma 2.12. *The functors $\tilde{\Psi} : \widetilde{\mathcal{A}ux} \rightarrow \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$ and $\tilde{\Phi} : \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) \rightarrow \widetilde{\mathcal{A}ux}$ send equivalent morphisms to equivalent morphisms. They descend to functors between quotient categories*

$$(63) \quad \begin{array}{ccc} \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) & \xrightarrow{\tilde{\Psi}} & \widetilde{\mathcal{A}ux} & & \widetilde{\mathcal{A}ux} & \xrightarrow{\tilde{\Phi}} & \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim & \xrightarrow{\tilde{\Psi}} & \widetilde{\mathcal{A}ux}/\sim & & \widetilde{\mathcal{A}ux}/\sim & \xrightarrow{\tilde{\Phi}} & \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim \end{array},$$

establishing an equivalence between $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim$ and $\widetilde{\mathcal{A}ux}/\sim$.

Proof. Note that for any morphism f in $\mathcal{OCov}_{\text{fin}}$ and any morphism \tilde{f} in $\text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$, we have the following equalities of algebra maps:

$$(64) \quad (\Psi f)_{\mathbb{P}^\infty(\mathbb{Z}/2)} = f, \quad \Phi \tilde{f} = \tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)}.$$

It follows that, if $(f, \alpha) \sim (g, \beta)$ in $\widetilde{\mathcal{A}ux}$, then

$$(65) \quad \widetilde{\Psi}(f, \alpha) = [\Psi f, \alpha^*] \sim [\Psi g, \beta^*] = \widetilde{\Psi}(g, \beta)$$

in $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$. Similarly, if $[\tilde{f}, \alpha^*] \sim [\tilde{g}, \beta^*]$ in $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))$, then

$$(66) \quad \widetilde{\Phi}[\tilde{f}, \alpha^*] = (\Phi \tilde{f}, \alpha) \sim (\Phi \tilde{g}, \beta) = \widetilde{\Phi}[\tilde{g}, \beta^*].$$

□

Summarizing the results of this section, we obtain the following commutative diagram of functors:

$$(67) \quad \begin{array}{ccccc} & & \mathcal{Cov}_{\text{fin}} & \xrightarrow{\quad} & \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim \\ & \nearrow \mathfrak{Z} & \uparrow & & \nearrow \widetilde{\Psi} \\ \mathcal{A}ux & \xrightarrow{\quad} & \widetilde{\mathcal{A}ux}/\sim & & \\ & \searrow \sim & & & \\ & & \text{Sh}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)) & \xrightarrow{\quad} & \widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2)). \\ & \nearrow \Psi & \uparrow & & \nearrow \widetilde{\Psi} \\ \mathcal{OCov}_{\text{fin}} & \xrightarrow{\quad} & \widetilde{\mathcal{A}ux} & & \end{array}$$

Using the above diagram, we immediately conclude the first main result of this article:

Theorem 2.13. *For any $(A, \Pi) \in \text{Ob}(\mathcal{Cov}_{\text{fin}})$, $F \in \text{Ob}(\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim)$, $f \in \text{Mor}(\mathcal{Cov}_{\text{fin}})$, and $[\tilde{f}, \alpha^*]_{\sim} \in \text{Mor}(\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim)$, the following assignments*

$$\begin{aligned} (A, \Pi) &\longmapsto \{U \mapsto A/R^\Pi(U)\}_{U \in \mathbf{0p}(\mathbb{P}^\infty(\mathbb{Z}/2))} \in \text{Ob}(\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim), \\ F &\longmapsto (F(\mathbb{P}^\infty(\mathbb{Z}/2)), \{\text{Res}_0(F), \text{Res}_1(F), \dots, \text{Res}_n(F), \dots\}) \in \text{Ob}(\mathcal{Cov}_{\text{fin}}), \\ f &\longmapsto [\Psi(f), \alpha_f^*]_{\sim} \in \text{Mor}(\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim), \\ [\tilde{f}, \alpha^*]_{\sim} &\longmapsto \tilde{f}_{\mathbb{P}^\infty(\mathbb{Z}/2)} \in \text{Mor}(\mathcal{Cov}_{\text{fin}}), \end{aligned}$$

are functors establishing an equivalence of categories between the category $\mathcal{Cov}_{\text{fin}}$ of finite coverings of algebras and the quotient category $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim$ of the category of finitely-supported flabby sheaves of algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$ with extended morphisms. Here $(A, \underline{\Pi})$ is the image of (A, Π) under an equivalence inverse to \mathfrak{Z} , and α_f is a tame surjection defined as in (55).

Observe that the equivalence of the above theorem is, essentially, the identity on morphisms. This is because, on both sides of the equivalence, morphisms considered as input data are only algebra

homomorphisms (see (62) and Definition 2.2). They do, however, satisfy quite different conditions to be considered morphisms in an appropriate category. Thus the essence of the theorem is to re-intertpret the natural defining conditions on an algebra homomorphism to be a morphism of coverings to more refined conditions that make it a morphism between sheaves. What we gain this way is a functorial description of coverings by the more potent concept of a sheaf. We know now that lattice operations applied to a covering will again yield a covering.

We end this section by stating Theorem 2.13 in the classical setting of the Gelfand-Neumark equivalence [16, Lem. 1] between the category of compact Hausdorff spaces and the opposite category of unital commutative C^* -algebras. Since the intersection of closed ideals in a C^* -algebra equals their product, the lattices of closed ideals in C^* -algebras are always distributive. Therefore, remembering that the epimorphisms of commutative unital C^* -algebras can be equivalently presented as the pullbacks of embeddings of compact Hausdorff spaces, we obtain:

Corollary 2.14. *The category of finite closed coverings of compact Hausdorff spaces (see the beginning of this section) is equivalent to the opposite of the quotient category $\widetilde{\text{Sh}}_{\text{fin}}(\mathbb{P}^\infty(\mathbb{Z}/2))/\sim$ of finitely-supported flabby sheaves of commutative unital C^* -algebras over $\mathbb{P}^\infty(\mathbb{Z}/2)$ with extended morphisms.*

3. QUANTUM PROJECTIVE SPACE $\mathbb{P}^N(\mathcal{T})$ FROM TOEPLITZ CUBES

In this section, the tensor product means the C^* -completed tensor product. Accordingly, we use the Heynemann-Sweedler notation for the completed tensor product. Since all C^* -algebras that we tensor are nuclear, this completion is unique. Therefore, it is also maximal, which guarantees the flatness of the completed tensor product. We use this property in our arguments.

3.1. Multipullback C^* -algebra.

As a starting point for our noncommutative deformation of a complex projective space, we take Diagram (44) from Section 1.5 and replace the algebra $C(D)$ of continuous functions on the unit disc by the Toeplitz algebra \mathcal{T} algebra considered as the algebra of continuous functions on a quantum disc [24]. Recall that the Toeplitz algebra is the universal C^* -algebra generated by z and z^* satisfying $z^*z = 1$. There is a well-known short exact sequence of C^* -algebras

$$(68) \quad 0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

Here σ is the so-called symbol map defined by mapping z to the unitary generator u of the algebra $C(S^1)$ of continuous functions on a circle. Note that the kernel of the symbol map is the algebra \mathcal{K} of compact operators.

Viewing S^1 as the unitary group $U(1)$, we obtain a compact quantum group structure on the algebra $C(S^1)$. Here the antipode is determined by $S(u) = u^{-1}$, the counit by $\varepsilon(u) = 1$, and finally the comultiplication by $\Delta(u) = u \otimes u$. Using this Hopf-algebraic terminology on the C^* -level makes sense due to the commutativity of $C(S^1)$. The coaction of $C(S^1)$ on \mathcal{T} comes from

the gauge action of $U(1)$ on \mathcal{T} that rescales z by the elements of $U(1)$, i.e., $z \mapsto \lambda z$. Explicitly, we have:

$$(69) \quad \rho : \mathcal{T} \longrightarrow \mathcal{T} \otimes C(S^1) \cong C(S^1, \mathcal{T}), \quad \rho(z) := z \otimes u, \quad \rho(z)(\lambda) = \lambda z, \quad \rho(t) := t^{(0)} \otimes t^{(1)},$$

Next, we employ the multiplication map m of $C(S^1)$ and the flip map

$$(70) \quad C(S^1) \otimes \mathcal{T}^{\otimes n} \ni f \otimes t_1 \otimes \cdots \otimes t_n \xrightarrow{\tau_n} t_1 \otimes \cdots \otimes t_n \otimes f \in \mathcal{T}^{\otimes n} \otimes C(S^1)$$

to extend ρ to the diagonal coaction $\rho_n : \mathcal{T}^{\otimes n} \longrightarrow \mathcal{T}^{\otimes n} \otimes C(S^1)$ defined inductively by

$$(71) \quad \rho_1 = \rho, \quad \rho_{n+1} = (\text{id}_{\mathcal{T}^{\otimes n+1}} \otimes m) \circ (\text{id}_{\mathcal{T}} \otimes \tau_n \otimes \text{id}_{C(S^1)}) \circ (\rho \otimes \rho_n).$$

Furthermore, for all $0 \leq i < j \leq N$, we define an isomorphism Ψ_{ij}

$$(72) \quad \chi_j \circ \Psi \circ \chi_{i+1}^{-1} : \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1} \xrightarrow{\Psi_{ij}} \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j}.$$

Here χ_j is given by

$$(73) \quad \text{id}_{\mathcal{T}^{\otimes j-1}} \otimes \tau_{N-j}^{-1} : \mathcal{T}^{\otimes N-1} \otimes C(S^1) \xrightarrow{\chi_j} \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j}$$

and Ψ by

$$(74) \quad (\text{id}_{\mathcal{T}^{\otimes N-1}} \otimes (S \circ m)) \circ (\rho_{N-1} \otimes \text{id}_{C(S^1)}) : \mathcal{T}^{\otimes N-1} \otimes C(S^1) \xrightarrow{\Psi} \mathcal{T}^{\otimes N-1} \otimes C(S^1).$$

Before proceeding further, let us prove the unipotent property of Ψ , which we shall need later on.

Lemma 3.1. $\Psi \circ \Psi = \text{id}_{\mathcal{T}^{\otimes N-1} \otimes C(S^1)}$

Proof. For any $\bigotimes_{1 \leq i < N} t_i \otimes h \in \mathcal{T}^{\otimes N} \otimes C(S^1)$, we compute:

$$(75) \quad \begin{aligned} (\Psi \circ \Psi) \left(\bigotimes_{1 \leq i < N} t_i \otimes h \right) &= \Psi \left(\bigotimes_{1 \leq i < N} t_i^{(0)} \otimes S \left(\prod_{1 \leq i < N} t_i^{(1)} h \right) \right) \\ &= \bigotimes_{1 \leq i < N} t_i^{(0)} \otimes S \left(\left(\prod_{1 \leq i < N} t_i^{(1)} \right) S \left(\prod_{1 \leq i < N} t_j^{(2)} h \right) \right) \\ &= \bigotimes_{1 \leq i < N} t_i^{(0)} \otimes S \left(\left(\prod_{1 \leq i < N} (t_i^{(1)} S(t_i^{(2)})) \right) S(h) \right) \\ &= \bigotimes_{1 \leq i < N} t_i \otimes h. \end{aligned}$$

□

Finally, to justify our construction of a quantum complex projective space, observe that the map Ψ_{ij} can be easily seen as an analogue of the pullback of the map Υ_{ij} of (42).

Definition 3.2. We define the C^* -algebra $C(\mathbb{P}^N(\mathcal{T}))$ as the limit of the diagram:

$$\begin{array}{ccccccc}
0 & \cdots & & i & \cdots & & j & \cdots & N \\
\mathcal{T}^{\otimes N} & \cdots & & \mathcal{T}^{\otimes N} & \cdots & & \mathcal{T}^{\otimes N} & \cdots & \mathcal{T}^{\otimes N} \\
& & & \downarrow \sigma_j & & & \downarrow \sigma_{i+1} & & \\
& & & \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} & \xleftarrow{\Psi_{ij}} & \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-i-1} & \cdots & \cdots &
\end{array}$$

Here $\sigma_k := \text{id}_{\mathcal{T}^{\otimes k-1}} \otimes \sigma \otimes \text{id}_{\mathcal{T}^{\otimes N-k}}$. We call $\mathbb{P}^N(\mathcal{T})$ a Toeplitz quantum complex projective space.

Note that by definition $C(\mathbb{P}^N(\mathcal{T})) \subseteq \prod_{i=0}^N \mathcal{T}^{\otimes N}$. We will denote the restrictions of the canonical projections on the components by

$$(76) \quad \pi_i : C(\mathbb{P}^N(\mathcal{T})) \longrightarrow \mathcal{T}^{\otimes N}, \quad \forall i \in \{0, \dots, N\}.$$

Since these maps are C^* -homomorphisms, the lattice generated by their kernels is automatically distributive. On the other hand, it follows from Lemma 3.4 that any element in the Toeplitz cube $\mathcal{T}^{\otimes n}$ can be complemented into a sequence that is an element of $C(\mathbb{P}^N(\mathcal{T}))$. This means that the maps (76) are surjective. Hence they form a covering of $C(\mathbb{P}^N(\mathcal{T}))$. By Theorem 2.13, this covering gives rise to the following sheaf of C^* -algebras:

$$(77) \quad \text{Op}(\mathbb{P}^\infty(\mathbb{Z}/2)) \ni U \xrightarrow{F} C(\mathbb{P}^N(\mathcal{T}))/R^\pi(U).$$

In particular, $F(\mathbb{A}_i) \cong \mathcal{T}^{\otimes N}$ and $F(\mathbb{A}_i \cap \mathbb{A}_j) \cong \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j}$ for all i and j . Furthermore, $F(\mathbb{A}_i \cup \mathbb{A}_j)$ is isomorphic to the pullback of two copies of $\mathcal{T}^{\otimes N}$:

$$(78) \quad
\begin{array}{ccc}
& F(\mathbb{A}_i \cup \mathbb{A}_j) & \\
& \swarrow & \searrow \\
\mathcal{T}^{\otimes N} & & \mathcal{T}^{\otimes N} \\
& \searrow \sigma_j & \swarrow \Psi_{ij} \circ \sigma_{i+1} \\
& \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j} &
\end{array}$$

The construction of $\mathbb{P}^N(\mathcal{T})$ is a generalization of the construction of the mirror quantum sphere [20, p. 734], i.e., $\mathbb{P}^1(\mathcal{T})$ is the mirror quantum sphere:

$$(79) \quad C(\mathbb{P}^1(\mathcal{T})) := \{(t_0, t_1) \in \mathcal{T} \times \mathcal{T} \mid \sigma(t_0) = S(\sigma(t_1))\}.$$

Removing S from this definition yields the C^* -algebra of the generic Podleś sphere [32]. The latter not only is not isomorphic with $C(\mathbb{P}^1(\mathcal{T}))$, but also is not Morita equivalent to $C(\mathbb{P}^1(\mathcal{T}))$. We conjecture that, by similar changes in maps Ψ_{ij} , also for $N > 1$ we can create non-equivalent quantum spaces.

3.2. The defining covering lattice of $\mathbb{P}^N(\mathcal{T})$ is free.

The goal of this subsection is to demonstrate that the distributive lattice of ideals generated by $\ker \pi_i$'s is free. To this end, we will need to know whether the tensor products $\mathcal{T}^{\otimes N}$ of Toeplitz algebras glue together to form $\mathbb{P}^N(\mathcal{T})$ in such a way that a partial gluing can be always extended to a full space. The following result gives the sufficient conditions:

Proposition 3.3. [12, Prop. 9] *Let $\{B_i\}_{i \in \{0, \dots, N\}}$ and $\{B_{ij}\}_{i, j \in \{0, \dots, N\}, i \neq j}$ be two families of C^* -algebras such that $B_{ij} = B_{ji}$ and let $\{\pi_j^i : B_i \rightarrow B_{ij}\}_{ij}$ be a family of surjective C^* -algebra maps. Also, let $\pi_i : B \rightarrow B_i$, $0 \leq i \leq N$, be the restrictions of the canonical projections on the components, where $B := \{(b_i)_i \in \prod_i B_i \mid \pi_j^i(b_i) = \pi_i^j(b_j)\}$. Assume that*

$$(1) \quad \pi_j^i(\ker \pi_k^i) = \pi_i^j(\ker \pi_k^j),$$

(2) *the isomorphisms $\pi_k^{ij} : B_i / (\ker \pi_j^i + \ker \pi_k^i) \longrightarrow B_{ij} / \pi_j^i(\ker \pi_k^i)$ defined as*

$$b_i + \ker \pi_j^i + \ker \pi_k^i \longmapsto \pi_j^i(b_i) + \pi_j^i(\ker \pi_k^i)$$

satisfy

$$(\pi_j^{ik})^{-1} \circ \pi_j^{ki} = (\pi_k^{ij})^{-1} \circ \pi_k^{ji} \circ (\pi_i^{jk})^{-1} \circ \pi_i^{kj}.$$

Then, if for $I \subsetneq \{0, \dots, N\}$ there exists an element $(b_i)_{i \in I} \in \prod_{i \in I} B_i$ such that $\pi_j^i(b_i) = \pi_i^j(b_j)$ for all $i, j \in I$, there also exists $(c_i)_{i \in \{0, \dots, N\}} \in \prod_{i \in \{0, \dots, N\}} B_i$ such that $\pi_j^i(c_i) = \pi_i^j(c_j)$ for all $i, j \in \{0, \dots, N\}$ and $c_i = b_i$ for all $i \in I$.

In the case of quantum projective spaces $\mathbb{P}^N(\mathcal{T})$, we can translate algebras and maps from Proposition 3.3 as follows:

$$(80) \quad B_i = \mathcal{T}^{\otimes N}, \quad B_{ij} = \mathcal{T}^{\otimes j-1} \otimes C(S^1) \otimes \mathcal{T}^{\otimes N-j}, \quad \text{where } i < j,$$

$$(81) \quad \pi_j^i = \begin{cases} \sigma_j & \text{when } i < j, \\ \Psi_{ji} \circ \sigma_{j+1} & \text{when } i > j. \end{cases}$$

It follows that

$$(82) \quad \ker \pi_j^i = \begin{cases} \ker \sigma_j = \mathcal{T}^{\otimes j-1} \otimes \mathcal{K} \otimes \mathcal{T}^{\otimes N-j} & \text{when } i < j, \\ \ker \sigma_{j+1} = \mathcal{T}^{\otimes j} \otimes \mathcal{K} \otimes \mathcal{T}^{\otimes N-j-1} & \text{when } i > j. \end{cases}$$

Since $\rho(\mathcal{K}) \subseteq \mathcal{K} \otimes C(S^1)$ and Ψ is an isomorphism by Lemma 3.1, it follows that

$$(83) \quad \Psi(\mathcal{T}^{\otimes j-1} \otimes \mathcal{K} \otimes \mathcal{T}^{\otimes N-j-1} \otimes C(S^1)) = \mathcal{T}^{\otimes j-1} \otimes \mathcal{K} \otimes \mathcal{T}^{\otimes N-j-1} \otimes C(S^1).$$

Now we can formulate and prove the following:

Lemma 3.4. *If $(b_i)_{i \in I} \in \prod_{i \in I \subsetneq \{0, \dots, N\}} \mathcal{T}^{\otimes N}$ satisfies $\pi_j^i(b_i) = \pi_i^j(b_j)$ for all $i, j \in I$, $i \neq j$, then there exists an element $b \in C(\mathbb{P}^N(\mathcal{T}))$ such that $\pi_i(b) = b_i$ for all $i \in I$.*

Proof. It is enough to check that the assumptions of Proposition 3.3 are satisfied. For the sake of brevity, we will omit the tensor symbols in the long formulas in what follows. We will also write \mathcal{S} instead of $C(S^1)$. Here we prove the first condition of Proposition 3.3:

- (1) $\pi_i^j(\ker \pi_k^j) = (\chi_j \circ \Psi \circ \chi_{i+1}^{-1} \circ \sigma_{i+1})(\ker \sigma_{k+1})$
 $= (\chi_j \circ \Psi \circ \chi_{i+1}^{-1})(\mathcal{T}^i \mathcal{S} \mathcal{T}^{k-i-1} \mathcal{K} \mathcal{T}^{N-k-1}) = \chi_j(\mathcal{T}^{k-1} \mathcal{K} \mathcal{T}^{N-k-1} \mathcal{S})$
 $= \mathcal{T}^{k-1} \mathcal{K} \mathcal{T}^{j-k-1} \mathcal{S} \mathcal{T}^{N-j} = \sigma_j(\ker \sigma_k) = \pi_j^i(\ker \pi_k^i)$, when $i < k < j$.
- (2) $\pi_i^j(\ker \pi_k^j) = (\chi_j \circ \Psi \circ \chi_{i+1}^{-1} \circ \sigma_{i+1})(\ker \sigma_k) = (\chi_j \circ \Psi \circ \chi_{i+1}^{-1})(\mathcal{T}^i \mathcal{S} \mathcal{T}^{k-i-2} \mathcal{K} \mathcal{T}^{N-k})$
 $= \chi_j(\mathcal{T}^{k-2} \mathcal{K} \mathcal{T}^{N-k} \mathcal{S}) = \mathcal{T}^{j-1} \mathcal{S} \mathcal{T}^{k-j-1} \mathcal{K} \mathcal{T}^{N-k} = \sigma_j(\ker \sigma_k) = \pi_j^i(\ker \pi_k^i)$,
when $i < j < k$.
- (3) $\pi_i^j(\ker \pi_k^j) = (\chi_j \circ \Psi \circ \chi_{i+1}^{-1} \circ \sigma_{i+1})(\ker \sigma_{k+1})$
 $= (\chi_j \circ \Psi \circ \chi_{i+1}^{-1})(\mathcal{T}^k \mathcal{K} \mathcal{T}^{i-k-1} \mathcal{S} \mathcal{T}^{N-i-1}) = \chi_j(\mathcal{T}^k \mathcal{K} \mathcal{T}^{N-k-2} \mathcal{S}) = \mathcal{T}^k \mathcal{K} \mathcal{T}^{j-k-2} \mathcal{S} \mathcal{T}^{N-j}$
 $= \sigma_j(\ker \sigma_{k+1}) = \pi_j^i(\ker \pi_k^i)$, when $k < i < j$.

For the second condition, note first that for any multivalued map $f : B_j \rightarrow B_i$, we define the function

$$(84) \quad [f]_k^{ij} : B_j / (\ker \pi_i^j + \ker \pi_k^j) \longrightarrow B_i / (\ker \pi_i^i + \ker \pi_k^i),$$

$$b_j + \ker \pi_i^j + \ker \pi_k^j \longmapsto f(b_j) + \ker \pi_i^i + \ker \pi_k^i.$$

whenever the assignment (84) is unique. In particular, since the condition (1) of Proposition 3.3 is fulfilled, we can write the map $\phi_k^{ij} := (\pi_k^{ij})^{-1} \circ \pi_k^{ji}$ as $[(\pi_j^i)^{-1} \circ \pi_i^j]_k^{ij}$. Explicitly, in our case, this map reads:

$$(85) \quad \phi_k^{ij} = \begin{cases} [\sigma_j^{-1} \circ \chi_j \circ \Psi \circ \chi_{i+1}^{-1} \circ \sigma_{i+1}]_k^{ij} & \text{when } i < j, \\ [\sigma_{j+1}^{-1} \circ \chi_{j+1} \circ \Psi \circ \chi_i^{-1} \circ \sigma_i]_k^{ij} & \text{when } i > j. \end{cases}$$

We need to prove that

$$(86) \quad \phi_k^{ij} = \phi_j^{ik} \circ \phi_i^{kj}, \quad \text{for all } i, j, k.$$

Since $(\phi_k^{ij})^{-1} = \phi_k^{ji}$, one can see readily that it is enough to limit ourselves to the case when $i < k < j$. Indeed, e.g., if $i < j < k$ then the equation (86) follows directly from $\phi_j^{ik} = \phi_k^{ij} \circ \phi_i^{jk}$. Next, let us denote the class of $(t_1 \otimes \cdots \otimes t_N) = \bigotimes_{1 \leq n \leq N} t_n \in \mathcal{T}^{\otimes N}$ in $\mathcal{T}^{\otimes N} / (\ker \pi_i^j + \ker \pi_k^j)$ by $[\bigotimes_{1 \leq n \leq N} t_n]_{ik}^j$. Then, using the Heynemann-Sweedler notation for completed tensor product, we compute:

$$(87) \quad \begin{aligned} \phi_k^{ij} \left(\left[\bigotimes_{1 \leq n \leq N} t_n \right]_{ik}^j \right) &= [\sigma_j^{-1} \circ \chi_j \circ \Psi \circ \chi_{i+1}^{-1} \circ \sigma_{i+1}]_k^{ij} \left(\left[\bigotimes_{1 \leq n \leq N} t_n \right]_{ik}^j \right) \\ &= [(\sigma_j^{-1} \circ \chi_j \circ \Psi) \left(\bigotimes_{\substack{1 \leq n \leq N \\ n \neq i+1}} t_n \otimes \sigma(t_{i+1}) \right)]_{jk}^i \\ &= [(\sigma_j^{-1} \circ \chi_j) \left(\bigotimes_{\substack{1 \leq n \leq N \\ n \neq i+1}} t_n^{(0)} \otimes S(\sigma(t_{i+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq i+1}} t_m^{(1)}) \right)]_{jk}^i \\ &= \left[\bigotimes_{\substack{1 \leq n \leq j \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S)(\sigma(t_{i+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq i+1}} t_m^{(1)}) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i. \end{aligned}$$

Applying the above formula twice (with changed non-dummy indices), we obtain:

$$\begin{aligned}
& (\phi_j^{ik} \circ \phi_i^{kj}) \left(\left[\bigotimes_{1 \leq n \leq N} t_n \right]_{ik}^j \right) = \phi_j^{ik} \left(\left[\bigotimes_{\substack{1 \leq n \leq j \\ n \neq k+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{k+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq k+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{ji}^k \right) \\
& = \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1}^{(0)}) \left((\sigma^{-1} \circ S) \left(\sigma(t_{k+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq k+1}} t_m^{(1)} \right) \right)^{(1)} \prod_{\substack{1 \leq w \leq N \\ w \neq i+1 \\ w \neq k+1}} t_w^{(0)(1)} \right) \right. \\
(88) \quad & \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_n^{(0)(0)} \otimes \left((\sigma^{-1} \circ S) \left(\sigma(t_{k+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq k+1}} t_m^{(1)} \right) \right)^{(0)} \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)(0)} \right]_{jk}^i.
\end{aligned}$$

Now, as $\sigma^{-1} : C(S^1) \rightarrow \mathcal{T}/\mathcal{K}$ is colinear, S is an anti-coalgebra map, and Δ is an algebra homomorphism, we can move the Heynemann-Sweedler indices inside the bold parentheses:

$$\begin{aligned}
& \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1}^{(0)}) S \left(\sigma(t_{k+1})^{(1)} \prod_{\substack{1 \leq m \leq N \\ m \neq k+1}} t_m^{(1)(1)} \right) \prod_{\substack{1 \leq w \leq N \\ w \neq i+1 \\ w \neq k+1}} t_w^{(0)(1)} \right) \right. \\
(89) \quad & \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_n^{(0)(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{k+1})^{(2)} \prod_{\substack{1 \leq m \leq N \\ m \neq k+1}} t_m^{(1)(2)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)(0)} \right]_{jk}^i.
\end{aligned}$$

Here we can renumber the Heynemann-Sweedler indices using the coassociativity of Δ . We can also use the anti-multiplicativity of S to move it inside the bold parentheses in the first line of the above calculation. Finally, we use the commutativity of $C(S^1)$ in order to reshuffle the argument of $\sigma^{-1} \circ S$ in the first line to obtain:

$$\begin{aligned}
& \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1}^{(0)}) S(t_{i+1}^{(1)}) S \left(\sigma(t_{k+1})^{(1)} \prod_{\substack{1 \leq w \leq N \\ w \neq i+1 \\ w \neq k+1}} (t_w^{(1)} S(t_w^{(2)})) \right) \right) \right. \\
(90) \quad & \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{k+1})^{(2)} t_{i+1}^{(2)} \prod_{\substack{1 \leq m \leq N \\ m \neq k+1 \\ m \neq i+1}} t_m^{(3)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i.
\end{aligned}$$

We can simplify the expression in the bold parentheses in the first line using $h^{(1)} S(h^{(2)}) = \varepsilon(h)$ and $\varepsilon(h^{(1)}) h^{(2)} = h$. This results in:

$$\begin{aligned}
& \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1}^{(0)}) S(t_{i+1}^{(1)}) S \left(\sigma(t_{k+1})^{(1)} \right) \right) \right. \\
(91) \quad & \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{k+1})^{(2)} t_{i+1}^{(2)} \prod_{\substack{1 \leq m \leq N \\ m \neq k+1 \\ m \neq i+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i.
\end{aligned}$$

By the colinearity of σ , we can substitute in the above expression

$$\begin{aligned}
& \sigma(t_{i+1}^{(0)}) \otimes t_{i+1}^{(1)} \mapsto \sigma(t_{i+1})^{(1)} \otimes \sigma(t_{i+1})^{(2)}, \\
(92) \quad & \sigma(t_{k+1})^{(1)} \otimes \sigma(t_{k+1})^{(2)} \mapsto \sigma(t_{k+1}^{(0)}) \otimes t_{k+1}^{(1)},
\end{aligned}$$

to derive:

$$(93) \quad \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1})^{(1)} S(\sigma(t_{i+1})^{(2)}) S(\sigma(t_{k+1}^{(0)})) \right) \right. \\ \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_r^{(0)} \otimes (\sigma^{-1} \circ S) \left(t_{k+1}^{(1)} \sigma(t_{i+1})^{(3)} \prod_{\substack{1 \leq m \leq N \\ m \neq k+1 \\ m \neq i+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i.$$

Applying again the antipode and counit properties yields the desired

$$(94) \quad \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(S(\sigma(t_{k+1}^{(0)})) \right) \right. \\ \left. \otimes \bigotimes_{k+2 \leq r \leq j} t_r^{(0)} \otimes (\sigma^{-1} \circ S) \left(t_{k+1}^{(1)} \sigma(t_{i+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq k+1 \\ m \neq i+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i \\ = \left[\bigotimes_{\substack{1 \leq n \leq k \\ n \neq i+1}} t_n^{(0)} \otimes t_{k+1}^{(0)} \otimes \bigotimes_{k+2 \leq r \leq j} t_r^{(0)} \otimes (\sigma^{-1} \circ S) \left(t_{k+1}^{(1)} \sigma(t_{i+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq k+1 \\ m \neq i+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i \\ = \left[\bigotimes_{\substack{1 \leq n \leq j \\ n \neq i+1}} t_n^{(0)} \otimes (\sigma^{-1} \circ S) \left(\sigma(t_{i+1}) \prod_{\substack{1 \leq m \leq N \\ m \neq i+1}} t_m^{(1)} \right) \otimes \bigotimes_{j+1 \leq s \leq N} t_s^{(0)} \right]_{jk}^i \\ = \phi_k^{ij} \left(\left[\bigotimes_{1 \leq n \leq N} t_n \right]_{ik}^j \right).$$

□

As an immediate consequence of Birkhoff's Representation Theorem (see our primer on lattices for more details), one sees that two finite distributive lattices are isomorphic if and only if their posets of meet irreducibles are isomorphic. In particular, consider a free distributive lattice generated by $\lambda_0, \dots, \lambda_N$. It is isomorphic to the lattice of upper sets of the set of all subsets of $\{0, \dots, N\}$ (e.g., see [19, Sect.2.2]). The elements of the form $\bigvee_{i \in I} \lambda_i$, where $\emptyset \neq I \subsetneq \{0, \dots, N\}$, are all distinct and meet irreducible. The first property can be deduced from the upper-set model of a finite free distributive lattice, and the latter holds for any finite distributive lattice. Indeed, suppose the contrary, i.e., that there exists a meet-irreducible element whose any presentation $\bigvee_{a \in \alpha} \bigwedge_{i \in a} \lambda_i$ is such that there is a set $a_0 \in \alpha$ that contains at least two elements. Now, the finiteness allows us to apply induction, and the distributivity combined with irreducibility allows us to make the induction step yielding the desired contradiction. Furthermore, using again the upper-set model of a finite free distributive lattice, one can easily check that the partial order of its meet-irreducibles elements is given by

$$(95) \quad \bigvee_{i \in I} \lambda_i \leq \bigvee_{j \in J} \lambda_j \quad \text{if and only if} \quad I \subseteq J, \quad \forall I, J \neq \emptyset, \quad I, J \subsetneq \{0, \dots, N\}.$$

Summarizing, we conclude that in order to prove that a given finitely generated distributive lattice is free it suffices to demonstrate that the joins of generators $\{\bigvee_{i \in I} \lambda_i\}_{\emptyset \neq I \subseteq \{0, \dots, N\}}$ are all distinct, meet irreducible, and satisfy (95).

Lemma 3.5. *For any nonempty subsets $I, J \subseteq \{0, \dots, N\}$, the ideals $\bigcap_{i \in I} \ker \pi_i$ are all distinct, and we have*

$$\bigcap_{i \in I} \ker \pi_i \supseteq \bigcap_{j \in J} \ker \pi_j \quad \text{if and only if} \quad I \subseteq J.$$

Proof. The “if”-implication is obvious. For the “only if”-implication, take $0 \neq x \in \mathcal{K}^{\otimes N}$ and, for any nonempty $I \subseteq \{0, \dots, N\}$, define

$$(96) \quad x_I := (x_i)_{i \in \{0, \dots, N\}} \in \bigcap_{i \in I} \ker \pi_i, \quad \text{where} \quad x_i := \begin{cases} x & \text{if } i \notin I, \\ 0 & \text{if } i \in I. \end{cases}$$

Let $I, J \subseteq \{0, \dots, N\}$ be nonempty, and assume that $I \setminus J$ is nonempty. Then it follows that

$$(97) \quad x_J \in \left(\bigcap_{j \in J} \ker \pi_j \right) \setminus \left(\bigcap_{i \in I} \ker \pi_i \right) \neq \emptyset.$$

This means that $\bigcap_{j \in J} \ker \pi_j \not\subseteq \bigcap_{i \in I} \ker \pi_i$, as desired. It follows that $\bigcap_{i \in I} \ker \pi_i$ are all distinct. \square

Lemma 3.6. *The ideals $\bigcap_{i \in I} \ker \pi_i$ are all meet (sum) irreducible for any $\emptyset \neq I \subseteq \{0, \dots, N\}$.*

Proof. We proceed by contradiction. Suppose that $\bigcap_{i \in I} \ker \pi_i$ is not meet irreducible for some $\emptyset \neq I \subseteq \{0, \dots, N\}$. By Lemma 3.5, $\bigcap_{i \in I} \ker \pi_i \neq \{0\}$ because $I \neq \{0, \dots, N\}$. Hence there exist ideals

$$(98) \quad a_\mu = \sum_{J \in \mathcal{J}_\mu} \bigcap_{j \in J} \ker \pi_j, \quad \mathcal{J}_\mu \subseteq \mathbf{2}^{\{0, \dots, N\}}, \quad \mu \in \{1, 2\},$$

such that

$$(99) \quad \bigcap_{i \in I} \ker \pi_i = a_1 + a_2, \quad \text{and} \quad a_1, a_2 \neq \bigcap_{i \in I} \ker \pi_i.$$

In particular, $a_\mu \subseteq \bigcap_{i \in I} \ker \pi_i$, $\mu \in \{1, 2\}$. On the other hand, if $I \in \mathcal{J}_\mu$, then $a_\mu \supseteq \bigcap_{i \in I} \ker \pi_i$. Hence $a_\mu = \bigcap_{i \in I} \ker \pi_i$, contrary to our assumption. It follows that, if $\bigcap_{i \in I} \ker \pi_i$ is not meet irreducible, then

$$(100) \quad \bigcap_{i \in I} \ker \pi_i = \sum_{J \in \mathcal{J}} \bigcap_{j \in J} \ker \pi_j, \quad \text{for some} \quad \mathcal{J} \subseteq \mathbf{2}^{\{0, \dots, N\}} \setminus \{I\}.$$

Suppose next that $I \setminus J_0$ is nonempty for some $J_0 \in \mathcal{J}$, and let $k \in I \setminus J_0$. Then

$$(101) \quad \{0\} = \pi_k \left(\bigcap_{i \in I} \ker \pi_i \right) = \pi_k \left(\sum_{J \in \mathcal{J}} \bigcap_{j \in J} \ker \pi_j \right) \supseteq \pi_k \left(\bigcap_{j \in J_0} \ker \pi_j \right).$$

However, by Lemma 3.5 we see that $(\bigcap_{j \in J_0} \ker \pi_j) \setminus \ker \pi_k$ is nonempty. Hence $\pi_k(\bigcap_{j \in J_0} \ker \pi_j)$ is not $\{0\}$, and we have a contradiction. It follows that for all $J_0 \in \mathcal{J}$ the set $I \setminus J_0$ is empty, i.e., $\forall J_0 \in \mathcal{J} : I \subsetneq J_0$.

Finally, let $m \in \{0, \dots, N\} \setminus I$, and let

$$(102) \quad T_m^I := t_1 \otimes \cdots \otimes t_N, \quad \text{where} \quad 0 \neq t_n \in \begin{cases} \mathcal{K} & \text{if } m < n \in I \text{ or } m > n - 1 \in I, \\ \mathcal{T} \setminus \mathcal{K} & \text{if } m < n \notin I \text{ or } m > n - 1 \notin I. \end{cases}$$

Note that $\pi_k^m(T_m^I) = 0$ if and only if $k \in I$. Hence, by Lemma 3.4, there exists $p_m \in \pi_m^{-1}(T_m^I) \cap \bigcap_{i \in I} \ker \pi_i$. Next, we define

$$(103) \quad \sigma_I^m := f_1 \otimes \cdots \otimes f_N, \quad \text{where} \quad f_n := \begin{cases} \text{id}_{\mathcal{T}} & \text{if } m < n \in I \text{ or } m > n - 1 \in I, \\ \sigma & \text{if } m < n \notin I \text{ or } m > n - 1 \notin I, \end{cases}$$

so that $\sigma_I^m(\pi_m(p_m)) \neq 0$. On the other hand, by our assumption (100), and the property that $J_0 \supsetneq I$ for all $J_0 \in \mathcal{J}$, we have

$$(104) \quad 0 \neq p_m \in \bigcap_{i \in I} \ker \pi_i \subseteq \sum_{J \supsetneq I} \bigcap_{j \in J} \ker \pi_j.$$

Furthermore, for any $x \in C(\mathbb{P}^N(\mathcal{T}))$, we have

$$(105) \quad \sigma_I^m(\pi_m(x)) = 0 \quad \text{if} \quad \pi_k^m(\pi_m(x)) = 0 \quad \text{for some} \quad k \notin I.$$

Now, for any $J \supsetneq I$, we choose $k_J \in J \setminus I$, so that

$$(106) \quad \pi_{k_J}^m \left(\pi_m \left(\bigcap_{j \in J \supsetneq I} \ker \pi_j \right) \right) \subseteq \pi_m^{k_J}(\pi_{k_J}(\ker \pi_{k_J})) = \{0\}.$$

Combining this with (105), we obtain $\sigma_I^m(\pi_m(\bigcap_{j \in J \supsetneq I} \ker \pi_j)) = \{0\}$ for all $J \supsetneq I$. Consequently, $\sigma_I^m(\pi_m(\sum_{J \supsetneq I} \bigcap_{j \in J} \ker \pi_j)) = \{0\}$, which contradicts (104), and ends the proof. \square

Summarizing, Lemma 3.5 and Lemma 3.6 combined with the Birkhoff Representation Theorem yield the second main result of this paper:

Theorem 3.7. *Let $C(\mathbb{P}^N(\mathcal{T})) \subset \prod_{i=0}^N \mathcal{T}^{\otimes N}$ be the C^* -algebra of the Toeplitz quantum projective space, defined as the limit of Diagram (3.2), and let*

$$\pi_i : C(\mathbb{P}^N(\mathcal{T})) \longrightarrow \mathcal{T}^{\otimes N}, \quad i \in \{0, \dots, N\},$$

be the family of restrictions of the canonical projections onto the components. Then the family of ideals $\{\ker \pi_i \mid 0 \leq i \leq N\}$ generates a free distributive lattice.

3.3. Other quantum projective spaces. In the Introduction, we compare our construction of quantum complex projective spaces with the construction coming from quantum groups. Let us complete this picture and end by describing other noncommutative versions of complex projective spaces that we found in the literature.

3.3.1. *Noncommutative projective schemes.* Projective spaces à la Artin-Zhang [4] and Rosenberg [33] are based on Gabriel’s Reconstruction Theorem [15, Ch. VI] (cf. [34]) and Serre’s Theorem [36, Prop. 7.8] (cf. [18, Vol. II, 3.3.5]). The former theorem describes how to reconstruct a scheme from its category of quasi-coherent sheaves. The latter establishes how to obtain the category of quasi-coherent sheaves over the projective scheme corresponding to a conical affine scheme. First, one constructs a graded algebra A of polynomials on this conical affine scheme and then, according to Serre’s recipe, one divides the category of graded A -modules by the subcategory of graded modules that are torsion. Such graded algebras corresponding to projective manifolds have finite global dimension, admit a dualizing module, and their Hilbert series have polynomial growth. All this means that they are, so called, *Artin–Schelter regular algebras*, or AS-regular algebras in short [2] (cf. [3]). This property makes sense for algebras which are not necessarily commutative, so that we think about noncommutative algebras of this sort as of generalized noncommutative projective manifolds. One important subclass of such well-behaving algebras are Sklyanin algebras [38]. Among other nice properties, they are quadratic Koszul, have finite Gelfand-Kirillov dimension [37], and are Cohen-Macaulay [28]. Another class of AS-regular algebras worth mentioning is the class of hyperbolic rings [33], which are also known as generalized Weyl algebras [6], or as generalized Laurent polynomial rings [13].

3.3.2. *Quantum deformations of Grassmanian and flag varieties.* In [41], Taft and Towber develop a direct approach to quantizing the Grassmanians, or more generally, flag varieties. They define a particular deformation of algebras of functions on the classical Grassmanians and flag varieties using an explicit (in terms of generators and relations) construction of affine flag schemes defined by Towber [42, 43]. Their deformation utilizes q -determinants [41, Defn. 1.3] (cf. [23, pg. 227] and [25, pg.312]) used to construct a q -deformed version of the exterior product [41, Sect. 2]. This yields a class of algebras known as quantum exterior algebras [7]. These quantum exterior algebras are different from Weyl algebras or Clifford algebras. They provide counter examples for a number of homological conjectures for finite dimensional algebras, even though they behave well cohomologically. See [7, Sect. 1] for more details.

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