INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

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Shigeru Kanemitsu, Takako Kuzumaki
Jerzy Urbanowicz

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On congruences for the sums $\sum_{i=1}^{[n/r]} \frac{\chi_n(i)}{i^k}$ of E. Lehmer's type*

Shigeru Kanemitsu† Takako Kuzumaki‡
Jerzy Urbanowicz§

Abstract

Let $n > 1$ be an odd natural number and let r $(1 < r < n)$ be a natural number relatively prime to n. Denote by χ_n the trivial character modulo n. We prove some new congruences for the sums $T_{r,k}(n) = \sum_{i=1}^{[n/r]} (\chi_n(i)/i^k) \pmod{n^{s+1}}$ for $s \in \{0, 1, 2\}$, for all divisors r of 24 and for some natural numbers $k \geq 1$, in particular for $k = 1$ or 2 in all the cases. These congruences are obtained by using an identity proved in [16], which was earlier successfully exploited in [16], [13] and [7] to solve some other problems. The congruences generalize those obtained by M. Lerch [12], E. Lehmer [11] and Z.-H. Sun [14] in the case when $n = p$ is an odd prime. We obtain 82 new congruences for $T_{r,k}(n)$. Two congruences for $T_{r,k}(n) \pmod{n^2}$ were proved in [1], resp. [9] for $(r, k) = (2, 1)$, resp. $(4, 2)$.

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1 Notation and introduction

Let $n > 1$ be odd and let $\chi_{0,n}$ (sometimes abbreviated as χ_n) be the trivial Dirichlet character modulo n (with $\chi_{0,1}$ designating the constant function $\chi_{0,1}(x) = 1$ for all integers x). For $r \geq 2$ prime to n denote by $q_r(n)$ the Euler quotient, i.e.,

$$q_r(n) = \frac{r^{\phi(n)} - 1}{n}.$$

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†Graduate School of Advanced Technology Kinki University Iizuka, Fukuoka, Japan; kanemitu@fuk.kindai.ac.jp
‡Department of Electrical Engineering, Faculty of Engineering, Gifu University, Gifu, Japan; kuzumaki@gifu-u.ac.jp
§Institute of Mathematics, Polish Academy of Sciences, and Institute of Computer Science, Polish Academy of Sciences, Warsaw, Poland; urbanowi@impan.pl

Here and throughout the paper ϕ is the Euler *phi*-function and $B_{n,\chi}$ denotes the n-th generalized Bernoulli number attached to the Dirichlet character χ. For definitions see [18], [8] or [17].

Given the discriminant d of a quadratic field, let χ_d denote its quadratic character (Kronecker symbol). We shall denote by $\chi_{d,n}$ the character χ_d modulo n.

It was proved in [4] that the numbers $B_{i,\chi_d}/i$ are rational integers unless $d = -4$ or $d = \pm p$, where p is an odd prime of a special form. If $d = -4$ and i is odd, then the numbers $E_{i-1} = -2B_{i,\chi_{-4}}/i$ are odd integers, called the Euler numbers. If $d = \pm p$, then the numbers B_{i,χ_d} have p in their denominators and $pB_{i,\chi_d} \equiv p-1 \pmod{p^{\mathrm{ord}_p(i)+1}}$.

We consider the ordinary Bernoulli numbers B_i(1) and the so-called D-numbers defined in [10] and [6] by $D_{i-1} = -3B_{i,\chi_{-3}}/i$ for i odd, having powers of 3 in their denominators. We also consider the rational integers $A_{i-1} = B_{i,\chi_8}/i$, $F_{i-1} = B_{i,\chi_{-3}\chi_{-4}}/i$ and $G_{i-1} = B_{i,\chi_{-3}\chi_{-8}}/i$, if $i \geq 2$ even, and $C_{i-1} = -B_{i,\chi_{-8}}/i$ and $H_{i-1} = -B_{i,\chi_{-3}\chi_8}/i$ if $i \geq 1$ odd.

In this paper we shall consider congruences for the character sums with negative weight

$$T_{r,k}(n) = \sum_{0<i<n/r} \frac{\chi_n(i)}{i^k}$$

modulo powers n^{s+1} for $n > 1$ odd and $s \in \{0,1,2\}$ where $\chi_n = \chi_{0,n}$ and r ($r \mid 24$ and $1 < r < n$) is coprime to n, and $k \geq 1$ is subject to the condition $k \leq n^s\phi(n)$. Note that since $\chi_n(i) = 0$ for $(i,n) > 1$, the sum is over $(i,n) = 1$.

The central role in this paper is played by an identity proved in [16]. Let χ be a Dirichlet character modulo M, N a positive integral multiple of M, and $r\,(>1)$ a positive integer prime to N. Then for any integer $m \geq 0$ we have

(1)

$$(m+1)r^m \sum_{0<n<N/r} \chi(n)n^m = -B_{m+1,\chi}r^m + \frac{\overline{\chi}(r)}{\phi(r)} \sum_{\psi\in G(r)} \overline{\psi}(-N)B_{m+1,\chi\psi}(N)\,,$$

where the sum on the right hand side is taken over all Dirichlet characters ψ modulo r. We denote by $G(r)$ the group of all such characters; then $\#G(r) = \phi(r)$. Here $B_{n,\chi}(X) = \sum_{i=0}^{n}\binom{n}{i}B_{n-i,\chi}X^i$ denotes the n-th generalized Bernoulli polynomial attached to χ. Since $r \mid 24$, the group $G(r)$ has exponent 2 and all characters modulo r are quadratic.

(1) Which are generalized Bernoulli numbers attached to the trivial primitive character $\chi_{0,1}$ (except when $i = 1$; then $B_{1,\chi_{0,1}} = 1/2 = -B_1$).

If the character χ modulo M is induced from a character $\widetilde{\chi}$ modulo some divisor of M then

$$B_{n,\chi} = B_{n,\widetilde{\chi}} \prod_{p|M} \left(1 - \widetilde{\chi}(p)p^{n-1}\right), \tag{2}$$

where the product is taken over all primes p dividing M.

If $(i,n) = 1$, then by Euler's theorem we have $i^{\phi(n)} \equiv 1 \,(\mathrm{mod}\, n)$, and more generally,

$$i^{\phi(n)n^s} \equiv 1 \,(\mathrm{mod}\, n^{s+1})$$

for $s \geq 0$.

Given r prime to n and integers $s \geq 0$, $k \geq 1$ we denote

$$S_{r,k,s}(n) = \sum_{0<i<n/r} \chi_n(i) i^{n^s\phi(n)-k} .$$

Then we have the congruence

$$T_{r,k}(n) \equiv S_{r,k,s}(n) \,(\mathrm{mod}\, n^{s+1}), \tag{3}$$

which allows us to study $T_{r,k}(n)$ through $S_{r,k,s}(n)$.

The main results of the paper are congruences for the sums $T_{r,k}(n)$ modulo n^{s+1} for $s \in \{0,1,2\}$. The congruences will be obtained by applying identity (1) to the sums $S_{r,k,s}(n)$.(2) They extend those proved by M. Lerch [12], E. Lehmer [11] and Z.-H. Sun [14] in the case when $n = p$ is an odd prime. In principle, the congruences in this particular case have a different form from those obtained for any natural odd n. Sometimes it is not easy to derive the former congruences from the latter. We shall do it in the second part of the paper.

Two such congruences modulo n^2 were earlier obtained, by using (1), in [1] for $r = 2$, $k = 1$ and in [9] for $r = 4$, $k = 2$. In the present paper we find 82 new congruences for the sums $T_{r,k}(n) \,(\mathrm{mod}\, n^{s+1})$ for $s \in \{0,1,2\}$, $r \mid 24$ and $k \geq 1$, in particular for $k = 1$ or 2. Most of our congruences for $T_{r,k}(n)$ have not been known earlier even in the particular case when $n = p$ is a prime. The machinery introduced in [16] is much more efficient than the methods exploited in [12], [11] and [14]. In an appendix to the paper we shall extend some congruences of E. Lehmer's type proved in [2] and [3].

(2) This identity was earlier successfully exploited in [16], [13] and [7] to solve some other problems. See also the book [17] devoted to the identity and related problems.

2 Some auxiliary formulae

The idea exploited in [1] and [9] to use identity (1) to extend classical congruences for the sums $T_{r,k}(n)$ seems to be very efficient. This identity allows us to obtain almost automatically many new congruences. Usually the proofs using (1) are much easier, more unified and much shorter than those applying other methods.

The general scheme of reasoning is uniform. To obtain congruences for the sums $T_{r,k}(n)$ modulo n^{s+1} we first determine, using (1), the sums $S_{r,k,s}(n)$ modulo n^{s+1}. We substitute in (1) $m = n^s\phi(n) - k \geq 0$ (and so $m + 1 = n^s\phi(n) - k + 1$) and $N = M = n$. Since $r \mid 24$ we assume that $n > 1$ is odd; then we have $(n, r) = 1$. If $3 \nmid r$ then we have $(n, r) = 1$. If $3 \mid r$, then we additionally assume that n is not divisible by 3. Note that, since $r \mid 24$, all generalized Bernoulli numbers occurring in $S_{r,k,s}(n)$ are rational.

Thus, throughout the paper, we write $m = n^s\phi(n) - k \geq 0$.(3) Consequently, we obtain

$$S_{r,k,s}(n) = S_1 + S_2\,, \tag{4}$$

where, by (2),

$$S_1 = -\frac{B_{m+1,\chi_{0,n}}}{m+1} = -\frac{B_{m+1}}{m+1}\prod_{p\mid n}\left(1 - p^m\right) \tag{5}$$

and

$$S_2 = \frac{1}{\phi(r)(m+1)r^m}\sum_{\psi\in G(r)}\psi(-n)B_{m+1,\chi_{0,n}\psi}(n)\,.$$

Note that $\chi_{0,n}$ is even. Thus, if $m \neq 0$ is even, then $B_{m+1} = 0$, and so $S_1 = 0$. If $m = 0$, then $1 - p^m = 0$, and so $S_1 = 0$ too. Otherwise, in view of (5), we have $S_1 \neq 0$. Furthermore,

$$\begin{aligned}
S_2 &= \frac{1}{\phi(r)(m+1)r^m}\sum_{\psi\in G(r)}\psi(-n)\sum_{i=0}^{m+1}\binom{m+1}{i}B_{i,\chi_{0,n}\psi}n^{m+1-i}\\
&= \frac{1}{\phi(r)(m+1)r^m}\sum_{i=0}^{m+1}\binom{m+1}{i}n^{m+1-i}\sum_{\psi\in G(r)}\psi(-n)B_{i,\chi_{0,n}\psi}\\
&= \frac{n^{m+1}}{\phi(r)(m+1)r^m}\sum_{\psi\in G(r)}\psi(-n)B_{0,\chi_{0,n}\psi}
\end{aligned}$$

(3) That is, $k \leq n^s\phi(n)$.

$$+\frac{1}{\phi(r)(m+1)r^m}\sum_{i=1}^{m+1}\binom{m+1}{i}n^{m+1-i}\sum_{\psi\in G(r)}\psi(-n)B_{i,\chi_{0,n}\psi}$$
$$=\frac{n^m\phi(n)}{(m+1)r^{m+1}}$$
$$+\frac{1}{\phi(r)(m+1)r^m}\sum_{i=0}^{m}\binom{m+1}{i+1}n^{m-i}\sum_{\psi\in G(r)}\psi(-n)B_{i+1,\chi_{0,n}\psi}$$

because $B_{0,\chi_{0,n}\psi}=0$ if ψ is not trivial modulo r and

$$B_{0,\chi_{0,n}\chi_{0,r}}=\frac{\phi(rn)}{rn}$$

otherwise, and hence (recall that $(r,n)=1$)

$$\frac{n^{m+1}}{\phi(r)(m+1)r^m}\sum_{\psi\in G(r)}\psi(-n)B_{0,\chi_{0,n}\psi}=\frac{n^m\phi(n)}{(m+1)r^{m+1}}\,.$$

Consequently,

$$S_2=\Theta_s+\frac{1}{\phi(r)r^m}\sum_{i=0}^{m}\binom{m}{i}n^{m-i}U_i(r)\,, \tag{6}$$

where

$$\Theta_s=\Theta_s(n,m,r)=\frac{n^m\phi(n)}{(m+1)r^{m+1}}$$

and

$$U_i(r)=\sum_{\psi\in G(r)}\psi(-n)\frac{B_{i+1,\chi_{0,n}\psi}}{i+1}\,.$$

2.1 $U_i(r)$ for $r \mid 24$

Let $n>1$ be odd and relatively prime to r. Here and subsequently, we set

$$\widetilde{B}_i=B_i\prod_{p\mid n}\left(1-p^{i-1}\right),$$
$$\widetilde{A}_i=(-1)^{\frac{n^2-1}{8}}A_i\prod_{p\mid n}\left(1-(-1)^{\frac{p^2-1}{8}}p^i\right),$$
$$\widetilde{C}_i=(-1)^{\frac{(n-1)(n+5)}{8}}C_i\prod_{p\mid n}\left(1-(-1)^{\frac{(p-1)(p+5)}{8}}p^i\right),$$

$$\widetilde{D}_i = (-1)^{\nu(n)} D_i \prod_{p|n} \bigl(1-(-1)^{\nu(p)} p^i\bigr),$$
$$\widetilde{E}_i = (-1)^{\frac{n-1}{2}} E_i \prod_{p|n} \bigl(1-(-1)^{\frac{p-1}{2}} p^i\bigr),$$
$$\widetilde{F}_i = (-1)^{\frac{n-1}{2}+\nu(n)} F_i \prod_{p|n} \bigl(1-(-1)^{\frac{p-1}{2}+\nu(p)} p^i\bigr),$$
$$\widetilde{G}_i = (-1)^{\frac{(n-1)(n+5)}{8}+\nu(n)} G_i \prod_{p|n} \bigl(1-(-1)^{\frac{(p-1)(p+5)}{8}+\nu(p)} p^i\bigr),$$
$$\widetilde{H}_i = (-1)^{\frac{n^2-1}{8}+\nu(n)} H_i \prod_{p|n} \bigl(1-(-1)^{\frac{p^2-1}{8}+\nu(p)} p^i\bigr),$$

where $\chi_{-3}(n) = (-1)^{\nu(n)}$, $\nu(n) = 0$, resp. 1 if $n \equiv 1$, resp. $-1 \pmod 3$.

In the following, we compute $U_i(r)$ for $r = 2$, 3, 4, 6, 8, 12 or 24.

1. Case $r = 2$

Then $\#G(2) = 1$ and $G(2) = \{\chi_{0,2}\}$. Then, by definition and identity (2),

$$U_i(2) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\bigl(1-2^i\bigr), & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases} \tag{7}$$

2. Case $r = 3$

Then $\#G(3) = 2$ and $G(3) = \{\chi_{0,3}, \chi_{-3}\}$. Then, by definition and identity (2),

$$U_i(3) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\bigl(1-3^i\bigr), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i, & \text{if } i \text{ is even.} \end{cases} \tag{8}$$

3. Case $r = 4$

Then $\#G(4) = 2$ and $G(4) = \{\chi_{0,4}, \chi_{-4}\}$. Thus, by definition and the same arguments as in the case $r = 3$ (note that both characters χ_{-3} and χ_{-4} are odd), in view of (2) we obtain

$$U_i(4) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\bigl(1-2^i\bigr), & \text{if } i \text{ is odd;} \\ \frac{1}{2}\widetilde{E}_i, & \text{if } i \text{ is even.} \end{cases} \tag{9}$$

4. Case $r = 6$

Then $\#G(6) = 2$ and $G(6) = \{\chi_{0,6}, \chi_{-3,6}\}$. Consequently, by (2) and the same arguments as in the previous case we obtain

$$U_i(6) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\big(1-2^i\big)\big(1-3^i\big), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i\big(1+2^i\big), & \text{if } i \text{ is even.} \end{cases} \tag{10}$$

5. Case $r = 8$

Then $\#G(8) = 4$ and $G(8) = \{\chi_{0,8}, \chi_{-4,8}, \chi_{-8}, \chi_8\}$. Therefore, in view of (2),

$$U_i(8) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\big(1-2^i\big) + \widetilde{A}_i, & \text{if } i \text{ is odd;} \\ \frac{1}{2}\widetilde{E}_i + \widetilde{C}_i, & \text{if } i \text{ is even.} \end{cases} \tag{11}$$

6. Case $r = 12$

Then $\#G(12) = 4$ and $G(12) = \{\chi_{0,12}, \chi_{-3,12}, \chi_{-4,12}, \chi_{(-3)(-4)}\}$. Consequently, by definition and (2),

$$U_i(12) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\big(1-2^i\big)\big(1-3^i\big) + \widetilde{F}_i, & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i\big(1+2^i\big) + \frac{1}{2}\widetilde{E}_i\big(1+3^i\big), & \text{if } i \text{ is even.} \end{cases} \tag{12}$$

7. Case $r = 24$

Then $\#G(24) = 8$ and

$$G(24) = \{\chi_{0,24}, \chi_{-3,24}, \chi_{-4,24}, \chi_{(-3)(-4),24}, \chi_{(-3)(-8)}, \chi_{(-3)8}, \chi_{-8,24}, \chi_{8,24}\}.$$

Consequently, in view of (2),

$$U_i(24) = \begin{cases} \frac{\widetilde{B}_{i+1}}{i+1}\big(1-2^i\big)\big(1-3^i\big) + \widetilde{F}_i + \widetilde{G}_i + \widetilde{A}_i\big(1+3^i\big), & \text{if } i \text{ is odd;} \\ \frac{1}{3}\widetilde{D}_i\big(1+2^i\big) + \frac{1}{2}\widetilde{E}_i\big(1+3^i\big) + \widetilde{H}_i + \widetilde{C}_i\big(1-3^i\big), & \text{if } i \text{ is even.} \end{cases} \tag{13}$$

2.2 The sums $S_{r,k,s}(n) \pmod{n^{s+1}}$ for $m > s$, $r \mid 24$, $s \le 2$

The generalized Bernoulli numbers attached to Dirichlet characters modulo r, with $r \mid 24$, are rational numbers. In what follows we consider congruences for $S_{r,k,s}(n)$ modulo n^{s+1} for $n > 1$ odd and $s \in \{0, 1, 2\}$. We assume that n is not divisible by 3 if $3 \mid r$; then r and $\phi(r)$ are coprime to n.

It is shown in the previous section that the numbers $U_i(r)$ are linear combinations of the numbers $\widetilde{A}_i$, $\widetilde{C}_i$, $\widetilde{D}_i$, $\widetilde{E}_i$, $\widetilde{F}_i$, $\widetilde{G}_i$, $\widetilde{H}_i$ and the quotients $\frac{\widetilde{B}_{i+1}}{i+1}$. Denote by $U_i^{odd}(r)$, resp. $U_i^{even}(r)$ the sum $U_i(r)$ taken over odd, resp. even characters ψ modulo r. Note that $U_i(r) = U_i^{odd}(r) + U_i^{even}(r)$ and $U_i^{odd}(r) = 0$ $U_i^{even}(r) = 0$ if i is odd or even, respectively.

First we recall some divisibility properties of the quotients $\frac{B_{i+1,\chi}}{i+1}$ for primitive Dirichlet characters χ of conductors $f_\chi \mid nr$. These quotients, multiplied by some Euler factors, are summands of U_i. We start with some elementary lemmas on the quotients $\frac{B_{i+1}}{i+1}$ of the ordinary Bernoulli numbers. Lemma 1 is called the von Staudt and Clausen theorem. Lemma 2 due to L. Carlitz is its generalization.

Lemma 1. (See [18, Theorem 5.10] or [8, Corollary to Theorem 3, p. 233]). *Let k be an even natural number and let p be a prime number. Then B_k contains p in its denominator if and only if $p-1 \mid k$ and $pB_k \equiv -1 \pmod{p}$.*

Lemma 2. (See [5].) *If $p^\nu(p-1) \mid k$, $\nu \geq 0$ then $pB_k \equiv p-1 \pmod{p^{\nu+1}}$.*

Lemma 3. (See [8, Proposition 15.2.4, p. 238]). *If $p-1 \nmid k$ then the quotients B_k/k are p-integral.*

Since conductors of non-trivial characters occurring in $U_i(r)$ are coprime to n, they are not powers of a prime divisor of n. In such cases we have a useful lemma:

Lemma 4. (See [6, Theorem 1.5].) *Let χ be a primitive Dirichlet character with conductor f_χ. If f_χ is not a power of a given prime number p, then the quotients $\frac{B_{n,\chi}}{n}$ $(n \geq 1)$ are p-integral.*

We set $NTU_i^{even}(r) = U_i^{even}(r) - \frac{\widetilde{B}_{i+1}}{i+1}\prod_{p|r}\left(1-p^i\right)$. By Lemma 4 we obtain:

Lemma 5. *Let r be coprime to p for a given prime number $p \mid n$. Then the numbers $U_i^{odd}(r)$ for i even and the numbers $NTU_i^{even}(r)$ for i odd are p-integral.*

Assume that $m = n^s\phi(n) - k > s$ for $s \in \{0,1,2\}$.(4) Since for odd $n > 1$ $\phi(n)$ is even, m and k are of the same parity. We divide each of the cases $s = 0, 1$ or 2 into two subcases:

(i) if k is even (example: $k = 2$),

(ii) if k is odd (example: $k = 1$).

Our purpose is to obtain some congruences for the sums $S_{r,k,s}(n)$ modulo n^{s+1} for $s \in \{0,1,2\}$, and next using congruence (3) to obtain congruences for the sums $T_{r,k}(n)$. We prove that the latter sums are congruent modulo n^{s+1} to linear combinations of the quotients $\widetilde{B}_m/m$ and some of the numbers

(4) Then $k < n^s\phi(n) - s$.

$\widetilde{A}_{m-1}$, $\widetilde{C}_m$, $\widetilde{C}_{m-2}$, $\widetilde{D}_m$, $\widetilde{D}_{m-2}$, $\widetilde{E}_m$, $\widetilde{E}_{m-2}$, $\widetilde{F}_{m-1}$, $\widetilde{G}_{m-1}$, $\widetilde{H}_m$, $\widetilde{H}_{m-2}$ if k is even, and of the quotients $\widetilde{B}_{m-1}/(m-1)$, $\widetilde{B}_{m+1}/(m+1)$ and some of the numbers $\widetilde{A}_m$, $\widetilde{A}_{m-2}$, $\widetilde{C}_{m-1}$, $\widetilde{D}_{m-1}$, $\widetilde{E}_{m-1}$, $\widetilde{F}_m$, $\widetilde{F}_{m-2}$, $\widetilde{G}_m$, $\widetilde{G}_{m-2}$, $\widetilde{H}_{m-1}$ if k is odd.(5)

We start with the study of the case $s = 2$. Next, similarly, we derive the remaining congruences modulo n^2 and modulo n. First we show when the numbers Θ_s (defined in (6)) are congruent to 0 modulo n^{s+1}.

Lemma 6. *Let $n > 1$ be odd and let $1 < r \leq n$ be coprime to n. Assume that $m > s$ and $p \mid n$ is a prime. Then the numbers Θ_s in* (6) *are p-integral and*

$$\Theta_s = \frac{n^m \phi(n)}{(m+1)r^{m+1}} \equiv 0 \pmod{n^{s+1}}$$

except when $s = 1$, $3 \| n$, $3 \nmid \phi(n)$ and $m = 2$.(6)

Proof. First we prove that the numbers Θ_s are p-integral for $m \geq s+1$. It suffices to show that $m\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq 0$. Let us define the function $g(x) = x - \log_p(x+1)$, which is increasing for $x \geq 1$. Since $\log_p(m+1) \geq \mathrm{ord}_p(m+1)$ and $\mathrm{ord}_p(n) \geq 1$ we obtain that

$$m\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq m - \log_p(m+1) = g(m) \geq g(s+1) > 0$$

because $g(3) = 3 - \log_p(5) > 0$, $g(2) = 2 - \log_p(4) > 0$ and $g(1) = 1 - \log_p(2) > 0$ for any prime p.

Let us consider the functions $f_s(x) = x - s - \log_p(x+1)$ for $x \geq 1$, which are increasing for $x \geq 1$.(7) Note that the congruence $\Theta_s \equiv 0 \pmod{n^{s+1}}$ for $m > s$ holds if and only if

$$(m-s)\mathrm{ord}_p(n) + \mathrm{ord}_p(\phi(n)) - \mathrm{ord}_p(m+1) > 0$$

for every $p \mid n$.

In view of $\log_p(m+1) \geq \mathrm{ord}_p(m+1)$ and $\mathrm{ord}_p(n) \geq 1$ the above follows from the inequality $f_s(m) > 0$ for $m \geq 3$ if $s = 1, 2$, and for $m \geq 1$ if $s = 0$ because

$$(m-s)\mathrm{ord}_p(n) - \mathrm{ord}_p(m+1) \geq (m-s) - \log_p(m+1) = f_s(m)$$

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(5)As well as of Euler's quotients $q_2(n)$ or $q_3(n)$ if $k = 1$.

(6)Then $\Theta_1 = n^2\phi(n)/3r^3$ and the exceptional $n's$ have the form $n = 3\prod_{i=1}^{u} p_i^{e_i}$ where $p_i \equiv 2 \pmod 3$ for $i = 1, \ldots, u$. Moreover $k = n\phi(n) - 2$ is even. Obviously, if $k \geq 2$ and $(k-1, n) = 1$, then the congruence $\Theta_1 \equiv 0 \pmod{n^{s+1}}$ is true because $m+1$ and n are coprime. We leave it to the reader to verify that the congruence holds if $k = 1$.

(7)The functions $g(x)$ and $f_s(x)$ are increasing since $g'(x) = f_s'(x) = 1 - \frac{1}{(x+1)\log p} > 0$ for $x \geq 1$.

and $f_s(m) \geq f_2(3) = 1 - \log_p(4) > 0$ if $s = 2$, $f_s(m) \geq f_1(3) = 2 - \log_p(4) > 0$ if $s = 1$ and $f_s(m) \geq f_0(1) = 1 - \log_p(2) > 0$ if $s = 0$ for every $p \mid n$. This gives the congruence $\Theta_s \equiv 0 \pmod{n^{s+1}}$ for $s = 0, 2$ and $m > s$ and $s = 1$ and $m \geq 3$.

In the case when $s = 1$ and $m = 2$ we have $f_1(2) = 1 - \log_p(3) > 0$ if $p \geq 5$, and so the congruence holds for $3 \nmid n$. We are left with the task of checking when the congruence holds for $s = 1$, $m = 2$ and $3 \mid n$. Then it is easily seen that the congruence $\Theta_1 = \frac{n^2\phi(n)}{3r^3} \equiv 0 \pmod{n^2}$ holds if and only if $\mathrm{ord}_3(\phi(n)) \geq 1$. This does not hold if and only if $s = 1$, $3 \| n$, $3 \nmid \phi(n)$, $m = 2$, as claimed. □

2.2.1 The case when $s = 2$

Assume that $m = n^2\phi(n) - k$ and $1 \leq k < n^2\phi(n) - 2$ $(m > 2)$. Then, by Lemma 6, $\Theta_2 \equiv 0 \pmod{n^3}$.

Case (i):

If $k \geq 2$ is even, then $m+1 = n^2\phi(n) - k + 1$ is odd. Consequently $S_1 = 0$ in (4). Thus, combining (4) and (6) gives $S_{r,k,2}(n) = \Theta_2 + S_2 \equiv S_2 \pmod{n^3}$, and

$$\begin{aligned} S_{r,k,2}(n) \equiv S_2 \equiv \frac{1}{\phi(r)r^m}\Big(&U_m^{odd}(r) + mnU_{m-1}^{even}(r) \\ &+ \binom{m}{2} n^2 U_{m-2}^{odd}(r) + \binom{m}{3} n^3 U_{m-3}^{even}(r)\Big) \pmod{n^3} \end{aligned} \tag{14}$$

because for every prime number $p \mid n$, by Lemma 5, the summands $U_m^{odd}(r)$, $\binom{m}{2} n^2 U_{m-2}^{odd}(r)$, $mnU_{m-1}^{even}(r)$ and $\binom{m}{3} n^3 U_{m-3}^{even}(r)$[8] are p-integral.

Case (ii):

If $k \geq 1$ is odd, then $m+1$ is even and $S_1 \neq 0$. Moreover, by Lemma 6, $\Theta_2 \equiv 0 \pmod{n^3}$. Thus, by (4), (5), (6), we obtain

$$\begin{aligned} S_{r,k,2}(n) \equiv -\frac{\widetilde{B}_{m+1}}{m+1} + \frac{1}{\phi(r)r^m}\Big(&U_m^{even}(r) \\ &+ mnU_{m-1}^{odd}(r) + \binom{m}{2} n^2 U_{m-2}^{even}(r)\Big) \pmod{n^3} \end{aligned}$$

[8] With m, $m-2$ even and $m-1$, $m-3$ odd.

since, by Lemmas 4 or 5, $\binom{m}{3} n^3 U_{m-3}^{odd}(r)^{(9)}$ is p-integral for any $p \mid n$ and divisible by n^3.

Consequently, if k is odd and r, $\phi(r)$ are relatively prime to n, we find, by Lemma 4, that

(15)

$$\begin{aligned} S_{r,k,2}(n) \equiv\ & \frac{\widetilde{B}_{m+1}}{m+1}\Big(-1 + \frac{1}{\phi(r) r^m} \prod_{q\mid r} \big(1-q^m\big)\Big) \\ & + \frac{1}{\phi(r) r^m}\Big(NTU_m^{even}(r) + mnU_{m-1}^{odd}(r) + \binom{m}{2} n^2 U_{m-2}^{even}(r)\Big) \pmod{n^3}. \end{aligned}$$

Note that for $p \mid n$, by Lemma 5, the summands $NTU_m^{even}(r)$, $\binom{m}{2} n^2 U_{m-2}^{even}(r)$ and $mnU_{m-1}^{odd}(r)^{(10)}$ are p-integral.

Moreover, if $p \mid n$ and $p-1 \mid m+1$, i.e., p is in the denominator of B_{m+1}, then by the little Fermat theorem, we have $q^m \equiv q^{-1} \pmod{p^{\mathrm{ord}_p(m+1)+1}}$ and $r^m \equiv r^{-1} \pmod{p^{\mathrm{ord}_p(m+1)+1}}$ (recall that r is coprime to n), and

$$-1 + \frac{1}{\phi(r) r^m} \prod_{q\mid r} \big(1-q^m\big) \equiv -1 + \frac{r}{\phi(r)} \prod_{q\mid r} \big(1-q^{-1}\big) = 0 \pmod{p^{\mathrm{ord}_p(m+1)+1}}.$$

Hence and from Lemma 2, it follows that for $p \mid n$ the first summand of the right hand side of (15) is p-integral in the case when $p-1 \mid m+1$. If $p-1 \nmid m+1$, then the same conclusion follows from Lemma 3.

2.2.2 The case when $s = 1$

Assume that $m = n\phi(n) - k$ and $1 \le k < n\phi(n) - 1$ ($m > 1$). Then, by Lemma 6, $\Theta_1 \equiv 0 \pmod{n^2}$ if $m > 2$. If $m = 2$ and $r \mid 8$, then the congruence holds if n is not divisible by 3 or divisible by 9. If $m = 2$ and $3 \| n$, then it is true for $3 \mid \phi(n)$.

Case (i):

If $k \ge 2$ is even, then analysis similar to that in the proof of (14) shows that

$$S_{r,k,1}(n) \equiv \frac{1}{\phi(r) r^m}\Big(U_m^{odd}(r) + mnU_{m-1}^{even}(r)\Big) \pmod{n^2} \tag{16}$$

if $m > 2$ or $m = 2$ and n is not exceptional in the sense of Lemma 6 since $\binom{m}{2} n^2 U_{m-2}^{odd}(r) + \binom{m}{3} n^3 U_{m-3}^{even}(r)$ is divisible by n^2. If $m = 2$ and n is

(9) With $m-3$ even.

(10) With m, $m-2$ odd and $m-1$ even.

exceptional, i.e. $3\|n$ and $3\nmid\phi(n)$, then we should add to the right hand side of (16) the correction $\Theta_1 = n^2\phi(n)/3r^3$, but we prefer to exclude the case when $m=2$, i.e., $k = n\phi(n)-2$.

Case (ii):

If $k\geq 1$ is odd, then by Lemma 6 we have $\Theta_1 \equiv 0 \pmod{n^2}$ and a similar argument to that in the proof of (15) shows that

(17)

$$S_{r,k,1}(n) \equiv \frac{\widetilde{B}_{m+1}}{m+1}\Big(-1+\frac{1}{\phi(r)r^m}\prod_{q|r}(1-q^m)\Big) + \frac{1}{\phi(r)r^m}\Big(NTU_m^{even}(r)+mnU_{m-1}^{odd}(r)+\binom{m}{2}n^2U_{m-2}^{even}(r)\Big) \pmod{n^2}.$$

2.2.3 The case when $s=0$

Assume that $m=\phi(n)-k$ and $1\leq k<\phi(n)$. Then, by Lemma 6, $\Theta_0 \equiv 0 \pmod{n}$.

Case (i):

If $k\geq 2$ is even, then in the same way as in the proof of (16) we obtain

$$S_{r,k,0}(n) \equiv \frac{1}{\phi(r)r^m}\Big(U_m^{odd}(r)+mnU_{m-1}^{even}(r)\Big) \pmod{n}. \tag{18}$$

Case (ii):

If $k\geq 1$ is odd, then by a similar argument to that in the proof of (17) we find

(19)

$$S_{r,k,0}(n) \equiv \frac{\widetilde{B}_{m+1}}{m+1}\Big(-1+\frac{1}{\phi(r)r^m}\prod_{q|r}(1-q^m)\Big)+\frac{1}{\phi(r)r^m}NTU_m^{even}(r) \pmod{n}$$

because $mnU_{m-1}^{odd}(r)+\binom{m}{2}n^2U_{m-2}^{even}(r)$ is divisible by n, which is an easy consequence of Lemmas 1 and 5.

3 The main results of the paper

In this section we compute the sums $T_{r,k}(n) \pmod{n^{s+1}}$ for $s\in\{0,1,2\}$ and all $r\mid 24$, using congruence (3) and congruences for the sums $S_{r,k,s}(n)$, namely congruences (14) and (15) if $s = 2$, (16) and (17) if $s = 1$, and (18) and (19) if $s = 0$.

We divide each of the three cases $s = 0$, 1 or 2 into seven subcases: $r = 2, 3, 4, 6, 8, 12, 24$, obtaining congruences for $T_{r,k}(n)$ for $1 \le k < n^s\phi(n) - s$. In the second part of the paper we shall derive from obtained congruences some congruences in the case when $n = p$ is an odd prime. Some of such congruences were proved by M. Lerch [12], E. Lehmer [11] and Z.-H. Sun [14], but most of them were not earlier known.

We substitute formulae (7–13) into congruences (14), (16) and (18) if k is even and congruences (15), (17) and (19) if k is odd. Consequently, after some calculations, we obtain Theorems 1–35.

In the theorems below, given any $k \ge 1$ and $\rho \in \mathbb{Z}$, we write

$$I(k,\rho) = \{n > 1\colon\ 2 \nmid n \text{ and } p \nmid n \text{ if } p-1 \mid k+\rho\}^{(11)}$$

and

$$Q_2(n) = -2q_2(n) + nq_2^2(n) - \frac{2}{3}n^2q_2^3(n),\ Q_3(n) = -\frac{3}{2}q_3(n) + \frac{3}{4}nq_3^2(n) - \frac{1}{2}n^2q_3^3(n)\,.$$

The sums $T_{r,1}(n)$ presented in Theorems $4, 9, 14, 19, 24, 29$ and 34 below are congruent to linear combinations of Euler's quotients $\widehat{EQ}_r(n)$ plus some generalized Bernoulli numbers where $\widehat{EQ}_2(n) = Q_2(n)$, $\widehat{EQ}_3(n) = Q_3(n)$, $\widehat{EQ}_4(n) = \frac{3}{2}Q_2(n)$, $\widehat{EQ}_6(n) = Q_2(n) + Q_3(n)$, $\widehat{EQ}_8(n) = 2Q_2(n)$, $\widehat{EQ}_{12}(n) = \frac{3}{2}Q_2(n) + Q_3(n)$ and $\widehat{EQ}_{24}(n) = 2Q_2(n) + Q_3(n)$. For $i = 2, 3$ set $Q_i''(n) = Q_i(n) \pmod{n^2}$ and $Q_i'(n) = Q_i(n) \pmod{n}$.

1. Case $r = 2$

Theorem 1. *Given an odd* $n > 1$ *and* $1 \le k < n^2\phi(n) - 2$, *write* $m = n^2\phi(n) - k$. *Then:*
(i)

$$T_{2,k}(n) \equiv \frac{1}{2}\big(2^{k+1}-1\big)n\widetilde{B}_m + \frac{1}{24}\binom{k+1}{2}\big(2^{k+3}-1\big)n^3\widetilde{B}_{m-2} \pmod{n^3}$$

if k *is even, and in particular,*

$$T_{2,k}(n) \equiv \frac{1}{2}\big(2^{k+1}-1\big)n\widetilde{B}_m \pmod{n^3}$$

(11) Note that if k and ρ are of the same parity and $n \in I(k,\rho)$, then $3 \nmid n$; e.g., $I(1,1) = \{n > 1\colon\ 2,3 \nmid n\}$, $I(3,1) = I(2,2) = \{n > 1\colon\ 2,3,5 \nmid n\}$ or $I(5,1) = I(4,2) = \{n > 1\colon\ 2,3,7 \nmid n\}$.

if $n \in I(k,2)$*;*
(ii)

$$T_{2,k}(n) \equiv 2^k\big(1-2^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} - \frac{k}{8}\big(2^{k+2}-1\big)n^2\widetilde{B}_{m-1} \pmod{n^3}$$

if k *is odd.*

Proof. If k is even, resp. odd, then it suffices to apply congruence (14), resp. (15). Substituting (7) into these congruences gives the theorem immediately. □

Theorem 2. *Given an odd* $n > 1$ *and* $1 \le k < n\phi(n) - 2$,[12] *write* $m = n\phi(n) - k$*. Then:*
(i)

$$T_{2,k}(n) \equiv \frac{1}{2}\big(2^{k+1}-1\big)n\widetilde{B}_m \pmod{n^2}$$

if k *is even;*
(ii) (cf. [14] if $n = p$ is an odd prime number)

$$T_{2,k}(n) \equiv 2^k\big(1-2^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} - \frac{k}{8}\big(2^{k+2}-1\big)n^2\widetilde{B}_{m-1} \pmod{n^2}$$

if k *is odd, and in particular,*

$$T_{2,k}(n) \equiv 2^k\big(1-2^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n^2}$$

if $n \in I(k,1)$*.*

Proof. Theorem 2 follows easily from (16), resp. (17) and (7), if k is even, resp. odd. □

Theorem 3. *Given an odd* $n > 1$ *and* $1 \le k < \phi(n)$*, write* $m = \phi(n) - k$*. Then:*
(i)

$$T_{2,k}(n) \equiv \frac{1}{2}\big(2^{k+1}-1\big)n\widetilde{B}_m \pmod{n}$$

if k *is even, and in particular,*

$$T_{2,k}(n) \equiv 0 \pmod{n}$$

[12] Theorem 2(i) is also true for $k = n\phi(n) - 2$ if we assume that n is not exceptional in the sense of Lemma 6; for exceptional n we should add the correction $\Theta_1 = \frac{1}{24}n^2\phi(n)$ to the right hand side of the congruence.

if $n \in I(k,0)$*;*
(ii)

$$T_{2,k}(n) \equiv\, 2^k\big(1-2^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1}\,(\mathrm{mod}\,n)$$

if k *is odd.*

Proof. Theorem 3 follows from (18), resp. (19) if k is even, resp. odd and from (7) in both cases. □

Theorem 4. *Let* $n>1$ *be odd. Then:*
(i) (cf. [14] if $n=p$ is an odd prime)

$$T_{2,1}(n) \equiv\, Q_2(n) - \frac{7}{8}n^2\widetilde{B}_{n^2\phi(n)-2}\,(\mathrm{mod}\,n^3);$$

(ii) (see [1] and cf. [14] if $n=p$ is an odd prime)

$$T_{2,1}(n) \equiv\, Q_2''(n)\,(\mathrm{mod}\,n^2)$$

if n *is not divisible by* 3*;*
(iii) (cf. [11] if $n=p$ is an odd prime)

$$T_{2,1}(n) \equiv\, Q_2'(n)\,(\mathrm{mod}\,n).$$

Proof. This is a particular case of Theorems 1–3(ii) for $k=1$. Then $m+1=n^s\phi(n)$ and, by $2^{\phi(n)}=nq_2(n)+1$, we have

$$2\big(1-2^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} = 2\big(1-(2^{\phi(n)})^{n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)} = 2\big(1-(1+nq_2(n))^{n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$\equiv\, \big(Q_2(n)+\alpha n^3\big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv\, Q_2(n)\,(\mathrm{mod}\,n^{s+1})$$

because $\alpha\in\mathbb{Z}$, $s\le 2$ and

$$\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv\, 1\,(\mathrm{mod}\,n^{s+1})\,. \tag{20}$$

Indeed, if $p_0 \mid n$ is a prime, then $(p_0-1)p_0^{(s+1)\mathrm{ord}_{p_0}(n)-1} \mid n^s\phi(n)$ and, by Lemma 2,

$$\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv\, \frac{n\big(p_0-1\big)}{p_0\phi(n)}\prod_{p\mid n,\,p\neq p_0}\big(1-p^{-1}\big) = 1\,\big(\mathrm{mod}\,p_0^{(s+1)\mathrm{ord}_{p_0}(n)}\big).$$

This completes the proof of (20) and of Theorem 4. □

Theorem 5. *Let $n > 1$ be odd. Then:*
(i)
$$T_{2,2}(n) \equiv \frac{7}{2} n \widetilde{B}_{n^2\phi(n)-2} + \frac{31}{8} n^3 \widetilde{B}_{n^2\phi(n)-4} \pmod{n^3},$$
and in particular
$$T_{2,2}(n) \equiv \frac{7}{2} n \widetilde{B}_{n^2\phi(n)-2} \pmod{n^3}$$
if $3, 5 \nmid n$;
(ii)
$$T_{2,2}(n) \equiv \frac{7}{2} n \widetilde{B}_{n\phi(n)-2} \pmod{n^2};$$
(iii)
$$T_{2,2}(n) \equiv 0 \pmod{n}$$
if $3 \nmid n$.

Proof. This is an immediate consequence of Theorems 1–3(i) for $k = 2$. □

2. Case $r = 3$

Theorem 6. *Given an odd $n > 1$ not divisible by 3 and $1 \le k < n^2\phi(n) - 2$, write $m = n^2\phi(n) - k$. Then:*
(i)
$$T_{3,k}(n) \equiv \frac{3^{k-1}}{2} \widetilde{D}_m + \frac{1}{6}\big(3^{k+1} - 1\big) n \widetilde{B}_m + \frac{3^{k-1}}{2} \binom{k+1}{2} n^2 \widetilde{D}_{m-2} \pmod{n^3}$$
if k is even and $n \in I(k, 2)$;
(ii)
$$T_{3,k}(n) \equiv \frac{3^k}{2}\big(1 - 3^{m+1}\big) \frac{\widetilde{B}_{m+1}}{m+1} - \frac{3^{k-1}}{2} k n \widetilde{D}_{m-1} - \frac{k}{36}\big(3^{k+2} - 1\big) n^2 \widetilde{B}_{m-1} \pmod{n^3}$$
if k is odd.

Proof. For k even, resp. odd we combine formula (8) with congruence (14), resp. (15). Hence the theorem follows at once. □

Theorem 7. *Given an odd $n > 1$ not divisible by 3 and $1 \le k < n\phi(n) - 1$, write $m = n\phi(n) - k$. Then:*
(i)
$$T_{3,k}(n) \equiv \frac{3^{k-1}}{2} \widetilde{D}_m + \frac{1}{6}\big(3^{k+1} - 1\big) n \widetilde{B}_m \pmod{n^2}$$
if k is even;

(ii)

$$T_{3,k}(n) \equiv \frac{3^k}{2}\big(1-3^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} - \frac{3^{k-1}}{2}kn\widetilde{D}_{m-1} \pmod{n^2}$$

if k is odd and $n \in I(k,1)$.

Proof. In this case we substitute (8) into congruences (16) if k is even or (17) if k is odd. □

Theorem 8. *Given an odd $n > 1$ not divisible by 3 and $1 \le k < \phi(n)$, write $m = \phi(n) - k$. Then:*
(i) (cf. [14] if $n = p$ is a prime)

$$T_{3,k}(n) \equiv \frac{3^{k-1}}{2}\widetilde{D}_m \pmod{n}$$

if k is even and $n \in I(k,0)$;
(ii) (cf. [14] if $n = p$ is a prime.)

$$T_{3,k}(n) \equiv \frac{3^k}{2}\big(1-3^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n}$$

if k is odd.

Proof. Theorem 8 follows from (8) and congruences (18) if k is even or (19) if k is odd. □

Theorem 9. *Let $n > 1$ be odd and not divisible by 3. Then:*
(i) (cf. [14] if $n = p$ is a prime)

$$T_{3,1}(n) \equiv Q_3(n) - \frac{1}{2}n\widetilde{D}_{n^2\phi(n)-2} - \frac{13}{18}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3};$$

(ii)

$$T_{3,1}(n) \equiv Q_3''(n) - \frac{1}{2}n\widetilde{D}_{n\phi(n)-2} \pmod{n^2};$$

(iii)

$$T_{3,1}(n) \equiv Q_3'(n) \pmod{n}.$$

Proof. This is a particular case of Theorems 6–8(ii) for $k = 1$. Then $m+1 = n^s\phi(n)$ and, by $3^{\phi(n)} = nq_3(n) + 1$ and (20), we obtain

$$\frac{3}{2}\big(1-3^{m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} = \frac{3}{2}\big(1-(3^{\phi(n)})^{n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)} = \frac{3}{2}\big(1-(1+nq_3(n))^{n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$\equiv \big(Q_3(n) + \beta n^3\big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv Q_3(n) \pmod{n^{s+1}}$$

because $\beta \in \mathbb{Z}$ and $s \le 2$. The rest of the proof is straightforward. □

Theorem 10. *Let $n > 1$ be odd and not divisible by 3. Then:*
(i)

$$T_{3,2}(n) \equiv \frac{3}{2}\widetilde{D}_{n^2\phi(n)-2} + \frac{13}{3}n\widetilde{B}_{n^2\phi(n)-2} + \frac{9}{2}n^2\widetilde{D}_{n^2\phi(n)-4} \pmod{n^3}$$

if n is not divisible by 5;
(ii)

$$T_{3,2}(n) \equiv \frac{3}{2}\widetilde{D}_{n\phi(n)-2} + \frac{13}{3}n\widetilde{B}_{n\phi(n)-2} \pmod{n^2};$$

(iii)

$$T_{3,2}(n) \equiv \frac{3}{2}\widetilde{D}_{\phi(n)-2} \pmod{n}.$$

Proof. This is a particular case of Theorems 6–8(i) for $k = 2$. □

3. Case $r = 4$

Theorem 11. *Given an odd $n > 3$ and $1 \le k < n^2\phi(n) - 2$, write $m = n^2\phi(n) - k$. Then:*
(i)

$$T_{4,k}(n) \equiv 2^{2k-2}\widetilde{E}_m + 2^{k-2}\big(2^{k+1} - 1\big)n\widetilde{B}_m + 2^{2k-2}\binom{k+1}{2}n^2\widetilde{E}_{m-2} \pmod{n^3}$$

if k is even and $n \in I(k, 2)$;
(ii)

$$T_{4,k}(n) \equiv 2^{2k-1}\big(1 - 2^m - 2^{2m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} - 2^{2k-2}kn\widetilde{E}_{m-1} - 2^{k-4}k\big(2^{k+2} - 1\big)n^2\widetilde{B}_{m-1} \pmod{n^3}$$

if k is odd.

Proof. This is an immediate consequence of (14) or (15). We apply formula (9). □

Theorem 12. *Given an odd $n > 3$ and $1 \le k < n\phi(n) - 2$,*[13] *write $m = n\phi(n) - k$. Then:*
(i)

$$T_{4,k}(n) \equiv 2^{2k-2}\widetilde{E}_m + 2^{k-2}\big(2^{k+1} - 1\big)n\widetilde{B}_m \pmod{n^2}$$

[13] Theorem 12(i) is also true for $k = n\phi(n) - 2$ if we assume that n is not exceptional in the sense of Lemma 6; for exceptional n we should add the correction $\Theta_1 = \frac{1}{192}n^2\phi(n)$ to the right hand side of the congruence.

if k is even;
(ii) (cf. [14] if $n = p$ is an odd prime)

$$T_{4,k}(n) \equiv 2^{2k-1}\big(1 - 2^m - 2^{2m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} - 2^{2k-2}kn\widetilde{E}_{m-1} \pmod{n^2})$$

if k is odd and $n \in I(k,1)$.

Proof. We substitute (9) into (16) or (17) and the theorem follows. □

Theorem 13. *Given an odd $n > 3$ and $1 \le k < \phi(n)-1$, write $m = \phi(n)-k$. Then:*
(i) (cf. [14] if $n = p$ is an odd prime)

$$T_{4,k}(n) \equiv 2^{2k-2}\widetilde{E}_m \pmod{n}$$

if k is even and $n \in I(k,0)$;
(ii) (cf. [14] if $n = p$ is an odd prime)

$$T_{4,k}(n) \equiv 2^{2k-1}\big(1 - 2^m - 2^{2m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} \pmod{n}$$

if k is odd.

Proof. Here we use congruences (18) or (19) together with formula (9). □

Theorem 14. *Let $n > 3$ be odd. Then:*
(i) (cf. [14] if $n = p$ is an odd prime)

$$T_{4,1}(n) \equiv \frac{3}{2}Q_2(n) - n\widetilde{E}_{n^2\phi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3};$$

(ii) (cf. [14] if $n = p$ is an odd prime)

$$T_{4,1}(n) \equiv \frac{3}{2}Q''_2(n) - n\widetilde{E}_{n\phi(n)-2} \pmod{n^2}$$

if n is not divisible by 3;
(iii)

$$T_{4,1}(n) \equiv \frac{3}{2}Q'_2(n) \pmod{n}.$$

Proof. This is a particular case of Theorems 11–13(ii) for $k = 1$. Then $m+1 = n^s\phi(n)$ and, by $2^{\phi(n)} = nq_2(n)+1$ and (20), we have

$$2\big(1 - 2^m - 2^{2m+1}\big)\frac{\widetilde{B}_{m+1}}{m+1} = \big(2 - (2^{\phi(n)})^{n^s} - (2^{\phi(n)})^{2n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$= \big((1-(1+nq_2(n))^{n^s}) + (1-(1+nq_2(n))^{2n^s})\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$\equiv \Big(\frac{1}{2}Q_2(n) + Q_2(n) + \gamma n^3\Big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv \frac{3}{2}Q_2(n) \pmod{n^{s+1}}$$

because $\gamma \in \mathbb{Z}$ and $s \le 2$. This gives the theorem at once since the rest of the proof is straightforward. □

Theorem 15. *Let $n > 3$ be odd. Then:*
(i)

$$T_{4,2}(n) \equiv 4\widetilde{E}_{n^2\phi(n)-2} + 7n\widetilde{B}_{n^2\phi(n)-2} + 12n^2\widetilde{E}_{n^2\phi(n)-4} \pmod{n^3}$$

if $3, 5 \nmid n$;
(ii) (see [9])

$$T_{4,2}(n) \equiv 4\widetilde{E}_{n\phi(n)-2} + 7n\widetilde{B}_{n\phi(n)-2} \pmod{n^2};$$

(iii)

$$T_{4,2}(n) \equiv 4\widetilde{E}_{\phi(n)-2} \pmod{n}^{(14)}$$

if n is not divisible by 3.

Proof. This is a particular case of Theorems 11–13(i) in case $k = 2$. □

4. Case $r = 6$

Theorem 16. *Given an odd $n > 5$ not divisible by 3 and $1 \le k < n^2\phi(n)-2$, write $m = n^2\phi(n) - k$. Then:*
(i)

$$T_{6,k}(n) \equiv \frac{3^{k-1}}{2}\big(2^k+1\big)\widetilde{D}_m + \frac{1}{12}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m + \frac{3^{k-1}}{8}\binom{k+1}{2}\big(2^{k+2}-1\big)n^2\widetilde{D}_{m-2} \pmod{n^3}$$

if k is even and $n \in I(k,2)$;
(ii)

$$T_{6,k}(n) \equiv 2^{k-1}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m-6^m\big) - \frac{3^{k-1}}{4}\big(2^{k+1}+1\big)kn\widetilde{D}_{m-1} - \frac{k}{144}\big(2^{k+2}-1\big)\big(3^{k+2}-1\big)n^2\widetilde{B}_{m-1} \pmod{n^3}$$

if k is odd.

(14) It was an open problem in [2, p. 204].

Proof. This is an immediate consequence of congruences (14) if k is even or (15) if k is odd and formula (10). □

Theorem 17. *Given an odd $n > 5$ not divisible by 3 and $1 \le k < n\phi(n) - 1$, write $m = n\phi(n) - k$. Then:*
(i)

$$T_{6,k}(n) \equiv \frac{3^{k-1}}{2}\big(2^k+1\big)\widetilde{D}_m + \frac{1}{12}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m \pmod{n^2}$$

if k is even;
(ii)

$$T_{6,k}(n) \equiv 2^{k-1}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m-6^m\big) - \frac{3^{k-1}}{4}\big(2^{k+1}+1\big)kn\widetilde{D}_{m-1} \pmod{n^2}$$

if k is odd and $n \in I(k,1)$.

Proof. Substituting (10) into congruences (16), resp. (17) gives the theorem if k is even, resp. odd at once. □

Theorem 18. *Given an odd $n > 5$ not divisible by 3 and $1 \le k < \phi(n)$, write $m = \phi(n) - k$. Then:*
(i) (cf. [14] if $n = p$ is an odd prime)

$$T_{6,k}(n) \equiv \frac{3^{k-1}}{2}\big(2^k+1\big)\widetilde{D}_m \pmod{n}$$

if k is even and $n \in I(k,0)$;
(ii) (cf. [14] if $n = p$ is an odd prime)

$$T_{6,k}(n) \equiv 2^{k-1}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m-6^m\big) \pmod{n}$$

if k is odd.

Proof. This follows from (18), resp. (19) and (10) for k even, resp. odd. □

Theorem 19. *Let $n > 5$ be odd and not divisible by 3. Then:*
(i)

$$T_{6,1}(n) \equiv Q_2(n) + Q_3(n) - \frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2} - \frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2} \pmod{n^3};$$

(ii)

$$T_{6,1}(n) \equiv Q_2''(n) + Q_3''(n) - \frac{5}{4}n\widetilde{D}_{n\phi(n)-2} \pmod{n^2};$$

(iii)

$$T_{6,1}(n) \equiv Q_2'(n) + Q_3'(n) \pmod{n}.$$

Proof. This is a particular case of Theorems 16–18(ii) for $k = 1$. Then $m+1 = n^s\phi(n)$ and, in view of $2^{\phi(n)} = nq_2(n)+1$, $3^{\phi(n)} = nq_3(n)+1$ and (20), we find that

$$\begin{aligned}
&3\bigl(1-2^m-3^m-6^m\bigr)\frac{\widetilde{B}_{m+1}}{m+1}\\
&=\frac{1}{2}\bigl(6-3(2^{\phi(n)})^{n^s}-2(3^{\phi(n)})^{n^s}-(2^{\phi(n)})^{n^s}(3^{\phi(n)})^{n^s}\bigr)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&=\frac{1}{2}\bigl(3(1-(1+nq_2(n))^{n^s})+2(1-(1+nq_3(n))^{n^s})\\
&+(1-(1+nq_2(n)^{n^s}(1+nq_3(n))^{n^s})\bigr)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&\equiv\frac{1}{2}\Bigl(\frac{3}{2}Q_2(n)+\frac{4}{3}Q_3(n)+\frac{1}{2}Q_2(n)+\frac{2}{3}Q_3(n)+\lambda n^3\Bigr)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)}\\
&\equiv Q_2(n)+Q_3(n)\pmod{n^{s+1}}
\end{aligned}$$

because $\lambda \in \mathbb{Z}$ and $s \le 2$. This gives the theorem. □

Theorem 20. *Let $n > 5$ be odd and not divisible by 3. Then:*
(i)

$$T_{6,2}(n) \equiv \frac{15}{2}\widetilde{D}_{n^2\phi(n)-2}+\frac{91}{6}n\widetilde{B}_{n^2\phi(n)-2}+\frac{153}{8}n^2\widetilde{D}_{n^2\phi(n)-4}\pmod{n^3};$$

(ii)

$$T_{6,2}(n) \equiv \frac{15}{2}\widetilde{D}_{n\phi(n)-2}+\frac{91}{6}n\widetilde{B}_{n\phi(n)-2}\pmod{n^2};$$

(iii)

$$T_{6,2}(n) \equiv \frac{15}{2}\widetilde{D}_{\phi(n)-2}\pmod{n}.$$

Proof. The theorem follows easily from Theorems 16–18(i) for $k = 2$. □

5. Case $r = 8$

Theorem 21. *Given an odd $n > 7$ and $1 \le k < n^2\phi(n)-2$, write $m = n^2\phi(n)-k$. Then:*
(i)

$$\begin{aligned}
T_{8,k}(n) \equiv\ &2^{3k-3}\widetilde{E}_m+2^{3k-2}\widetilde{C}_m+2^{2k-3}\bigl(2^{k+1}-1\bigr)n\widetilde{B}_m-2^{3k-2}kn\widetilde{A}_{m-1}\\
&+2^{3k-3}\binom{k+1}{2}n^2\widetilde{E}_{m-2}+2^{3k-2}\binom{k+1}{2}n^2\widetilde{C}_{m-2}\pmod{n^3}
\end{aligned}$$

if k is even and $n \in I(k,2)$;
(ii)

$$\begin{aligned} T_{8,k}(n) \equiv\ & 2^{3k-2}\big(1 - 2^m - 2^{3m+2}\big)\frac{\widetilde{B}_{m+1}}{m+1} + 2^{3k-2}\widetilde{A}_m \\ & - 2^{3k-3}kn\widetilde{E}_{m-1} - 2^{3k-2}kn\widetilde{C}_{m-1} \\ & - 2^{2k-5}kn^2\big(2^{k+2}-1\big)\widetilde{B}_{m-1} + 2^{3k-2}\binom{k+1}{2}n^2\widetilde{A}_{m-2} \pmod{n^3} \end{aligned}$$

if k is odd.

Proof. This follows from congruence (14), resp. (15) for k even, resp. odd and formula (11). □

Theorem 22. *Given an odd $n > 7$ and $1 \le k < n\phi(n) - 1$,*[15] *write $m = n\phi(n) - k$. Then:*
(i)

$$\begin{aligned} T_{8,k}(n) \equiv\ & 2^{3k-3}\widetilde{E}_m + 2^{3k-2}\widetilde{C}_m + 2^{2k-3}\big(2^{k+1}-1\big)n\widetilde{B}_m \\ & - 2^{3k-2}kn\widetilde{A}_{m-1} \pmod{n^2} \end{aligned}$$

if k is even;
(ii)

$$\begin{aligned} T_{8,k}(n) \equiv\ & 2^{3k-2}\big(1 - 2^m - 2^{3m+2}\big)\frac{\widetilde{B}_{m+1}}{m+1} + 2^{3k-2}\widetilde{A}_m - 2^{3k-3}kn\widetilde{E}_{m-1} \\ & - 2^{3k-2}kn\widetilde{C}_{m-1} \pmod{n^2} \end{aligned}$$

if k is odd and $n \in I(k,1)$.

Proof. This follows from (11), and (16), resp. (17) if k is even, resp. odd. □

Theorem 23. *Given an odd $n > 7$ and $1 \le k < \phi(n)$, write $m = \phi(n) - k$. Then:*
(i)

$$T_{8,k}(n) \equiv 2^{3k-3}\widetilde{E}_m + 2^{3k-2}\widetilde{C}_m \pmod{n}$$

if k is even and $n \in I(k,0)$;
(ii)

$$T_{8,k}(n) \equiv 2^{3k-2}\big(1 - 2^m - 2^{3m+2}\big)\frac{\widetilde{B}_{m+1}}{m+1} + 2^{3k-2}\widetilde{A}_m \pmod{n}$$

if k is odd.

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[15] Theorem 22(i) is also true for $k = n\phi(n) - 2$ if we assume that n is not exceptional in the sense of Lemma 6; for exceptional n we should add the correction $\Theta_1 = \frac{1}{1536}n^2\phi(n)$ to the right hand side of the congruence.

Proof. This is an immediate consequence of (11) and congruences (18), resp. (19) if k is even, resp. odd. □

Theorem 24. *Let $n > 7$ be odd. Then:*
(i)

$$T_{8,1}(n) \equiv 2Q_2(n) + 2\widetilde{A}_{n^2\phi(n)-1} - n\widetilde{E}_{n^2\phi(n)-2} - 2n\widetilde{C}_{n^2\phi(n)-2} - \frac{7}{8}n^2\widetilde{B}_{n^2\phi(n)-2} + 2n^2\widetilde{A}_{n^2\phi(n)-3} \pmod{n^3};$$

(ii)

$$T_{8,1}(n) \equiv 2Q_2''(n) + 2\widetilde{A}_{n\phi(n)-1} - n\widetilde{E}_{n\phi(n)-2} - 2n\widetilde{C}_{n\phi(n)-2} \pmod{n^2}$$

if n is not divisible by 3*;*
(iii)

$$T_{8,1}(n) \equiv 2Q_2'(n) + 2\widetilde{A}_{\phi(n)-1} \pmod{n}.$$

Proof. This is a particular case of Theorems 21–23(ii) for $k = 1$. Then $m + 1 = n^s\phi(n)$ and, by virtue of $2^{\phi(n)} = nq_2(n) + 1$ and (20), we obtain

$$2\big(1 - 2^m - 2^{3m+2}\big)\frac{\widetilde{B}_{m+1}}{m+1} = \big(2 - (2^{\phi(n)})^{n^s} - (2^{\phi(n)})^{3n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$= \big(2 - (1 + nq_2(n))^{n^s} - (1 + nq_2(n))^{3n^s}\big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}$$

$$\equiv \Big(\frac{1}{2}Q_2(n) + \frac{3}{2}Q_2(n) + \xi n^3\Big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)} \equiv 2Q_2(n) \pmod{n^{s+1}}$$

because $\xi \in \mathbb{Z}$ and $s \leq 2$. This gives the theorem at once. □

Theorem 25. *Let $n > 7$ be odd. Then, we have:*
(i)

$$T_{8,2}(n) \equiv 8\widetilde{E}_{n^2\phi(n)-2} + 16\widetilde{C}_{n^2\phi(n)-2} + 14n\widetilde{B}_{n^2\phi(n)-2} - 32n\widetilde{A}_{n^2\phi(n)-3} + 24n^2\widetilde{E}_{n^2\phi(n)-4} + 48n^2\widetilde{C}_{n^2\phi(n)-4} \pmod{n^3}$$

if $3, 5 \nmid n$;
(ii)

$$T_{8,2}(n) \equiv 8\widetilde{E}_{n\phi(n)-2} + 16\widetilde{C}_{n\phi(n)-2} + 14n\widetilde{B}_{n\phi(n)-2} - 32n\widetilde{A}_{n\phi(n)-3} \pmod{n^2};$$

(iii)

$$T_{8,2}(n) \equiv 8\widetilde{E}_{\phi(n)-2} + 16\widetilde{C}_{\phi(n)-2} \pmod{n}$$

if n is not divisible by 3.

Proof. Apply Theorems 21–23(i) for $k = 2$. □

6. Case $r = 12$

Theorem 26. *Given an odd $n > 11$ not divisible by 3 and $1 \le k < n^2\phi(n)-2$, write $m = n^2\phi(n) - k$. Then:*
(i)

$$\begin{aligned} T_{12,k}(n) \equiv\ & 2^{k-2}3^{k-1}\big(2^k+1\big)\widetilde{D}_m + 2^{2k-3}\big(3^k+1\big)\widetilde{E}_m \\ &+ \frac{2^{k-3}}{3}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m - 2^{2k-2}3^k kn\widetilde{F}_{m-1} \\ &+ 2^{k-4}3^{k-1}\binom{k+1}{2}\big(2^{k+2}+1\big)n^2\widetilde{D}_{m-2} \\ &+ \frac{2^{2k-3}}{9}\binom{k+1}{2}\big(3^{k+2}+1\big)n^2\widetilde{E}_{m-2} \pmod{n^3} \end{aligned}$$

if k is even and $n \in I(k,2)$;
(ii)

$$\begin{aligned} T_{12,k}(n) \equiv\ & 2^{2k-2}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m+6^m-4\cdot 12^m\big) + 2^{2k-2}3^k\widetilde{F}_m \\ &- 2^{k-3}3^{k-1}\big(2^{k+1}+1\big)kn\widetilde{D}_{m-1} - \frac{2^{2k-3}}{3}\big(3^{k+1}+1\big)kn\widetilde{E}_{m-1} \\ &- \frac{2^{k-5}}{9}\big(2^{k+2}-1\big)\big(3^{k+2}-1\big)kn^2\widetilde{B}_{m-1} \\ &+ 2^{2k-2}3^k\binom{k+1}{2}n^2\widetilde{F}_{m-2} \pmod{n^3} \end{aligned}$$

if k is odd.

Proof. Apply congruences (14), resp. (15) and formula (12). □

Theorem 27. *Given an odd $n > 11$ not divisible by 3 and $1 \le k < n\phi(n)-1$, write $m = n\phi(n) - k$. Then:*
(i)

$$\begin{aligned} T_{12,k}(n) \equiv\ & 2^{k-2}3^{k-1}\big(2^k+1\big)\widetilde{D}_m + 2^{2k-3}\big(3^k+1\big)\widetilde{E}_m \\ &+ \frac{2^{k-3}}{3}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m - 2^{2k-2}3^k kn\widetilde{F}_{m-1} \pmod{n^2} \end{aligned}$$

if k is even;

(ii)

$$T_{12,k}(n) \equiv 2^{2k-2}3^k \frac{\widetilde{B}_{m+1}}{m+1}\bigl(1-2^m-3^m+6^m-4\cdot 12^m\bigr)+2^{2k-2}3^k\widetilde{F}_m \\ -2^{k-3}3^{k-1}\bigl(2^{k+1}+1\bigr)kn\widetilde{D}_{m-1}-\frac{2^{2k-3}}{3}\bigl(3^{k+1}+1\bigr)kn\widetilde{E}_{m-1} \pmod{n^2}$$

if k is odd and $n \in I(k,1)$.

Proof. The above congruences follow from congruences (16), resp. (17), if k is even, resp. odd. We also use formula (12). □

Theorem 28. *Given an odd $n > 11$ not divisible by 3 and $1 \le k < \phi(n)$, write $m = \phi(n) - k$. Then:*
(i)

$$T_{12,k}(n) \equiv 2^{k-2}3^{k-1}\bigl(2^k+1\bigr)\widetilde{D}_m + 2^{2k-3}\bigl(3^k+1\bigr)\widetilde{E}_m \pmod{n}$$

if k is even and $n \in I(k,0)$;
(ii)

$$T_{12,k}(n) \equiv 2^{2k-2}3^k \frac{\widetilde{B}_{m+1}}{m+1}\bigl(1-2^m-3^m+6^m-4\cdot 12^m\bigr)+2^{2k-2}3^k\widetilde{F}_m \pmod{n}$$

if k is odd.

Proof. We proceed in the same way as in the proof of the previous theorem. Now we use congruences (18), resp. (19) if k is even, resp. odd, and formula (12). □

Theorem 29. *Let $n > 11$ be odd not divisible by 3. Then:*
(i)

$$T_{12,1}(n) \equiv \frac{3}{2}Q_2(n)+Q_3(n)+3\widetilde{F}_{n^2\phi(n)-1}-\frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2}-\frac{5}{3}n\widetilde{E}_{n^2\phi(n)-2} \\ -\frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2}+3n^2\widetilde{F}_{n^2\phi(n)-3} \pmod{n^3};$$

(ii)

$$T_{12,1}(n) \equiv \frac{3}{2}Q_2''(n)+Q_3''(n)+3\widetilde{F}_{n\phi(n)-1}-\frac{5}{4}n\widetilde{D}_{n\phi(n)-2}-\frac{5}{3}n\widetilde{E}_{n\phi(n)-2} \pmod{n^2};$$

(iii)

$$T_{12,1}(n) \equiv \frac{3}{2}Q_2'(n)+Q_3'(n)+3\widetilde{F}_{\phi(n)-1} \pmod{n}.$$

Proof. This is a particular case of Theorems 26–28(ii) for $k = 1$. Then $m+1 = n^s\phi(n)$ and, by virtue of $2^{\phi(n)} = nq_2(n)+1$, $3^{\phi(n)} = nq_3(n)+1$ and (20), we have

$$\begin{aligned}
&3\big(1-2^m-3^m+6^m-4\cdot 12^m\big)\frac{\widetilde{B}_{m+1}}{m+1}\\
&= \Big(3-\frac{3}{2}(2^{\phi(n)})^{n^s}-(3^{\phi(n)})^{n^s}+\frac{1}{2}(2^{\phi(n)})^{n^s}(3^{\phi(n)})^{n^s}-(2^{\phi(n)})^{2n^s}(3^{\phi(n)})^{n^s}\Big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&= \Big(\frac{3}{2}\big(1-(1+nq_2(n))^{n^s}\big)+\big(1-(1+nq_3(n))^{n^s}\big)\\
&-\frac{1}{2}\big(1-(1+nq_2(n))^{n^s}(1-(1+nq_3(n))^{n^s}\big)\\
&+\big(1-(1+nq_2(n))^{2n^s}(1+nq_3(n))^{n^s}\big)\Big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&\equiv \Big(\frac{3}{4}Q_2(n)+\frac{2}{3}Q_3(n)-\frac{1}{4}Q_2(n)-\frac{1}{3}Q_3(n)+Q_2(n)+\frac{2}{3}Q_3(n)+\eta n^3\Big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)}\\
&\equiv \frac{3}{2}Q_2(n)+Q_3(n) \pmod{n^{s+1}}
\end{aligned}$$

because $\eta\in\mathbb{Z}$ and $s\le 2$. The rest of the proof is straightforward. □

Theorem 30. *Let* $n>11$ *be odd and not divisible by* 3*. Then:*
(i)

$$\begin{aligned}
T_{12,2}(n) \equiv\ & 15\widetilde{D}_{n^2\phi(n)-2}+20\widetilde{E}_{n^2\phi(n)-2}+\frac{91}{3}n\widetilde{B}_{n^2\phi(n)-2}-72n\widetilde{F}_{n^2\phi(n)-3}\\
&+\frac{153}{4}n^2\widetilde{D}_{n^2\phi(n)-4}+\frac{164}{3}n^2\widetilde{E}_{n^2\phi(n)-4} \pmod{n^3}
\end{aligned}$$

if n *is not divisible by* 5*;*
(ii)

$$T_{12,2}(n) \equiv\ 15\widetilde{D}_{n\phi(n)-2}+20\widetilde{E}_{n\phi(n)-2}+\frac{91}{3}n\widetilde{B}_{n\phi(n)-2}-72n\widetilde{F}_{n\phi(n)-3} \pmod{n^2};$$

(iii)

$$T_{12,2}(n) \equiv\ 15\widetilde{D}_{\phi(n)-2}+20\widetilde{E}_{\phi(n)-2} \pmod{n}.$$

Proof. This follows at once from Theorems 26–28(i) for $k=2$. □

7. Case $r = 24$

Theorem 31. *Given an odd* $n > 23$ *not divisible by* 3 *and* $1 \le k < n^2\phi(n)-2$, *write* $m = n^2\phi(n) - k$. *Then:*
(i)

$$\begin{aligned} T_{24,k}(n) \equiv\ & 2^{2k-3}3^{k-1}\big(2^k+1\big)\widetilde{D}_m + 2^{3k-4}\big(3^k+1\big)\widetilde{E}_m + 2^{3k-3}3^k\widetilde{H}_m \\ & + 2^{3k-3}\big(3^k-1\big)\widetilde{C}_m + \frac{2^{2k-4}}{3}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m - 2^{3k-3}3^k kn\widetilde{F}_{m-1} \\ & - 2^{3k-3}3^k kn\widetilde{G}_{m-1} - \frac{2^{3k-3}}{3}\big(3^{k+1}+1\big)kn\widetilde{A}_{m-1} \\ & + 2^{2k-5}3^{k-1}\binom{k+1}{2}\big(2^{k+2}+1\big)n^2\widetilde{D}_{m-2} + \frac{2^{3k-4}}{9}\binom{k+1}{2}\big(3^{k+2}+1\big)n^2\widetilde{E}_{m-2} \\ & + 2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{H}_{m-2} + \frac{2^{3k-3}}{9}\big(3^{k+2}-1\big)\binom{k+1}{2}n^2\widetilde{C}_{m-2} \pmod{n^3} \end{aligned}$$

if k *is even and* $n \in I(k,2)$;
(ii)

$$\begin{aligned} T_{24,k}(n) \equiv\ & 2^{3k-3}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m+6^m-8\cdot 24^m\big) + 2^{3k-3}3^k\widetilde{F}_m \\ & + 2^{3k-3}3^k\widetilde{G}_m + 2^{3k-3}\big(3^k+1\big)\widetilde{A}_m - 2^{2k-4}3^{k-1}\big(2^{k+1}+1\big)kn\widetilde{D}_{m-1} \\ & - \frac{2^{3k-4}}{3}\big(3^{k+1}+1\big)kn\widetilde{E}_{m-1} - 2^{3k-3}3^k kn\widetilde{H}_{m-1} - \frac{2^{3k-3}}{3}\big(3^{k+1}-1\big)kn\widetilde{C}_{m-1} \\ & - \frac{2^{2k-6}}{9}\big(2^{k+2}-1\big)\big(3^{k+2}-1\big)kn^2\widetilde{B}_{m-1} + 2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{F}_{m-2} \\ & + 2^{3k-3}3^k\binom{k+1}{2}n^2\widetilde{G}_{m-2} + \frac{2^{3k-3}}{9}\binom{k+1}{2}\big(3^{k+2}+1\big)n^2\widetilde{A}_{m-2} \pmod{n^3} \end{aligned}$$

if k *is odd.*

Proof. This follows from congruences (14), resp. (15) if k is even, resp. odd with the use of (13). □

Theorem 32. *Given an odd* $n > 23$ *not divisible by* 3 *and* $1 \le k < n\phi(n)-1$, *write* $m = n\phi(n) - k$. *Then:*
(i)

$$\begin{aligned} T_{24,k}(n) \equiv\ & 2^{2k-3}3^{k-1}\big(2^k+1\big)\widetilde{D}_m + 2^{3k-4}\big(3^k+1\big)\widetilde{E}_m + 2^{3k-3}3^k\widetilde{H}_m \\ & + 2^{3k-3}\big(3^k-1\big)\widetilde{C}_m + \frac{2^{2k-4}}{3}\big(2^{k+1}-1\big)\big(3^{k+1}-1\big)n\widetilde{B}_m \\ & - 2^{2k-3}3^k kn\widetilde{F}_{m-1} - 2^{3k-3}3^k kn\widetilde{G}_{m-1} \end{aligned}$$

$$- \frac{2^{3k-3}}{3}\big(3^{k+1}+1\big)kn\widetilde{A}_{m-1} \pmod{n^2}$$

if k is even;
(ii)

$$\begin{aligned}T_{24,k}(n) \equiv\ & 2^{3k-3}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m+6^m-8\cdot 24^m\big)+2^{3k-3}3^k\widetilde{F}_m\\ &+2^{3k-3}3^k\widetilde{G}_m+2^{3k-3}\big(3^k+1\big)\widetilde{A}_m-2^{2k-4}3^{k-1}\big(2^{k+1}+1\big)kn\widetilde{D}_{m-1}\\ &-\frac{2^{3k-4}}{3}\big(3^{k+1}+1\big)kn\widetilde{E}_{m-1}-2^{3k-3}3^k kn\widetilde{H}_{m-1}\\ &-\frac{2^{3k-3}}{3}\big(3^{k+1}-1\big)kn\widetilde{C}_{m-1} \pmod{n^2}\end{aligned}$$

if k is odd and $n \in I(k,1)$.

Proof. The same reasoning as in the proof of the previous theorem applies to congruences modulo n^2. Now we use congruences (16), resp. (17) and formula (13). □

Theorem 33. *Given an odd $n > 23$ not divisible by 3 and $1 \le k < \phi(n)$, write $m = \phi(n) - k$. Then:*
(i)

$$\begin{aligned}T_{24,k}(n) \equiv\ & 2^{2k-3}3^{k-1}\big(2^k+1\big)\widetilde{D}_m+2^{3k-4}\big(3^k+1\big)\widetilde{E}_m\\ &+2^{3k-3}3^k\widetilde{H}_m+2^{3k-3}\big(3^k-1\big)\widetilde{C}_m \pmod{n}\end{aligned}$$

if k is even and $n \in I(k,0)$;
(ii)

$$\begin{aligned}T_{24,k}(n) \equiv\ & 2^{3k-3}3^k\frac{\widetilde{B}_{m+1}}{m+1}\big(1-2^m-3^m+6^m-8\cdot 24^m\big)\\ &+2^{3k-3}3^k\widetilde{F}_m+2^{3k-3}3^k\widetilde{G}_m+2^{3k-3}\big(3^k+1\big)\widetilde{A}_m \pmod{n}\end{aligned}$$

if k is odd.

Proof. This follows from congruences (18), resp. (19) if k is even, resp. odd. We make use of formula (13). □

Theorem 34. *Let $n > 23$ be odd and not divisible by 3. Then:*
(i)

$$T_{24,1}(n) \equiv\ 2Q_2(n)+Q_3(n)+3\widetilde{F}_{n^2\phi(n)-1}+3\widetilde{G}_{n^2\phi(n)-1}+4\widetilde{A}_{n^2\phi(n)-1}$$

$$
\begin{aligned}
&-\frac{5}{4}n\widetilde{D}_{n^2\phi(n)-2}-\frac{5}{3}n\widetilde{E}_{n^2\phi(n)-2}-3n\widetilde{H}_{n^2\phi(n)-2}-\frac{8}{3}n\widetilde{C}_{n^2\phi(n)-2}\\
&-\frac{91}{72}n^2\widetilde{B}_{n^2\phi(n)-2}+3n^2\widetilde{F}_{n^2\phi(n)-3}+3n^2\widetilde{G}_{n^2\phi(n)-3}+\frac{28}{9}n^2\widetilde{A}_{n^2\phi(n)-3}(\operatorname{mod} n^3);
\end{aligned}
$$

(ii)

$$
\begin{aligned}
T_{24,1}(n) \equiv\ & 2Q_2''(n)+Q_3''(n)+3\widetilde{F}_{n\phi(n)-1}+3\widetilde{G}_{n\phi(n)-1}+4\widetilde{A}_{n\phi(n)-1}\\
&-\frac{5}{4}n\widetilde{D}_{n\phi(n)-2}-\frac{5}{3}n\widetilde{E}_{n\phi(n)-2}-3n\widetilde{H}_{n\phi(n)-2}-\frac{8}{3}n\widetilde{C}_{n\phi(n)-2}\ (\operatorname{mod} n^2);
\end{aligned}
$$

(iii)

$$
T_{24,1}(n) \equiv\ 2Q_2'(n)+Q_3'(n)+3\widetilde{F}_{\phi(n)-1}+3\widetilde{G}_{\phi(n)-1}+4\widetilde{A}_{\phi(n)-1}\ (\operatorname{mod} n).
$$

Proof. This is a particular case of Theorems 31–33(ii) for $k=1$. Then $m+1=n^s\phi(n)$ and, in view of $2^{\phi(n)}=nq_2(n)+1$, $3^{\phi(n)}=nq_3(n)+1$ and (20), we have

$$
\begin{aligned}
&3\big(1-2^m-3^m+6^m-8\cdot 24^m\big)\frac{\widetilde{B}_{m+1}}{m+1}\\
&=\Big(3-\frac{3}{2}(2^{\phi(n)})^{n^s}-(3^{\phi(n)})^{n^s}+\frac{1}{2}(2^{\phi(n)})^{n^s}(3^{\phi(n)})^{n^s}-(2^{\phi(n)})^{3n^s}(3^{\phi(n)})^{n^s}\Big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&=\Big(\frac{3}{2}\big(1-(1+nq_2(n))^{n^s}\big)+\big(1-(1+nq_3(n))^{n^s}\big)\\
&-\frac{1}{2}\big(1-(1+nq_2(n))^{n^s}(1+nq_3(n))^{n^s}\big)\\
&+\big(1-(1+nq_2(n))^{3n^s}(1+nq_3(n))^{n^s}\big)\Big)\frac{\widetilde{B}_{n^s\phi(n)}}{n^s\phi(n)}\\
&\equiv\ \Big(\frac{3}{4}Q_2(n)+\frac{2}{3}Q_3(n)-\frac{1}{4}Q_2(n)-\frac{1}{3}Q_3(n)+\frac{3}{2}Q_2(n)+\frac{2}{3}Q_3(n)+\omega n^3\Big)\frac{n\widetilde{B}_{n^s\phi(n)}}{\phi(n)}\\
&\equiv\ 2Q_2(n)+Q_3(n)\ (\operatorname{mod} n^{s+1})
\end{aligned}
$$

because $\omega\in\mathbb{Z}$ and $s\le 2$. This proves the theorem. □

Theorem 35. *Let $n>23$ be odd and not divisible by* 3*. Then:*
(i)

$$
\begin{aligned}
T_{24,2}(n) \equiv\ & 30\widetilde{D}_{n^2\phi(n)-2}+40\widetilde{E}_{n^2\phi(n)-2}+72\widetilde{H}_{n^2\phi(n)-2}+64\widetilde{C}_{n^2\phi(n)-2}\\
&+\frac{182}{3}n\widetilde{B}_{n^2\phi(n)-2}-144n\widetilde{F}_{n^2\phi(n)-3}-144n\widetilde{G}_{n^2\phi(n)-3}-\frac{448}{3}n\widetilde{A}_{n^2\phi(n)-3}\\
&+\frac{153}{2}n^2\widetilde{D}_{n^2\phi(n)-4}+\frac{328}{3}n^2\widetilde{E}_{n^2\phi(n)-4}+216n^2\widetilde{H}_{n^2\phi(n)-4}
\end{aligned}
$$

$$+\frac{640}{3}n^2\widetilde{C}_{n^2\phi(n)-4} \pmod{n^3}$$

if n *is not divisible by* 5*;*
(ii)

$$\begin{aligned} T_{24,2}(n) \equiv\ & 30\widetilde{D}_{n\phi(n)-2} + 40\widetilde{E}_{n\phi(n)-2} + 72\widetilde{H}_{n\phi(n)-2} + 64\widetilde{C}_{n\phi(n)-2} \\ & + \frac{182}{3}n\widetilde{B}_{n\phi(n)-2} - 36n\widetilde{F}_{n\phi(n)-3} - 144n\widetilde{G}_{n\phi(n)-3} \\ & - \frac{448}{3}n\widetilde{A}_{n\phi(n)-3} \pmod{n^2}; \end{aligned}$$

(iii)

$$T_{24,2}(n) \equiv 30\widetilde{D}_{\phi(n)-2} + 40\widetilde{E}_{\phi(n)-2} + 72\widetilde{H}_{\phi(n)-2} + 64\widetilde{C}_{\phi(n)-2} \pmod{n}$$

Proof. This follows easily from Theorems 31–33(i) for $k = 2$. □

4 Concluding remarks

Let $p \geq 3$ be a prime number and let r be a natural number such that $1 < r < p$. In the next part of the paper we are going to prove some new congruences for the sums $T_{r,k}(p) = \sum_{i=1}^{[p/r]}(1/i^k)$ modulo p^{s+1} for $s = 0, 1$ or 2, all divisors r of 24 and $k \geq 1$, in particular for $k = 1$ or 2 in all the cases. We shall use the congruences proved in the present paper in the case when $n = p$ is an odd prime as well as Kummer's congruences for the generalized Bernoulli numbers.

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