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On a Kurzweil type theorem via ubiquity

by

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Abstract. Kurzweil's theorem (1955) is concerned with zero-one laws for well approximable targets in inhomogeneous Diophantine approximation under the badly approximable assumption. In this article, we prove the divergent part of a Kurzweil type theorem via a suitable construction of ubiquitous systems when the badly approximable assumption is relaxed. Moreover, we also discuss some counterparts of Kurzweil's theorem.

1. Introduction. Kurzweil's theorem [Kur55] in inhomogeneous Diophantine approximation is concerned with well approximable target vectors. We start by introducing related definitions and notations. Given a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and an $m \times n$ matrix $A \in M_{m,n}(\mathbb{R})$, we say that $\mathbf{b} \in \mathbb{R}^m$ is *ψ-approximable for A* if there exist infinitely many solutions $\mathbf{q} \in \mathbb{Z}^n$ to the inequality

$$\|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} < \psi(\|\mathbf{q}\|).$$

Denote by $W_A(\psi)$ the set of such vectors in the unit cube $[0,1]^m$. Here and hereafter, $\|\mathbf{x}\| = \max_{1\le i\le m} |x_i|$ and $\|\mathbf{x}\|_{\mathbb{Z}} = \min_{\mathbf{n}\in\mathbb{Z}^m} \|\mathbf{x} - \mathbf{n}\|$ for $\mathbf{x} \in \mathbb{R}^m$. We say that $A \in M_{m,n}(\mathbb{R})$ is *badly approximable* if

$$\liminf_{\|\mathbf{q}\|\to\infty} \|\mathbf{q}\|^{n/m} \|A\mathbf{q}\|_{\mathbb{Z}} > 0.$$

Kurzweil proved the following zero-one law for $W_A(\psi)$.

Theorem 1.1 ([Kur55]). *If $A \in M_{m,n}(\mathbb{R})$ is badly approximable, then for any decreasing $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ we have*

$$|W_A(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty. \end{cases}$$

Here and hereafter, $|\cdot|$ stands for Lebesgue measure on $\mathbb{R}^m$.

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We remark that Kurzweil showed that in fact the badly approximable condition is an equivalent condition for the zero-one law, not a sufficient condition.

In this article, we will consider similar results when the badly approximable condition is relaxed. We say that $A \in M_{m,n}(\mathbb{R})$ is *singular* if for any $\epsilon > 0$ for all large enough $X \geq 1$ there exists $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q}\|_{\mathbb{Z}} < \epsilon X^{-n/m} \quad \text{and} \quad 0 < \|\mathbf{q}\| < X.$$

Otherwise we call it *non-singular* (or *regular* following [Cas57]). One can check that $A \in M_{m,n}(\mathbb{R})$ is singular if and only if for any $\epsilon > 0$ for all large enough $\ell \in \mathbb{Z}_{\geq 1}$ there exists $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q}\|_{\mathbb{Z}} < \epsilon 2^{-\frac{n}{m}\ell} \quad \text{and} \quad 0 < \|\mathbf{q}\| < 2^{\ell}. \tag{1.1}$$

Hence $A \in M_{m,n}(\mathbb{R})$ is non-singular if and only if there exists $\epsilon > 0$ such that the set

$$L(\epsilon) := \{\ell \in \mathbb{Z}_{\geq 1} : \text{there is no solution } \mathbf{q} \in \mathbb{Z}^n \text{ to (1.1) with } \ell\}$$

is unbounded. We call $L(\epsilon)$ the *ϵ-return sequence for A.*

Remark 1.2. (1) Note that $A \in M_{m,n}(\mathbb{R})$ is badly approximable if and only if there exists $\epsilon > 0$ such that $L(\epsilon) = \mathbb{Z}_{\geq 1}$.

(2) In a dynamical point of view as in [Dan85], the set $L(\epsilon)$ corresponds to return times to a compact set related to ϵ of a certain diagonal flow in the space of lattices.

The following is the main theorem of this article.

Theorem 1.3. *Let $A \in M_{m,n}(\mathbb{R})$ be non-singular with ϵ-return sequence $L(\epsilon) = \{\ell_i\}_{i \geq 1}$. For any decreasing $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $0 \leq s \leq m$, the s-dimensional Hausdorff measure of $W_A(\psi)$ is given by*

$$\mathcal{H}^s(W_A(\psi)) = \mathcal{H}^s([0,1]^m) \quad \text{if} \quad \sum_{i=1}^{\infty} 2^{\ell_i n} \psi(2^{\ell_i})^s = \infty.$$

For $\delta > 0$, let $\psi_\delta(q) = \delta q^{-n/m}$. Denote $\mathbf{Bad}_A(\delta) = [0,1]^m \setminus W_A(\psi_\delta)$ and $\mathbf{Bad}_A = \bigcup_{\delta > 0} \mathbf{Bad}_A(\delta)$. Theorem 1.3 with $\psi = \psi_\delta$ and $s = m$ directly implies the following corollary.

Corollary 1.4. *If $A \in M_{m,n}(\mathbb{R})$ is non-singular, then for any $\delta > 0$, the set $\mathbf{Bad}_A(\delta)$ has Lebesgue measure zero, hence $\mathbf{Bad}_A$ has Lebesgue measure zero.*

Remark 1.5. (1) Some historical remarks about Corollary 1.4 are in order. The one-dimensional result of the corollary was proved in [Kim07] using irrational rotations and the Ostrowski representation. The corollary in full generality was proved in [Sha13] using a certain mixing property in homogeneous dynamics. The simultaneous version (i.e. $n = 1$) of the corollary was proved in [Mos] using a certain well-distribution property. Our method relies on a suitable construction of a ubiquitous system.

(2) A zero-one law for Lebesgue measure of $W_A(\psi)$ in the one-dimensional case was investigated in [FK16]. According to the results there, Theorem 1.3 is not optimal. It seems interesting to obtain zero-one laws for $W_A(\psi)$ in the multidimensional case.

(3) A weighted version of Kurzweil's theorem was investigated in [Har12]. There have been several recent results on weighted ubiquity and weighted transference theorems (see [CG$^+$20, G20, WW21]). It seems plausible that these results can be utilized to obtain a weighted version of Theorem 1.3.

As stated in [BB$^+$09, Section 9], using Theorem 1.1 and the Mass Transference Principle of [BV06], one can deduce a Hausdorff measure version of Kurzweil's theorem. Theorem 1.3 which relies on the ubiquity method also implies the following corollary.

COROLLARY 1.6. *If $A \in M_{m,n}(\mathbb{R})$ is badly approximable, then for any decreasing $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $0 \le s \le m$, we have*

$$\mathcal{H}^s(W_A(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^s < \infty, \\ \mathcal{H}^s([0,1]^m) & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q)^s = \infty. \end{cases}$$

Moreover, the convergence part holds for every $A \in M_{m,n}(\mathbb{R})$.

Proof. The convergence part will be proved in Section 2. Since the sums

$$\sum_{\ell=1}^{\infty} 2^{\ell n}\psi(2^{\ell})^s \quad \text{and} \quad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^s$$

are convergent (or divergent) simultaneously, the divergence part follows from Theorem 1.3 and Remark 1.2(1). ∎

We explore some counterparts of Kurzweil's theory. We denote by $w(A, \mathbf{b})$ the supremum of the real numbers w for which, for arbitrarily large X, the inequalities

$$\|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} < X^{-w} \quad \text{and} \quad \|\mathbf{q}\| < X$$

have an integral solution $\mathbf{q} \in \mathbb{Z}^n$. We also denote by $\widehat{w}(A)$ the supremum of the real numbers w for which, for all sufficiently large X, the inequalities

$$\|A\mathbf{q}\|_{\mathbb{Z}} < X^{-w} \quad \text{and} \quad \|\mathbf{q}\| < X$$

have a non-zero integral solution $\mathbf{q} \in \mathbb{Z}^n$. If $\widehat{w}({}^tA) > m/n$, then by [BL05, Theorem], for almost all $\mathbf{b} \in \mathbb{R}^m$,

$$w(A, \mathbf{b}) = \frac{1}{\widehat{w}({}^tA)} < \frac{n}{m}.$$

Thus for any $\delta > 0$, the Lebesgue measure of $\mathbf{Bad}_A(\delta)$ is full. This is opposite to Corollary 1.4. Note that if $\widehat{w}({}^tA) > m/n$, then ${}^tA$ is singular, hence A is singular. So, they do not conflict with each other.

If we consider the case $\widehat{w}({}^tA) = m/n$ and ${}^tA$ is singular, then we cannot deduce from [BL05, Theorem] that $\mathbf{Bad}_A(\delta)$ is of full Lebesgue measure for any $\delta > 0$. We will give a certain sufficient condition for $\mathbf{Bad}_A(\delta)$ being of full Lebesgue measure for any $\delta > 0$.

If $\mathrm{rk}_{\mathbb{Z}}({}^tA\mathbb{Z}^m + \mathbb{Z}^n) < m+n$, then $\widehat{w}({}^tA) = \infty$, hence we may assume that $\mathrm{rk}_{\mathbb{Z}}({}^tA\mathbb{Z}^m + \mathbb{Z}^n) = m+n$. Then following [BL05, Section 3], there exists a sequence $(\mathbf{y}_k)_{k\geq 1}$ of best approximations for ${}^tA$ in $\mathbb{Z}^m$. Denote $Y_k = \|\mathbf{y}_k\|$ and $M_k = \|{}^tA\mathbf{y}_k\|_{\mathbb{Z}}$.

Theorem 1.7. *If*

$$\sum_{k\geq 2} \max\left((Y_k^{m/n} M_{k-1})^{\frac{n}{m+n}}, (Y_{k+1}^{m/n} M_k)^{\frac{n}{m+n}}\right) < \infty,$$

then for any $\delta > 0$, the Lebesgue measure of $\mathbf{Bad}_A(\delta)$ is full.

Remark 1.8. (1) The summability assumption implies $Y_{k+1}^{m/n} M_k \to 0$ as $k \to \infty$, hence ${}^tA$ is singular.

(2) This theorem is stronger than the previous observation because if $\widehat{w}({}^tA) > m/n$, then there is $\gamma > 0$ such that $Y_{k+1}^{\frac{m}{n}+\gamma} M_k < 1$ for all sufficiently large $k \geq 1$. Hence,

$$\sum_{k\geq 2} \max\left((Y_k^{m/n} M_{k-1})^{\frac{n}{m+n}}, (Y_{k+1}^{m/n} M_k)^{\frac{n}{m+n}}\right) < \sum_{k\geq 1} (Y_{k+1}^{-\gamma})^{\frac{n}{m+n}} < \infty$$

since Y_k increases at least geometrically (see [BL05, Lemma 1]).

(3) It was proved in [BK$^+$21, KKL] that A is *singular on average* if and only if there exists $\delta > 0$ such that $\mathbf{Bad}_A(\delta)$ has full Hausdorff dimension. Thus it seems interesting to obtain an equivalent Diophantine property of A for $\mathbf{Bad}_A(\delta)$ being of full Lebesgue measure.

The structure of this paper is as follows: In Section 2, we prove the convergence part of Corollary 1.6. In Section 3, we introduce some preliminaries for the proof of Theorem 1.3 including ubiquitous systems, the Transference Principle, and a Weyl type uniform distribution. We prove Theorems 1.3 and 1.7 in Sections 4 and 5, respectively.

2. Convergence part: a warm up. In this section, we prove the convergence part of Corollary 1.6. We will use the following Hausdorff measure version of the Borel–Cantelli lemma [BD99, Lemma 3.10].

Lemma 2.1 (Hausdorff–Cantelli). *Let $\{B_i\}_{i\geq 1}$ be a sequence of subsets in $\mathbb{R}^m$. For any $0 \leq s \leq k$,*

$$\mathcal{H}^s\Big(\limsup_{i\to\infty} B_i\Big) = 0 \quad \text{if} \quad \sum_i \operatorname{diam}(B_i)^s < \infty.$$

Note that

$$W_A(\psi) = \limsup_{\|\mathbf{q}\|\to\infty} B(A\mathbf{q}, \psi(\|\mathbf{q}\|)),$$

where $B(A\mathbf{q}, \psi(\|\mathbf{q})\|)$ denotes the ball in $\mathbb{R}^m$ of radius $\psi(\|\mathbf{q}\|)$ centered at $A\mathbf{q}$ modulo 1, and

$$\sum_{\mathbf{q}\in\mathbb{Z}^n} \operatorname{diam}(B(A\mathbf{q}, \psi(\|\mathbf{q}\|)))^s < \infty \iff \sum_{q=1}^{\infty} q^{n-1}\psi(q)^s < \infty.$$

Hence, using the Hausdorff–Cantelli lemma, we see that for any $0 \leq s \leq m$,

$$\mathcal{H}^s(W_A(\psi)) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^s < \infty.$$

This proves the convergence part of Corollary 1.6.

3. Preliminaries for divergence part

3.1. Ubiquity systems. The proof of Theorem 1.3 is based on the ubiquity framework developed in [BDV06], which provides a very general and abstract method for calculating the Hausdorff measure of a large class of limsup sets. In this subsection, we define ubiquitous systems that suit our situation.

We consider $\mathbb{T}^m$ with the supremum norm $\|\cdot\|$. With notation of [BDV06] we set

$$J = \{\mathbf{q} \in \mathbb{Z}^n\}, \qquad R_{\mathbf{q}} = A\mathbf{q} \in \mathbb{T}^m, \qquad \mathcal{R} = \{R_{\mathbf{q}} : \mathbf{q} \in J\}, \qquad \beta_{\mathbf{q}} = \|\mathbf{q}\|.$$

Let $l = \{l_i\}$ and $u = \{u_i\}$ be positive increasing sequences such that

$$l_i < u_i \quad \text{and} \quad \lim_{i\to\infty} l_i = \infty.$$

For a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, we define

$$\Delta_l^u(\psi, i) := \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, l_i<\|\mathbf{q}\|\leq u_i} B(A\mathbf{q}, \psi(\|\mathbf{q}\|)).$$

It follows that

$$W_A(\psi) = \limsup_{i\to\infty} \Delta_l^u(\psi, i) := \bigcap_{M=1}^{\infty} \bigcup_{i=M}^{\infty} \Delta_l^u(\psi, i).$$

Throughout, $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ will denote a function satisfying $\lim_{r\to\infty} \rho(r) = 0$ and is referred to as a *ubiquitous function*. Let

$$\Delta_l^u(\rho, i) := \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, l_i<\|\mathbf{q}\|\le u_i} B(A\mathbf{q}, \rho(u_i)).$$

DEFINITION 3.1 (Local ubiquity). Let B be an arbitrary ball in $\mathbb{T}^m$. Suppose that there exist a ubiquitous function ρ and an absolute constant $\kappa > 0$ such that

$$|B \cap \Delta_l^u(\rho, i)| \ge \kappa|B| \quad \text{for } i \ge i_0(B). \tag{3.1}$$

Then the pair $(\mathcal{R}, \beta)$ is said to be a *locally ubiquitous system* relative to (ρ, l, u).

Finally, a function h is said to be *u-regular* if there exists a positive constant $\lambda < 1$ such that for i sufficiently large,

$$h(u_{i+1}) \le \lambda h(u_i).$$

With notation of [BDV06], the Lebesgue measure on $\mathbb{T}^m$ is of type (M2) with $\delta = m$ and the intersection conditions are also satisfied with $\gamma = 0$. These conditions are not stated here but these extra conditions exist and need to be established for the more abstract ubiquity.

Combining [BDV06, Corollaries 2 and 4], we have the following theorem.

THEOREM 3.2 ([BDV06]). *Suppose that $(\mathcal{R}, \beta)$ is a local ubiquitous system relative to (ρ, l, u) and assume further that ρ is u-regular. Then for any $0 \le s \le m$,*

$$\mathcal{H}^s(W_A(\psi)) = \mathcal{H}^s(\mathbb{T}^m) \quad \text{if} \quad \sum_{i=1}^{\infty} \frac{\psi(u_i)^s}{\rho(u_i)^m} = \infty.$$

3.2. Transference principle. We need the following transference principle between homogeneous and inhomogeneous Diophantine approximation. See [Cas57, Chapter V, Theorem VI]).

THEOREM 3.3 (Transference principle). *Suppose that there is no solution $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that*

$$\|A\mathbf{q}\|_{\mathbb{Z}} < C \quad \text{and} \quad \|\mathbf{q}\| < X.$$

Then for any $\mathbf{b} \in \mathbb{R}^m$, there exists $\mathbf{q} \in \mathbb{Z}^n$ such that

$$\|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} \le C_1 \quad \text{and} \quad \|\mathbf{q}\| \le X_1,$$

where

$$C_1 = \tfrac{1}{2}(h+1)C, \qquad X_1 = \tfrac{1}{2}(h+1)X, \qquad h = X^{-n}C^{-m}.$$

This principle implies the following corollary.

Corollary 3.4. *Let $A \in M_{m,n}(\mathbb{R})$ be non-singular and let $L(\epsilon) = \{\ell_i\}_{i\geq 1}$ be the ϵ-return sequence for A. Then for any $\mathbf{b} \in \mathbb{R}^m$, there exists $\mathbf{q} \in \mathbb{Z}^n$ such that*

$$\|A\mathbf{q} - \mathbf{b}\|_{\mathbb{Z}} \leq \tfrac{1}{2}(\epsilon^{-m} + 1)\epsilon 2^{-\frac{n}{m}\ell_i} \quad \text{and} \quad \|\mathbf{q}\| \leq \tfrac{1}{2}(\epsilon^{-m} + 1)2^{\ell_i}.$$

Proof. This follows directly from Theorem 3.3 with $C = \epsilon 2^{-\frac{n}{m}\ell_i}$ and $X = 2^{\ell_i}$. ∎

3.3. Weyl type uniform distribution. In this subsection, we establish a Weyl type uniform distribution result for the sequence $\{A\mathbf{q}\}_{\mathbf{q}\in\mathbb{Z}^n} \subset \mathbb{T}^m$. For $A \in M_{m,n}(\mathbb{R})$, Kronecker's theorem (see e.g. [Cas57, Chapter III, Theorem IV]) asserts that the sequence $\{A\mathbf{q}\}_{\mathbf{q}\in\mathbb{Z}^n}$ is dense in $\mathbb{T}^m$ if and only if the subgroup

$$G({}^tA) := {}^tA\mathbb{Z}^m + \mathbb{Z}^n \subset \mathbb{R}^n$$

has maximal rank $m+n$ over $\mathbb{Z}$. If A is non-singular, then ${}^tA$ is non-singular, hence $G({}^tA)$ has maximal rank $m+n$ over $\mathbb{Z}$. By Kronecker's theorem, the sequence $\{A\mathbf{q}\}_{\mathbf{q}\in\mathbb{Z}^n}$ is dense in $\mathbb{T}^m$.

But the density result is not enough for our purpose. We need the following Weyl type uniform distribution result.

Proposition 3.5. *If $G({}^tA)$ has maximal rank $m+n$ over $\mathbb{Z}$, then the sequence $\{A\mathbf{q}\}_{\mathbf{q}\in\mathbb{Z}^n}$ is uniformly distributed in the following sense: for any ball $B \subset \mathbb{T}^m$,*

$$\frac{\#\{A\mathbf{q} \in B : \|\mathbf{q}\| \leq N\}}{\#\{\mathbf{q} \in \mathbb{Z}^n : \|\mathbf{q}\| \leq N\}} \to |B| \quad \text{as } N \to \infty.$$

Proof. We first claim that for any $\mathbf{c} \in \mathbb{Z}^m \setminus \{0\}$,

$$\frac{1}{\#\{\mathbf{q} \in \mathbb{Z}^n : \|\mathbf{q}\| \leq N\}} \sum_{\mathbf{q}\in\mathbb{Z}^n:\|\mathbf{q}\|\leq N} e^{2\pi i(\mathbf{c}\cdot A\mathbf{q})} \to 0 \quad \text{as } N \to \infty.$$

Indeed, by the maximal rank assumption, we have ${}^tA\mathbf{c} \in \mathbb{R}^n \setminus \mathbb{Q}^n$. Without loss of generality, we may assume that the first coordinate, say α, of ${}^tA\mathbf{c}$ is irrational. It follows from $\mathbf{c} \cdot A\mathbf{q} = {}^tA\mathbf{c} \cdot \mathbf{q}$ that

$$\begin{aligned}\frac{1}{N^n}\Big|\sum_{\mathbf{q}\in\mathbb{Z}^n:\,\|\mathbf{q}\|\leq N} e^{2\pi i(\mathbf{c}\cdot A\mathbf{q})}\Big| &= \frac{1}{N^n}\Big|\sum_{\mathbf{q}\in\mathbb{Z}^n:\,\|\mathbf{q}\|\leq N} e^{2\pi i({}^tA\mathbf{c}\cdot\mathbf{q})}\Big| \\ &\ll \frac{1}{N^n} N^{n-1} \sum_{q_1=-N}^{N} e^{2\pi i\alpha q_1} \\ &\leq \frac{1}{N}\,\frac{2}{|e^{2\pi i\alpha} - 1|}.\end{aligned}$$

Since $\#\{\mathbf{q} \in \mathbb{Z}^n : \|\mathbf{q}\| \leq N\} \asymp N^n$, this proves the claim.

Following the proof of Weyl's classical criterion (see e.g. [KN74, Theorem 2.1]), we can deduce that for any ball $B \subset \mathbb{T}^m$ we have

$$\frac{\#\{A\mathbf{q} \in B : \|\mathbf{q}\| \leq N\}}{\#\{\mathbf{q} \in \mathbb{Z}^n : \|\mathbf{q}\| \leq N\}} \to |B| \quad \text{as } N \to \infty. \ \blacksquare$$

REMARK 3.6. The above proposition is slightly different from the multidimensional Weyl criterion. We do not take every partial sum but "radial" partial sum.

4. Proof of Theorem 1.3. Let A be non-singular and let $L(\epsilon) = \{\ell_i\}_{i\geq 1}$ be the ϵ-return sequence. With the notations of Subsection 3.1, we take sequences $l = l(\epsilon) = \{l_i\}$ and $u = u(\epsilon) = \{u_i\}$ as follows:

$$u_i = \tfrac{1}{2}(\epsilon^{-m} + 1)2^{\ell_i} \quad \text{and} \quad l_i = c_1 u_i,$$

with some positive constant $c_1 = c_1(\epsilon) < 1$, which will be determined later.

We first establish the following local ubiquity result with the set-up of Subsection 3.1.

THEOREM 4.1. *The pair $(\mathcal{R}, \beta)$ is a locally ubiquitous system relative to $(\rho(r) = c_2 r^{-n/m}, l, u)$ with constant $c_2 = \epsilon\big(\frac{1}{2}(\epsilon^{-m} + 1)\big)^{1+\frac{n}{m}}$.*

Proof. Fix any ball $B = B(x, r_0)$ in $\mathbb{T}^m$. By Corollary 3.4, we have

$$B = B \cap \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, \|\mathbf{q}\|\leq u_i} B\big(A\mathbf{q}, \tfrac{1}{2}(\epsilon^{-m} + 1)\epsilon 2^{-\frac{n}{m}\ell_i}\big).$$

Since

$$\tfrac{1}{2}(\epsilon^{-m} + 1)\epsilon 2^{-\frac{n}{m}\ell_i} = \epsilon(\tfrac{1}{2}(\epsilon^{-m} + 1))^{1+\frac{n}{m}} u_i^{-\frac{n}{m}} = \rho(u_i),$$

it follows that

$$|B| \leq \Big|B \cap \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, \|\mathbf{q}\|\leq l_i} B(A\mathbf{q}, \rho(u_i))\Big| + \Big|B \cap \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, l_i<\|\mathbf{q}\|\leq u_i} B(A\mathbf{q}, \rho(u_i))\Big| \tag{4.1}$$

By Proposition 3.5 with $2B = B(x, 2r_0)$, there is an absolute constant $C > 0$ independent of the choice of B such that for all large enough $i \geq 1$,

$$\#\{A\mathbf{q} \in 2B : \|\mathbf{q}\| \leq l_i\} \leq C l_i^n |B|.$$

Thus for all $i \geq 1$ large enough such that $\rho(u_i) < r_0$, we have

$$\Big|B \cap \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, \|\mathbf{q}\|\leq l_i} B(A\mathbf{q}, \rho(u_i))\Big| \leq C l_i^n |B| (2\rho(u_i))^m = (2c_2)^m C c_1^n |B|.$$

By taking $0 < c_1 < 1$ such that $(2c_2)^m C c_1^n < 1/2$, which depends only on ϵ, it follows from (4.1) that for all large enough $i \geq 1$,

$$|B \cap \Delta_l^u(\rho, i)| = \Big| B \cap \bigcup_{\mathbf{q}\in\mathbb{Z}^n:\, l_i < \|\mathbf{q}\| \leq u_i} B(A\mathbf{q}, \rho(u_i))\Big|$$
$$\geq |B| - (2c_2)^m C c_1^n |B| > \tfrac{1}{2}|B|. \ \blacksquare$$

Proof of Theorem 1.3. It follows from $\ell_{i+1} \geq \ell_i + 1$ that for any $i \geq 1$,

$$\rho(u_{i+1}) = \tfrac{1}{2}(\epsilon^{-m}+1)\epsilon 2^{-\frac{n}{m}\ell_{i+1}} \leq 2^{-n/m}\tfrac{1}{2}(\epsilon^{-m}+1)\epsilon 2^{-\frac{n}{m}\ell_i} = 2^{-n/m}\rho(u_i),$$

hence ρ is u-regular. Since the sums

$$\sum_{i=1}^{\infty} \frac{\psi(u_i)^s}{\rho(u_i)^m} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\psi(2^{\ell_i})^s}{\rho(2^{\ell_i})^m}$$

are convergent (or divergent) simultaneously, Theorems 3.2 and 4.1 imply Theorem 1.3. $\blacksquare$

5. Proof of Theorem 1.7. In order to prove Theorem 1.7, we basically follow the proof of [BL05, Theorem].

As in the introduction, let $(\mathbf{y}_k)_{k\geq 1}$ be a sequence of best approximations for ${}^tA$. Let $Y_k = \|\mathbf{y}_k\|$, $M_k = \|{}^tA\mathbf{y}_k\|_{\mathbb{Z}}$, and

$$\gamma_k = \max\big((Y_k^{m/n} M_{k-1})^{\frac{n}{m+n}}, (Y_{k+1}^{m/n} M_k)^{\frac{n}{m+n}}\big)$$

for each $k \geq 2$. For any $\alpha > 0$, consider the set

$$B_\alpha(\{\gamma_k\}) = \{\mathbf{b} \in [0,1]^m : \|\mathbf{b}\cdot\mathbf{y}_k\|_{\mathbb{Z}} > \alpha\gamma_k \text{ for all large enough } k \geq 2\}.$$

PROPOSITION 5.1. *For any $\alpha > 0$,*

$$B_\alpha(\{\gamma_k\}) \subset \mathbf{Bad}_A\left(\frac{\alpha - n}{m}\right).$$

Proof. Consider the following two sequences:

$$U_k = \left(\frac{Y_k}{\gamma_k}\right)^{m/n} \quad \text{and} \quad V_k = \frac{\gamma_k}{M_k}.$$

We first claim that (1) $V_k \to \infty$ as $k \to \infty$; (2) $U_k < V_k$; (3) $U_{k+1} \leq V_k$. Indeed, (1) is clear, (2) follows from $M_k < M_{k-1}$ and $Y_k < Y_{k+1}$, and (3) follows from

$$U_{k+1} \leq \left(\frac{Y_{k+1}}{(Y_{k+1}^{m/n} M_k)^{\frac{n}{m+n}}}\right)^{m/n} = \frac{(Y_{k+1}^{m/n} M_k)^{\frac{n}{m+n}}}{M_k} \leq V_k.$$

Hence the union $\bigcup_{k\geq 2}[U_k, V_k)$ covers all sufficiently large numbers.

Now fix any $\mathbf{b} \in B_\alpha(\{\gamma_k\})$ and let $\mathbf{q} \in \mathbb{Z}^n$ be an integral vector with sufficiently large norm. Then we can find an index $k \geq 2$ with

$$U_k \leq \|\mathbf{q}\| < V_k. \tag{5.1}$$

Using the inequality in [BL05, (16)], we have

$$\alpha\gamma_k < \|\mathbf{b}\cdot\mathbf{y}_k\|_{\mathbb{Z}} \le mY_k\|A\mathbf{q}-\mathbf{b}\|_{\mathbb{Z}} + n\|\mathbf{q}\|M_k,$$

hence it follows from (5.1) that

$$\|\mathbf{q}\|^{n/m}\|A\mathbf{q}-\mathbf{b}\|_{\mathbb{Z}} > \frac{\alpha\gamma_k - nV_kM_k}{mY_k}U_k^{n/m} = \frac{\alpha-n}{m}.$$

This proves the claim. ∎

Proof of Theorem 1.7. It follows from the assumption $\sum_{k\ge 2}\gamma_k < \infty$ and the Borel–Cantelli lemma that $|B_\alpha(\{\gamma_k\})| = 1$ for any $\alpha > 0$. Given any $\delta > 0$, Proposition 5.1 with $\alpha = m\delta + n$ implies Theorem 1.7. ∎

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