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A characterization of analytic functions of several real variables

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Abstract. We present a characterization of analytic functions of several real variables by their behavior in the 2-dimensional case.

Our purpose is to prove the following characterization of analytic functions of n real variables.

Theorem 1. *Let $f: U \to \mathbb{R}$ be a function defined on an open subset U of $\mathbb{R}^n$ with $n \geq 2$. Assume that for each affine plane $P \subset \mathbb{R}^n$, the restriction $f|_{U\cap P}$ of f to $U \cap P$ is analytic. Then f is analytic.*

It is well known that the above statement with the planes replaced by lines is false: the function h defined by

$$h(x,y) = \frac{x^3}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0), \quad \text{and} \quad h(0,0) = 0,$$

is analytic on each affine line in $\mathbb{R}^2$, but h is not analytic at 0.

We presented Theorem 1 at the Pierre Lelong Seminar in Paris in 1971, but we never formally published it (however, the proof is contained in the informal lecture notes [2] in the more general setting of mappings between topological vector spaces).

Before formulating the next result, needed in the proof of Theorem 1, let us recall that a function $f: U \to \mathbb{R}$ is said to have the pth Gateaux derivative $\delta_u^p f$ at a point $u \in U$ (or f is of class G^p at u) if

(i) For every $h \in \mathbb{R}^n$ the function

$$f_h: W_h \ni t \mapsto f(u+th) \in \mathbb{R},$$

defined in a neighborhood W_h of 0 in $\mathbb{R}$, has the pth derivative at 0.

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(ii) The function

$$\delta_u^k f\colon \mathbb{R}^n \ni h \mapsto \frac{d^k}{dt^k} f(u+th)\Big|_{t=0} \in \mathbb{R}$$

is a homogeneous polynomial of degree k, for $k = 0, 1, \ldots, p$ (that includes the possibility $\delta_u^k f = 0$ for some k).

A function which, at each point of U, is of class G^p, $p = 0, 1, 2, \ldots$, is said to be of class G^∞.

The proof of Theorem 1 is based on the following result.

THEOREM 2. *Let $f\colon U \to \mathbb{R}$ be a function defined on an open subset U of $\mathbb{R}^n$. Assume that f is of class G^∞ and that for each affine line $L \subset \mathbb{R}^n$, the restriction $f|_{U\cap L}$ is analytic. Then f is analytic.*

Proof of Theorem 1. Let us prove first that f is of class G^∞. Fix a point u in U and a positive integer k. Define $\delta_u^k f$ as in (ii) above, which is possible, f being analytic on each $U \cap L$, where L is an affine line in $\mathbb{R}^n$. Since for each affine plane $P \subset \mathbb{R}^n$ the function $f|_{U\cap P}$ is analytic, it follows that $\delta_u^k f$ is a (homogeneous) polynomial on each vector plane V in $\mathbb{R}^n$. In turn, it follows that $\delta_u^k f$ is a polynomial with respect to each variable separately. Lemma 1 in [3] then implies that $\delta_u^k f$ is a polynomial on $\mathbb{R}^n$, necessarily homogeneous of degree k (or $\delta_u^k f = 0$). The function f is therefore G^∞ on U and analytic on each affine line in $\mathbb{R}^n$. By Theorem 2, f is analytic. ∎

CONJECTURE. *Let M be a real analytic manifold of dimension at least* 3. *Assume that for a function $f\colon M \to \mathbb{R}$, the restriction $f|_P$ to each compact* 2*-dimensional analytic submanifold P of M is analytic. Then f is analytic.*

Applying Theorem 1, it is easy to see that the conjecture is valid for the n-sphere $\mathbb{S}^n$ and for the projective space $\mathbb{RP}^n$.

Recently Theorem 1 was used in [5] to obtain its algebraic version, which says that a function $r\colon \mathbb{R}^n \to \mathbb{R}$ is regular if r is regular on each affine plane in $\mathbb{R}^n$.

There is however a sharp contrast with the situation in the C^∞ case. The following counter-example was kindly communicated to us by W. Pawłucki.

COUNTER-EXAMPLE. There exists a function $g\colon \mathbb{R}^3 \to \mathbb{R}$ which is C^∞ on each affine plane in $\mathbb{R}^3$, but is not even continuous at 0.

Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function identically 0 outside of $I =]0,1[$ and strictly positive on I, $\alpha(1/2) = 1$. Let

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid x > 0,\ y \in]x^2, 2x^2[,\ z \in]x^2, 2x^2[\}$$

be the "rhinoceros-horn-like" set, with the tip at $0 \in \mathbb{R}^3$. Observe that for each vector plane $V \subset \mathbb{R}^3$ the point 0 of V is an isolated point of $V \cap \overline{T}$,

$\overline{T}$ being the closure of T. Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$g(x,y,z) = \begin{cases} 0 & \text{for } (x,y,z) \notin T, \\ \alpha(y/x^2 - 1)\alpha(z/x^2 - 1) & \text{for } (x,y,z) \in T. \end{cases}$$

The function g is C^∞ on $\mathbb{R}^3 \setminus \{0\}$. Clearly, its restriction to each affine plane not passing through 0 is C^∞. The same holds for each vector plane $V \subset \mathbb{R}^3$, because $g|_V$ is identically null in a neighborhood of 0 in V (for a sufficiently small neighborhood W of 0 in V, one has $W \cap \overline{T} = \{0\}$). Finally, g is not continuous at 0 because it is constantly 1 on the whole arc $A = \{(t, \frac{3}{2}t^2, \frac{3}{2}t^2) \mid t > 0\} \subset T$, $0 \in \overline{A}$ and $g(0) = 0$. Note that g is G^∞ on $\mathbb{R}^3$ and $\delta_0^k f = 0$ for all k.

To prove Theorem 2 we need three lemmas.

Lemma 1. *Let $A = A_1 \times \cdots \times A_n \subset \mathbb{C}^n$ where A_i are compact, connected subsets of $\mathbb{C}$, not reduced to a single point, and $\omega > 1$. Then there exist a $\delta > 0$ and a positive integer k such that for any $s > k$ and any polynomial $p\colon \mathbb{C}^n \to \mathbb{C}$ of degree $\le s$ the following implication holds:*

$$(|p(x)| \le M \text{ for } x \in A) \ \Rightarrow\ (|p(x)| \le M\omega^s \text{ for } \operatorname{dist}(x, A) < \delta),$$

where $\operatorname{dist}(x, A)$ *the distance from x to A.*

This lemma is due to F. Leja [6] for $n = 1$, and to the second named author for $n > 1$ [7]. An easy proof is given in [1, p. 278].

Lemma 2. *If $P\colon \mathbb{C}^n \to \mathbb{C}$ is a homogeneous polynomial and $|P(z)| \le M$ for $\|z - z_0\| \le r$, then $|P(z)| \le M$ for $\|z\| \le r$.*

Proof. Let $\|h\| \le r$, $h \in \mathbb{C}^n$. The maximum principle applied to the holomorphic function $\varphi\colon \mathbb{C} \ni t \mapsto P(tz_0 + h) \in \mathbb{C}$ implies that there is a $t_0 \in \mathbb{C}$, $|t_0| = 1$, such that $|\varphi(t)| \le |\varphi(t_0)|$ for $|t| \le 1$. Hence

$$|P(h)| = |\varphi(0)| \le |\varphi(t_0)| = |P(t_0 z_0 + h)| = |P(z_0 + t_0^{-1}h)| \le M. \ \blacksquare$$

Lemma 3. *Let $\sum_{l=0}^\infty P_l$ be a series of homogeneous polynomials in n variables with $\deg P_l = l$. Let $\mathbb{S}^{n-1} = \{a \in \mathbb{R}^n \mid \|a\| = 1\}$ and assume that there exists an open nonempty subset Ω of $\mathbb{S}^{n-1}$ such that for every $a \in \Omega$ one can find $\rho = \rho_a > 0$ such that the series $\sum_{l=0}^\infty P_l(x)$ converges at $x = \rho a$. Then there exist $c, r > 0$ such that*

$$|P_l(z)| \le c/2^l, \quad z \in \mathbb{C}^n,\ \|z\| \le r,\ l \ge 1, \tag{1}$$

i.e. the function $z \mapsto \sum_{l=0}^\infty P_l(z)$ is holomorphic in the ball $\|z\| < r$, $z \in \mathbb{C}^n$.

Proof. Given an $a \in \Omega$, we may choose $\rho_a > 0$ such that

$$|P_l(ta)| \le 1, \quad |t| \le \rho_a,\ l \ge 1.$$

For every $k = 1, 2, \ldots$, the set

$$E_k = \{a \in \Omega \mid |P_l(ta)| \le 1,\ |t| \le 1/k,\ l \ge 1\}$$

is closed in Ω, $E_k \subset E_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty} E_k$. By the Baire theorem there exists an open nonempty set $\Omega^* \subset \Omega$ such that $\Omega^* \subset E_k$ for all k large enough, say $k \geq k_0$. Therefore

$$|P_l(ta)| \leq 1, \qquad |t| \leq r_0 = 1/k_0,\ a \in \Omega^*,\ l \geq 1.$$

The set $\{ta \in \mathbb{R}^n \mid a \in \Omega^*,\ 0 < t < r_0\}$ is open in $\mathbb{R}^n$ and it contains an n-cube $A = I_1 \times \cdots \times I_n$, where $I_j \subset \mathbb{R}$ are compact intervals.

According to Lemma 1, there is an open ball B in $\mathbb{C}^n$ and a constant $c > 0$ such that

$$(2) \qquad |P_l(z)| \leq c2^l \quad \text{for all } z \in B \text{ and all } l \geq 1.$$

By Lemma 2 one can assume that B is centered at $0 \in \mathbb{C}^n$. Then by (2) and the homogeneity of P_l one gets

$$|P_l(z)| \leq c/2^l \quad \text{for all } z \in \tfrac{1}{4}B \text{ and all } l \geq 1,$$

which shows (1) if we take $r =$ the radius of $\frac{1}{4}B$. ∎

Proof of Theorem 2. Let x_0 be a point in U. Given an $a \in \mathbb{S}^{n-1}$, the function $t \mapsto f(x_0 + ta)$ is analytic at $t = 0 \in \mathbb{R}$, so there exists a $\rho_a > 0$ such that

$$f(x_0 + ta) = \sum_{l=0}^{\infty} P_l(a)t^l \quad \text{for } t \in (-\rho_a, \rho_a),$$

where $l!P_l = \delta_{x_0}^l f$ is the lth Gateaux derivative of f at x_0. By Lemma 3 the series $\sum_{l=0}^{\infty} P_l$ is uniformly convergent in a ball $\|z\| < r$, $z \in \mathbb{C}^n$, and its sum $\tilde{f}$ is a holomorphic function there. But $\tilde{f}(x_0 + ta) = f(x_0 + ta)$ for $|t| < \rho_a$. By the identity property of analytic functions, $\tilde{f}(x_0 + ta) = f(x_0 + ta)$ for $|t| < \rho$, where $\rho = \min(r, \operatorname{dist}(x_0, \partial U))$. It follows that $\tilde{f}(x_0 + x) = f(x_0 + x)$ for all x in $\mathbb{R}^n$ with $\|x\| < \rho$, i.e. f is analytic at x_0. ∎

Remark. Theorem 2 is proved in [4, Th. 7.5] for mappings between topological vector spaces, but the proof is more complicated.

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