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The non-p-part of the fine Selmer group in a $\mathbb{Z}_p$-extension

by

Adithya Chakravarthy (Toronto)

Abstract. Fix two distinct primes p and ℓ. Let A be an abelian variety over $\mathbb{Q}(\zeta_\ell)$, the cyclotomic field of ℓth roots of unity. Suppose that $A(\mathbb{Q}(\zeta_\ell))[\ell] \neq 0$. We show that there exists a number field L and a $\mathbb{Z}_p$-extension L_∞/L where the ℓ-primary fine Selmer group of A grows arbitrarily quickly. This is a fine Selmer group analogue of a theorem of Washington that there are certain (non-cyclotomic) $\mathbb{Z}_p$-extensions where the ℓ-part of the class group can grow arbitrarily quickly. We also prove this for a wide class of non-commutative p-adic Lie extensions. Finally, we include several examples to illustrate this theorem.

1. Introduction. We begin with a fundamental theorem of Iwasawa, which serves as the starting point of Iwasawa theory. Let K be a number field and let K_∞/K be a $\mathbb{Z}_p$-extension: a Galois extension with Galois group isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers. For such an extension K_∞/K, there exists a unique sequence of fields

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\infty$$

such that each K_n/K is a cyclic extension of degree p^n. Iwasawa [Iwa59] proved the following now famous theorem about the growth of class numbers in such towers.

Theorem (Iwasawa). *Let K be a number field and let K_∞/K be a $\mathbb{Z}_p$-extension with layers K_n. Suppose that p^{e_n} is the exact power of p dividing the class number of K_n. Then there exist integers μ, λ, ν such that*

$$e_n = \mu p^n + \lambda n + \nu$$

for all sufficiently large values of n.

A large part of classical Iwasawa theory is devoted to studying the invariants μ and λ in the above formula. In a beautiful paper, Iwasawa showed

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[Iwa73, Theorem 1] that there are $\mathbb{Z}_p$-extensions for which the μ-invariant can be *arbitrarily* large.

THEOREM 1.1 (Iwasawa). *Let $N \geq 1$. There exists a number field L and a $\mathbb{Z}_p$-extension L_∞/L such that $\mu \geq N$.*

Now that we have discussed the p-part of the class group in a $\mathbb{Z}_p$-extension, we turns to its ℓ-part, where $\ell \neq p$ are *distinct* primes. The fundamental theorem in this area is due to Washington [Was78]:

THEOREM 1.2 (Washington). *Let $\ell \neq p$ be primes. Let K be an abelian extension of the field $\mathbb{Q}$ of rational numbers. Let K^{cyc}/K be the cyclotomic $\mathbb{Z}_p$-extension of K and let ℓ^{e_n} be the exact power of ℓ dividing the class number of K_n. Then e_n is bounded as $n \to \infty$.*

Based on this, one might reasonably guess that the ℓ-part of the class group is bounded in an *arbitrary* $\mathbb{Z}_p$-extension. But this turns out to be false, as proven in [Was75, Theorem 6].

THEOREM 1.3 (Washington). *Let $N \geq 1$. There exists a number field L and a $\mathbb{Z}_p$-extension L_∞/L such that*

$$e_n \geq Np^n,$$

where ℓ^{e_n} is the exact power of ℓ dividing the class number of L_n.

The purpose of this article is to discuss analogues of Theorems 1.1 and 1.3 for fine Selmer groups of elliptic curves. Let E be an elliptic curve over a number field F. The Mordell–Weil Theorem says the group $E(F)$ of rational points is a finitely generated abelian group. This arithmetic of this group is essentially controlled by the Selmer group of E/F. In [Maz72], Mazur introduced the Iwasawa theory of Selmer groups in $\mathbb{Z}_p$-extensions of F. The notion of the fine Selmer group was formally introduced by Coates and Sujatha in [CS05], even though it had been studied by Rubin [Rub00] and Perrin-Riou [PR93, PR95] under various guises in the late 80's and early 90's. In [CS05], Coates and Sujatha showed that these fine Selmer groups have stronger finiteness properties than classical Selmer groups.

In [Kun20, Theorem 4.2], Kundu proved an analogue of Theorem 1.1 for fine Selmer groups: If A is an abelian variety and $N \geq 1$ is an integer, then there exists a number field L and a $\mathbb{Z}_p$-extension L_∞/L such that the μ-invariant of the fine Selmer group of A over L_∞ is at least N. In other words, μ-invariants of fine Selmer groups can be arbitrarily large. And in [KL23, Theorem B], Kundu and Lei proved a fine Selmer group analogue of Theorem 1.2: In the *cyclotomic* $\mathbb{Z}_p$-extension, the ℓ-part of the fine Selmer group of A stabilizes.

The purpose of the present paper is to prove a fine Selmer group analogue of Theorem 1.3. If ℓ is a prime and A is an abelian variety over a number

field F, let $R_{\ell^\infty}(A/F)$ denote the ℓ-primary fine Selmer group of A over F. (See Section 5.1 for the precise definition.) If G is an abelian group, the *ℓ-rank* of G is defined by $r_\ell(G) = \dim_{\mathbb{Z}/\ell\mathbb{Z}} G[\ell]$. Note that it is possible to have $r_\ell(G) = \infty$; this is true for example if G contains infinitely many copies of $\mathbb{Z}/\ell\mathbb{Z}$. (See [LM16, Section 3] as a reference for this definition.) Here is our main result:

THEOREM 1.4. *Let $\ell \neq p$ be primes. Let $\mathbb{Q}(\zeta_\ell)$ be the cyclotomic field of ℓth roots of unity and let A be an abelian variety over $\mathbb{Q}(\zeta_\ell)$. Suppose that $A(\mathbb{Q}(\zeta_\ell))[\ell] \neq 0$. For every integer $N \geq 1$, there exists a finite extension $L/\mathbb{Q}(\zeta_\ell)$ and a $\mathbb{Z}_p$-extension L_∞/L such that*

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq Nq^n$$

for all $n \geq 0$, where $q = \min\{\ell, p\}$. In particular, $r_\ell(R_{\ell^\infty}(A/L_n)) \to \infty$ as $n \to \infty$.

More generally, we can also prove this theorem for many non-commutative p-adic Lie extensions. The relevant statement requires a definition:

DEFINITION 1.5. A pro-p group Γ is *uniform of dimension d* if it is topologically finitely generated by d generators and there exists a unique filtration by the p-descending central series of Γ. In other words, we have

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots$$

such that each Γ_{n+1} is normal in Γ_n and $\Gamma_n/\Gamma_{n+1} \simeq (\mathbb{Z}/p\mathbb{Z})^d$.

If F is a number field and $\mathfrak{p}$ is a prime ideal of F, then the *$\mathfrak{p}$-class group* of F is the quotient of $\mathrm{Cl}(F)$ by the subgroup generated by the ideal class of $\mathfrak{p}$.

ASSUMPTION 1. Let Γ be a uniform pro-p group with a fixed-point-free automorphism of order m, where $m > 2$ is a prime different from p. Assume that:

(1) there exists a $\mathbb{Z}/m\mathbb{Z}$-extension of number fields F/F_0, where F_0 is totally imaginary,
(2) the field F contains the pth roots of unity,
(3) there is a unique prime $\mathfrak{p}$ of F lying over the rational prime p,
(4) the p-part of the $\mathfrak{p}$-class group of F is trivial.

THEOREM 1.6. *Let $\ell \neq p$ be primes. Let $\mathbb{Q}(\zeta_\ell)$ be the cyclotomic field of ℓth roots of unity and let A be an abelian variety over $\mathbb{Q}(\zeta_\ell)$. Suppose that $A(\mathbb{Q}(\zeta_\ell))[\ell] \neq 0$. Let Γ be a uniform pro-p group with a fixed-point-free automorphism of order m. If $m > 2$, suppose that Assumption 1 holds.*

Then for every integer $N \geq 1$, there exists a finite extension $L/\mathbb{Q}(\zeta_\ell)$ and a Γ-extension L_∞/L such that

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq Nq^n$$

for all $n \geq 0$, where $q = \min\{\ell, p\}$. In particular, $r_\ell(R_{\ell^\infty}(A/L_n)) \to \infty$ as $n \to \infty$.

Let $\mathrm{Sel}_{\ell^\infty}(A/F)$ denote the ℓ-primary (usual) Selmer group of A over F. (See Section 5.1 for the precise definition.) Since the fine Selmer group is a subgroup of the usual Selmer group, we have the following

COROLLARY 1.7. *Retain the notations and assumptions of Theorem* 1.6. *For every integer $N \geq 1$, there exists a finite extension $L/\mathbb{Q}(\zeta_\ell)$ and a Γ-extension L_∞/L such that*

$$r_\ell(\mathrm{Sel}_{\ell^\infty}(A/L_n)) \geq Nq^n$$

for all $n \geq 0$, where $q = \min\{\ell, p\}$. In particular, $r_\ell(\mathrm{Sel}_{\ell^\infty}(A/L_n)) \to \infty$ as $n \to \infty$.

We give several numerical examples at the end of the paper to show that Assumption 1 often holds.

Strategy. We follow Iwasawa's approach in [Iwa73], which inspired Washington's approach in [Was75, Section VI]. We construct a $\mathbb{Z}_p^d$-extension L_∞/L where the ℓ-rank of the *class group* is unbounded.

THEOREM 1.8. *Let $\ell \neq p$ be primes. Let Γ be a uniform pro-p group. If $m > 2$, then suppose Assumption* 1 *holds. For every integer $N \geq 1$, there exists a finite extension $L/\mathbb{Q}(\zeta_\ell)$ and a Γ-extension L_∞/L such that for all $n \geq 0$*

$$r_\ell(\mathrm{Cl}(L_n)) \geq Np^n.$$

In particular, $r_\ell(\mathrm{Cl}(L_n)) \to \infty$ as $n \to \infty$.

We then use results of Lim–Murty [LM16] to show that the ℓ-rank of the fine Selmer group is close in size to the ℓ-rank of the class group. Putting these two results together, we conclude that the ℓ-part of the fine Selmer group is unbounded in L_∞/L, proving Theorem 1.6.

2. Construction of the Γ-extension K_∞/K.

In this section we will construct a Γ-extension K_∞/K where infinitely many primes split completely. Here Γ will be a uniform pro-p group with a fixed-point-free automorphism τ of order m. The construction proceeds differently in the cases $m = 2$ and $m > 2$.

2.1. The case $m = 2$.

By [RZ00, Corolary 4.6.10], if Γ is a uniform pro-p group with a fixed-point-free automorphism τ of order $m = 2$, then $\Gamma \simeq \mathbb{Z}_p^d$ for some $d \geq 1$.

To motivate the next proposition, recall that if K is an imaginary quadratic field, then there is a $\mathbb{Z}_p$-extension K_∞/K called the *anticyclotomic $\mathbb{Z}_p$-extension* of K. This extension has the property that infinitely many

primes of K split completely in K_∞. The next proposition generalizes this to $\mathbb{Z}_p^d$-extensions for $d \geq 1$.

PROPOSITION 2.1. *Let K be a CM field such that $K/\mathbb{Q}$ is a $\mathbb{Z}/2d\mathbb{Z}$-extension. Suppose that there is only one prime in K lying over p. Then there is a $\mathbb{Z}_p^d$-extension K_∞/K such that infinitely many primes of K split completely in K_∞.*

Proof. This fact is well-known, but for lack of a reference we sketch the proof here. The construction below is reproduced from [Lon12, Section 2]. Let K^+ be the maximal totally real subfield of K. For any integral ideal $\mathfrak{c} \subseteq \mathcal{O}_{K^+}$, let $\mathcal{O}_\mathfrak{c} = \mathcal{O}_{K^+} + \mathfrak{c}\mathcal{O}_K$ be the order of conductor $\mathfrak{c}$ in K. The *ring class field* $K[\mathfrak{c}]/K$ of K of conductor $\mathfrak{c}$ is the Galois extension of K such that there is an isomorphism via the Artin map:

$$\mathrm{Gal}(K[c]/K) \simeq \mathrm{Cl}(\mathcal{O}_\mathfrak{c}).$$

Let $\mathfrak{p}$ be the unique prime of K^+ lying over p. Put $K[\mathfrak{p}^\infty] = \bigcup_{n=1}^\infty K[\mathfrak{p}^n]$. Define K_∞ to be the unique subfield of $K[\mathfrak{p}^\infty]$ satisfying

$$\mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p^{[K_\mathfrak{p}^+:\mathbb{Q}_p]} = \mathbb{Z}_p^d.$$

Suppose that ℓ is a rational prime which is inert in K. Then the ideal class of ℓ is trivial in $\mathrm{Cl}(K)$ and hence class field theory shows that ℓ splits completely in any ring class field $K[\mathfrak{c}]$ of conductor coprime to ℓ. (See [Nek07, Section 2.6.3].) In particular, if ℓ is inert in $K/\mathbb{Q}$ and ℓ is coprime to p then ℓ splits completely in K_∞. By the Chebotarev density theorem, there are infinitely many such primes. ∎

2.2. The case $m > 2$. Here is the main result:

PROPOSITION 2.2. *Let Γ be a uniform pro-p group. Suppose Assumption 1 holds. Then there exists a Galois extension K/F and a Γ-extension K_∞/K such that infinitely many primes of K split completely in K_∞.*

Now let F be a number field and let $F_{\max,p}$ be the maximal pro-p extension of F unramified outside the primes above p. The number field F is called *p-rational* if $\mathrm{Gal}(F_{\max,p}/F)$ is pro-p free.

LEMMA 2.3. *Let F be a number field with a primitive pth root of unity. Then F is p-rational if and only if there exists a unique prime $\mathfrak{p}$ above p and the p-part of the $\mathfrak{p}$-class group of F is trivial.*

Proof. This is part of [Gra, Theorem IV.3.5]. ∎

LEMMA 2.4. *Keep the notations and assumptions from Proposition 2.2. Let n be an integer such that $[F_0 : \mathbb{Q}]p^n \geq 2d$ and let K_0 (resp. K) be the nth layer of the cyclotomic $\mathbb{Z}_p$-extension of F_0 (resp. F). Then there exists an intermediate field $K \subset K_\infty \subset K_{\max,\,p}$ such that K_∞ is Galois over K_0*

with Galois group $\Gamma \rtimes \langle\tau\rangle$. Suppose τ acts fixed-point-freely on Γ. Then every place of K_0 which is inert in K/K_0 splits completely in K_∞/K.

Proof. This follows from [HM19, Propositions 3.6 and 3.7] if F is p-rational. But Lemma 2.3 gives conditions for F to be p-rational and F satisfies those conditions. ∎

Proposition 2.2 now follows from Lemma 2.4, because by the Chebotarev density theorem there are infinitely many primes that are inert in the cyclic extension K/K_0.

3. Construction of the Γ-extension L_∞/L. Fix $N \geq 1$. In this section we will construct a Γ-extension L_∞/L with the properties in Theorem 1.8, i.e. such that

$$r_\ell(\mathrm{Cl}(L_n)) \geq Np^n \quad \text{for all } n \geq 0.$$

PROPOSITION 3.1. *Let $N \geq 1$ be an integer. Let Γ be a uniform pro-p group of dimension d with a fixed-point-free automorphism of order m. If $m > 2$, suppose Assumption 1 holds. Let K_∞/K denote the Γ-extension from Proposition 2.1 (resp. Proposition 2.2) if $m = 2$ (resp. $m > 2$). There exists a finite extension L/K and a Γ-extension L_∞/L satisfying the following:*

(1) *The extension L_∞ contains K_∞. Furthermore, for all $n \geq 0$, the number of primes ramifying in L_n/K_n is at least*

$$(N + md\ell(\ell-1))p^n.$$

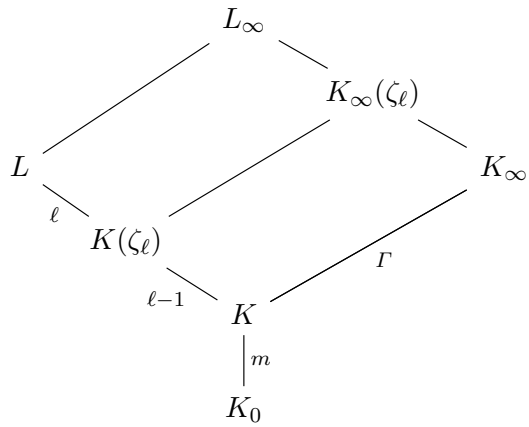
(2) *We have $[L_n : \mathbb{Q}] \geq m\ell(\ell-1)dp^n$ for all $n \geq 0$.*

Proof. We only prove the case $m > 2$; the proof for $m = 2$ is identical and left to the reader. By Proposition 2.2, infinitely many primes of K split completely in K_∞/K. Let $t \geq 1$ be an integer (to be chosen later) and let $v_1, \ldots, v_t$ be primes in K that split completely in K_∞/K.

CLAIM 3.2. *There exists $\alpha \in K$ such that $\mathrm{ord}_{v_i}(\alpha) = 1$ for all $i = 1, \ldots, t$.*

Proof of claim. Consider the ideal class $I = [v_1 \cdot \ldots \cdot v_t] \in \mathrm{Cl}(K)$. Then II^{-1} is trivial in $\mathrm{Cl}(K)$ so there exists a fractional ideal w of K such that $v_1 \cdot \ldots \cdot v_t \cdot w$ is a principal ideal; write $v_1 \cdot \ldots \cdot v_t \cdot w = (\alpha)$ for some $\alpha \in K$. Then $\mathrm{ord}_{v_i}(\alpha) \geq 1$ for all $i = 1, \ldots, t$. We can ensure that $\mathrm{ord}_{v_i}(\alpha)$ is *exactly* 1 by dividing w through by v_i if necessary. This proves the claim. ∎

Now let $\alpha \in K$ be such that $\mathrm{ord}_{v_i}(\alpha) = 1$ for all $i = 1, \ldots, t$. Put $L = K(\alpha^{1/\ell}, \zeta_\ell)$. Then $L/K(\zeta_\ell)$ is a cyclic degree ℓ extension where $v_1, \ldots, v_t$ ramify. Put $L_\infty = K_\infty L$. Then L_∞/L is a Γ-extension. We summarize this in a diagram:



Let K_n (resp. L_n) be the nth layer of the Γ-extension K_∞ (resp. L_∞). The primes $v_1, \dots, v_t$ ramify in L/K. Furthermore, all the primes of K_n lying over $v_1, \dots, v_t$ must ramify in L_n as well. Since each v_i splits completely, there are tp^n such primes of K_n. Therefore, the number of primes of L_n/K_n that ramify is at least tp^n. Now set

$$t := N + md\ell(\ell - 1).$$

This proves property (1).

To prove property (2), we just count degrees in the above field diagram, noting that $[L_n : L] = dp^n$. This completes the proof of Proposition 3.1. ∎

4. Growth of class groups in L_∞/L: the proof of Theorem 1.8. We want to show that the ℓ-part of the class group in the Γ-extension L_∞/L is unbounded. Our main tool to do this is the so-called *ambiguous class number formula*.

Definition 4.1. Let ℓ be a prime. Let K be a number field and L/K be a cyclic $\mathbb{Z}/\ell\mathbb{Z}$-extension. Let $\sigma \in \mathrm{Gal}(L/K)$ be a generator. An ideal class $[\mathfrak{a}] \in \mathrm{Cl}(L)$ is called *strongly ambiguous* if $[\mathfrak{a}]^{\sigma-1} = (1)$.

The subgroup of $\mathrm{Cl}(L)$ consisting of strongly ambiguous classes is denoted by $\mathrm{Am}_{\mathrm{st}}(L/K)$.

The following is given in [Kun20, Proposition 4.5].

Proposition 4.2 (Ambiguous class number formula). *Let ℓ be a prime. Let K be a number field and L/K be a cyclic $\mathbb{Z}/\ell\mathbb{Z}$-extension such that σ is a generator of the Galois group* $\mathrm{Gal}(L/K)$. *Then*

$$r_\ell(\mathrm{Am}_{\mathrm{st}}(L/K)) \geq T - [L : \mathbb{Q}],$$

where T is the number of ramified primes in L/K.

Proof of Theorem 1.8. Observe that for each $n \geq 1$, the extension L_n/K_n is cyclic of degree ℓ. Applying Proposition 4.2 to L_n/K_n, we have

$$r_\ell(\mathrm{Am}_{\mathrm{st}}(L_n/K_n)) \geq T - [L_n : \mathbb{Q}],$$

and the number of primes ramifying in L_n/K_n is at least

$$(N + 2d\ell(\ell-1))p^n.$$

We have $[L_n : \mathbb{Q}] \geq m\ell(\ell-1)dp^n$ for all $n \geq 0$.

By Proposition 3.1(2), we have $T \geq (N + md\ell(\ell-1))p^n$ and $[L_n : \mathbb{Q}] \geq md\ell(\ell-1)p^n$. And since $\mathrm{Am}_{\mathrm{st}}(L_n/K_n)$ is a subgroup of $\mathrm{Cl}(L_n)$, we have

$$r_\ell(\mathrm{Cl}(L_n)) \geq r_\ell(\mathrm{Am}_{\mathrm{st}}(L_n/K_n)).$$

Combining these, we obtain

$$\begin{aligned} r_\ell(\mathrm{Cl}(L_n)) &\geq r_\ell(\mathrm{Am}_{\mathrm{st}}(L_n/K_n)) \geq T - [L_n : \mathbb{Q}] \\ &\geq (N + md\ell(\ell-1))p^n - md\ell(\ell-1)p^n = Np^n \end{aligned}$$

for all $n \geq 0$. This completes the proof of Theorem 1.8. ∎

5. Application to fine Selmer groups: the proof of Theorems 1.6

5.1. Review of fine Selmer group. Let F be a number field and p be a prime. Let A be an abelian variety over F and let S be a finite set of primes containing $S_p \cup S_{\mathrm{bad}} \cup S_\infty$. Denote by F_S the maximal extension of F unramified outside S.

The usual p^∞-*Selmer group* of A is defined by

$$\mathrm{Sel}_{p^\infty}(A/F) = \ker\Big(\sum H^1(F, A[p^\infty]) \to \prod_v H^1(F_v, A)[p^\infty]\Big).$$

Here v runs through all the primes of F. The *fine Selmer group* of A is defined by the exact sequence

$$0 \to R_{p^\infty}(A/F) \to \mathrm{Sel}_{p^\infty}(A/F) \to \bigoplus_{v|p} A(F_v) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p.$$

5.2. Proof of Theorem 1.6. Let F be a number field and S a finite set of places of F. The *S-class group* of F, denoted $\mathrm{Cl}_S(F)$, is the quotient of $\mathrm{Cl}(F)$ by the subgroup generated by the ideal classes of prime ideals in S. The following proposition is proven in [LM16, Lemma 4.3].

Proposition 5.1. *Let A be an abelian variety over a number field F. Let S be a finite set of primes containing $S_\ell \cup S_{\mathrm{bad}} \cup S_\infty$. Suppose that $A(F)[\ell] \neq 0$. Then*

$$r_\ell(R_{\ell^\infty}(A/F)) \geq r_\ell(\mathrm{Cl}_S(F)) \cdot r_\ell(A(F)[\ell]) - 2\dim(A),$$

where $\dim(A)$ *denotes the dimension of the abelian variety A.*

We will first relate the S-class group of F to the class group of F.

LEMMA 5.2. *Let L be a number field and let ℓ be a rational prime. Let S be a finite set of places of L containing the primes above ℓ. Let s_0 be the number of finite primes in S. Then for all n,*

$$|r_\ell(\mathrm{Cl}(L_n)) - r_\ell(\mathrm{Cl}_S(L_n))| \geq 2s_0\ell^n.$$

Proof. We reproduce this proof from [Kun20, Lemma 4.6, Step A]. Let s_n be the number of finite primes of L_n lying over a prime of S. Consider the following short exact sequence for all n [NSW13, Lemma 10.3.12]:

$$\mathbb{Z}^{s_0} \to \mathrm{Cl}(L_n) \to \mathrm{Cl}_S(L_n).$$

Taking ℓ-ranks of this sequence, by [LM16, Lemma 3.2] we obtain

$$|r_\ell(\mathrm{Cl}(L_n)) - r_\ell(\mathrm{Cl}_S(L_n))| \geq 2s_0\ell^n. \quad \blacksquare$$

Proof of Theorem 1.6. Recall that we have primes $\ell \neq p$ and an abelian variety A defined over $\mathbb{Q}(\zeta_\ell)$. Let $S = S_p \cup S_{\mathrm{bad}} \cup S_\infty$. Let s_0 be the number of finite places of S. We can apply Theorem 1.8 (replacing N with $N + 2s_0$) to construct a Γ-extension L_∞/L such that

$$r_\ell(\mathrm{Cl}(L_n)) \geq (N + 2s_0)p^n$$

for all $n \geq 0$. Proposition 5.1 tells us that for all $n \geq 0$,

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq r_\ell(\mathrm{Cl}_S(L_n)) \cdot r_\ell(A(L_n)[\ell]) - 2\dim(A).$$

Since we have assumed $A(\zeta_\ell)[\ell] \neq 0$, it follows that $r_\ell(A(L_n)[\ell]) \geq 1$ for all $n \geq 1$. This gives us

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq r_\ell(\mathrm{Cl}_S(L_n)) - 2\dim(A).$$

Applying Lemma 5.2, we get

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq (r_\ell(\mathrm{Cl}(L_n)) - 2s_0\ell^n) - 2d \geq N + 2s_0p^n - 2s_0\ell^n - 2\dim(A).$$

Suppose $\ell < p$. Then we have

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq N + 2s_0\ell^n - 2s_0\ell^n - 2d = N\ell^n - 2\dim(A).$$

Now suppose $\ell > p$. Then

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq N + 2s_0p^n - 2s_0p^n - 2d = Np^n - 2\dim(A).$$

Either way, we get

$$r_\ell(R_{\ell^\infty}(A/L_n)) \geq Nq^n - 2\dim(A) \geq Nq^n,$$

where $q = \min\{\ell, p\}$. This completes the proof of Theorem 1.6. $\blacksquare$

6. Examples

EXAMPLE 1. Let $E = 11a1$. Then $E(\mathbb{Q}(\zeta_5))[5] \neq 0$, so we can pick $\ell = 5$ and $p = 3$. Let $N = 2$. We will construct a $\Gamma = \mathbb{Z}_3$-extension L_∞/L such that

$$r_5(R_{5^\infty}(E/L_n)) \geq 2 \cdot 3^n \quad \text{for all } n \geq 0.$$

We pick $K = \mathbb{Q}(\zeta_3)$ because there is a unique prime of F lying over $p = 3$. Let K_∞/K be the anticyclotomic $\mathbb{Z}_3$-extension of K. We have $t = N + 2d\ell(\ell-1) = 42$. We want to find $t = 42$ primes $v_1, \ldots, v_t$ of F that split completely in K_∞. It is enough to find 42 primes different from $p = 3$ that are inert in $\mathbb{Q}(\zeta_3)$. We list these primes below:

$2, 5, 11, 17, 23, 29, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 131, 137, 149, 167, 173, 179, 191,$
$197, 227, 233, 239, 251, 257, 263, 269, 281, 293, 311, 317, 347, 353, 359, 383, 389, 401, 419.$

Let α be the product of these 42 primes,

$$\alpha = 5564821300878169567266781038470220470547296829866818046142810483994789051955010075838675 10.$$

Now put $L = K(\zeta_5, \sqrt[5]{\alpha})$, and let $L_\infty = K_\infty L$. Then L_∞/L is a $\mathbb{Z}_3$-extension. Theorem 1.8 says that

$$r_5(\mathrm{Cl}(L_n)) \geq 2 \cdot 3^n \quad \text{for all } n \geq 0.$$

And Theorem 1.4 says that

$$r_5(R_{5\infty}(E/L_n)) \geq 2 \cdot 3^n \quad \text{for all } n \geq 0.$$

Example 2. We now look at $\mathbb{Z}_p^d$-extensions for $d = 3$. As in the previous example, let $E = 11a1$, $\ell = 5$ and $p = 3$. Set $N = 10$. We will construct a $\mathbb{Z}_3^3$-extension L_∞/L such that

$$r_\ell(R_{\ell\infty}(E/L_n)) \geq 10 \cdot 3^n \quad \text{for all } n \geq 0.$$

Consider the CM field $K = \mathbb{Q}(\zeta_9)$. Let K_∞ be the $\mathbb{Z}_3^3$-extension of K given in Proposition 2.1. We have $t = N + 2d\ell(\ell-1) = 90$. We want to find $t = 90$ primes $v_1, \ldots, v_t$ that are inert in K. By the well-known splitting laws for primes in cyclotomic fields, every rational prime which is $\equiv 2, 5$ modulo 9 is inert in K. Here is a list of 90 such primes:

$2, 5, 11, 23, 29, 41, 47, 59, 83, 101, 113, 131, 137, 149, 167, 173, 191, 227, 239, 257, 263, 281,$
$293, 311, 317, 347, 353, 383, 389, 401, 419, 443, 461, 479, 491, 509, 563, 569, 587, 599, 617, 641,$
$653, 659, 677, 743, 761, 797, 821, 839, 857, 887, 911, 929, 941, 947, 977, 983, 1013, 1019, 1031,$
$1049, 1091, 1103, 1109, 1163, 1181, 1193, 1217, 1229, 1283, 1289, 1301, 1307, 1319, 1361,$
$1373, 1409, 1427, 1433, 1451, 1481, 1487, 1499, 1523, 1553, 1559, 1571, 1607, 1613.$

Let α be the product of these primes:

$$\alpha = 3026691567190856771201105872323454284465474656097714712640878306872219738 23946203120683121105279988012699117394288474908858414443287091308966386167902242957859532761609270923483095428112544069874627622945451584053107032 9013191741865236750170.$$

Now put $L = K(\zeta_5, \sqrt[5]{\alpha})$ and let $L_\infty = K_\infty L$. Then L_∞/L is a $\mathbb{Z}_3^2$-extension. Theorem 1.8 says that

$$r_5(\mathrm{Cl}(L_n)) \geq 10 \cdot 3^n \quad \text{for all } n \geq 0.$$

And Theorem 1.6 says that

$$r_5(R_{5^\infty}(E/L_n)) \geq 10 \cdot 3^n \quad \text{for all } n \geq 0.$$

Example 3. We discuss a non-commutative example of Γ. The nilpotent uniform groups of dimension $d = 3$ are parametrized, up to isomorphism, by a number $s \in \mathbb{N}$. They are given by (see [GSK09, Section 7, Theorem 7.4])

$$\Gamma(s) = \langle x, y, z : [x, z] = [y, z] = 1,\ [x, y] = z^{p^s} \rangle.$$

The groups $\Gamma(s)$ are non-abelian; they fit in the exact sequence

$$1 \to \mathbb{Z}_p \to \Gamma(s) \to \mathbb{Z}_p^2 \to 1.$$

If $p \equiv 1$ modulo 3, the group $\Gamma(1)$ has an automorphism τ of order 3 which has no fixed points (see [HM19, Proposition 4.1]). Therefore $m = 3$.

Put $\Gamma = \Gamma(1)$. Let $E = 19a1$. Since $E(\mathbb{Q})[3] \neq 0$, let $\ell = 3$, and $p = 7$. Let $N = 6$. We will construct a Γ-extension L_∞/L such that

$$r_5(R_{5^\infty}(E/L_n)) \geq 6 \cdot 3^n \quad \text{for all } n \geq 0.$$

We construct a degree $m = 3$ extension F/F_0 as in Assumption 1. Let $F_0 = \mathbb{Q}(\zeta_7)$. Let $F = F_0(\theta)$, where θ is a root of the irreducible cubic polynomial

$$x^3 - x^2 - 4x - 1.$$

Then F/F_0 is a degree 3 cyclic extension. According to the LMFDB database [LMF23], the class number of F is 13 and there is a unique prime $\mathfrak{p}$ lying over $p = 7$. In particular, the p-part of the $\mathfrak{p}$-class group of F is trivial so F is p-rational. Therefore, F/F_0 satisfies the hypotheses of Proposition 3.1.

Set $t := N + md\ell(\ell-1) = 6+3\cdot 3\cdot 3\cdot 2 = 60$. We need to find 60 primes of $F_0 = \mathbb{Q}(\zeta_7)$ which are inert in F. Consider the following 10 rational primes:

$$43, 127, 491, 673, 953, 1499, 1583, 2129, 2311, 2591. \tag{1}$$

By a calculation in Sage, each of these 10 rational primes splits completely in F_0 into 6 factors and each of these factors is inert in F. For example, 43 factors in F_0 as

$$\begin{aligned} 43 = {} & (\zeta_7^5 + 2\zeta_7^3 + \zeta_7^2 + 1) \cdot (\zeta_7^5 + \zeta_7^4 + 2\zeta_7^2 + \zeta_7) \\ & \cdot (2\zeta_7^5 + \zeta_7^4 + 2\zeta_7^3 + \zeta_7^2 + 2\zeta_7 + 1) \cdot (-2\zeta_7^5 - \zeta_7^4 - \zeta_7^3 - 2\zeta_7^2 - 2\zeta_7 - 1) \\ & \cdot (2\zeta_7^4 + \zeta_7^3 + \zeta_7^2 + \zeta_7) \cdot (\zeta_7^5 + \zeta_7^4 + \zeta_7^3 + 2\zeta_7^2), \end{aligned}$$

and each of the factors is inert in F. In total, there are $10 \cdot 6 = 60$ primes of F_0 lying over the 10 rational primes (1), each of which is inert in F.

Let α be the product of these 60 primes:

$$\begin{aligned} \alpha = {} & 784025037792166554050235760891167382653206060626833429989912309772985 94 \\ & 366840200239211889414161610945783214748072276266387591561420797021082 39 \\ & 313497652801991067685041337071171617321114788409671453358754013644971. \end{aligned}$$

We apply Lemma 2.4. Since $[F_0 : \mathbb{Q}] = 6 \geq 2d$, we set $n = 0$, so that $K_0 = F_0$ and $K = F$. The lemma says that there exists a Γ-extension K_∞/K such that every inert prime in K/K_0 splits completely in K_∞.

Put $L = K(\alpha^{1/3}, \zeta_3)$ and $L_\infty = LK_\infty$. Then L_∞/L is a Γ-extension. Theorem 1.8 says that

$$r_5(\mathrm{Cl}(L_n)) \geq 6 \cdot 3^n \quad \text{for all } n \geq 0.$$

And Theorem 1.6 says that

$$r_5(R_{5^\infty}(E/L_n)) \geq 6 \cdot 3^n \quad \text{for all } n \geq 0.$$

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Adithya Chakravarthy
University of Toronto
Toronto, ON, M5S 2E4, Canada
E-mail: adithyachakra21@gmail.com