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# On the differentiation of integrals with respect to translation invariant convex density bases

by

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**Abstract.** For a translation invariant convex density basis $B$ it is shown that its Busemann–Feller extension $B_{\mathrm{BF}}$ has properties close to $B$, namely $B_{\mathrm{BF}}$ differentiates the same class of non-negative functions as $B$, and the integral of an arbitrary non-negative function $f \in L(\mathbb{R}^n)$ at almost every point $x \in \mathbb{R}^n$ has the same type limits of indeterminacy with respect to the bases $B$ and $B_{\mathrm{BF}}$. This theorem provides a certain general principle of extending results obtained for Busemann–Feller bases to results for bases without the Busemann–Feller property. Applications of the theorem are given.

**1. Definitions and notation.** A mapping $B$ defined on $\mathbb{R}^n$ is called a *differentiation basis* (briefly, *basis*) if for each $x \in \mathbb{R}^n$ the value $B(x)$ is a collection of bounded measurable sets of positive measure which contain $x$ and there exists a sequence $(R_k)$ of sets from $B(x)$ with $\lim_{k\to\infty} \operatorname{diam} R_k = 0$.

Let $B$ be a basis. For $f \in L(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the upper and lower limits of the integral means $|R|^{-1} \int_R f$, where $R$ is an arbitrary set from $B(x)$ and $\operatorname{diam} R \to 0$, are called the *upper and lower derivatives with respect to $B$ of the integral of $f$ at the point $x$*, and are denoted by $\overline{D}_B(\int f, x)$ and $\underline{D}_B(\int f, x)$, respectively. If the two derivatives coincide, then their common value is called the *derivative of $\int f$ at $x$* and is denoted by $D_B(\int f, x)$. We say that $B$ *differentiates* $\int f$ (or $\int f$ *is differentiable with respect to* $B$) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each $f$ in a class $F \subset L(\mathbb{R}^n)$ of functions, we say that $B$ *differentiates* $F$.

A basis $B$ is called:

- *homothecy invariant* if for every $x \in \mathbb{R}^n$, every $R \in B(x)$ and every homothecy $H : \mathbb{R}^n \to \mathbb{R}^n$ we have $H(R) \in B(H(x))$;

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- *translation invariant* if for every $x \in \mathbb{R}^n$, every $R \in B(x)$ and every translation $T : \mathbb{R}^n \to \mathbb{R}^n$ we have $T(R) \in B(T(x))$;
- *formed by sets from a collection* $\Delta$ if for every $x \in \mathbb{R}^n$ and $R \in B(x)$ we have $R \in \Delta$;
- *convex* if it is formed by convex sets;
- *Busemann–Feller* if $(x \in \mathbb{R}^n,\ R \in B(x),\ y \in R) \Rightarrow R \in B(y)$;
- *subbasis of a basis* $B'$ (notation: $B \subset B'$) if $B(x) \subset B'(x)$ for every $x \in \mathbb{R}^n$;
- *density basis* if $B$ differentiates $\int \chi_E$ for every bounded measurable set $E \subset \mathbb{R}^n$.

Note that each homothecy invariant basis is also translation invariant.

In what follows, the dimension of the space $\mathbb{R}^n$ is assumed to be greater than 1.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis for which $\mathbf{I}(x)$ consists of all $n$-dimensional intervals containing $x$. Differentiation with respect to $\mathbf{I}$ is called *strong differentiation*. Note that $\mathbf{I}$ is a density basis; moreover, by the well-known result of Jessen, Marcinkiewicz and Zygmund (see, e.g., [2, p. 50]), $\mathbf{I}$ differentiates the class $L(1+\ln^+ L)^{n-1}(\mathbb{R}^n)$.

*The maximal operator* $M_B$ and *the truncated maximal operator* $M_B^r$ ($r \in (0,\infty]$) *corresponding to a basis* $B$ are defined as follows:

$$M_B(f)(x) = \sup_{R\in B(x)} \frac{1}{|R|}\int_R |f|, \qquad M_B^r(f)(x) = \sup_{R\in B(x),\,\operatorname{diam} R<r} \frac{1}{|R|}\int_R |f|,$$

where $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Obviously, $M_B = M_B^\infty$.

Let us say that a basis $B$ is *measurable* if for any $f \in L(\mathbb{R}^n)$ the functions $\overline{D}_B(\int f,\cdot)$, $\underline{D}_B(\int f,\cdot)$ and $M_B^r(f)$ ($r \in (0,\infty]$) are measurable. It is easy to check that if $B$ is a translation invariant basis then $B$ is measurable.

For measurable bases $B$ and $B'$ we say that $B'$ *locally majorizes* $B$ (written $B \le B'$) if there exist $c \ge 1$ and $\delta > 0$ such that

$$|\{M_B^r(f) \ge \lambda\}| \le c|\{M_{B'}^{cr}(f) \ge \lambda/c\}| \tag{1.1}$$

for all $f \in L(\mathbb{R}^n)$, $r \in (0,\delta)$ and $\lambda \in (0,\infty)$. If there exists $c \ge 1$ such that (1.1) holds for all $f \in L(\mathbb{R}^n)$, $r \in (0,\infty]$ and $\lambda \in (0,\infty)$ then we say that $B'$ *majorizes* $B$ (written $B \preceq B'$).

It is easy to see that $B_1 \le B_2 \le B_3 \Rightarrow B_1 \le B_3$ and $B_1 \preceq B_2 \preceq B_3 \Rightarrow B_1 \preceq B_3$.

For a basis $B$, we will denote by $\overline{B}$ the collection $\bigcup_{x\in\mathbb{R}^n} B(x)$. Following [6] let us call $\overline{B}$ the *spread* of $B$.

For a basis $B$ let $B_{\mathrm{BF}}$ denote the basis defined as follows: $B_{\mathrm{BF}}(x) = \{R \in \overline{B} : R \ni x\}$ ($x \in \mathbb{R}^n$). Let us call $B_{\mathrm{BF}}$ the *Busemann–Feller extension* of a basis $B$. It is easy to see that: 1) $B \subset B_{\mathrm{BF}}$; 2) $B_{\mathrm{BF}}$ is the smallest Busemann–Feller basis containing $B$; 3) $B_{\mathrm{BF}}$ is the largest basis having the same spread as $B$; 4) if $B$ is homothecy invariant [translation invariant, convex] then $B_{\mathrm{BF}}$ is also homothecy invariant [translation invariant, convex].

REMARK 1.1. If $B$ is a translation invariant convex basis then for every $f \in L(\mathbb{R}^n)$ the upper and lower derivatives with respect to $B$ have the following "separation" property: $\underline{D}_B(\int f, x) \le f(x) \le \overline{D}_B(\int f, x)$ for almost every $x \in \mathbb{R}^n$. Indeed, there exists a translation invariant basis $B' \subset B$ for which $B'(0) = \{R_k\}$, where $R_1 \supset R_2 \supset \cdots$ and $\operatorname{diam} R_k \to 0$. The basis $B'$ differentiates $L(\mathbb{R}^n)$. Consequently, for every $f \in L(\mathbb{R}^n)$ we have $\underline{D}_B(\int f, x) \le \underline{D}_{B'}(\int f, x) = f(x) = \overline{D}_{B'}(\int f, x) \le \overline{D}_B(\int f, x)$ almost everywhere.

REMARK 1.2. In [10, Lemma 2] it was observed that if $B$ is a density basis then for every non-negative function $f \in L(\mathbb{R}^n)$ we have $\underline{D}_B(\int f, x) = f(x)$ at almost every $x \in \mathbb{R}^n$.

REMARK 1.3. From Remarks 1.1 and 1.2 it follows that if $B$ is a translation invariant convex density basis then the integral of an arbitrary non-negative function $f \in L(\mathbb{R}^n)$ at almost every point $x \in \mathbb{R}^n$ may have only the following three types of *limits of indeterminacy* with respect to the basis $B$:

- $\underline{D}_B(\int f, x) = f(x) = \overline{D}_B(\int f, x) < \infty$;
- $\underline{D}_B(\int f, x) = f(x) < \overline{D}_B(\int f, x) < \infty$;
- $\underline{D}_B(\int f, x) = f(x) < \overline{D}_B(\int f, x) = \infty$.

**2. Results.** Suppose we are given two bases $B$ and $B'$ with the same spread, i.e. $\overline{B} = \overline{B'}$. It seems natural to consider whether the differential properties of $B$ and $B'$ are similar. In fact, for translation invariant convex density bases the differential properties are quite close.

THEOREM 2.1. *If $B$ is a translation invariant convex density basis then its Busemann–Feller extension $B_{\mathrm{BF}}$ differentiates the same class of non-negative functions as $B$; moreover, the integral of an arbitrary non-negative function $f \in L(\mathbb{R}^n)$ at almost every point $x \in \mathbb{R}^n$ has the same type of limits of indeterminacy with respect to the bases $B$ and $B_{\mathrm{BF}}$.*

Theorem 2.1 provides a certain general principle of extending results obtained for Busemann–Feller bases to results for bases without the Busemann–Feller property. Some such applications of Theorem 2.1 will be given in the last section of the paper.

REMARK 2.2. Theorem 2.1 is not true for an arbitrary density basis. Namely, Hagelstein and Parissis [4, proof of Theorem 3] constructed a translation invariant density basis $B$ in $\mathbb{R}$ whose Busemann–Feller extension is not a density basis.

Remark 2.3. Let $B$ be a basis in $\mathbb{R}^2$ for which $B(x)$ ($x \in \mathbb{R}^2$) consists of all two-dimensional intervals of the type $[x_1, x_1 + t_1] \times [x_2, x_2 + t_2]$. Obviously, $B$ is a homothecy invariant convex density basis and $B_{\mathrm{BF}} = \mathbf{I}$. Zerekidze [13] constructed a function $f \in L(\mathbb{R}^2)$ whose integral is differentiable with respect to $B$ but not with respect to $\mathbf{I}$. Thus, the requirement of non-negativity of functions in Theorem 2.1 is essential.

We deduce Theorem 2.1 from the following two results.

Theorem 2.4. *If $B$ and $B'$ are measurable density bases locally majorizing each other then the integral of an arbitrary non-negative function $f \in L(\mathbb{R}^n)$ at almost every point $x \in \mathbb{R}^n$ has the same type of limits of indeterminacy with respect to $B$ and $B'$.*

Theorem 2.5. *If $B$ is a translation invariant convex density basis then its Busemann–Feller extension $B_{\mathrm{BF}}$ is also a density basis, and $B$ and $B_{\mathrm{BF}}$ locally majorize each other.*

## 3. Proof of Theorem 2.4

Lemma 3.1. *Let $B$ and $B'$ be measurable bases and suppose $B \le B'$. Then for every $f \in L(\mathbb{R}^n)$ and every $\lambda \in (0, \infty)$,*

$$\big|\{\overline{D}_B(\textstyle\int f, \cdot) \ge \lambda\} \setminus \{\overline{D}_{B'}(\textstyle\int f, \cdot) \ge \lambda/c\}\big| = 0,$$

*where $c$ is the constant from* (1.1).

*Proof.* Letting $r \to 0$ in (1.1) we see that for every non-negative $g \in L(\mathbb{R}^n)$ with bounded support,

$$\big|\{\overline{D}_B(\textstyle\int g, \cdot) \ge \lambda\}\big| \le c\big|\{\overline{D}_{B'}(\textstyle\int g, \cdot) \ge \lambda/c\}\big|. \tag{3.1}$$

Since the value of the upper derivative depends only on the local behaviour of the function, we can conclude from (3.1) that for every non-negative $g \in L(\mathbb{R}^n)$ and any cube $Q$,

$$\begin{aligned}\big|\{\overline{D}_B(\textstyle\int g, \cdot) \ge \lambda\} \cap Q\big| &= \big|\{\overline{D}_B(\textstyle\int g\chi_Q, \cdot) \ge \lambda\}\big| \\ &\le c\big|\{\overline{D}_{B'}(\textstyle\int g\chi_Q, \cdot) \ge \lambda/c\}\big| = c\big|\{\overline{D}_{B'}(\textstyle\int g, \cdot) \ge \lambda/c\} \cap Q\big|.\end{aligned} \tag{3.2}$$

Let us now assume that the assertion of the lemma does not hold, that is, $|\{\overline{D}_B(\int f, \cdot) \ge \lambda\} \setminus \{\overline{D}_{B'}(\int f, \cdot) \ge \lambda/c\}| > 0$. Then there exists a point $x$ that is a density point for the sets $\{\overline{D}_B(\int f, \cdot) \ge \lambda\}$ and $\{\overline{D}_{B'}(\int f, \cdot) < \lambda/c\}$. Hence, considering a sufficiently small cube $Q$ with centre at $x$, we obtain an inequality contradicting (3.2). ∎

Lemma 3.2. *Let $B$ and $B'$ be measurable bases and suppose $B \le B'$. Then for every non-negative function $f \in L(\mathbb{R}^n)$,*

$$\big|\{\overline{D}_B(\textstyle\int f, \cdot) = \infty\} \setminus \{\overline{D}_{B'}(\textstyle\int f, \cdot) = \infty\}\big| = 0.$$

*Proof.* Letting first $r \to 0$ and then $\lambda \to \infty$ in (1.1), we see that $|\{\overline{D}_B(\int g, \cdot) = \infty\}| \le c|\{\overline{D}_{B'}(\int g, \cdot) = \infty\}|$ for every non-negative $g \in L(\mathbb{R}^n)$ with bounded support. The rest of the proof is analogous to that of Lemma 3.1. ∎

For $f \in L(\mathbb{R}^n)$ and $a > 0$ denote $f_{[a]} = f\chi_{\{|f|\le a\}}$ and $f^{[a]} = f\chi_{\{|f|>a\}}$.

LEMMA 3.3. *Let $B$ be a density basis. Then for all non-negative $f \in L(\mathbb{R}^n)$ and $\lambda \in (0,\infty)$,*

$$\Big|\big\{\overline{D}_B(\textstyle\int f, \cdot) \ge f + \lambda\big\} \bigtriangleup \displaystyle\bigcap_{k=1}^{\infty} \big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge \lambda\big\}\Big| = 0.$$

*Proof.* Since each density basis differentiates the integral of every bounded summable function (see, e.g., [2, p. 72]), we know that $B$ differentiates $\int f_{[k]}$ for every $k \in \mathbb{N}$. Therefore it is easy to see that for every $k \in \mathbb{N}$,

$$\big|\big\{\overline{D}_B(\textstyle\int f, \cdot) \ge f + \lambda\big\} \bigtriangleup \big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge f^{[k]} + \lambda\big\}\big| = 0. \tag{3.3}$$

For almost every $x$ we have $f^{[k]}(x) = 0$ if $k$ is large enough. Hence it is easy to see that

$$\Big|\Big(\varliminf_{k\to\infty} \big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge f^{[k]} + \lambda\big\}\Big) \bigtriangleup \displaystyle\bigcap_{k=1}^{\infty} \big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge \lambda\big\}\Big| = 0. \tag{3.4}$$

From (3.3) and (3.4) we deduce the validity of the lemma. ∎

LEMMA 3.4. *Let $B$ and $B'$ be measurable density bases and suppose $B \le B'$. Then for all non-negative $f \in L(\mathbb{R}^n)$ and $\lambda \in (0,\infty)$,*

$$\big|\big\{\overline{D}_B(\textstyle\int f, \cdot) \ge f + \lambda\big\} \setminus \big\{\overline{D}_{B'}(\textstyle\int f, \cdot) \ge f + \lambda/c\big\}\big| = 0,$$

*where $c$ is the constant from* (1.1).

*Proof.* By Lemma 3.3 we have

$$\Big|\big\{\overline{D}_B(\textstyle\int f, \cdot) \ge f + \lambda\big\} \bigtriangleup \displaystyle\bigcap_{k=1}^{\infty} \big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge \lambda\big\}\Big| = 0, \tag{3.5}$$

$$\Big|\big\{\overline{D}_{B'}(\textstyle\int f, \cdot) \ge f + \lambda/c\big\} \bigtriangleup \displaystyle\bigcap_{k=1}^{\infty} \big\{\overline{D}_{B'}(\textstyle\int f^{[k]}, \cdot) \ge \lambda/c\big\}\Big| = 0. \tag{3.6}$$

On the other hand by Lemma 3.1 for every $k \in \mathbb{N}$ we have

$$\big|\big\{\overline{D}_B(\textstyle\int f^{[k]}, \cdot) \ge \lambda\big\} \setminus \big\{\overline{D}_{B'}(\textstyle\int f^{[k]}, \cdot) \ge \lambda/c\big\}\big| = 0. \tag{3.7}$$

From (3.5)–(3.7) we easily obtain the assertion of the lemma. ∎

LEMMA 3.5. *Let $B$ and $B'$ be measurable density bases and suppose $B \le B'$. Then for every non-negative $f \in L(\mathbb{R}^n)$,*

$$\big|\big\{\overline{D}_B(\textstyle\int f, \cdot) > f\big\} \setminus \big\{\overline{D}_{B'}(\textstyle\int f, \cdot) > f\big\}\big| = 0.$$

*Proof.* We have $\{\overline{D}_B(\int f,\cdot) > f\} = \bigcup_{m=1}^{\infty}\{\overline{D}_B(\int f,\cdot) \geq f + 1/m\}$, and $\{\overline{D}_{B'}(\int f,\cdot) > f\} = \bigcup_{m=1}^{\infty}\{\overline{D}_{B'}(\int f,\cdot) \geq f+1/(cm)\}$, where $c$ is the constant from (1.1). Now using Lemma 3.4 for $f$ and for every $\lambda = 1/m$ ($m \in \mathbb{N}$) we conclude the proof. ∎

Taking into account Remark 1.3, from Lemmas 3.2 and 3.5 we obtain Theorem 2.4.

**4. Proof of Theorem 2.5.** Given a basis $B$ and a non-degenerate linear mapping $M : \mathbb{R}^n \to \mathbb{R}^n$, let $B_M$ be the basis for which $B_M(M(x)) = \{M(R) : R \in B(x)\}$ ($x \in \mathbb{R}^n$).

Lemma 4.1. *If $B$ is a density basis then for every non-degenerate linear mapping $M : \mathbb{R}^n \to \mathbb{R}^n$ the basis $B_M$ also has the density property.*

*Proof.* Let $E$ be a bounded measurable set. Then for every $x \in \mathbb{R}^n$ and $R \in B(x)$ we have

$$\frac{|M(R)\cap M(E)|}{|M(R)|} = \frac{|M(R\cap E)|}{|M(R)|} = \frac{|J_M|\,|R\cap E|}{|J_M|\,|R|} = \frac{|R\cap E|}{|R|}.$$

Here $J_M$ denotes the Jacobian of $M$. Hence, for every $x \in \mathbb{R}^n$ we have the equalities $\underline{D}_{B_M}(\int \chi_{M(E)}, M(x)) = \underline{D}_B(\int \chi_E, x)$ and $\overline{D}_{B_M}(\int \chi_{M(E)}, M(x)) = \overline{D}_B(\int \chi_E, x)$. Consequently, since $B$ is a density basis, we conclude that $\int \chi_{M(E)}$ is differentiable with respect to $B_M$. Clearly, this implies that $B_M$ is a density basis. ∎

We will need the following characterization of translation invariant density bases proved by Hagelstein and Parissis [4].

Theorem A. *Let $B$ be a translation invariant basis. Then the following properties are equivalent:*

(1) *$B$ is a density basis;*
(2) *for each $\lambda \in (0,1)$ there exist positive constants $r(B,\lambda)$ and $c(B,\lambda)$ such that $|\{M_B^{r(B,\lambda)}(\chi_E) \geq \lambda\}| \leq c(B,\lambda)|E|$ for each measurable set $E$.*

Note that the conclusion of Theorem A for Busemann–Feller translation invariant bases was proved in [8].

Below without loss of generality assume that for a convex basis $B$ the sets forming $B$ (i.e. sets from the spread $\overline{B}$) are closed.

For a set $A$ with centre of symmetry $x$ and for $\alpha > 0$ we denote by $\alpha A$ the dilation of $A$ with coefficient $\alpha$: $\alpha A = \{x + \alpha(y - x) : y \in A\}$.

F. John proved (see, e.g., [2, p. 139]) that for any bounded closed convex set $E$ in $\mathbb{R}^n$ with positive measure there exists a closed ellipsoid $T$ for which $T \subset E \subset nT$. This assertion easily implies the following lemma (see [10, Lemma 3] for details).

Lemma 4.2. *For any bounded closed convex set $E$ in $\mathbb{R}^n$ with positive measure there exists a closed $n$-dimensional rectangle $R$ such that $R \subset E \subset n^2 R$.*

For every set $E$ from $B(0)$, using Lemma 4.2 we can find closed rectangles $R_*(E)$ and $R^*(E)$ such that $R^*(E) = n^2 R_*(E)$ and $R_*(E) \subset E \subset R^*(E)$.

Let $B_*$ and $B^*$ be the translation invariant bases with values at the origin defined as follows: $B_*(0) = \{\{0\} \cup R_*(E) : E \in B(0)\}$ and $B^*(0) = \{R^*(E) : E \in B(0)\}$. Note that the sets forming $B_*$ in general are not rectangles. The reason is that for a set $E \in B(0)$ the rectangle $R_*(E)$ may not contain the origin. It is easy to check that $B_* \le B \le B_{\mathrm{BF}} \le B^*_{\mathrm{BF}}$; moreover, for every $f \in L(\mathbb{R}^n)$, $r \in (0,\infty]$ and $x \in \mathbb{R}^n$ we have the pointwise estimates $M^r_{B_*}(f)(x) \le n^{2n} M^{n^2 r}_B(f)(x)$, $M^r_B(f)(x) \le M^r_{B_{\mathrm{BF}}}(f)(x)$, $M^r_{B_{\mathrm{BF}}}(f)(x) \le n^{2n} M^{n^2 r}_{B^*_{\mathrm{BF}}}(f)(x)$.

For a basis $B$ and a positive number $c$, denote by $cB$ the basis $B_M$ where $M$ is the homothecy with centre at the origin and with coefficient $c$.

For a basis $B$, denote by $B_{\mathrm{sym}}$ the basis $B_M$ where $M$ is the symmetry through the origin.

Lemma 4.3. *For all $f \in L(\mathbb{R}^n)$, $r \in (0,\infty]$ and $\lambda \in (0,\infty)$,*

$$\{M^r_{B^*_{\mathrm{BF}}}(f) \ge \lambda\} \subset \{M^{c_1 r}_{(c_2 B_*)_{\mathrm{sym}}}(\chi_{\{M^{c_3 r}_{c_4 B_*}(f) \ge \lambda/c_5\}}) \ge 1/c_6\},$$

*where $c_1,\dots,c_6$ are constants greater than 1 depending only on $n$.*

*Proof.* Suppose $M^r_{B^*_{\mathrm{BF}}}(f)(x) \ge \lambda$. Let $R^* \in B^*_{\mathrm{BF}}(x)$ be a rectangle for which $\operatorname{diam} R^* < r$ and $\int_{R^*} |f| \ge \lambda |R^*|/2$. Denote $R_* = n^{-2} R^*$. It is easy to see that there is a point $x_* \in R^*$ for which $\{x_*\} \cup R_* \in B_*(x_*)$. Without loss of generality assume that $R^*$ is an interval of the type $[0,t_1] \times \cdots \times [0,t_n]$ and the point $x_*$ is to the "left" of the centre of $R_*$, i.e. $x_{*,1} \le t_1/2, \dots, x_{*,n} \le t_n/2$. Let us consider the interval $R$ with centre at the origin which is the translate of $4R^*$. It is easy to see that there is a point $y$ lying to the left of the origin for which $\{y\} \cup R \in (4n^2 B_*)(y)$. Obviously, $y$ belongs to the rectangle $n^2 R$. Now let us consider the set $P$ of all points $y+t$ with $t$ to the left of the origin and $R + t \supset R^*$. It is easy to check that $P$ is the translate of the rectangle $R^*$. Consider any $y + t \in P$. Since $\{y\} \cup R \in (4n^2 B_*)(y)$, we have $\{y+t\} \cup (R+t) \in (4n^2 B_*)(y+t)$. Hence, taking into account that $\operatorname{diam}(\{y+t\} \cup (R+t)) = \operatorname{diam}(\{y\} \cup R) \le \operatorname{diam}(n^2 R) = n^2 \operatorname{diam} R = n^2 4 \operatorname{diam} R^* < 4n^2 r$ we have

$$M^{4n^2 r}_{4n^2 B_*}(f)(y+t) \ge \frac{1}{|R|} \int_{R+t} |f| \ge \frac{1}{4^n |R^*|} \int_{R^*} |f| \ge \frac{\lambda}{2 \cdot 4^n}.$$

Thus,

$$P \subset \{M^{4n^2 r}_{4n^2 B_*}(f) \ge \lambda/(2 \cdot 4^n)\}. \tag{4.1}$$

Let $S:\mathbb{R}^n\to\mathbb{R}^n$ be the symmetry through the centre of symmetry of $R^*\cup P$. Clearly, $S(x)\in P$. Then $\{S(x)\}\cup R+S(x)-y\in(4n^2B_*)(S(x))$ and $R+S(x)-y\supset R^*$. Consequently, taking into account the definition of the basis $(4n^2B_*)_{\mathrm{sym}}$ we find that $\{x\}\cup(S(R)+x-S(y))\in(4n^2B_*)_{\mathrm{sym}}(x)$ and $S(R)+x-S(y)\supset P$. Therefore

$$(4.2)\qquad M^{4n^2r}_{(4n^2B_*)_{\mathrm{sym}}}(\chi_P)(x)\geq\frac{1}{|S(R)|}\int\limits_{S(R)+x-S(y)}\chi_P=\frac{1}{4^n|R^*|}|P|=\frac{1}{4^n}.$$

From (4.1) and (4.2) the conclusion follows. ∎

Lemma 4.4. *For every $f\in L(\mathbb{R}^n)$, $r\in(0,\infty]$ and $\lambda\in(0,\infty)$,*

$$\{M^r_{B^*_{\mathrm{BF}}}(f)\geq\lambda\}\subset\{M^{3r}_{3B^*_{\mathrm{BF}}}(\chi_{\{M^r_{B_*}(f)\geq\lambda/\alpha_1\}})\geq 1/\alpha_2\},$$

*where $\alpha_1$ and $\alpha_2$ are constants greater than 1 depending only on $n$.*

*Proof.* Suppose $M^r_{B^*_{\mathrm{BF}}}(f)(x)\geq\lambda$. Let $R^*\in B^*_{\mathrm{BF}}(x)$ be a rectangle for which $\operatorname{diam}R^*<r$ and $\int_{R^*}|f|\geq\lambda|R^*|/2$. Denote $R_*=n^{-2}R^*$. It is easy to see that there is a point $x_*\in R^*$ for which $\{x_*\}\cup R_*\in B_*(x_*)$. Let us decompose $R^*$ into rectangles which are translates of $\frac{1}{2}R_*$. One of those rectangles, say $I$, obviously satisfies $\int_I|f|\geq\lambda|I|/2$. Denote $P=\{x_*+t: t\in\mathbb{R}^n,\ R_*+t\supset I\}$. It is easy to see that $P$ is the translate of $I$ and $P\subset 3R^*$. Let us consider any $x_*+t\in P$. Since $\{x_*\}\cup R_*\in B_*(x_*)$, we have $\{x_*+t\}\cup(R_*+t)\in B_*(x_*+t)$. Hence, taking into account that $\operatorname{diam}(\{x_*+t\}\cup(R_*+t))=\operatorname{diam}(\{x_*\}\cup R_*)\leq\operatorname{diam}R^*<r$ we have

$$M^r_{B_*}(f)(x_*+t)\geq\frac{1}{|R_*|}\int\limits_{R_*+t}|f|\geq\frac{1}{2^n|I|}\int\limits_I|f|\geq\frac{\lambda}{2^{n+1}}.$$

Thus,

$$(4.3)\qquad P\subset\{M^r_{B_*}(f)\geq\lambda/2^{n+1}\}.$$

By the inclusion $P\subset 3R^*$ we have

$$(4.4)\qquad M^{3r}_{3B^*_{\mathrm{BF}}}(\chi_P)(x)\geq\frac{1}{|3R^*|}\int\limits_{3R^*}\chi_P=\frac{|P|}{6^nn^{2n}|I|}=\frac{1}{6^nn^{2n}}.$$

From (4.3) and (4.4) we obtain the conclusion. ∎

Now let us move directly to the proof of Theorem 2.5. Since $B$ is a density basis, by the majorization $B_*\leq B$ and Theorem A we conclude that $B_*$ is also a density basis. Further, by Lemma 4.1, $c_4B_*$ and $(c_2B_*)_{\mathrm{sym}}$ are density bases. Consequently, using Lemma 4.3 and Theorem A it is easy to see that $B^*_{\mathrm{BF}}$ is a density basis. Hence, by Lemma 4.1 the basis $3B^*_{\mathrm{BF}}$ also has the density property. Therefore by Lemma 4.4 and Theorem A we conclude that $B^*_{\mathrm{BF}}\leq B_*$. Now, using Theorem A and the relations $B_*\leq B\leq B_{\mathrm{BF}}\leq B^*_{\mathrm{BF}}$

we conclude that $B_{\mathrm{BF}}$ is a density basis and the bases $B_*$, $B$, $B_{\mathrm{BF}}$ and $B^*_{\mathrm{BF}}$ locally majorize each other. The theorem is proved.

REMARK 4.5. Taking into account the properties of the basis $B^*_{\mathrm{BF}}$, from the proof of Theorem 2.5 we obtain the following result: For every translation invariant convex density basis $B$ there exists a Busemann–Feller translation invariant basis $B'$ formed by $n$-dimensional rectangles such that the integrals of any non-negative function $f \in L(\mathbb{R}^n)$ at almost every point $x \in \mathbb{R}^n$ have the same type of limits of indeterminacy with respect to the bases $B$ and $B'$.

REMARK 4.6. Let us recall the following well-known result (see, e.g., [2, p. 77]): If $B$ is a density basis, $f \in L(\mathbb{R}^n)$ and $B$ differentiates $\int |f|$, then $B$ also differentiates $\int f$. Taking into account this result, from Theorem 2.1 we obtain the following assertion: Let $B$ be a translation invariant convex density basis and $\varphi(L)(\mathbb{R}^n)$ be some integral class. Then $B$ differentiates $\varphi(L)(\mathbb{R}^n)$ if and only if $B_{\mathrm{BF}}$ differentiates $\varphi(L)(\mathbb{R}^n)$.

REMARK 4.7. The following characterization of homothecy invariant density bases (see [2, p. 69]) holds:

THEOREM B. *Let $B$ be a homothecy invariant basis. Then the following properties are equivalent:*

(1) *$B$ a density basis;*
(2) *for each $\lambda \in (0,1)$ there exists a positive constant $c(B,\lambda)$ such that $|\{M_B(\chi_E) \ge \lambda\}| \le c(B,\lambda)|E|$ for each measurable set $E$.*

Using Theorem B instead of Theorem A, it can be proved that if a basis $B$ in Theorem 2.5 is homothecy invariant then $B$ and $B_{\mathrm{BF}}$ majorize each other.

## 5. Some applications

**I.** Besicovitch [1] proved the following theorem about the limits of indeterminacy for strong differentiation: For every $f \in L(\mathbb{R}^2)$ and almost every point $x \in \mathbb{R}^2$,

$$\underline{D}_{\mathbf{I}}(\textstyle\int f, x) \neq f(x) \ \Rightarrow\ \underline{D}_{\mathbf{I}}(\textstyle\int f, x) = -\infty, \tag{5.1}$$

$$\overline{D}_{\mathbf{I}}(\textstyle\int f, x) \neq f(x) \ \Rightarrow\ \overline{D}_{\mathbf{I}}(\textstyle\int f, x) = \infty. \tag{5.2}$$

An analog of this result for the multi-dimensional case was obtained by Ward [12]. Note that the multi-dimensional extension may also be obtained using the version of F. Riesz "rising sun" lemma proved in [7].

Guzmán [3, p. 389] posed the following problem: *To what bases can Besicovitch's result be extended*?

We say that a basis $B$ has the *Besicovitch property* [the *weak Besicovitch property*] if for every $f \in L(\mathbb{R}^n)$ [for every non-negative $f \in L(\mathbb{R}^n)$] and almost every $x \in \mathbb{R}^n$ the lower and upper derivatives $\underline{D}_B(\int f, x)$ and $\overline{D}_B(\int f, x)$ satisfy the implications analogous to (5.1) and (5.2).

Remark 5.1. If $B$ is not a density basis then it is easy to see that there is a measurable bounded set $E$ for which the set $\{x \in E : \underline{D}_B(\int \chi_E, x) < 1\}$ is not of measure zero. Consequently, if a basis has the weak Besicovitch property then it is a density basis.

We have the following characterization of homothecy invariant convex bases with the weak Besicovitch property.

Theorem 5.2. *A homothecy invariant convex basis has the weak Besicovitch property if and only if it is a density basis.*

Theorem 5.2 can be obtained by using Theorem 2.1 and the following result proved in [10]: A Busemann–Feller homothecy invariant convex basis has the weak Besicovitch property if and only if it is a density basis.

Note that a result analogous to Theorem 5.2 for Busemann–Feller homothecy invariant convex bases formed by central-symmetric sets was proved by Guzmán and Menárgues [2, p. 106].

Remark 5.3. The analogue of Theorem 5.2 for translation invariant convex bases is not valid. Namely, [9] exhibits an example of a Busemann–Feller translation invariant basis formed by two-dimensional intervals which fails the weak Besicovitch property.

Remark 5.4. It is unknown whether there is valid a characterization of homothecy invariant convex bases with the Besicovitch property analogous to Theorem 5.2. Moreover, the question is open even for homothecy invariant bases formed by $n$-dimensional intervals. A partial result in this direction is obtained in [11].

**II.** A basis $B$ is called *regular* if there is $c \geq 1$ such that for every set $R$ from the spread of $B$ there exists a cubic interval $Q$ with $R \subset Q$ and $|Q| \leq c|R|$. Note that (see, e.g., [2, p. 25]) each regular basis differentiates $L(\mathbb{R}^n)$.

Let $B$ be a homothecy invariant density basis. The *halo function* $\varphi_B$ of the basis $B$ is defined as follows: $\varphi_B(t) = 0$ if $0 \leq t \leq 1$ and

$$\varphi_B(t) = \sup_E \frac{|\{M_B(\chi_E) \geq 1/t\}|}{|E|} \quad \text{if } t > 1,$$

where the supremum is taken over all bounded measurable sets $E$ with positive measure.

The *halo conjecture* (see [2, pp. 177–178]) asserts that any Busemann–Feller homothecy invariant density basis $B$ differentiates the class $\varphi_B(L)(\mathbb{R}^n)$.

In [2, p. 206] Moriyón characterized Busemann–Feller homothecy invariant density bases formed by open sets which differentiate $L(\mathbb{R}^n)$ and also justified the halo conjecture for bases with halo function satisfying the condition $\varphi_B(t) \sim t$. Namely, Moriyón proved the following theorem: Let $B$ be a Busemann–Feller homothecy invariant basis formed by open sets. Then the following statements are equivalent:

- $B$ differentiates $L(\mathbb{R}^n)$;
- the halo function of $B$ satisfies $c_1 t \le \varphi_B(t) \le c_2 t$ $(t \ge 0)$, where $c_1$ and $c_2$ are positive constants;
- $B$ is a regular basis.

Taking into account Theorem B and Remarks 4.6 and 4.7, from Theorems 2.1 and 2.5 we can conclude that the analog of Moriyón's result is true for every homothecy invariant convex basis.

**III.** Hagelstein and Stokolos [5] proved that every Busemann–Feller homothecy invariant convex density basis differentiates the class $L^p(\mathbb{R}^n) \cap L(\mathbb{R}^n)$ for sufficiently large $p$. Using Theorem 2.1 one can extend this result to every homothecy invariant convex density basis.

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