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Uniform quasi-multiplicativity of locally constant cocycles and applications

by

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Abstract. We show that every locally constant cocycle $\mathcal{A}$ is k-quasi-multiplicative under the irreducibility assumption. More precisely, we show that if $\mathcal{A}^t$ and $\mathcal{A}^{\wedge m}$ are irreducible for every $t \mid d$ and $1 \leq m \leq d-1$, then $\mathcal{A}$ is k-uniformly spannable for some $k \in \mathbb{N}$, which implies that $\mathcal{A}$ is k-quasi-multiplicative. We apply our results to show that the unique subadditive equilibrium Gibbs state is ψ-mixing and calculate the Hausdorff dimension of cylindrical shrinking target sets and recurrence sets.

1. Introduction and statement of the results. A *matrix cocycle* $\mathcal{A}$ on a compact metric space X is a continuous map $\mathcal{A}\colon X \to \mathrm{GL}_d(\mathbb{R})$ over a topological dynamical system (X,T). For $n \in \mathbb{N}$ and $x \in X$, we define the product of $\mathcal{A}$ along the length n orbit of X as

$$\mathcal{A}^n(x) := \mathcal{A}(T^{n-1}(x)) \dots \mathcal{A}(x).$$

The submultiplicativity of the norm $\|\cdot\|$ implies that $\|\mathcal{A}\|$ is submultiplicative in the sense that for any $m,n \in \mathbb{N}$,

$$0 \leq \|\mathcal{A}^{n+m}(x)\| \leq \|\mathcal{A}^m(T^n(x))\|\,\|\mathcal{A}^n(x)\|.$$

Such a submultiplicative sequence gives rise to a norm potential $\Phi_{\mathcal{A}} := \{\log \|\mathcal{A}^n\|\}_{n\in\mathbb{N}}$.

Let $\ell \in \mathbb{N}$ be given. The *one-sided shift* Σ_ℓ of ℓ symbols is the space $\{1,\dots,\ell\}^{\mathbb{N}}$. Let $\sigma\colon \Sigma_\ell \to \Sigma_\ell$ be the left shift map defined by $\sigma i = i_1 i_2 \dots$ for all $i = i_0 i_1 \dots \in \Sigma_\ell$, and for simplicity denote it by (Σ_ℓ, σ). We will focus on locally constant cocycles $\mathcal{A}$, which are matrix cocycles $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ over a one-sided shift (Σ_ℓ,σ) that depends only on the zeroth symbol x_0 of $x = (x_i)_{i\in\mathbb{N}}$. Assume that $(A_1,\dots,A_\ell) \in \mathrm{GL}_d(\mathbb{R})^\ell$ generate a locally constant

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cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$. We say that $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *irreducible* if there does not exist a proper subspace $V \subset \mathbb{R}^d$ such that $A_iV \subset V$ for $i = 1, \ldots, \ell$. We also say that $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *strongly irreducible* if there does not exist a finite collection $V_1, \ldots, V_m$ of non-zero proper subspaces V_j such that $A_i(\bigcup_{j=1}^m V_j) = \bigcup_{j=1}^m V_j$ for every $i = 1, \ldots, \ell$.

For any length n word $I = i_0 \ldots i_{n-1}$ (see Section 2 for the definition), we denote

$$\mathcal{A}_I := A_{i_{n-1}} \ldots A_{i_0}.$$

Denoting by $\mathcal{L}$ the set of all finite words, we say a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ generated by $(A_1, \ldots, A_\ell)$ is *quasi-multiplicative* if there exist $k \in \mathbb{N}$ and $c > 0$ such that for any $I, J \in \mathcal{L}$, there exists $K = K(I, J) \in \mathcal{L}$ with $|K| \leq k$ such that

$$\|\mathcal{A}_{IKJ}\| \geq c\|\mathcal{A}_I\| \, \|\mathcal{A}_J\|.$$

Notice that quasi-multiplicativity of $\mathcal{A}$ resembles Bowen's specification property [6] in some respects. Feng [11] showed that the quasi-multiplicativity property implies the uniqueness of the Gibbs equilibrium measure for the norm of the cocycle $\mathcal{A}$. Feng [12] also proved that if $\mathcal{A}$ is a locally constant $\mathrm{GL}_d(\mathbb{R})$-cocycle over a full shift generated by an irreducible set of matrices, then $\mathcal{A}$ is quasi-multiplicative. Recently, there has been further research in quasi-multiplicativity (see [19, 27, 3, 26] for instance). Unfortunately, the lack of control on the length of the connecting word K from quasi-multiplicativity is a limitation in studying important applications such as the Bernoulli property on a class of subadditive equilibrium states and shrinking target sets and recurrence sets; see Section 4. When the connecting word K in the quasi-multiplicativity property has a fixed length $k \in \mathbb{N}$, we say $\mathcal{A}$ is *k-quasi-multiplicative*; see Definition 2.2.

There are a few results along this line in the literature. In the same setting of locally constant cocycles, Bárány and Troscheit [3, Proposition 2.5] and Morris [26, Theorem 7] proved that $\mathcal{A}$ is k-quasi-multiplicative when $\mathcal{A}$ is (strongly) irreducible and proximal. Note that if $\mathcal{A}$ is strongly irreducible, then $\mathcal{A}^t$ is irreducible for every $1 \leq t \leq d$. In this paper, we generalize their results. A distinction in our result from similar results is that we only require versions of irreducibility as our assumptions, while many recent similar results additionally need some form of proximality to obtain the k-quasi-multiplicativity property. In fact, our result is inspired by a recent result of Bochi and Garibaldi [4, Proposition 3.9] in a more general setting; see Remark 2.6 for further comments on their work.

We say that a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *k-uniformly spannable* (see Subsection 2.2 for more information) if there exists $k \in \mathbb{N}$

such that for any non-zero vector $u \in \mathbb{R}^d$,

$$V_{u,k} = \mathbb{R}^d,$$

where $V_{u,k} := \mathrm{Span}\{\mathcal{A}_I u : I \in \mathcal{L} \text{ with } |I| = k\} \subseteq \mathbb{R}^d$.

We will also make use of the exterior product cocycle $\mathcal{A}^{\wedge m}$ for $1 \leq m \leq d-1$, where $\mathcal{A}^{\wedge m}(x)$ is considered as a linear transformation on $(\mathbb{R}^d)^{\wedge m}$. Our main result is as follows:

THEOREM 1.1. *Let $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ be a locally constant cocycle. Suppose $\mathcal{A}^t$ and $\mathcal{A}^{\wedge m}$ are irreducible for every $t \mid d$ and $1 \leq m \leq d-1$. Then $\mathcal{A}$ is k-uniformly spannable for some $k \in \mathbb{N}$.*

As spannable cocycles are quasi-multiplicative, the following is an immediate corollary of the above theorem.

COROLLARY 1.2. *Let $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ be a locally constant cocycle. Suppose $\mathcal{A}^t$ and $\mathcal{A}^{\wedge m}$ are irreducible for every $t \mid d$ and $1 \leq m \leq d-1$. Then $\mathcal{A}$ is k-quasi-multiplicative for some $k \in \mathbb{N}$.*

Corollary 1.2 has nice applications in subadditive thermodynamic formalism and number theory, which we discuss in more detail in Section 4.

Earlier, in Section 2, we introduce the relevant notation and prove Corollary 1.2, and in Section 3, we prove Theorem 1.1.

2. Preliminaries

2.1. Set-up. For each $n \in \mathbb{N}$, we define $\mathcal{L}(n)$ to be the set of all length n words of Σ_ℓ, and we define $\mathcal{L} := \bigcup_{n\in\mathbb{N}} \mathcal{L}(n)$ to be the set of all words. If $i = i_0 i_1 \ldots \in \mathcal{L}$, then we define $i|_n = i_0 \ldots i_{n-1}$ for all $n \in \mathbb{N}$. The empty word $i|_0$ is denoted by $\varnothing$. The length of $i \in \mathcal{L}$ is denoted by $|i|$. The longest common prefix of $i, j \in \mathcal{L} \cup \Sigma_\ell$ is denoted by $i \wedge j$. The concatenation of two words $i \in \mathcal{L} \cup \Sigma_\ell$ and $j \in \mathcal{L}$ is denoted by ji. If $i \in \mathcal{L}(n)$ for some n, then we set $[i] = \{j \in \Sigma_\ell : j|_n = i\}$. The set $[i]$ is called a *cylinder set*. A *cylinder* containing $x = (x_i)_{i\in\mathbb{Z}} \in \Sigma_\ell$ of length $n \in \mathbb{N}$ is defined by $[x]_n := \{(y_i)_{i\in\mathbb{N}} \in \Sigma_\ell : x_i = y_i \text{ for all } 0 \leq i \leq n-1\}$. The shift space Σ_ℓ is compact in the topology generated by the cylinder sets. Moreover, the cylinder sets are open and closed in this topology and they generate the Borel σ-algebra. We denote by $\mathcal{M}(\Sigma_\ell, \sigma)$ the space of all σ-invariant Borel probability measures on Σ_ℓ.

2.2. Quasi-multiplicativity and spannability

DEFINITION 2.1. We say a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *quasi-multiplicative* if there exist $k \in \mathbb{N}$ and $c > 0$ such that for any $I, J \in \mathcal{L}$, there exists $K = K(I,J) \in \mathcal{L}$ with $|K| \leq k$ such that

$$\|\mathcal{A}_{IKJ}\| \geq c\|\mathcal{A}_I\|\,\|\mathcal{A}_J\|. \tag{2.1}$$

Definition 2.2. We say a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *k-quasi-multiplicative* for some $k \in \mathbb{N}$ if there exists $c > 0$ such that for any $I, J \in \mathcal{L}$, there exists $K = K(I, J) \in \mathcal{L}(k)$ such that (2.1) holds.

We will elaborate more on the applications of k-quasi-multiplicativity in Section 4. In the remaining part of this subsection, we describe a notion of spannability, closely related to quasi-multiplicativity. In what follows, we will repeatedly make use of the following notation: given a locally constant cocycle $\mathcal{A}$, a vector $u \in \mathbb{R}^d$ and an integer $k \in \mathbb{N}$, we define

$$V_{u,k} := \mathrm{Span}\,\{\mathcal{A}_I u : I \in \mathcal{L}(k)\} \subseteq \mathbb{R}^d. \tag{2.2}$$

Definition 2.3. We say a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is *spannable* if for any non-zero vector $u \in \mathbb{R}^d$, there exists $k = k(u) \in \mathbb{N}$ such that $V_{u,k} = \mathbb{R}^d$. If $k = k(u)$ can be chosen uniformly in u, then we say $\mathcal{A}$ is *k-uniformly spannable.*

Remark 2.4. We note that if $V_{u,k}$ is equal to the entire space $\mathbb{R}^d$ for some $u \in \mathbb{R}^d$ and $k \in \mathbb{N}$, then by continuity so is $V_{v,k}$ for all $v \in \mathbb{R}^d$ in a small neighborhood of u. Moreover, if $V_{u,k}$ is equal to $\mathbb{R}^d$, then so is $V_{u,k+1}$. In particular, if $\mathcal{A}$ is k-uniformly spannable for some k, then it is k'-uniformly spannable for any $k' \geq k$.

Throughout the paper, when we measure the angle between non-zero vectors, we mean the angle between the lines spanned by the vectors. Similarly, when we measure the angle between a non-zero vector v and a hyperplane $\mathbb{W}$, we mean the minimum angle $\measuredangle(v, w)$ over all $w \in \mathbb{W} \setminus \{0\}$. The following statement can be found in [8, Proposition 8]; it states that spannability implies quasi-multiplicativity.

Proposition 2.5. *Suppose a locally constant cocycle $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ is k-uniformly spannable. Then $\mathcal{A}$ is k-quasi-multiplicative.*

Proof. The proof is similar to [8, Proposition 8] for fiber-bunched cocycles. We give a sketch of the proof here for the convenience of the readers.

For any $A \in \mathrm{GL}_d(\mathbb{R})$, let $v_{A,1} \in \mathbb{R}^d$ be a unit vector such that $\|Av_{A,1}\| = \|A\|$, and let $v_{A,2}$ be the unit vector in the direction of $Av_{A,1}$.

By k-uniform spannability, we begin by finding $\varepsilon > 0$ such that for given arbitrary $I, J \in \mathcal{L}$, we can find $K \in \mathcal{L}(k)$ such that $\measuredangle(\mathcal{A}_K v_{\mathcal{A}_I,2}, (v_{\mathcal{A}_J,1})^\perp) \geq \varepsilon$. By [27, Lemma 4.5], this translates into the existence of $c > 0$ satisfying the k-quasi-multiplicativity condition (2.1). ∎

Proof of Corollary 1.2. The assertion follows from the combination of Theorem 1.1 and Proposition 2.5. ∎

We end this subsection by commenting on a class of cocycles that generalize the class of locally constant cocycles and by comparing how the notions defined above relate to such cocycles.

Remark 2.6. Beyond locally constant cocycles, there exists a subset of Hölder continuous cocycles that are nearly conformal. We call them *fiber-bunched cocycles*. The most useful property of these cocycles is the existence of holonomies and we often tend to treat them as generalizations of locally constant cocycles; see [5, 27, 22].

The properties introduced above, such as quasi-multiplicativity and spannability, can be successfully extended, and sufficient conditions have been found which imply such properties. For instance, the above-mentioned result of Feng [11] on the uniqueness of the equilibrium state using quasi-multiplicativity remains valid for fiber-bunched cocycles. Moreover, Park [27] showed that typicality, an assumption introduced by Bonatti and Viana [5] to replicate the effect of strong irreducibility and non-compactness from the classical work of Furstenberg [14], implies quasi-multiplicativity. The k-uniform spannability introduced above was motivated by the work of Bochi and Garibaldi [4] who showed that irreducibility, along with an extra assumption on how close the cocycle is from being conformal, implies uniform spannability. They use the term *uniform spannability*, when translated to our setting of locally constant cocycles, to roughly mean $\bigcup_{k=1}^{n} V_{u,k} = \mathbb{R}^d$ for some $n \in \mathbb{N}$. In order to distinguish from their version of uniform spannability, we have decided to use *k-spannability* to denote the stronger statement that $V_{u,k} = \mathbb{R}^d$.

3. Proof of Theorem 1.1. In the proof, whenever we write $V = W$ for two m-dimensional subspaces of $\mathbb{R}^d$, we mean they are equal considered as elements of the Grassmannian $\mathrm{Gr}(m, \mathbb{R}^d)$. Moreover, for $I \in \mathcal{L}$ we define

$$\mathcal{A}_I V := \mathrm{Span}\,\{\mathcal{A}_I v : v \in V\}.$$

Suppose for a contradiction that there does not exist $k \in \mathbb{N}$ such that $\mathcal{A}$ is k-uniformly spannable, meaning that for every $k \in \mathbb{N}$ there exists $u = u_k \in \mathbb{R}^d$ such that $V_{u,k}$ is a proper subspace of $\mathbb{R}^d$. Define an open set

$$S_k := \{u \in \mathbb{R}^d : V_{u,k} = \mathbb{R}^d\}.$$

Then its complement $T_k := \mathbb{R}^d \setminus S_k$ is a closed non-empty set for every $k \in \mathbb{N}$. Moreover, it is clear from the definition that $S_k \subseteq S_{k+1}$, meaning that T_k satisfies the reverse inclusion $T_{k+1} \subseteq T_k$. Therefore, the nested intersection $\bigcap_{k\in\mathbb{N}} \mathbb{P}T_k$ is necessarily non-empty. In particular, we can choose a vector $u \in \mathbb{R}^d$ which belongs to T_k, meaning that $V_k := V_{u,k}$ is a proper subspace of $\mathbb{R}^d$, for every $k \in \mathbb{N}$.

As $n \mapsto \dim V_n$ is a bounded non-decreasing function, the dimension of V_k has to stabilize to some $m \in \mathbb{N}$ strictly smaller than d. By dropping the first few subspaces from the sequence $\{V_k\}_{k\in\mathbb{N}}$ we may assume that $\dim V_k = \gamma$

for all $k \in \mathbb{N}$. Moreover, from the definition (2.2) of V_k, we have

$$V_{k+n} = \mathcal{A}_I V_k = \mathcal{A}_J V_k \tag{3.1}$$

for all $k, n \in \mathbb{N}$ and $I, J \in \mathcal{L}(n)$. This is the defining characteristic of the sequence $\{V_k\}_{k\in\mathbb{N}}$. Now, by choosing a possibly different sequence of subspaces we may assume that the common dimension γ of $\{V_k\}_{k\in\mathbb{N}}$ is the smallest such number; that is, if $\{W_k\}_{k\in\mathbb{N}}$ is another sequence of subspaces of common dimension satisfying (3.1), then the dimension is at least γ.

LEMMA 3.1. *For any $i \neq j$, the subspaces V_i and V_j either coincide or intersect trivially.*

Proof. Suppose there exists $i \neq j$ such that $W := V_i \cap V_j$ is a non-trivial proper subspace of both V_i and V_j. Then for any $n \in \mathbb{N}$ and $I \in \mathcal{L}(n)$ we have

$$\mathcal{A}_I W = \mathcal{A}_I(V_i \cap V_j) = V_{i+n} \cap V_{j+n},$$

where the resulting subspace $V_{i+n} \cap V_{j+n}$ does not depend on I. In particular, this allows us to define a sequence $\{W_k\}_{k\in\mathbb{N}}$ of subspaces satisfying (3.1) by $W_k := \mathcal{A}_I W$ for any $I \in \mathcal{L}(k)$, contrary to the minimality assumption on $\{V_k\}_{k\in\mathbb{N}}$. ∎

Following the lemma, we will now consider two separate cases, conclude that both lead to a contradiction, and hence deduce that $\mathcal{A}$ must be k-uniformly spannable for some $k \in \mathbb{N}$.

CASE 1: *$V_i = V_j$ for some distinct $i, j \in \mathbb{N}$.* Since all V_k have the same dimension, the sequence $\{V_k\}_{k\in\mathbb{N}}$ must be periodic with some period $t \in \mathbb{N}$; that is, $t \in \mathbb{N}$ is a smallest integer such that $V_1 = V_{t+1}$. Moreover, V_i and V_j have a trivial intersection for distinct $1 \le i, j \le t$. Recalling that each V_k has dimension γ, the subspace $W := \operatorname{Span}\{v \in V_i : i = 1, \dots, t\}$ is a γt-dimensional subspace of $\mathbb{R}^d$ preserved under $\mathcal{A}$. Since $\mathcal{A}$ is irreducible by assumption, this implies that W must be the entire subspace $\mathbb{R}^d$, and that t divides d. However, this implies that $\mathcal{A}^t$, which preserves the non-trivial subspace V_1 (or any one of V_k), is reducible, a contradiction.

CASE 2: *Any two distinct subspaces from $\{V_k\}_{k\in\mathbb{N}}$ have a trivial intersection.* This is the choice $m = \gamma$. We will show that we also arrive at a contradiction in this case.

We begin by choosing a decomposable vector $w_k = v_{1,k} \wedge \cdots \wedge v_{m,k} \in (\mathbb{R}^d)^{\wedge m}$, where $\operatorname{Span}\{v_{1,k}, \dots, v_{m,k}\}$ coincides with V_k. Since $A_i V_k = A_j V_k = V_{k+1}$ from (3.1) for any A_i, A_j in the image of $\mathcal{A}$, decomposable vectors $A_i^{\wedge m} w_k$ and w_{k+1} differ by a multiplicative constant. Moreover, each w_k is an eigenvector of $B_{i,j} := (A_i^{\wedge m})^{-1} A_j^{\wedge m}$. We now fix any $i \neq j$ such that $B := B_{i,j}$ is not a scalar multiple of the identity transformation. Such a choice is possible because otherwise the $A_i^{\wedge m}$ would be scalar multiples of one another, which contradicts the assumption that $\mathcal{A}^{\wedge m}$ is irreducible for all $1 \leq m \leq d-1$. Now let λ_k be the corresponding eigenvalue of w_k.

Considered as a subspace of $(\mathbb{R}^d)^{\wedge m}$, let $W_k := \mathrm{Span}\,\{w_1, \ldots, w_k\}$ for each $k \in \mathbb{N}$. We claim that the subspace W_N with $N := \binom{d}{m}$ is equal to the entire $(\mathbb{R}^d)^{\wedge m}$ because otherwise there would exist some $k < N$ such that $W_k = W_{k+1}$, meaning that w_{k+1} belongs to W_k. However, this would imply that $\mathcal{A}^{\wedge m}$ preserves a proper subspace W_k of $(\mathbb{R}^d)^{\wedge m}$ because we can inductively show that w_l (which is a scalar multiple of $A_i^{\wedge m} w_{l-1}$ for any A_i) belongs to W_k for every $l > k$, and this contradicts the assumption that $\mathcal{A}^{\wedge m}$ is irreducible for all $1 \leq m \leq d-1$.

Therefore, W_N coincides with $(\mathbb{R}^d)^{\wedge m}$, and $\{w_1, \ldots, w_N\}$ forms a basis of $(\mathbb{R}^d)^{\wedge m}$. Now choose any $l > N$ such that w_l can be written as

$$w_l = c_1 w_1 + \cdots + c_N w_N.$$

Applying B on both sides and using the fact that each w_k is an eigenvector of B with eigenvalue λ_k, we get $\lambda_l w_l = \sum_{i=1}^N c_i \lambda_i w_i$. Substituting the above equation for w_l into $\lambda_l w_l$ and equating coefficients gives $\lambda_l = \lambda_1 = \cdots = \lambda_N$, so B is a scalar multiple of the identity transformation, a contradiction.

Since both cases lead to a contradiction, the proof of Theorem 1.1 is complete.

4. Applications

4.1. Gibbs matrix equilibrium states have the Bernoulli property. For any matrix cocycle $\mathcal{A}\colon X \to \mathrm{GL}_d(\mathbb{R})$ over a topological dynamical system (X, T), and $s \geq 0$, the potential $\Phi_{\mathcal{A}}^s := \{s \log \|\mathcal{A}^n\|\}_{n \in \mathbb{N}}$ is subadditive. Therefore, the subadditive thermodynamic formalism applies. For instance, the *subadditive variational principle* (see [9]) states that

$$P(\Phi_{\mathcal{A}}^s) = \sup_{\mu \in \mathcal{M}(X,T)} \Big\{ h_\mu(T) + s \lim_{n\to\infty} \frac{1}{n} \int \log \|\mathcal{A}^n(x)\| \, d\mu \Big\}.$$

Invariant measures achieving the supremum are called *equilibrium states* of $\Phi_{\mathcal{A}}^s$. If the entropy map is upper semicontinuous, the supremum is always attained. The thermodynamic interpretation of the parameter s is as the inverse temperature of a system (see [30]), while the equilibrium measure μ_s of $\Phi_{\mathcal{A}}^s$ describes the equilibrium of the system at temperature $1/s$. The $s \to \infty$ limit is therefore a *zero temperature limit*, and an accumulation point of the μ_s can be interpreted as a ground state (see, e.g., [10, 17, 18, 7, 23, 21]).

Feng and Käenmäki [13] showed that if $\mathcal{A}$ is a locally constant $\mathrm{GL}_d(\mathbb{R})$-cocycle over a full shift generated by an irreducible set of matrices, then $\Phi_{\mathcal{A}}^s$ has a unique equilibrium state μ_s for all $s \geq 0$. Moreover, μ_s has the subadditive *Gibbs property*: there exists $C_0 > 1$ such that for any $x \in \Sigma_\ell$

and $n \in \mathbb{N}$, we have

$$C_0^{-1} \leq \frac{\mu_s([x]_n)}{e^{-nP(\Phi^s_{\mathcal{A}})}\|\mathcal{A}^n(x)\|^s} \leq C.$$

Note that if μ satisfies a Gibbs inequality with respect to $\Phi^s_{\mathcal{A}}$, then μ is fully supported on Σ_ℓ.

Let $\sigma\colon \Sigma_\ell \to \Sigma_\ell$ be a left shift map. We say that an invariant measure $\mu \in \mathcal{M}(\Sigma_\ell, \sigma)$ is *totally ergodic* if for every $n \in \mathbb{N}$, μ is ergodic with respect to σ^n. We also define $\hat{\Sigma}_\ell := \{1, \ldots, \ell\}^{\mathbb{Z}}$ equipped with a norm d, $\hat{\sigma}(\{x_k\}_{k\in\mathbb{Z}}) := (x_{k+1})_{\mathbb{Z}}$, and $\mathcal{M}(\hat{\Sigma}_\ell, \hat{\sigma})$ to be the space of all $\hat{\sigma}$-invariant Borel probability measures on $\hat{\Sigma}_\ell$.

We define the natural projection $\pi\colon \hat{\Sigma}_\ell \to \Sigma_\ell$ by $\pi(\{x_k\}_{k\in\mathbb{Z}}) := \{x_k\}_{k=0}^{\infty}$; it is continuous and surjective. Clearly $\hat{\mu} \mapsto \pi_*\hat{\mu}$ defines a continuous function $\mathcal{M}(\hat{\Sigma}_\ell, \hat{\sigma}) \to \mathcal{M}(\Sigma_\ell, \sigma)$, and since shift-invariant measures on Σ_ℓ and on $\hat{\Sigma}_\ell$ are both characterized by their values on cylinder sets, this map is bijective. Let $\mu \in \mathcal{M}(\Sigma_\ell, \sigma)$; we will write $\hat{\mu}$ for the unique element of $\mathcal{M}(\hat{\Sigma}_\ell, \hat{\sigma})$ such that $\mu = \pi_*\hat{\mu}$, and we call $\hat{\mu}$ the *natural extension* of the measure μ. Since properties such as mixing, ergodicity and total ergodicity can be characterized in terms of correlations between cylinder sets, it is not difficult to see that each of those properties holds for an invariant measure $\mu \in \mathcal{M}(\Sigma_\ell, \sigma)$ if and only if the corresponding property holds for $\hat{\mu} \in \mathcal{M}(\hat{\Sigma}_\ell, \hat{\sigma})$. We say that a measure $\hat{\mu}$ on $\hat{\Sigma}_\ell$ is a *Bernoulli measure* if it has the form $\hat{\mu} = (\sum_{i=1}^{\ell} p_i\delta_i)^{\mathbb{Z}}$ for some probability vector $(p_1, \ldots, p_\ell)$; and $\hat{\mu}$ has the *Bernoulli property* if there exist a Bernoulli measure $\hat{v}$ on $\hat{\Sigma}_\ell$ and a measure-space isomorphism $\phi\colon \hat{\Sigma}_\ell \to \hat{\Sigma}_\ell$ such that $\phi \circ \hat{\sigma} = \hat{\sigma} \circ \phi$ and $\phi_*\hat{\mu} = \hat{v}$ (See [26, Section 7] for more details). It is clear that every Bernoulli measure has the Bernoulli property, but the converse is in general false.

Morris [24, 25] showed that the total ergodicity of equilibrium states implies mixing and that the failure of mixing can be characterized by certain structures of the cocycle. Recently, he improved his own result by showing that total ergodicity implies the Bernoulli property [26]. Piraino [29, Theorem 3.3] showed that for any $s \geq 0$, the unique Gibbs state μ_s for $\mathcal{A}$ has the Bernoulli property when $\mathcal{A}$ is proximal and strongly irreducible. By using Corollary 1.2, we generalize their results.

THEOREM 4.1. *Let $\mathcal{A}\colon \Sigma_\ell \to \mathrm{GL}_d(\mathbb{R})$ be a locally constant cocycle. Suppose that $\mathcal{A}^t$ and $\mathcal{A}^{\wedge m}$ are irreducible for every $t \mid d$ and $1 \leq m \leq d-1$. Then for any $s > 0$, the unique Gibbs state μ_s for the norm potential $\Phi^s_{\mathcal{A}}$ is ψ-mixing:*

$$\lim_{n\to\infty} \sup_{I,J\in\mathcal{L}} \left| \frac{\mu_s([I] \cap \sigma^{-n-|I|}[J])}{\mu_s([I])\mu_s([J])} - 1 \right| = 0,$$

and its natural extension $\hat{\mu}$ has the Bernoulli property.

Proof. Denote $\mu := \mu_s$. Since μ is the unique Gibbs equilibrium state for $\Phi^s_{\mathcal{A}}$, there exists $C_0 > 0$ such that

$$C_0^{-1}\|\mathcal{A}_I\|^s \le e^{|I|P(\Phi^s_{\mathcal{A}})}\mu([I]) \le C_0\|\mathcal{A}_I\|^s \tag{4.1}$$

for every $I \in \mathcal{L}$. By Corollary 1.2, there exist $m \in \mathbb{N}$ and a constant $C_1 > 0$ such that for all $I, J \in \mathcal{L}$ there exists $K \in \mathcal{L}(m)$ such that

$$\|\mathcal{A}_{IKJ}\| \ge C_1\|\mathcal{A}_I\|\,\|\mathcal{A}_J\|. \tag{4.2}$$

Therefore, by (4.1) and (4.2), for every $I, J \in \mathcal{L}$ we have

$$\begin{aligned}
C_1\mu([I])\mu([J]) &\le C_0^2 C_1 e^{-(|I|+|J|)P(\Phi^s_{\mathcal{A}})}\|\mathcal{A}_I\|^s\|\mathcal{A}_J\|^s \\
&\le C_0^2 e^{-(|I|+|J|)P(\Phi^s_{\mathcal{A}})}\|\mathcal{A}_{IKJ}\|^s \le C_0^3 e^{|K|P(\Phi^s_{\mathcal{A}})}\mu([IKJ]) \\
&\le C_0^3 e^{mP(\Phi^s_{\mathcal{A}})}\sum_{|K|=m}\mu([IKJ]) \\
&= C_0^3 e^{mP(\Phi^s_{\mathcal{A}})}\mu([I]\cap\sigma^{-m-|I|}[J])
\end{aligned}$$

so that

$$\mu([I]\cap\sigma^{-m-|I|}[J]) \ge \kappa\mu([I])\mu([J]), \tag{4.3}$$

where $\kappa := C_0^{-3}C_1 e^{-mP(\Phi^s_{\mathcal{A}})}$.

By (4.3), for any $n \ge m$ we have

$$\begin{aligned}
\mu([I]\cap\sigma^{-n-|I|}[J]) &= \sum_{|K'|=n-m}\mu\big([IK']\cap\sigma^{-m-|K'|-|I|}[J]\big) \\
&\ge \kappa\sum_{|K'|=n-m}\mu([IK'])\mu([J]) \\
&= \kappa\mu([J])\sum_{|K'|=n-m}\mu([IK']) = \kappa\mu([I])\mu([J]).
\end{aligned}$$

Thus by an approximation argument we deduce that

$$\liminf_{n\to\infty}\mu(X\cap\sigma^{-n}Y) \ge \kappa\mu(X)\mu(Y)$$

for all X, Y Borel measurable. The above inequality implies that μ is totally ergodic. Then, the assertion follows from [26, Theorem 1]. ∎

4.2. Shrinking target sets and recurrence sets. Let $T\colon X \to X$ be a topological dynamical system on a compact metric space (X, d). Assume that μ is a T-invariant ergodic measure. By the Birkhoff ergodic theorem, for any ball B in X of positive μ-measure, the set

$$S := \{x \in X : T^n x \in B \text{ for infinitely many } n \in \mathbb{N}\}$$

has full μ-measure.

Now, in from the above definition of S we allow both the center and the radius of B to vary with n; given a function $h\colon \mathbb{N} \to \mathbb{R}_+$ tending to 0 as

$n \to \infty$ and a sequence $\{z_n\}_{n\geq 1}$ of points in X, the set S can be generalized to

$$S(h) = \{x \in X : T^n x \in B(z_n, h(n)) \text{ for infinitely many } n \in \mathbb{N}\}.$$

This set $S(h)$ is called the *shrinking target.* Then one can ask how large the size of $S(h)$ is in the sense of measure and in the sense of dimension. This was called the *shrinking target problem* by Hill and Velani [15]; it concerns what happens if the target B shrinks in time and more generally if the target also moves around in time. The points in $S(h)$ can be thought of as trajectories which hit a shrinking, moving target infinitely often.

The shrinking target problem has intricate links to number theory when one uses naturally arising sets in Diophantine approximation as shrinking targets; see e.g. [1, 2, 28].

The above works mostly concern conformal dynamics or dynamical systems in $\mathbb{R}^1$, and transition to higher-dimensional non-conformal dynamics presents severe challenges. To overcome the extreme challenges that affinities pose, a common approach is to "randomize" the affine maps by considering typical translation parameters. Koivusalo and Ramirez [20] give an expression for the Hausdorff dimension of a self-affine shrinking target problem. They show that for a fixed symbolic target with exponentially shrinking diameter and well-behaved affine maps, the Hausdorff dimension is typically given by the zero of an appropriate pressure function. Strong assumptions are made on the affine system, as well as on the fixed target.

Let $\mathcal{A} = (A_1, \dots, A_\ell)$ be a collection of non-singular 2×2 contracting matrices. Let $\mathbf{t} = \{t_1, \dots, t_\ell\}$ be a collection of ℓ vectors in $\mathbb{R}^2$. Let $D_{\mathbf{t}} = \{f_i(x) = A_i x + t_i\}_{i=1}^{\ell}$ be an iterated function system formed by affine maps on $\mathbb{R}^2$.

For a finite word $\mathbf{i} = (i_1, \dots, i_n) \in \mathcal{L}$, set $f_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_n}$. There exists a unique non-empty compact set $\Lambda \subset \mathbb{R}^2$ such that

$$\Lambda = \bigcup_{i=1}^{\ell} f_i(\Lambda);$$

see [16]. Suppose that $\ell \geq 2$. Let us denote by $\pi_{\mathbf{t}}$ the natural projection from Σ_ℓ to the attractor of Λ, that is,

$$\pi_{\mathbf{t}}(\mathbf{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \sum_{k=1}^{\infty} A_{i_1} \dots A_{i_{k-1}} t_{i_k}.$$

Clearly, $\pi_{\mathbf{t}}(\mathbf{i}) = f_{i_1}(\pi_{\mathbf{t}}(\sigma(\mathbf{i})))$.

We define the *singular value function* as follows:

$$\varphi^s(A) = \begin{cases} \|A\|^s & \text{if } 0 \le s < 1, \\ \|A\|\,\|A^{-1}\|^{-(s-1)} & \text{if } 1 \le s < 2, \\ |\det(A)|^{s/2} & \text{if } 2 \le s < \infty. \end{cases}$$

The *pressure* of the self-affine system is defined as

$$P(\log \varphi^s(\mathcal{A})) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{I \in \mathcal{L}(n)} \varphi^s(\mathcal{A}_I),$$

where the existence of the limit is guaranteed by the submultiplicativity of $\varphi^s(A)$. For simplicity, we denote $P(s) := P(\log \varphi^s(\mathcal{A}))$.

Let $\{J_k\}_{k\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ be a sequence of target cylinders. We are interested in the shrinking target set

$$S_{\mathbf{t}}(\{J_k\}_{k\in\mathbb{N}}) = \pi_{\mathbf{t}}\{\mathrm{i} \in \Sigma_\ell : \sigma^k \mathrm{i} \in [J_k] \text{ for infinitely many } k \in \mathbb{N}\}.$$

For our sequence of target cylinders, we define the following *inverse lower pressure*:

$$\alpha(s) := \liminf_{k\to\infty} \frac{-1}{|J_k|} \log \varphi^s(\mathcal{A}_{J_k})$$

Let

$$s_0 := \inf\{s > 0 : P(s) \le \alpha(s)\}.$$

Unfortunately, the uncertainty of the length of the connecting word K in the quasi-multiplicativity property does not let us study shrinking target sets and recurrence sets effectively. In order to study them, Bárány and Troscheit [3] prove that $\mathcal{A}^{\wedge s}$ is uniformly k-quasi-multiplicative for every $s \in (0, d)$ when $\mathcal{A}$ is fully strongly irreducible and fully proximal. In the two-dimensional case, we can improve their results [3, Corollaries 2.7 and 2.8] by using Corollary 1.2.

LEMMA 4.2. *Assume that $\mathcal{A} = (A_1, \ldots, A_\ell)$ is a collection of non-singular 2×2 matrices. Suppose that $\mathcal{A}$ is k-quasi-multiplicative. Then, for every $s \in [0, 2]$, there exist $k \in \mathbb{N}$ and $C > 0$ such that for any $I, J \in \mathcal{L}$, there exists $K = K(I, J) \in \mathcal{L}(k)$ such that*

$$\varphi^s(\mathcal{A}_{IKJ}) \ge C\varphi^s(\mathcal{A}_I)\varphi^s(\mathcal{A}_J).$$

Proof. Since $\mathcal{A}$ is k-quasi-multiplicative, there exist $k \in \mathbb{N}$ and $c > 0$ such that for any $I, J \in \mathcal{L}$, there exists $K \in \mathcal{L}(k)$ such that

$$\|\mathcal{A}_{IKJ}\| \ge c\|\mathcal{A}_I\|\,\|\mathcal{A}_J\|.$$

When $s \in [0, 1]$, the proof follows from k-quasi-multiplicativity of $\mathcal{A}$ by raising to the power s.

Now, let $1 < s \leq 2$. Notice that $\|A\|\,\|A^{-1}\|^{-(s-1)} = |\det(A)|^{s-1}\|A\|^{2-s}$ for any $A \in \mathrm{GL}_2(\mathbb{R})$. Then the proof follows by multiplying, raising the k-quasi-multiplicativity relation for $\mathcal{A}$ to the power $2-s$, and raising the determinant to the power $s-1$. ∎

COROLLARY 4.3. *Assume that $\mathcal{A} = (A_1, \dots, A_\ell)$ is a collection of 2×2 contracting matrices and $\{J_k\}_{k\in\mathbb{N}}$ is a sequence of target cylinders. Suppose that $\mathcal{A}$ and $\mathcal{A}^2$ are irreducible and $\|A_i\| < 1/2$ for all $i \in \{1, \dots, \ell\}$. Then*

$$\dim_H S_{\mathbf{t}}(\{J_k\}_k) = \min\{2, s_0\} \quad \textit{for Lebesgue-almost every } \mathbf{t}.$$

Moreover, $\mathcal{L}_2(S_{\mathbf{t}}(\{J_k\}_k)) > 0$ for Lebesgue-almost every $\mathbf{t}$ if $s_0 > 2$.

Proof. By Corollary 1.2, $\mathcal{A}$ is k-quasi-multiplicative. Then the statement follows from the combination of [3, Theorem 2.2] and Lemma 4.2. ∎

Let $\psi\colon \mathbb{N} \to \mathbb{N}$ and $\beta := \liminf_{n\to\infty} \frac{\psi(n)}{n}$. We consider the recurrence set

$$R_{\mathbf{t}}(\psi) := \pi_{\mathbf{t}}\{\mathrm{i} \in \Sigma_\ell : \sigma^k \mathrm{i} \in [\mathrm{i}|_{\psi(k)}] \text{ for infinitely many } k \in \mathbb{N}\}.$$

We define the *square-pressure function*

$$P_2(s) := \lim_{n\to\infty} \frac{-1}{n} \log \sum_{i\in\mathcal{L}(n)} (\varphi^s(\mathcal{A}_i))^2,$$

where the limit exists because of the subadditivity of $\varphi^s(\mathcal{A})$. This function is continuous in s, strictly increasing, and satisfies $P_2(0) = -\log N$ and $P_2(s) \to \infty$ as $s \to \infty$.

COROLLARY 4.4. *Let $\mathcal{A} = (A_1, \dots, A_\ell)$ be a collection of 2×2 contracting matrices. Suppose that $\mathcal{A}$ and $\mathcal{A}^2$ are irreducible and $\|A_i\| < 1/2$ for all $i \in \{1, \dots, \ell\}$. Let $\psi : \mathbb{N} \to \mathbb{N}$ with $\beta = \liminf_{n\to\infty} \psi(n)/n < 1$. Then*

$$\dim_H R_{\mathbf{t}}(\psi) = \min\{2, r_0\} \quad \textit{for Lebesgue-almost every } \mathbf{t},$$

where r_0 is the unique solution of the equation

$$(1-\beta)P(r_0) = \beta P_2(r_0).$$

Moreover, $\mathcal{L}_2(R_{\mathbf{t}}((\psi)_k)) > 0$ for Lebesgue-almost every $\mathbf{t}$ if $r_0 > 2$.

Proof. By Corollary 1.2, $\mathcal{A}$ is k-quasi-multiplicative. Thus, the proof follows from the combination of [3, Theorem 2.4] and Lemma 4.2. ∎

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