

## Partial strong compactness and squares

by

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**Abstract.** We analyze the connection between some properties of partially strongly compact cardinals: the completion of filters of certain size and instances of the compactness of  $\mathcal{L}_{\kappa,\kappa}$ . Using this equivalence we show that if any  $\kappa$ -complete filter on  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter and  $\lambda^{<\kappa} = \lambda$  then  $\square(\mu)$  fails for all regular  $\mu \in [\kappa, 2^\lambda]$ . As an application, we improve the lower bound for the consistency strength of  $\kappa$ -compactness, a case which was explicitly considered by Mitchell.

**1. Introduction.** Strongly compact cardinals, one of the most intriguing large cardinal notions, were defined by Tarski (for a complete historical overview see [8, Chapter 4]). Even though they are very natural and well studied, some of their basic properties are still quite mysterious.

Strongly compact cardinals are characterized by many different global principles. Taking local versions of those principles one obtains different, and sometimes non-equivalent, large cardinal axioms. We will use the following definitions as our versions for local strong compactness:

DEFINITION 1.1. Let  $\kappa$  be a regular cardinal and let  $\lambda$  be a cardinal.

- $\kappa$  has the  $\lambda$ -filter extension property if any  $\kappa$ -complete filter  $\mathcal{F}$  on  $\lambda$  can be extended to a  $\kappa$ -complete ultrafilter.
- $\mathcal{L}_{\kappa,\kappa}$ -compactness for languages of size  $\lambda$  holds if for every language  $\mathcal{L}$  with  $\lambda$  many non-logical symbols and every collection  $\Phi$  of  $\mathcal{L}_{\kappa,\kappa}$ -sentences in the language  $\mathcal{L}$ , if every subcollection  $\Phi' \subseteq \Phi$  of size  $< \kappa$  has a model then  $\Phi$  has a model.

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- A cardinal  $\kappa$  is  $\lambda$ -strongly compact if there is a fine  $\kappa$ -complete ultrafilter on  $P_\kappa\lambda$  <sup>(1)</sup>.

We will say that a collection of formulas  $\Phi$  is  $\kappa$ -consistent if any subset of it of size  $< \kappa$  has a model. Thus  $\mathcal{L}_{\kappa,\kappa}$ -compactness is the statement that  $\kappa$ -consistency implies consistency.

In this paper we would like to give a level-by-level analysis of those properties. Let  $\lambda = \lambda^{<\kappa}$  be a cardinal. Then:

$2^\lambda$ -strong compactness  $\implies$   $\lambda$ -filter extension

$\iff \mathcal{L}_{\kappa,\kappa}$ -compactness for languages of size  $2^\lambda \implies \lambda$ -strong compactness.

The first implication can be found in [8, Proposition 4.1] and the last one is [8, Theorem 22.17]. The equivalence is Theorem 2.2 below, which is the main technical result of this paper.

Following Gitik [5], we say that a cardinal  $\kappa$  is  $\kappa$ -compact if it has the  $\kappa$ -filter extension property. This case was explicitly considered by Mitchell [10], since its characterization uses only measures on  $\kappa$ . Mitchell asked about the possibility of existence of a cardinal  $\kappa$  which is  $\kappa$ -compact in a model of the form  $L[\mathcal{U}]$ .

Using Theorem 2.2 and Lemma 2.4, we deduce the following:

**THEOREM.** *If  $\kappa$  has the  $\kappa$ -filter extension property then  $\square(\kappa)$  and  $\square(\kappa^+)$  fail. In particular, if  $\kappa$  is  $\kappa$ -compact then there is an inner model with class many Woodin cardinals and class many strong cardinals.*

We remark that the  $\kappa$ -compactness of  $\kappa$  has a similar effect on properties of cardinals up to  $2^\kappa$  to that of  $2^\kappa$ -strong compactness (see Corollary 2.6). Some of those implications cannot be obtained from weaker reflection principles such as stationary reflection. Instances of Rado's conjecture, which follow from  $\kappa$ -compactness of  $\kappa$  as well, can be used to provide some of the combinatorial reflection properties of  $\kappa^+$ . Yet, some of the consequences of  $\kappa$ -compactness are unknown to follow from some instances of Rado's conjecture, for example instances of compactness of the chromatic number of graphs or simultaneous reflection of stationary sets.

The results of this paper suggest that  $\kappa$ -compactness is essentially a property of the cardinal  $2^\kappa$  rather than of  $\kappa$  itself. Indeed, large cardinal properties which are formulated in terms of ultrafilters on  $\kappa$  tend to be weaker and do not entail any compactness properties at the level of  $\kappa^+$ . For example, measurability and even superstrength or 1-extendibility of a cardinal  $\kappa$  are

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<sup>(1)</sup> In [8, Chapter 22], the term  $\lambda$ -compactness is used to denote what we call  $\lambda$ -strong compactness. We prefer the more cumbersome name in order to avoid inconsistency with the term  $\kappa$ -compact, which refers to a cardinal  $\kappa$  that has the  $\kappa$ -filter extension property. Our terminology also differs from the terminology of Bagaria and Magidor [2], where it refers to the degree of completeness of the extended ultrafilters.

compatible with  $\square_\kappa$  (see, for example, [3]). Moreover, in [5], Gitik shows that if one restricts the  $\kappa$ -filter extension property to some classes of natural filters, then the consistency strength drops significantly. The least large cardinal axiom which implies the failure of squares at its successor seems to be *subcompactness* (see [12]), and the results of this paper suggest that  $\kappa$ -compactness is related to a strong version of subcompactness.

The paper is organized as follows. In Section 2 we prove the equivalence between levels of the filter extension property and levels of  $\mathcal{L}_{\kappa,\kappa}$ -compactness. We remark that bounded instances of  $\mathcal{L}_{\kappa,\kappa}$ -compactness are equivalent to the existence of certain elementary embeddings. From this we conclude that many local reflection phenomena which are derived from the existence of strongly compact cardinals can be obtained from the  $\kappa$ -filter extension property for  $\kappa$  and improve the known lower bounds for its consistency strength (see [10] and [5]).

In Section 3, we provide some upper bounds for the consistency strength of the  $\lambda$ -filter extension property.

**2. Equivalence of compactness and filter extension property.** In this section we will demonstrate the equivalence between the  $\lambda$ -filter extension property and  $\mathcal{L}_{\kappa,\kappa}$ -compactness for languages of size  $2^\lambda$ , for  $\lambda = \lambda^{<\kappa}$ .

The following theorem was proved for  $\kappa = \lambda = \omega$  by Fichtenholz and Kantorovich and for  $\kappa = \omega \leq \lambda$  by Hausdorff. Hausdorff's proof generalizes to the case of arbitrary  $\kappa$ , assuming  $\lambda = \lambda^{<\kappa}$ . We include a proof for completeness.

LEMMA 2.1 (Fichtenholz–Kantorovich, Hausdorff). *Let  $\kappa \leq \lambda$  be infinite cardinals with  $\lambda = \lambda^{<\kappa}$ . There is a family  $\mathcal{I} \subseteq \mathcal{P}(\lambda)$  with  $|\mathcal{I}| = 2^\lambda$  such that for every pair of disjoint collections  $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$  with  $|\mathcal{A}|, |\mathcal{B}| < \kappa$  and  $\mathcal{A} \neq \emptyset$ ,*

$$\left| \bigcap_{X \in \mathcal{A}} X \setminus \bigcup_{Y \in \mathcal{B}} Y \right| = \lambda.$$

*Proof.* Clearly, it is sufficient to find an independent set in  $\mathcal{P}(J)$  for some  $|J| = \lambda$ . Let

$$J = \{\langle X, Z \rangle \mid X \in P_\kappa \lambda, Z \subseteq \mathcal{P}(X)\}.$$

Since  $\lambda^{<\kappa} = \lambda$ , we can compute  $\lambda \leq |J| \leq \lambda^{<\kappa} \cdot 2^{<\kappa} = \lambda$ .

For a set  $A \subseteq \lambda$ , let  $I(A) = \{\langle X, Z \rangle \mid A \cap X \in Z\}$ . Let us show that  $\mathcal{I} = \{I(A) \mid A \subseteq \lambda\}$  is as required.

If  $A \neq B$  are subsets of  $\lambda$  then there is some ordinal  $\gamma$  in  $A \triangle B$ . So  $\{\gamma\}, \{\{\gamma\}\} \in I(A) \triangle I(B)$  and in particular  $I(A) \neq I(B)$ . Thus  $|\mathcal{I}| = 2^\lambda$ .

Let  $\mathcal{A} = \{I(A_\alpha) \mid \alpha < \rho\} \subseteq \mathcal{I}$  and  $\mathcal{B} = \{I(B_\beta) \mid \beta < \zeta\} \subseteq \mathcal{I}$  with  $\rho, \zeta < \kappa$  and  $A_\alpha \neq B_\beta$  for all  $\alpha, \beta$ .

We need to show that for  $D = \bigcap \mathcal{A} \setminus \bigcup \mathcal{B}$  we have  $|D| = \lambda$  (if  $\mathcal{A} = \emptyset$ , we define  $\bigcap \mathcal{A} = J$ ). Indeed, if  $X \in P_\kappa \lambda$  is so large that

$$\{A_\alpha \cap X \mid \alpha < \rho\} \cap \{B_\beta \cap X \mid \beta < \zeta\} = \emptyset$$

then  $\langle X, \{A_\alpha \cap X \mid \alpha < \rho\} \rangle \in D$ . Since there are  $\lambda$  many possibilities for  $X$ , we conclude that  $|D| = \lambda$ . ■

**THEOREM 2.2.** *Let  $\kappa, \lambda$  be cardinals and assume  $\lambda^{<\kappa} = \lambda$ . Then  $\kappa$  has the  $\lambda$ -filter extension property if and only if  $\mathcal{L}_{\kappa, \kappa}$ -compactness holds for languages of size  $2^\lambda$ .*

*Proof.* Using Henkin's classical construction we can reduce the problem of compactness of  $\mathcal{L}_{\kappa, \kappa}$  to compactness of the propositional logic  $\mathcal{L}_{\kappa, 1}$  (without quantifiers, but with conjunctions and disjunctions of size  $< \kappa$ ):

Let  $\mathcal{L}$  be a language of size  $2^\lambda$ . For each  $\mathcal{L}_{\kappa, \kappa}$ -formula in  $\mathcal{L}$  of the form  $\exists_{\alpha < \rho} x_\alpha \psi(\langle x_\alpha \mid \alpha < \rho \rangle)$  we add to  $\mathcal{L}$  a sequence  $\langle c_\alpha^\psi \mid \alpha < \rho \rangle$  of  $\rho$  many constants. We would like to interpret those constants so that

$$\exists_{\alpha < \rho} x_\alpha \psi(\langle x_\alpha \mid \alpha < \rho \rangle) \iff \psi(\langle c_\alpha^\psi \mid \alpha < \rho \rangle).$$

By repeating the process  $\kappa$  many times, we may assume that any formula that starts with an existential quantifier has corresponding constants in  $\mathcal{L}$ . Those constants witness the validity of the formula if it is true, and have arbitrary values if it is false.

Let us introduce an atomic propositional formula  $[[\varphi]]$  for every  $\mathcal{L}_{\kappa, \kappa}$ -formula  $\varphi$  over  $\mathcal{L}$ . Let  $\Psi$  be the collection of all formulas of the form:

- (1)  $[[\varphi]]$  is true if  $\varphi$  is a tautology.
- (2)  $[[\varphi]] = \neg[[\neg\varphi]]$ .
- (3) For every  $\rho < \kappa$  and every  $\rho$ -sequence  $\langle \varphi_\alpha \mid \alpha < \rho \rangle$  of formulas,

$$\left[ \left[ \bigwedge_{\alpha < \rho} \varphi_\alpha \right] \right] = \bigwedge_{\alpha < \rho} [[\varphi_\alpha]].$$

- (4) For every  $\rho < \kappa$  and every formula  $\varphi$ ,

$$[[\exists_{\alpha < \rho} x_\alpha \varphi(\langle x_\alpha \mid \alpha < \rho \rangle)]] = [[\varphi(\langle c_\alpha^\varphi \mid \alpha < \rho \rangle)]]$$

and for every sequence  $\langle t_\alpha \mid \alpha < \rho \rangle$  of terms,

$$[[\varphi(\langle t_\alpha \mid \alpha < \rho \rangle)]] \rightarrow [[\varphi(\langle c_\alpha^\varphi \mid \alpha < \rho \rangle)]].$$

For a given evaluation of all the variables  $\{[[\varphi]] \mid \varphi \text{ is an } \mathcal{L}_{\kappa, \kappa}\text{-formula}\}$ , we define an equivalence relation between terms by  $t \sim t'$  if and only if the truth value of  $[[t = t']]$  is true. Let us denote by  $[t]_\sim$  the equivalence class of the term  $t$ . For every relation  $R$  of arity  $n$  in  $\mathcal{L}$ , we define  $R([t_0]_\sim, \dots, [t_{n-1}]_\sim)$  if  $[[R(t_0, \dots, t_{n-1})]]$  is true. For every function  $F$  of arity  $n$  in  $\mathcal{L}$  we define  $F([t_0]_\sim, \dots, [t_{n-1}]_\sim) = [s]_\sim$  if  $[[F(t_0, \dots, t_{n-1}) = s]]$  is true.

For any  $\kappa$ -consistent theory  $T$  over the language  $\mathcal{L}$  with the logic  $\mathcal{L}_{\kappa,\kappa}$ , one can translate the problem of the consistency of  $T$  into the problem of constructing an assignment which is consistent with the collection  $\Phi$  of propositional variables which consists of  $\{[\varphi] \mid \varphi \in T\}$  and  $\Psi$ . Clearly,  $\Phi$  is still  $\kappa$ -consistent and any consistent assignment for it provides a model for  $T$ .

From now on, let us focus on propositional theories. Let us fix a  $\kappa$ -independent family  $\mathcal{I} = \{A_\delta \mid \delta < 2^\lambda\}$  in  $\mathcal{P}(\lambda)$ . Such a family exists by Lemma 2.1. Let  $\mathcal{B} \subseteq \mathcal{P}(\lambda)$  be the  $\kappa$ -complete Boolean algebra generated by  $\mathcal{I}$ . The independence of  $\mathcal{I}$  is equivalent to the fact that  $\mathcal{B}$  is isomorphic to the  $\kappa$ -complete free Boolean algebra with  $2^\lambda$  generators.

Let us define an embedding  $\iota$  from the Lindenbaum–Tarski algebra of formulas in  $\mathcal{L}_{\kappa,1}$  with atoms  $\langle a_\gamma \mid \gamma < 2^\lambda \rangle$  into  $\mathcal{B}$  by setting  $\iota(a_\gamma) = A_\gamma$  (and inductively,  $\iota(\neg\varphi) = \lambda \setminus \iota(\varphi)$  and  $\iota(\bigwedge_{i < \eta} \varphi_i) = \bigcap_{i < \eta} \iota(\varphi_i)$ ).

For a function  $s: \Gamma \rightarrow 2$ , where  $\Gamma \subseteq 2^\lambda$  with  $|\Gamma| < \kappa$ , let us define

$$A(s) = \left( \bigcap_{\alpha \in \Gamma, s(\alpha)=1} A_\alpha \right) \setminus \left( \bigcup_{\beta \in \Gamma, s(\beta)=0} A_\beta \right)$$

with the convention that  $\bigcap_{A \in \emptyset} A = \lambda$  (similarly to the convention in the proof of Lemma 2.1). By the independence of  $\mathcal{I}$ , we have  $A(s) \neq \emptyset$  for all  $s \in {}^\Gamma 2$  and  $\Gamma \in P_\kappa 2^\lambda$ . Let us define the value of  $s(\varphi)$  to be the truth value of  $\varphi$  after assigning to each variable  $a_\gamma$  in  $\Gamma$  the truth value  $s(\gamma)$ . This value is well defined only when  $\Gamma$  contains all the indices of the variables that appear in  $\varphi$ .

For a formula  $\varphi$ , let  $\Gamma_\varphi$  denote the set of indices of variables that appear in  $\varphi$ .

CLAIM 2.3. *For every  $\Gamma \supseteq \Gamma_\varphi$  with  $|\Gamma| < \kappa$ ,*

$$\iota(\varphi) = \bigcup \{A(s) \mid s \in {}^\Gamma 2, s(\varphi) = 1\}.$$

*Proof.* Let us show first that for every  $\Gamma \subseteq 2^\lambda$  with  $|\Gamma| < \kappa$ , the collection

$$\{A(s) \mid s \in {}^\Gamma 2\}$$

is a partition of  $\lambda$ . Indeed, if  $s, s'$  are different assignments then there is  $\gamma$  such that  $s(\gamma) \neq s'(\gamma)$ . In particular,  $A(s) \subseteq A_\gamma$  and  $A(s') \subseteq \lambda \setminus A_\gamma$ , and so they are disjoint. In order to show that the union of this collection is  $\lambda$ , let us pick  $\delta \in \lambda$ . Let  $s_\delta: \Gamma \rightarrow 2$  be defined by  $s_\delta(\gamma) = 1$  if  $\delta \in A_\gamma$ , and  $s_\delta(\gamma) = 0$  otherwise. Clearly,  $\delta \in A(s_\delta)$ .

Thus, for a given formula  $\varphi$  and for any  $\Gamma \supseteq \Gamma_\varphi$  with  $|\Gamma| < \kappa$ ,

$$\bigcup \{A(s) \mid s \in {}^\Gamma 2, s(\varphi) = 1\} = \bigcup \{A(s) \mid s \in {}^{\Gamma_\varphi} 2, s(\varphi) = 1\}.$$

We argue by induction on the complexity of formulas. For atomic formulas the claim is true by the definition of  $\iota(a_\gamma)$ . For  $\varphi = \neg\psi$ , and every assignment with domain  $\Gamma_\psi = \Gamma_\varphi$  and  $s(\varphi) = 1 - s(\psi)$ ,

$$\begin{aligned} \iota(\varphi) &= \lambda \setminus \iota(\psi) = \lambda \setminus \left( \bigcup_{s \in \Gamma_\psi 2, s(\psi)=1} A(s) \right) \\ &= \bigcup_{s \in \Gamma_\psi 2, s(\psi)=0} A(s) = \bigcup_{s \in \Gamma_\varphi 2, s(\varphi)=1} A(s) \end{aligned}$$

where the third equality is based on the observation above that the set of  $A(s)$ , where  $s$  ranges over all assignments for  $\Gamma$ , is a partition of  $\lambda$ .

For  $\varphi = \bigwedge_{\alpha < \rho} \psi_\alpha$ , let  $\Gamma_\alpha$  be  $\Gamma_{\psi_\alpha}$ . Let  $\Gamma = \Gamma_\varphi = \bigcup_{\alpha < \rho} \Gamma_\alpha$ . Clearly,  $|\Gamma| < \kappa$ . For all  $\alpha < \rho$ ,

$$\begin{aligned} \iota(\psi_\alpha) &= \bigcup \{A(s) \mid s \in \Gamma_\alpha 2, s(\psi_\alpha) = 0\} \\ &= \bigcup \{A(s) \mid s \in \Gamma 2, s(\psi_\alpha) = 0\} \end{aligned}$$

and thus (since the sets  $\{A(s) \mid s \in \Gamma 2\}$  are pairwise disjoint)

$$\iota(\varphi) = \bigcap_{\alpha < \rho} \iota(\psi_\alpha) = \bigcup_{s \in \Gamma 2, \forall \alpha < \rho, s(\psi_\alpha)=0} A(s) = \bigcup_{s \in \Gamma 2, s(\varphi)=0} A(s). \blacksquare$$

In particular, if  $\varphi$  is consistent (i.e. there is an assignment  $s$  such that  $s(\varphi) = 1$ ) then  $\iota(\varphi) \neq \emptyset$ , and if  $\varphi$  is inconsistent then  $\iota(\varphi) = \emptyset$ . This implies that if  $\varphi, \psi$  are formulas such that  $\varphi \leftrightarrow \psi$  is a tautology then  $\iota(\varphi) = \iota(\psi)$ , and thus  $\iota$  is well defined on the Lindenbaum–Tarski Boolean algebra.

Let  $\Phi$  be a collection of  $\mathcal{L}_{\kappa,1}$  propositional formulas, with variables  $\{a_\delta \mid \delta < 2^\lambda\}$ , such that any subcollection of  $< \kappa$  formulas has a consistent assignment.

Let  $\mathcal{F}$  be the  $\kappa$ -complete filter generated by  $\iota(\varphi)$ ,  $\varphi \in \Phi$ . Then  $\mathcal{F}$  is a proper filter, since any collection of  $< \kappa$  formulas from  $\Phi$  is consistent. Since  $\iota$  respects Boolean operations of length  $< \kappa$ , if  $\{\varphi_\alpha \mid \alpha < \rho\} \subseteq \Phi$  and  $\rho < \kappa$  then

$$\bigcap_{\alpha < \rho} \iota(\varphi_\alpha) = \iota\left(\bigwedge_{\alpha < \rho} \varphi_\alpha\right),$$

and since the formula  $\bigwedge_{\alpha < \rho} \varphi_\alpha$  is consistent, the intersection is non-empty.

Let  $\mathcal{U} \supseteq \mathcal{F}$  be a  $\kappa$ -complete ultrafilter. Then  $\mathcal{U}$  defines an assignment on the variables  $\{a_\delta \mid \delta < 2^\lambda\}$ : we set  $a_\delta$  to be true if and only if  $A_\delta \in \mathcal{U}$ . Let  $S \in {}^{2^\lambda}2$  be this assignment. Since  $\mathcal{U}$  is  $\kappa$ -complete, by induction on the complexity of formulas we can see that for every formula  $\varphi$ ,  $(S \upharpoonright \Gamma_\varphi)(\varphi)$  is 1 if and only if  $\iota(\varphi) \in \mathcal{U}$ .

Let us prove the other direction. Assume that  $\mathcal{L}_{\kappa,\kappa}$ -compactness holds for languages of size  $2^\lambda$  and  $\mathcal{F}$  is a  $\kappa$ -complete ultrafilter. Define a constant  $a_X$

for every  $X \in \mathcal{P}(\lambda)$ , and let  $U$  be a unary predicate. Let  $\Phi$  be the following collection of formulas in the language  $\mathcal{L}$  over the logic  $\mathcal{L}_{\kappa,\omega}$ :

- (1)  $U(a_X)$  for all  $X \in \mathcal{F}$ .
- (2)  $(\bigwedge_{i < \eta} U(a_{X_i})) \rightarrow U(a_Y)$  for all sequences  $\langle X_i \mid i < \eta < \kappa \rangle$  of subsets of  $\lambda$ , where  $Y \supseteq \bigcap X_i$ .
- (3)  $U(a_X) \leftrightarrow \neg U(a_{\lambda \setminus X})$ .

Clearly, any model of  $\Phi$  will define a  $\kappa$ -complete ultrafilter that extends  $\mathcal{F}$ . ■

The following lemma is a generalization of the characterization of weakly compact cardinals using elementary embeddings, as in Hauser [6]. The proof is a direct modification of the proof for weakly compact cardinals (see for example [8, Theorem 4.5]).

LEMMA 2.4. *Let  $\kappa \leq \lambda$  be uncountable cardinals and assume that  $\mathcal{L}_{\kappa,\kappa}$ -compactness holds for languages of size  $\lambda^{<\kappa}$ . Then, for every transitive model  $M$  with  $|M| = \lambda^{<\kappa}$ ,  ${}^{<\kappa}M \subseteq M$ , and  $\lambda \subseteq M$ , there is a transitive model  $N$  and an elementary embedding  $j: M \rightarrow N$  such that:*

- (1)  $\text{crit } j = \kappa$ , and in particular  $j$  is  $<\kappa$ -continuous.
- (2) There is  $s \in N$  with  $j''\lambda \subseteq s$  and  $|s|^N < j(\kappa)$ .

*Proof.* Let  $\mathcal{L}$  be a language with a constant symbol  $c_x$  for all  $x \in M$ , two additional constants  $d, s$  and a binary relation  $E$ .

Let us consider the set of formulas that consists of all  $\mathcal{L}_{\kappa,\kappa}$ -elementary diagrams of  $M$  (using the constants  $c_x$  and membership as  $E$ ). Let us add the following formulas:

- (1)  $c_\alpha E d$  for all  $\alpha < \kappa$ .
- (2)  $d E c_\kappa$ .
- (3)  $c_\alpha E s$  for all  $\alpha < \lambda$ .
- (4)  $|s| < c_\kappa$  (with the standard set-theoretical meaning).

Note that any collection of  $< \kappa$  formulas has a model. Namely, take  $s$  to be a set of cardinality  $< \kappa$  that contains all ordinals  $\alpha$  such that the formula “ $c_\alpha \in s$ ” appears in the collection, and take  $d$  to be an arbitrary ordinal below  $\kappa$  which is larger than all ordinals smaller than  $\kappa$  that were mentioned in the collection.

Thus,  $\Phi$  is  $\kappa$ -consistent and therefore it has a model. The membership relation of this model,  $E$ , is well founded since  $M$  is well founded. Let  $N$  be the transitive collapse of the model obtained. Each element  $x$  of  $M$  has a corresponding constant in the language,  $c_x$ . Let  $j(x)$  be  $c_x^N$ , the member of  $N$  which is evaluated as  $c_x$ . The embedding  $j$  is elementary, since the elementary diagram of  $M$  was included in  $\Phi$ .

The critical point of  $j$  is at least  $\kappa$ , since for all  $\alpha < \kappa$ , the assertion “ $x E c_\alpha \Rightarrow \bigvee_{\beta < \alpha} x = c_\beta$ ” is an  $\mathcal{L}_{\kappa,\kappa}$ -sentence that appears in the elementary

diagram of  $M$ . Thus, in  $N$ , there is no new ordinal below  $\kappa$ . But clearly,  $d$  is evaluated as a new ordinal below  $j(\kappa)$ , and thus  $\kappa = \text{crit } j$ .

The existence of  $s$  is clear by construction. The continuity of  $j$  follows from the closure of  $M$  under sequences of length  $< \kappa$ . ■

**COROLLARY 2.5.** *Let  $\kappa \leq \lambda$  be uncountable cardinals with  $\lambda^{<\kappa} = \lambda$ . Then  $\kappa$  has the  $\lambda$ -filter extension property if and only if for every transitive model  $M$  with  $2^\lambda \subseteq M$  and  $|M| = 2^\lambda$ , there is a transitive model  $N$  and an elementary embedding  $j: M \rightarrow N$  such that  $\text{crit } j = \kappa$  and there is  $s \in N$  with  $|s|^N < j(\kappa)$  and  $j'' 2^\lambda \subseteq s$ .*

*Proof.* The direct implication follows from Theorem 2.2 and Lemma 2.4. The converse is obtained in exactly the same way as in the standard deduction of the filter extension property from strong compactness:

Let  $\mathcal{F}$  be a  $\kappa$ -complete filter and let  $M$  be a sufficiently nice transitive model such that  $\mathcal{F} \in M$ , and  $2^\lambda, \mathcal{P}(\lambda) \subseteq M$ . In  $N$ ,  $j'' \mathcal{F}$  is covered by a set  $S$  of size  $< j(\kappa)$ . Without loss of generality,  $S \subseteq j(\mathcal{F})$ . Therefore,  $\bigcap S \neq \emptyset$ . Let us pick any  $t \in \bigcap S$  and define  $\mathcal{U} = \{X \subseteq \lambda \mid t \in j(X)\}$ . Then  $\mathcal{U}$  is clearly a  $\kappa$ -complete ultrafilter that extends  $\mathcal{F}$ . ■

Using the elementary embedding from Lemma 2.4, one can conclude that the  $\lambda$ -filter extension property has many of the standard consequences of the existence of a partial strongly compact cardinal (see [13]).

**COROLLARY 2.6.** *Let  $\kappa \leq \lambda$  and assume that the  $\lambda$ -filter extension property holds and  $\lambda^{<\kappa} = \lambda$ . Let  $\mu \in [\kappa, 2^\lambda]$  be a regular cardinal.*

- (1) *Every collection of  $< \kappa$  stationary subsets of  $S_{<\kappa}^\mu$  has a common reflection point.*
- (2)  $\square(\mu, < \kappa)$  *fails.*
- (3) *If  $G$  is a graph of size  $\mu$ , and  $\rho < \kappa$  is a cardinal such that every subgraph of  $G$  of size  $< \kappa$  has chromatic number at most  $\rho$ , then  $G$  has chromatic number at most  $\rho$ .*

Let us focus on the case of  $\lambda = \kappa$  (the case of  $\kappa$ -compactness). In [10], Mitchell asked about the consistency strength of the existence of an uncountable cardinal  $\kappa$  that has the  $\kappa$ -filter extension property. Mitchell conjectured that the consistency strength of this property is in the realm of  $o(\kappa) = \kappa^{++}$ . In [5], Gitik coined the term  $\kappa$ -compact for this property and investigated many aspects of it. In particular, he showed that, counter-intuitively,  $\kappa$ -compactness is much stronger than  $o(\kappa) = \kappa^{++}$ . Indeed, he showed that if  $\kappa$  is  $\kappa$ -compact then there is an inner model with a Woodin cardinal. In Corollary 2.7 below we will improve Gitik's lower bound.

Note that any of the assertions in Corollary 2.6 implies the failure of  $\square(\mu)$  for any regular cardinal  $\mu$  in the interval  $[\kappa, 2^\kappa]$ , and in particular the failure



of  $\square(\kappa^+)$ , if  $\kappa$  is  $\kappa$ -compact. This will be used in Corollary 2.7 to derive significant consistency strength from the assumption of  $\kappa$ -compactness:

**COROLLARY 2.7.** *If  $\kappa$  is  $\kappa$ -compact then there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

*Proof.* By Corollary 2.6, if  $\kappa$  is  $\kappa$ -compact, then  $\square(\kappa)$  and  $\square(\kappa^+)$  both fail. By [7], the failure of  $\square(\kappa)$  together with  $\square_\kappa$  for a countably closed cardinal  $\kappa \geq \aleph_3$  implies the existence of an inner model with a proper class of Woodin cardinals and a proper class of strong cardinals. ■

One can use the characterization in Corollary 2.5 to obtain consistency results from the assumption of  $\kappa$ -compactness which are usually obtained by using  $\kappa^+$ -strongly compact cardinals.

For example, let us consider a bounded variant of Rado's conjecture. Let us say that a tree  $T$  is  $\mu$ -special if it is a union of  $\leq \mu$  antichains. Clearly, if a tree is  $\mu$ -special then its height is at most  $\mu^+$ , and any tree with height  $< \mu^+$  is  $\mu$ -special. Thus, in this context we are interested only in trees of height  $\mu^+$ .

Let us use Fuchino's notations from [4]:  $RC(\mu, <\kappa, \lambda)$  is the assertion that a tree  $T$  (of height  $\mu^+$ ) with  $|T| = \lambda$  is  $\mu$ -special if and only if every subtree  $T' \subseteq T$  of size  $< \kappa$  is  $\mu$ -special.

By Corollary 2.6, if  $\kappa$  is  $\kappa$ -compact then  $RC(\rho, <\kappa, \kappa^+)$  holds for all  $\rho < \kappa$ . By applying exactly the same proof as in [14], but replacing the supercompact (or strongly compact) embedding with the elementary embeddings from Lemma 2.4, one can force  $\kappa$  to be  $\omega_2$  and obtain the consistency of the local version of Rado's conjecture from  $\kappa$ -compactness. Note that the elementary embedding which is used in the proof depends on the non-special tree which we prove to have a non-special small subtree.

**REMARK 2.8.** Let  $\kappa$  be  $\kappa$ -compact. The Lévy collapse  $\text{Col}(\omega_1, <\kappa)$  forces that  $RC(\aleph_0, <\aleph_2, \aleph_3)$  holds.

By the arguments of [15, Theorem 10],  $RC(\mu, <\kappa, \kappa^+)$  already implies the failure of  $\square(\kappa^+)$ . It is unclear how weaker square principles are affected by Rado's conjecture. In [16], Torres-Pérez and Wu show that Rado's conjecture implies the failure of  $\square(\lambda, \omega)$  for all regular  $\lambda > \omega_1$ . Nevertheless, it is unclear whether the conjunction  $\bigwedge_{\mu < \kappa} RC(\mu, <\kappa, \kappa^+)$  implies the failure of  $\square(\kappa^+, \omega_1)$ .

In [17], Zhang investigates a weaker variant of Rado's conjecture,  $RC^b$ . This principle is still sufficiently strong to imply stationary reflection for subsets of cofinality  $\omega$  and the failure of  $\square(\mu)$  for all regular  $\mu > \omega_1$ . Zhang shows that this version does not imply simultaneous stationary reflection for pairs of stationary subsets of  $S_\omega^{\omega_2}$ . It is still open whether Rado's conjecture implies any instance of simultaneous stationary reflection.

**3. Consistency strength.** Let us start with a definition of a large cardinal notion in the realm of supercompactness that plays an important role in the analysis of the filter extension property.

DEFINITION 3.1. Let  $\kappa \leq \lambda$  be cardinals. We say that  $\kappa$  is  $\lambda$ - $\Pi_1^1$ -subcompact if for every  $A \subseteq H(\lambda)$  and every  $\Pi_1^1$ -statement  $\Phi$ , if  $\langle H(\lambda), \in, A \rangle \models \Phi$  then there are  $\rho < \kappa$ ,  $\bar{\lambda} < \kappa$  and  $B \subseteq H(\bar{\lambda})$  such that

$$\langle H(\bar{\lambda}), \in, B \rangle \models \Phi$$

and there is an elementary embedding

$$j: \langle H(\bar{\lambda}), \in, B \rangle \rightarrow \langle H(\lambda), \in, A \rangle$$

with critical point  $\rho$  such that  $j(\rho) = \kappa$ .

For  $\kappa = \lambda$ , since  $\kappa \notin H(\lambda)$ , we require that  $j = \text{id}$ . Thus, a cardinal  $\kappa$  is  $\kappa$ - $\Pi_1^1$ -subcompact if and only if it is weakly compact.

The important case  $\lambda = \kappa^+$  was introduced by Neeman and Steel [11, Section 1]. In their paper this type of cardinal is called  $\Pi_1^2$ -subcompact.

The following theorem improves [5, Theorem 1.1] slightly.

THEOREM 3.2. *Let  $\kappa \leq \lambda$  be a  $\lambda^+$ - $\Pi_1^1$ -subcompact cardinal. Then  $\kappa$  has the  $\lambda$ -filter extension property. In particular, any  $\kappa^+$ - $\Pi_1^1$ -subcompact cardinal is  $\kappa$ -compact.*

*Proof.* Let  $\mathcal{F}$  be a  $\kappa$ -complete filter on  $\lambda$ . Assume that there is no  $\kappa$ -complete ultrafilter  $\mathcal{U}$  extending  $\mathcal{F}$ . This is a  $\Pi_1^1$ -statement in  $H(\lambda^+)$  (with parameter  $\mathcal{F}$ ). Let us denote this statement by  $\Phi$ .

Thus, by  $\lambda^+$ - $\Pi_1^1$ -subcompactness, there is  $\rho < \bar{\lambda} < \kappa$  and an elementary embedding

$$j: \langle H(\bar{\lambda}^+), \in, \bar{\mathcal{F}} \rangle \rightarrow \langle H(\lambda^+), \in, \mathcal{F} \rangle$$

such that there is no  $\rho$ -complete ultrafilter on  $\mathcal{P}(\bar{\lambda})$  extending  $\bar{\mathcal{F}}$ .

Let us look at  $j''\bar{\mathcal{F}}$ . This is a subset of  $\mathcal{F}$  of size  $2^{\bar{\lambda}} < \kappa$ . Thus, there is an element  $s \in \bigcap_{A \in \bar{\mathcal{F}}} j(A)$ . Let  $\bar{\mathcal{U}}$  be the measure generated by  $s$ , namely,  $X \in \bar{\mathcal{U}}$  iff  $s \in j(X)$ .

This measure is  $\rho$ -complete, as every sequence  $\langle X_i \mid i < \eta < \rho \rangle$  of  $< \rho$  sets is a member of  $H(\bar{\lambda}^+)$  and thus one can apply  $j$  to it to obtain

$$s \in \bigcap_{i < \eta} j(X_i) = j\left(\bigcap_{i < \eta} X_i\right).$$

We conclude that  $\bigcap_{i < \eta} X_i \in \bar{\mathcal{U}}$ , so  $\bar{\mathcal{U}}$  is a  $\rho$ -complete ultrafilter which extends  $\bar{\mathcal{F}}$ , contrary to the assumption that  $H(\rho^+)$  satisfies  $\Phi$ . ■

By standard reflection arguments (originated in [9]), if  $\kappa \leq \lambda$  and  $\kappa$  is  $\lambda$ -supercompact then  $\kappa$  is also  $\lambda$ - $\Pi_1^1$ -subcompact. Moreover, if  $\lambda = \kappa^{+\alpha}$  for

$\alpha < \kappa$ , then the set of all  $\rho < \kappa$  such that  $\rho$  is  $\rho^{+\alpha}$ - $\Pi_1^1$ -subcompact belongs to the normal measure on  $\kappa$  which is derived from any  $\lambda$ -supercompact embedding.

Let  $V$  be a model of GCH and level-by-level equivalence of strong compactness and supercompactness, as in [1], and assume that there is a cardinal  $\kappa$  which is  $\kappa^+$ -strongly compact in  $V$ . Then in  $V$  there are many cardinals  $\rho < \kappa$  which are  $\rho$ -compact and not  $\rho^+$ -strongly compact.

By [11, Theorem 4.6], if there is a weakly iterable premise  $\mathcal{Q}$  such that  $(\kappa^+)^V = (\delta^+)^{\mathcal{Q}}$  for some cardinal  $\delta$  and  $\square(\kappa^+)$  fails in  $V$ , then  $\delta$  is  $\delta^+$ - $\Pi_1^1$ -subcompact in  $\mathcal{Q}$ . In particular, if  $\mathcal{Q}$  is a weakly iterable premise such that  $\mathcal{Q} \models$  “ $\kappa$  is  $\kappa$ -compact” then  $\mathcal{Q} \models$  “ $\kappa$  is  $\kappa^+$ - $\Pi_1^1$ -subcompact”.

Thus, it is natural to conjecture:

CONJECTURE 3.3. The existence of  $\kappa$  which is  $\kappa$ -compact is equiconsistent with the existence of a cardinal  $\delta$  which is  $\delta^+$ - $\Pi_1^1$ -subcompact.

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