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Identities involving (doubly) symmetric polynomials and integrals over Grassmannians

by

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Abstract. We obtain identities involving symmetric and doubly symmetric polynomials. These identities provide a way of handling expressions appearing in the Atiyah–Bott–Berline–Vergne formula for Grassmannians. As corollaries, we obtain formulas for integrals over Grassmannians of characteristic classes of the tautological bundles. Moreover, we provide a proof of the Martin formula for the classical Grassmannian.

1. Introduction. Throughout we always assume that all polynomials are over a field and $\lambda_1, \ldots, \lambda_n$ are indeterminates. The Lagrange interpolation formula says that a polynomial $P(x)$ of degree not greater than $n-1$ in one variable can be written as

$$P(x) = \sum_{i=1}^{n} P(\lambda_i) L_i(x), \quad \text{where} \quad L_i(x) = \prod_{j \neq i} \frac{x - \lambda_j}{\lambda_i - \lambda_j}. \tag{1.1}$$

This implies the identity

$$\sum_{i=1}^{n} \frac{P(\lambda_i)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} = c_n, \tag{1.2}$$

where c_n is the coefficient of x^{n-1} in the polynomial $P(x)$.

The first goal of this paper is to generalize the identity (1.2) to multivariate symmetric polynomials. For convenience, we shall write $[n]$ for the set $\{1, \ldots, n\}$. For each subset $I = \{i_1, \ldots, i_k\} \subset [n]$, we denote $\lambda_I = (\lambda_{i_1}, \ldots, \lambda_{i_k})$ and $I^c = [n] \setminus I$. Recall that a polynomial $P(x_1, \ldots, x_k)$ is

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said to be *symmetric* if it is invariant under permutations of $x_1, \ldots, x_k$. We obtain the following result.

THEOREM 1.1. *Let $P(x_1, \ldots, x_k)$ be a symmetric polynomial of degree not greater than $k(n-k)$ in k variables ($k < n$). Then*

$$\sum_{I \subset [n],\, |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I,\, j \in I^c} (\lambda_i - \lambda_j)} = \frac{c(k,n)}{k!},$$

where $c(k,n)$ is the coefficient of $x_1^{n-1} \ldots x_k^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_k) \prod_{j \neq i} (x_i - x_j).$$

More generally, we also obtain an identity involving doubly symmetric polynomials. Recall that a polynomial $P(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ is said to be *doubly symmetric* if it is invariant under permutations of $x_1, \ldots, x_k$ and under permutations of $y_1, \ldots, y_{n-k}$. We obtain the following result.

THEOREM 1.2. *Let $P(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$ be a doubly symmetric polynomial of degree not greater than $k(n-k)$. Then*

$$\sum_{I \subset [n],\, |I|=k} \frac{P(\lambda_I, \lambda_{I^c})}{\prod_{i \in I,\, j \in I^c} (\lambda_i - \lambda_j)} = \frac{d(k,n)}{k!(n-k)!},$$

where $d(k,n)$ is the coefficient of $x_1^{n-1} \ldots x_k^{n-1} y_1^{n-1} \ldots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_k, y_1, \ldots, y_{n-k}) \prod_{j \neq i} (x_i - x_j) \prod_{j \neq i} (y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^{k} (y_i - x_j).$$

The second goal of this paper is to give a method of dealing with integrals over Grassmannians. The idea is as follows. Localization in equivariant cohomology allows us to express integrals in terms of some data attached to the fixed points of a torus action. In particular, for Grassmannians, we obtain interesting formulas with nontrivial relations involving rational functions. Let $G(k,n)$ be the Grassmannian of k-dimensional linear spaces in $\mathbb{C}^n$. Consider the following integrals:

$$\int_{G(k,n)} \Phi(\mathcal{S}), \qquad \int_{G(k,n)} \Psi(\mathcal{Q}), \qquad \int_{G(k,n)} \Delta(\mathcal{S}, \mathcal{Q}),$$

where $\Phi(\mathcal{S}), \Psi(\mathcal{Q})$ are some characteristic classes of the tautological subbundle $\mathcal{S}$ and the quotient bundle $\mathcal{Q}$ on $G(k,n)$, and $\Delta(\mathcal{S}, \mathcal{Q})$ is a characteristic class of both $\mathcal{S}$ and $\mathcal{Q}$.

Using localization in equivariant cohomology, Weber [W12] and Zielenkiewicz [Zi14] expressed the integrals as iterated residues at infinity of holomorphic functions. Our identities provide another method for dealing with such expressions.

COROLLARY 1.3 (compare with [W12, (4)]). *Suppose that $\Phi(\mathcal{S})$ is represented by a symmetric polynomial $P(x_1,\dots,x_k)$ of degree not greater than $k(n-k)$ in k variables $x_1,\dots,x_k$ which are the Chern roots of $\mathcal{S}$ and $\Psi(\mathcal{Q})$ is represented by a symmetric polynomial $Q(y_1,\dots,y_{n-k})$ of degree not greater than $k(n-k)$ in $n-k$ variables $y_1,\dots,y_{n-k}$ which are the Chern roots of $\mathcal{Q}$. The following statements hold:*

(a) *We have*

$$\int_{G(k,n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)}\frac{c(k,n)}{k!},$$

where $c(k,n)$ is the coefficient of $x_1^{n-1}\dots x_k^{n-1}$ in the polynomial

$$P(x_1,\dots,x_k)\prod_{j\neq i}(x_i-x_j).$$

(b) *We have*

$$\int_{G(k,n)} \Psi(\mathcal{Q}) = \frac{c(k,n)}{(n-k)!},$$

where $c(k,n)$ is the coefficient of $y_1^{n-1}\dots y_{n-k}^{n-1}$ in the polynomial

$$Q(y_1,\dots,y_{n-k})\prod_{j\neq i}(y_i-y_j).$$

COROLLARY 1.4 (compare with [Zi14, Formula 1]). *Suppose that $\Delta(\mathcal{S},\mathcal{Q})$ is represented by a doubly symmetric polynomial $P(x_1,\dots,x_k,y_1,\dots,y_{n-k})$ of degree not greater than $k(n-k)$ in n variables which are the Chern roots of $\mathcal{S}$ and $\mathcal{Q}$ respectively. Then*

$$\int_{G(k,n)} \Delta(\mathcal{S},\mathcal{Q}) = (-1)^{k(n-k)}\frac{d(k,n)}{k!(n-k)!},$$

where $d(k,n)$ is the coefficient of $x_1^{n-1}\dots x_k^{n-1}y_1^{n-1}\dots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1,\dots,x_k,y_1,\dots,y_{n-k})\prod_{j\neq i}(x_i-x_j)\prod_{j\neq i}(y_i-y_j)\prod_{i=1}^{n-k}\prod_{j=1}^{k}(y_i-x_j).$$

The results of Corollaries 1.3 and 1.4 are special cases of the formulas derived in [W12, Zi14]. However, our proofs are new and unrelated. The formulas in the corollaries are derived from Theorems 1.1 and 1.2, which are proven in a purely algebraic fashion, using the division algorithm for multivariate polynomials and iterated Lagrange interpolation. Indeed, the main motivation for these results is to establish a relationship between the

Atiyah–Bott–Berline–Vergne formula and Lagrange interpolation, which is a novelty.

The third goal of this paper is connected with the Martin formula for symplectic quotients. In an unpublished paper, Martin [M00, Theorem B] proved an integration formula, which expresses integrals on the symplectic quotient $X/\!\!/G$ of a Hamiltonian G-manifold X in terms of those on the associated symplectic quotient $X/\!\!/T$, where $T \subset G$ is a maximal torus, in the case where $X/\!\!/G$ is a compact manifold. For the case of the classical Grassmannian, the Martin formula reduces to statement (a) of Corollary 1.3. This will be clarified in the final section of this paper. Therefore, the proof of Corollary 1.3, which is purely algebraic, is also an algebraic proof of the Martin formula for the classical Grassmannian.

The rest of the paper is organized as follows: The proofs of the identities are presented in Section 2. Section 3 gives a brief review of the localization formula in equivariant cohomology and the proof of the corollaries. Section 4 is devoted to Martin's formula.

2. Proof of the identities. Set

$$F(x_1,\ldots,x_k) = P(x_1,\ldots,x_k)\prod_{j\neq i}(x_i - x_j).$$

By the assumption, the degree of F is not greater than

$$k(n-k) + k(k-1) = k(n-1).$$

Since P is symmetric, so is F. By the division algorithm for multivariate polynomials (see [CLO07, Theorem 3]), there exist polynomials $F_i(x_1,\ldots,x_k)$, $i = 1,\ldots,k$, and $R(x_1,\ldots,x_k)$ such that

$$R(x_1,\ldots,x_k) = F(x_1,\ldots,x_k) - \sum_{i=1}^{k} F_i(x_1,\ldots,x_k)\prod_{j=1}^{n}(x_i - \lambda_j),$$

and all partial degrees of R are not greater that $n-1$. By the Lagrange interpolation formula,

$$R(x_1,\ldots,x_k) = \sum_{i_1=1}^{n} R(\lambda_{i_1}, x_2,\ldots,x_k)L_{i_1}(x_1).$$

By the Lagrange interpolation formula for the polynomials $R(\lambda_{i_1}, x_2,\ldots,x_k)$, we have

$$R(x_1,\ldots,x_k) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} R(\lambda_{i_1},\lambda_{i_2}, x_3,\ldots,x_k)L_{i_1}(x_1)L_{i_2}(x_2).$$

Continuing, we obtain

$$R(x_1,\dots,x_k)=\sum_{i_1,\dots,i_k=1}^{n} R(\lambda_I)\prod_{l=1}^{k} L_{i_l}(x_l).$$

For each $I=\{i_1,\dots,i_k\}$, we have $R(\lambda_I)=F(\lambda_I)$, and if $i_s=i_t$ for some $s\neq t$, then $R(\lambda_I)=0$. Since the degree of F is not greater than $k(n-1)$, the coefficient of $x_1^{n-1}\dots x_k^{n-1}$ in R is equal to that in F. Thus the coefficient of $x_1^{n-1}\dots x_k^{n-1}$ in F is

$$k!\sum_{I\subset[n],\,|I|=k}\frac{F(\lambda_I)}{\prod_{i\in I,\,j\neq i}(\lambda_i-\lambda_j)}.$$

For each $I\subset[n]$, we have

$$F(\lambda_I)=P(\lambda_I)\prod_{i,j\in I,\,j\neq i}(\lambda_i-\lambda_j),$$

and

$$\prod_{i\in I,\,j\neq i}(\lambda_i-\lambda_j)=\prod_{i\in I,\,j\in I^c}(\lambda_i-\lambda_j)\prod_{i,j\in I,\,j\neq i}(\lambda_i-\lambda_j).$$

This implies that the coefficient of $x_1^{n-1}\dots x_k^{n-1}$ in F is equal to

$$k!\sum_{I\subset[n],\,|I|=k}\frac{P(\lambda_I)}{\prod_{i\in I,\,j\in I^c}(\lambda_i-\lambda_j)}.$$

Thus Theorem 1.1 is proved.

The proof of Theorem 1.2 is very similar and is omitted.

Remark 2.1. If $P(x_1,\dots,x_k)$ is a symmetric polynomial whose partial degrees are not greater than $n-k$, then we have the following formula, which was proved by Chen and Louck [CL96, Theorem 2.1]:

$$P(x_1,\dots,x_k)=\sum_{I\subset[n],\,|I|=k} P(\lambda_I)\frac{\prod_{x\in X,\,j\in I^c}(x-\lambda_j)}{\prod_{i\in I,\,j\in I^c}(\lambda_i-\lambda_j)}.$$

By the interpolation formula of Chen and Louck, we get

$$\sum_{I\subset[n],\,|I|=k}\frac{P(\lambda_I)}{\prod_{i\in I,\,j\in I^c}(\lambda_i-\lambda_j)}=d(k,n),$$

where $d(k,n)$ is the coefficient of $x_1^{n-k}\dots x_k^{n-k}$ in P. This is in fact a special case of Theorem 1.1. It was proved in [Ze82] that $k!$ is the coefficient of $x_1^{k-1}\dots x_k^{k-1}$ in $\prod_{j\neq i}(x_i-x_j)$. If the partial degrees of P are not greater than $n-k$, then we get

$$c(k,n)=d(k,n)k!.$$

REMARK 2.2. Theorem 1.2 is a generalization of Theorem 1.1. Indeed, if $P(x_1, \ldots, x_k)$ is a symmetric polynomial, then it is also doubly symmetric. Theorem 1.2 says that

$$\sum_{I\subset[n],\,|I|=k} \frac{P(\lambda_I)}{\prod_{i\in I,\, j\in I^c}(\lambda_i - \lambda_j)} = \frac{d(k,n)}{k!(n-k)!},$$

where $d(k,n)$ is the coefficient of $x_1^{n-1} \ldots x_k^{n-1} y_1^{n-1} \ldots y_{n-k}^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_k) \prod_{j\neq i}(x_i - x_j) \prod_{j\neq i}(y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^{k} (y_i - x_j).$$

It was proved in [Ze82] that $(n-k)!$ is the coefficient of $y_1^{n-k-1} \ldots y_{n-k}^{n-k-1}$ in $\prod_{j\neq i}(y_i - y_j)$. Thus $(n-k)!$ is also the coefficient of $y_1^{n-1} \ldots y_{n-k}^{n-1}$ in

$$\prod_{j\neq i}(y_i - y_j) \prod_{i=1}^{n-k} \prod_{j=1}^{k} (y_i - x_j).$$

This means that

$$\frac{d(k,n)}{(n-k)!} = c(k,n),$$

which is the coefficient of $x_1^{n-1} \ldots x_k^{n-1}$ in

$$P(x_1, \ldots, x_k) \prod_{j\neq i}(x_i - x_j),$$

as stated in Theorem 1.1.

3. Localization in equivariant cohomology. In this section, we recall some basic definitions and results in equivariant cohomology. For more details, we refer to [AB84, BV82, Bo60, Br98, CK99, EG98]. Throughout we consider cohomology with coefficients in the complex field $\mathbb{C}$.

Let $T = (\mathbb{C}^*)^n$ be an algebraic torus of dimension n, classified by the principal T-bundle $ET \to BT$ whose total space ET is contractible. Let X be a compact space endowed with a T-action. Put $X_T = X \times_T ET$, which is itself a bundle over BT with fiber X. Recall that the T-equivariant cohomology of X is defined to be $H_T^*(X) = H^*(X_T)$, where $H^*(X_T)$ is the ordinary cohomology of X_T. Note that $H_T^*(\text{point}) = H^*(BT)$. By pullback via the map $X \to$ point, we see that $H_T^*(X)$ is an $H^*(BT)$-module. Thus we may consider $H^*(BT)$ as the coefficient ring for equivariant cohomology.

A *T-equivariant vector bundle* is a vector bundle E on X together with a lifting of the action on X to an action on E which is linear on fibers. Note that E_T is a vector bundle over X_T.

The T-*equivariant Chern classes* $c_i^T(E) \in H_T^*(X)$ are defined to be the Chern classes $c_i(E_T)$. If E has rank r, then the top Chern class $c_r^T(E)$ is called the T-*equivariant Euler class* of E and is denoted $e^T(E) \in H_T^*(X)$. More generally, the T-*equivariant characteristic class* $c^T(E) \in H_T^*(X)$ is defined to be the characteristic class $c(E_T)$.

Let $\chi(T)$ be the character group of the torus T. For each $\rho \in \chi(T)$, let $\mathbb{C}_\rho$ denote the one-dimensional representation of T determined by ρ. Then $L_\rho = (\mathbb{C}_\rho)_T$ is a line bundle over BT, and the assignment $\rho \mapsto -c_1(L_\rho)$ defines a group isomorphism $f : \chi(T) \simeq H^2(BT)$, which induces a ring isomorphism $\mathrm{Sym}(\chi(T)) \simeq H^*(BT)$. We call $f(\rho)$ the *weight* of ρ. In particular, we denote by λ_i the weight of ρ_i defined by $\rho_i(x_1, \dots, x_n) = x_i$. We thus obtain an isomorphism

$$H_T^*(\text{point}) = H^*(BT) \simeq \mathbb{C}[\lambda_1, \dots, \lambda_n].$$

Let $\mathcal{R}_T \simeq \mathbb{C}(\lambda_1, \dots, \lambda_n)$ be the field of fractions of $\mathbb{C}[\lambda_1, \dots, \lambda_n]$. An important result in equivariant cohomology is the localization theorem. Historically, localization in equivariant cohomology was studied by Borel [Bo60] and then further by Quillen [Q71], Atiyah–Bott [AB84], and Berline–Vergne [BV82]. Among many formulations of the localization theorem, we choose the one by Atiyah and Bott [AB84].

Theorem 3.1 (Atiyah–Bott [AB84]). *Let X^T be the fixed point locus of the torus action. Then the inclusion $i : X^T \hookrightarrow X$ induces an isomorphism*

$$i^* : H_T^*(X) \otimes \mathcal{R}_T \simeq H_T^*(X^T) \otimes \mathcal{R}_T.$$

Moreover, Atiyah and Bott [AB84] gave an explicit formula for the inverse isomorphism. If X is a compact manifold and X^T is finite, then the localization theorem can be rephrased as follows:

Theorem 3.2 (Atiyah–Bott [AB84], Berline–Vergne [BV82]). *Suppose that X is a compact manifold endowed with a torus action and the fixed point locus X^T is finite. For $\alpha \in H_T^*(X)$, we have*

$$\int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{e_p}, \tag{3.1}$$

where e_p is the T-equivariant Euler class of the tangent bundle at the fixed point p, and $\alpha|_p$ is the restriction of α to the point p.

For many applications, the Atiyah–Bott–Berline–Vergne formula can be formulated in more down-to-earth terms. We are mainly interested in the computation of integrals over Grassmannians.

Proof of Corollary 1.3. Consider the action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ given in coordinates by

$$(a_1, \dots, a_n) \cdot (x_1, \dots, x_n) = (a_1 x_1, \dots, a_n x_n).$$

This induces a torus action on the Grassmannian $G(k,n)$ with isolated fixed points p_I corresponding to coordinate k-planes in $\mathbb{C}^n$. Each fixed point p_I is indexed by a subset $I \subset [n]$ of size k. By the Atiyah–Bott–Berline–Vergne formula, we have

$$\int_{G(k,n)} \Phi(\mathcal{S}) = \sum_{p_I} \frac{\Phi^T(\mathcal{S}|_{p_I})}{e_{p_I}}, \qquad \int_{G(k,n)} \Psi(\mathcal{Q}) = \sum_{p_I} \frac{\Psi^T(\mathcal{Q}|_{p_I})}{e_{p_I}}.$$

For each p_I, the torus actions on the fibers $\mathcal{S}|_{p_I}$ and $\mathcal{Q}|_{p_I}$ have the characters ρ_i for $i \in I$ and ρ_j for $j \in I^c$ respectively. Combining this with the assumption implies that the T-equivariant characteristic classes at p_I are

$$\Phi^T(\mathcal{S}|_{p_I}) = P(\lambda_I) \quad \text{and} \quad \Psi^T(\mathcal{Q}|_{p_I}) = Q(\lambda_{I^c}).$$

Since the tangent bundle is isomorphic to $\mathcal{S}^\vee \otimes \mathcal{Q}$, the characters of the torus action on the tangent bundle at p_I are

$$\{\rho_i^{-1}\rho_j \mid i \in I,\, j \in I^c\}.$$

Thus the T-equivariant Euler class of the tangent bundle at p_I is

$$e_{p_I} = \prod_{i \in I,\, j \in I^c} (\lambda_j - \lambda_i) = (-1)^{k(n-k)} \prod_{i \in I,\, j \in I^c} (\lambda_i - \lambda_j).$$

Therefore, we obtain

$$\int_{G(k,n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)} \sum_{I \subset [n],\, |I|=k} \frac{P(\lambda_I)}{\prod_{i \in I,\, j \in I^c} (\lambda_i - \lambda_j)}$$

and

$$\int_{G(k,n)} \Psi(\mathcal{Q}) = \sum_{I \subset [n],\, |I|=k} \frac{Q(\lambda_{I^c})}{\prod_{i \in I,\, j \in I^c} (\lambda_j - \lambda_i)} = \sum_{I \subset [n],\, |I|=n-k} \frac{Q(\lambda_I)}{\prod_{i \in I,\, j \in I^c} (\lambda_i - \lambda_j)}.$$

Combining this with Theorem 1.1 yields Corollary 1.3. ∎

The proof of Corollary 1.4 is very similar and is omitted.

4. Martin's formula. In this section, we present the Martin formula for symplectic quotients, which expresses integrals on a symplectic quotient in terms of those on its associated symplectic quotient. In the case of the classical Grassmannian, we show that statement (a) of Corollary 1.3 can be deduced from the Martin formula. For more details of the construction, we refer to [M00, Section 7].

Let X be a compact manifold endowed with a G-action, where G is a reductive algebraic group with a maximal torus $T \subset G$. Assume that the moment maps μ_G and μ_T for the G- and T-actions on X exist. The symplectic quotients $X/\!\!/G$ and $X/\!\!/T$ are defined to be the topological quotients $\mu_G^{-1}(0)/G$ and $\mu_T^{-1}(0)/T$ respectively. Moreover, both $\mu_G^{-1}(0)$ and $\mu_T^{-1}(0)$

are assumed to be compact manifolds, on which the respective G- and T-actions are free. It follows that $X/\!\!/G$ and $X/\!\!/T$ are compact manifolds. There is a natural inclusion $i : \mu_G^{-1}(0)/T \hookrightarrow \mu_T^{-1}(0)/T$ and a projection $\pi : \mu_G^{-1}(0)/T \twoheadrightarrow \mu_G^{-1}(0)/G$. We say $\tilde{a} \in H^*(X/\!\!/T)$ is a *lift* of $a \in H^*(X/\!\!/G)$ if $\pi^* a = i^* \tilde{a}$. The set of roots is denoted by $\triangle$. We denote by L_α the line bundle on $X/\!\!/T$ associated with $\alpha \in \triangle$, and set

$$e = \prod_{\alpha \in \triangle} c_1(L_\alpha).$$

Given a cohomology class $a \in H^*(X/\!\!/G)$ with lift $\tilde{a} \in H^*(X/\!\!/T)$, it was proved by Martin in [M00, Theorem B] that

$$\int_{X/\!\!/G} a = \frac{1}{|W|} \int_{X/\!\!/T} \tilde{a} \cup e, \tag{4.1}$$

where $|W|$ is the order of the Weyl group of G.

The Grassmannian $G(k,n)$ can be described as the symplectic quotient of the set of complex $n \times k$ matrices by the unitary group,

$$G(k,n) = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)/\!\!/U(k).$$

The associated symplectic quotient by the maximal torus $T \subset U(k)$ turns out to be the k-fold product $(\mathbb{CP}^{n-1})^k$. Its cohomology ring is generated by elements $u_1, \ldots, u_k$, where u_i is the positive generator of the cohomology ring of the ith copy of $\mathbb{CP}^{n-1}$.

The Weyl group of $U(k)$ is the symmetric group S_k on k elements. The roots α of $U(k)$ can be enumerated by pairs of positive integers (i,j) with $1 \le i,j \le k$ and $i \ne j$. The cohomology class corresponding to the root (i,j) is the class $u_j - u_i$ and their product is

$$e = \prod_{i \ne j} (u_j - u_i).$$

In the last paragraph of [M00, Section 7], Martin described the tautological subbundle $\mathcal{S}$ on the Grassmannnian $G(k,n)$ in terms of the symplectic quotient construction as

$$\mathcal{S} \cong \mu_{U(k)}^{-1}(0) \times_{U(k)} \mathbb{C}^k_{\mathrm{def.}},$$

where $\mathbb{C}^k_{\mathrm{def.}}$ denotes the defining representation of $U(k)$. Thus the dual $\mathcal{S}^\vee$ is constructed from the dual of the defining representation, and when we restrict this dual representation to the maximal torus, it decomposes into k one-dimensional representations, which have associated line bundles on $(\mathbb{CP}^{n-1})^k$ with Euler classes $u_1, \ldots, u_k$. This implies that we can identify the Chern classes of $\mathcal{S}^\vee$ as elementary symmetric polynomials of the u_i. Under the notation and assumptions of Corollary 1.3, the lift of $\Phi(\mathcal{S})$ is

$(-1)^{k(n-k)}P(u_1,\dots,u_k)$ and the Martin formula (4.1) gives

$$\int_{G(k,n)} \Phi(\mathcal{S}) = \frac{1}{k!}\int_{(\mathbb{CP}^{n-1})^k} (-1)^{k(n-k)}P(u_1,\dots,u_k)\prod_{i\neq j}(u_j-u_i).$$

It is known that the cohomology ring of $(\mathbb{CP}^{n-1})^k$ is

$$H^*((\mathbb{CP}^{n-1})^k) \cong \mathbb{C}[u_1,\dots,u_k]/(u_1^n,\dots,u_k^n).$$

This implies that

$$\int_{(\mathbb{CP}^{n-1})^k} P(u_1,\dots,u_k)\prod_{i\neq j}(u_j-u_i) = c(k,n).$$

We thus have

$$\int_{G(k,n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)}\frac{c(k,n)}{k!}.$$

This means that statement (a) of Corollary 1.3 can be derived from the Martin formula together with the description of the cohomology ring of the product of projective spaces and of the push-forward for projective spaces. Equivalently, we can start with statement (a) of Corollary 1.3 and using the description of the cohomology and push-forward on the product of projective spaces arrive at the Martin formula for the classical Grassmannian. The only part of Martin's proof which does not follow from Corollary 1.3 is the description of the tautological subbundle on the Grassmannian, but this is a classical result. Thus we have provided another proof of the Martin formula for the classical Grassmannian.

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