

Dilated floor functions having nonnegative commutator I. Positive and mixed sign dilations

by

JEFFREY C. LAGARIAS and DAVID HARRY RICHMAN (Ann Arbor, MI)

1. Introduction. The *floor function* $\lfloor x \rfloor$ rounds a real number down to the nearest integer. It is a basic operation of discretization. Given a real parameter α , we define the *dilated floor function* $f_\alpha(x) = \lfloor \alpha x \rfloor$; it performs discretization at the length scale α^{-1} . Dilated floor functions have recently played a role in describing Ehrhart quasi-polynomials for dilates of rational simplices, treating them as step-polynomials [1, Example 13], [3, Definition 23], [2].

It is a fundamental question to understand the interaction of discretization at two different scales. This problem arises in computer graphics, for example, when one rescales figures already discretized. Mathematically it leads to study of compositions of two such functions $f_\alpha \circ f_\beta(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor$. A number of identities relating dilated floor functions with different dilation factors are given in Graham, Knuth and Patashnik [11, Chap. 3]. They raised research problems concerning compositions of dilated floor functions [11, Research problem 50, p. 101], some later addressed in Graham and O’Bryant [12]. Compositional iterates of a single dilated floor function were studied in 1994 by Fraenkel [10], and iterates of floor functions and fractional part functions in 1995 by Høaland and Knuth [13].

Dilated floor functions generally do not commute under composition of functions; the order of taking successive discretizations matters. In other words, the compositional commutator

$$[f_\alpha, f_\beta](x) := f_\alpha \circ f_\beta(x) - f_\beta \circ f_\alpha(x) = \lfloor \alpha \lfloor \beta x \rfloor \rfloor - \lfloor \beta \lfloor \alpha x \rfloor \rfloor$$

is generally not the zero function. The commutator $[f_\alpha, f_\beta]$ is always a

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11B83, 11D07, 11Z05, 26D07, 52C05.

Key words and phrases: floor function, Beatty sequence.

Received 2 June 2018; revised 5 September 2018.

Published online 19 November 2018.

bounded generalized polynomial in the sense of Bergelson and Leibman [7], who studied ergodic properties of generalized polynomials under iteration.

Recently the authors together with T. Murayama [14] obtained necessary and sufficient conditions for two dilated floor functions to commute under composition, i.e. to have $[f_\alpha, f_\beta] = 0$.

THEOREM 1.0 (Commuting dilated floor functions). *The complete set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that*

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \text{for all } x \in \mathbb{R}$$

consists of:

- (i) *three continuous families (α, α) , $(\alpha, 0)$, $(0, \beta)$ for all $\alpha \in \mathbb{R}$ resp. $\beta \in \mathbb{R}$,*
- (ii) *the infinite discrete family*

$$\left\{ (\alpha, \beta) = \left(\frac{1}{m}, \frac{1}{n} \right) : m, n \geq 1 \right\},$$

where m, n are positive integers. (The families overlap when $m = n$.)

We let S_0 denote this set of dilation factors (α, β) satisfying $[f_\alpha, f_\beta] = 0$. The interesting feature of this result is the existence of the discrete family of “exceptional” solutions given in (ii), which appear to have number-theoretic significance (see Cardinal [8, Lemma 6]).

This paper addresses the more general problem of determining the set S of all values $(\alpha, \beta) \in \mathbb{R}^2$ that satisfy the one-sided inequality

$$(1.1) \quad \lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \text{for all } x \in \mathbb{R}.$$

In what follows we will often abbreviate (1.1) by writing $[f_\alpha, f_\beta] \geq 0$, with the understanding that this inequality should hold identically for all $x \in \mathbb{R}$. When $\alpha = 0$ or $\beta = 0$ it is easy to see the inequality is satisfied, as both sides are equal to zero. If α and β differ in sign, the situation is also easy to analyze and the answer does not depend on the magnitudes $|\alpha|, |\beta|$; the inequality holds exactly when $\alpha < 0$ and $\beta > 0$ (see Section 3 for details). The cases when α and β have the same sign, however, lead to an intricate and interesting answer.

1.1. Classification theorems. The nonnegative commutator set S contains the coordinate axes, where $\alpha = 0$ or $\beta = 0$, by inspection, since $\lfloor \alpha \lfloor \beta x \rfloor \rfloor = \lfloor \beta \lfloor \alpha x \rfloor \rfloor = 0$. The description of the nonnegative commutator set S for nonzero α, β is given by the following three classification theorems, according to the signs of α and β . The first case is for mixed sign dilations, which is straightforward.

THEOREM 1.1 (Mixed sign dilations classification). *Suppose dilation factors α and β have opposite signs.*

- (a) If $\alpha < 0$ and $\beta > 0$, then the commutator relation $[f_\alpha, f_\beta] \geq 0$ is satisfied.
- (b) If $\alpha > 0$ and $\beta < 0$, then the commutator relation $[f_\alpha, f_\beta] \geq 0$ is not satisfied.

In part (b) we have $[f_\alpha, f_\beta] \leq 0$, as follows from part (a). Furthermore, by Theorem 1.0 the commutator $[f_\alpha, f_\beta]$ is never identically zero when α and β have opposite signs, so part (b) follows from part (a).

The main part of the analysis is the classification theorem for same sign dilations, which splits in two cases: positive dilations and negative dilations. Positive dilations are classified in the following result.

THEOREM 1.2 (Positive dilations classification). *Given dilation factors $\alpha, \beta > 0$, the inequality*

$$[\alpha[\beta x]] \geq [\beta[\alpha x]] \quad \text{for all } x \in \mathbb{R}$$

holds if and only if there are integers $m, n \geq 0$, not both 0, such that

$$(1.2) \quad m\alpha\beta + n\alpha = \beta.$$

We describe the positive dilation part of the set S geometrically in more detail in Section 2.1, and we prove Theorem 1.2 in Sections 5 and 6.

The negative dilation case has a more complicated classification. We state it here, and will prove it in Part II [15].

THEOREM 1.3 (Negative dilations classification). *Given dilation factors $\alpha, \beta < 0$, the inequality*

$$[\alpha[\beta x]] \geq [\beta[\alpha x]] \quad \text{for all } x \in \mathbb{R}$$

holds if and only if one (or more) of the following conditions holds:

- (i) *There are integers $m \geq 0, n \geq 1$ such that*

$$(1.3) \quad m\alpha\beta - n\beta = -\alpha.$$

- (ii) *There are coprime integers $p, q \geq 1$ such that*

$$(1.4) \quad \alpha = -\frac{q}{p}, \quad -\frac{1}{p} \leq \beta < 0.$$

- (iii) *There are coprime integers $p, q \geq 1$ and integers $m \geq 0, n \geq 1, r \geq 2$ such that*

$$(1.5) \quad \alpha = -\frac{q}{p}, \quad \beta = -\frac{1}{p} \left(1 + \frac{1}{r} \left(\frac{m}{p} + \frac{n}{q} - 1 \right) \right)^{-1}$$

with $0 < m/p + n/q < 1$.

These classification theorems describe the solution set S as a countable union of real semi-algebraic sets having dimensions 2, 1, or 0. The mixed dilation case consists of a dimension 2 component ($\alpha < 0$ and $\beta > 0$). The positive dilation cases and negative dilation cases of S both contain

1-dimensional families of solutions, which are each parts of real algebraic curves, consisting either of straight half-lines or arcs of rectangular hyperbolas. In the positive dilation case these families cover S . In the negative dilation case there is an additional family of 1-dimensional curves (case (ii) of Theorem 1.3), which consist of finite length vertical line segments (of varying length) at all rational values of α . Finally, the negative dilation case has in addition a countable number of 0-dimensional components (case (iii) of Theorem 1.3). These have rational coordinates, so we term these points “sporadic rational solutions.”

Figure 1.1 gives a schematic plot of these solutions.

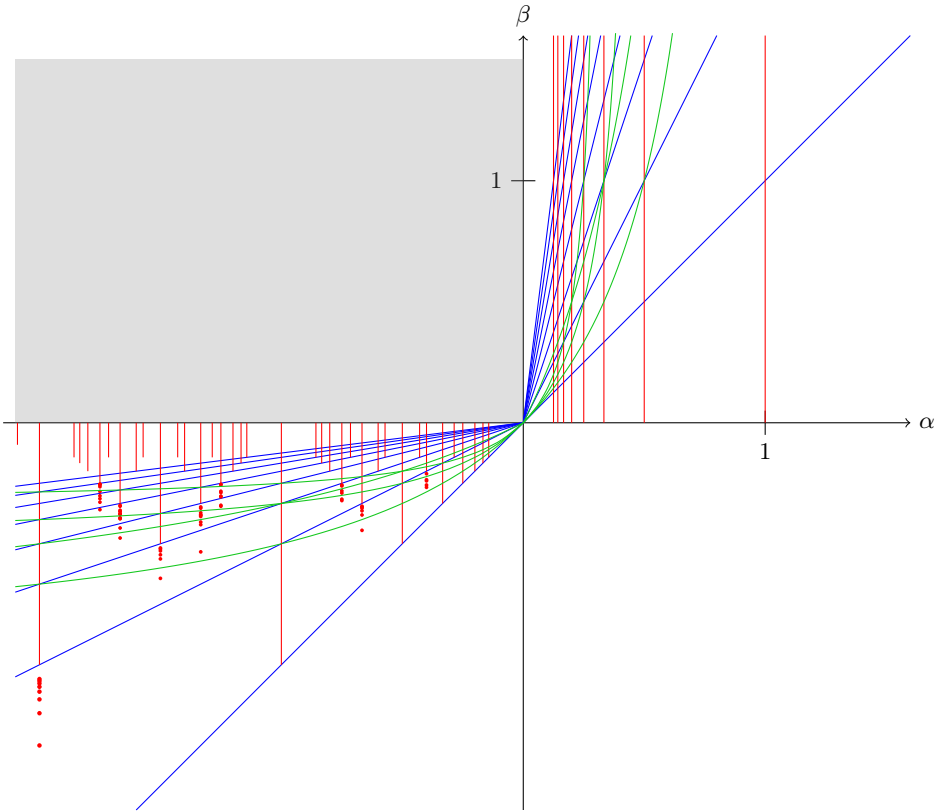


Fig. 1.1. All solutions S to the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$

The structure of S exhibits new phenomena compared to the set S_0 of commuting dilations characterized in Theorem 1.0. We may recover S_0 from the set S by intersecting S with its reflection across the line of slope 1 through the origin.

1.2. The set S is closed. We deduce a topological property of the nonnegative commutator relation from the classification theorems.

THEOREM 1.4 (Closed set property of S). *The set S of all pairs of dilation factors (α, β) which satisfy the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$ is a closed subset of \mathbb{R}^2 .*

The property that S is a closed set is not unexpected but is also not obvious because the maps f_α, f_β are discontinuous functions of x . We deduce it only as a consequence of the complete classification of the solution set S given in Theorems 1.2 and 1.3.

1.3. Preorder on \mathbb{R}^* induced by S . A significant fact established during the proofs is a transitivity property of the nonnegative commutator relation, which encodes a compatibility property of commutators of three pairs of functions. This property yields an induced preorder on the set of nonzero real numbers, which seems of independent interest. It was pointed out to us by David Speyer as following from our classification arguments.

THEOREM 1.5 (Nonnegative commutator transitivity). *The nonnegative commutator relation is transitive on nonzero dilated floor functions, meaning that for nonzero dilation factors α, β, γ ,*

$$(1.6) \quad [f_\alpha, f_\beta] \geq 0 \text{ and } [f_\beta, f_\gamma] \geq 0 \text{ imply that } [f_\alpha, f_\gamma] \geq 0.$$

This transitivity property is a special property of the class of dilated floor functions; one can find examples of nondecreasing functions f, g, h on the real line satisfying $[f, g](x) \geq 0$ and $[g, h](x) \geq 0$, for which $[f, h](x) \geq 0$ does not hold.

It follows from transitivity that this relation determines a preorder on the set of nonzero real numbers. This preorder is not a partial order; in particular all the elements $\{1/n : n \geq 1\}$ are equivalent, and this is the only nontrivial equivalence class by Theorem 1.0. We obtain a partial order by identifying equivalent elements. The induced partial order extends the divisibility relation on positive integers. The elements in the nontrivial equivalence class are all positive, so the preorder is already a partial order when restricted to negative values of α .

The proof of the preorder property (1.6) is given in Section 3; it does not require establishing the detailed classification results in Theorems 1.2 and 1.3.

2. Main results: positive dilations. This paper determines when $[f_\alpha, f_\beta] \geq 0$ holds for positive dilations and mixed sign dilations. It proves the closed set property for the solution set S on the closed subset of \mathbb{R}^2 where at least one of $\alpha \geq 0$ or $\beta \geq 0$ holds. It proves the preorder theorem for all nonzero dilation parameters. The main part of the analysis concerns

the positive dilations case, where both $\alpha, \beta > 0$. The remainder of this section states further results for positive dilations, and gives the organization of the rest of the paper.

2.1. Geometric description of S : positive dilations. We give a more detailed geometric description of the solutions S in the positive dilations case in (α, β) coordinates.

We picture the solution set S for positive dilations in (α, β) -coordinates in Figure 2.1.

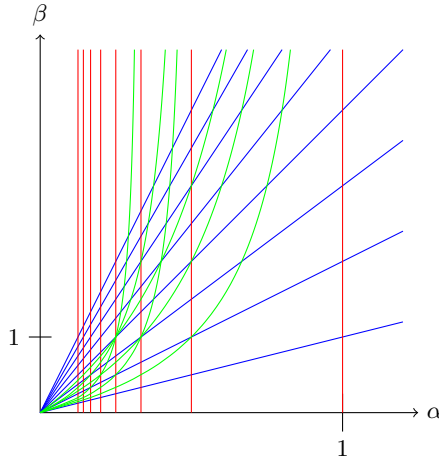


Fig. 2.1. Positive dilation solutions of S pictured in (α, β) -coordinates

The positive dilation solutions in S can be rewritten as solutions to

$$(2.1) \quad m\alpha + n\frac{\alpha}{\beta} = 1$$

with $m, n \geq 0$. These solutions comprise three distinct families of curves:

Case (i-a): if $m = 0$, then (2.1) cuts out an oblique line through the origin with integer slope n ;

Case (i-b): if $n = 0$, then (2.1) cuts out a vertical line $\alpha = 1/m$;

Case (i-c): if both $m, n \geq 1$, then (2.1) cuts out a rectangular hyperbola, which approaches the origin with integer slope n , where it is tangent to one of the $m = 0$ solutions, and which has vertical asymptote $\alpha = 1/m$, so it is tangent at infinity to one of the $n = 0$ solutions.

2.2. Symmetries of S : positive dilations. In the process of proving the classification theorems we establish several symmetries of the nonnegative commutator set S in the positive dilation case.

THEOREM 2.1 (Symmetries of the set S : positive dilations). *For positive dilations $\alpha, \beta > 0$, the set S is mapped into itself under the following symmetries:*

- (i) *For any integer $m \geq 1$, if $(\alpha, \beta) \in S$, then $(\alpha, m\beta) \in S$.*
- (ii) *For any integer $m \geq 1$, if $(\alpha, \beta) \in S$, then $(\frac{1}{m}\alpha, \frac{1}{m}\beta) \in S$.*

There is an additional symmetry of a different type on the positive solutions of S , under the birational transformation

$$(2.2) \quad (\alpha, \beta) \mapsto \left(\frac{\alpha}{\beta}, \frac{1}{\beta} \right).$$

This birational transformation acts as an involution on the open first quadrant of the plane.

THEOREM 2.2 (First quadrant birational symmetry of the set S). *On the region of positive dilations, the set S is invariant under the birational symmetry (2.2). That is, if $\alpha, \beta > 0$ and $(\alpha, \beta) \in S$, then $(\alpha/\beta, 1/\beta) \in S$.*

The set S viewed under a certain change of coordinates is related to disjointness of Beatty sequences, as we discuss below. In terms of the change of variables $(\mu, \nu) = (1/\alpha, \beta/\alpha)$, the birational transformation (2.2) becomes transposition $(\mu, \nu) \mapsto (\nu, \mu)$.

2.3. Relation to disjoint Beatty sequences: positive dilations.

The problem of classifying which positive values (α, β) correspond to a non-negative commutator $[f_\alpha, f_\beta]$ has a close parallel with the problem of classifying disjoint Beatty sequences.

We recall the definition of Beatty sequences. Let $\mathbb{N}^+ = \{1, 2, \dots\}$ denote the positive integers.

DEFINITION 2.3. Given a positive real number u , the *Beatty sequence* $\mathcal{B}^+(u)$ on \mathbb{N}^+ is the set

$$\mathcal{B}^+(u) := \{[nu] : n \in \mathbb{N}^+\}.$$

This definition was motivated by a problem posed in the American Mathematical Monthly by Samuel Beatty [5] in 1926, which may be restated in the following form.

THEOREM 2.4 (“Beatty’s Theorem”). *Let u, v be positive irrational numbers satisfying*

$$\frac{1}{u} + \frac{1}{v} = 1.$$

Then the Beatty sequences $\mathcal{B}^+(u)$ and $\mathcal{B}^+(v)$ partition the positive integers \mathbb{N}^+ , i.e.

$$\mathcal{B}^+(u) \cap \mathcal{B}^+(v) = \emptyset \quad \text{and} \quad \mathcal{B}^+(u) \cup \mathcal{B}^+(v) = \mathbb{N}^+.$$

Beatty's problem was solved in 1927 by L. Ostrowski and J. Hyslop, and by A. C. Aitken (and not by Beatty; see [5]). The result was also obtained independently in 1927 by Uspensky [24]. Moreover, it had already been noted in 1894 by Lord Rayleigh [20, p. 122] in the context of a vibrating string. A converse to Theorem 2.4 holds, explicitly noted by Fraenkel [9, p. 6]: the sets $\mathcal{B}^+(u)$ and $\mathcal{B}^+(v)$ partition the set \mathbb{N}^+ only if u, v are irrational and satisfy $1/u + 1/v = 1$.

The more general question "When are two Beatty sequences $\mathcal{B}^+(u)$ and $\mathcal{B}^+(v)$ disjoint?" was posed and answered by Skolem [21, Theorem 8] in 1957. A proof was given in a follow-up 1957 result of Bang [4, Theorem 9], and another proof was given in Niven [17, Theorem 3.11]. For further generalizations of Beatty's partition theorem for \mathbb{N}^+ , consult O'Bryant [18].

PROPOSITION 2.5 (Skolem–Bang). *For $u, v > 0$ the Beatty sequences $\mathcal{B}^+(u)$, $\mathcal{B}^+(v)$ are disjoint if and only if both u and v are irrational, and there exist integers $m, n \geq 1$ such that*

$$(2.3) \quad \frac{m}{u} + \frac{n}{v} = 1.$$

This result has a strong parallel with the conclusion of Theorem 1.2, after making the change of variables $(\mu, \nu) = (1/\alpha, \beta/\alpha)$. We call these new variables (μ, ν) -coordinates for the nonnegative commutator problem. (The inverse change of variables is $(\alpha, \beta) = (1/\mu, \nu/\mu)$.)

THEOREM 2.6 (Theorem 1.2 in (μ, ν) -coordinates). *Given $\mu, \nu > 0$, the inequality*

$$\left\lfloor \frac{1}{\mu} \left\lfloor \frac{\nu}{\mu} x \right\rfloor \right\rfloor \geq \left\lfloor \frac{\nu}{\mu} \left\lfloor \frac{1}{\mu} x \right\rfloor \right\rfloor \quad \text{for all } x \in \mathbb{R}$$

holds if and only if there are integers $m, n \geq 0$, not both zero, such that

$$(2.4) \quad \frac{m}{\mu} + \frac{n}{\nu} = 1.$$

The criterion (2.4) of this theorem is similar to the criterion (2.3) in the disjoint Beatty sequences result of Skolem and Bang. There are some differences: the criterion of Theorem 2.6 admits a wider range of m, n than for disjoint Beatty sequences and it removes the irrationality restriction on the parameters (u, v) resp. (μ, ν) .

The proof of Theorem 2.6 requires a new ingredient not present in establishing criterion (2.3) for disjoint Beatty sequences, which is needed to handle rational μ and ν satisfying (2.4). It involves a relation with the two-dimensional Diophantine Frobenius problem, appearing in case (2-b) of the proof of Theorem 6.3 in Section 6.2.

The circle of ideas used to prove Theorem 2.6 permits us to prove a parallel Beatty sequence result, formulated in terms of an extension of the notion of Beatty sequences from \mathbb{N}^+ to \mathbb{Z} .

DEFINITION 2.7. Let u be a positive real number.

(1) The *Beatty sequence* $\mathcal{B}(u)$ over \mathbb{Z} is

$$\mathcal{B}(u) := \{\lfloor nu \rfloor : n \in \mathbb{Z}\} = \{\lfloor x \rfloor : x \in u\mathbb{Z}\}.$$

(2) The *reduced Beatty sequence* $\mathcal{B}_0(u)$ over \mathbb{Z} is

$$\mathcal{B}_0(u) := \{\lfloor nu \rfloor : n \in \mathbb{Z} \text{ and } nu \notin \mathbb{Z}\} = \{\lfloor x \rfloor : x \in u\mathbb{Z} \setminus \mathbb{Z}\}.$$

All Beatty sequences over \mathbb{Z} contain $0 = \lfloor 0 \rfloor$. For $0 < u < 1$ one has $\mathcal{B}_0(u) = \mathcal{B}(u) = \mathbb{Z}$. If $u > 1$ is irrational then the reduced Beatty sequence is almost equal to the full one: $\mathcal{B}_0(u) = \mathcal{B}(u) \setminus \{0\}$. If $u = r/s > 1$ is rational, given in lowest terms, then $\mathcal{B}_0(u) = \mathcal{B}(u) \setminus r\mathbb{Z}$. In particular if u is an integer then $\mathcal{B}_0(u) = \emptyset$, while if u is not an integer, then $\mathcal{B}_0(u)$ is an infinite set.

We obtain the following criterion for disjointness of reduced (full) Beatty sequences, which parallels the Skolem–Bang result.

THEOREM 2.8 (Disjointness of reduced Beatty sequences over \mathbb{Z}). *For $u, v > 0$ the reduced Beatty sequences $\mathcal{B}_0(u)$ and $\mathcal{B}_0(v)$ over \mathbb{Z} are disjoint if and only if there exist integers $m, n \geq 0$, not both zero, such that*

$$(2.5) \quad \frac{m}{u} + \frac{n}{v} = 1.$$

To deduce this result, we use Proposition 5.2 of this paper, which asserts that for positive dilations $(\alpha, \beta > 0)$ the nonnegative commutator property

$$\lfloor \alpha \lfloor \beta x \rfloor \rfloor \geq \lfloor \beta \lfloor \alpha x \rfloor \rfloor \quad \text{for all } x \in \mathbb{R}$$

holds if and only if the quantities $\mu = 1/\alpha$ and $\nu = \beta/\alpha$ satisfy

$$\mathcal{B}_0(\mu) \cap \mathcal{B}_0(\nu) = \emptyset.$$

Theorem 2.8 follows on combining Proposition 5.2 with Theorem 2.6, along with the observation that the change of variable from (α, β) -coordinates to (μ, ν) -coordinates is a bijection from the positive quadrant of \mathbb{R}^2 to itself.

2.4. Outline of proofs. In Section 3 we determine when the commutator relation $\lfloor f_\alpha, f_\beta \rfloor \geq 0$ holds in the case when the dilation factors α, β have different signs. We also deduce the transitivity of the nonnegative commutator relation (Theorem 1.5), which implies there is an induced preorder on the set of nonzero dilations.

In Section 4 we introduce a family of rounding functions depending on a dilation factor α , which are variations of the floor function that all have average slope 1. This family of functions may be of independent interest. Using rounding functions, we derive the criterion Proposition 4.3 for the nonnegative commutator relation. We derive ordering relations among different rounding functions, and from these derive the symmetries of S on the positive dilation region (Theorem 2.1).

In Sections 5 and 6 we prove Theorem 1.2. We address the two directions of the “if and only if” statement of this theorem separately in Section 5 (sufficiency of criterion (1.2)) and Section 6 (necessity of (1.2)). The specific statements are given in Theorem 5.4 and Theorem 6.3. The two parts of the proof use different coordinate systems for the dilation parameters. The symmetry of exchanging μ and ν in the (μ, ν) -coordinate system for disjoint Beatty sequences extends to a birational symmetry of S in the (α, β) -coordinates for positive dilations (Theorem 2.2).

In Section 7 we prove the closure Theorem 1.4 for S in the case where at least one parameter α or β is nonnegative. The proof for the remaining case of Theorem 1.4 in the negative dilations region is deferred to Part II [15].

2.5. Notation. (1) The moduli space parametrizing dilations, also termed “parameter space,” has coordinates denoted (α, β) . The proofs use several different birationally transformed coordinate systems for parameter space. The coordinate system (α, β) is used in Sections 3 and 4, the (μ, ν) -coordinate system is used in Section 5, and the (σ, τ) -coordinate system is used in Section 6. All of these coordinates are positive real-valued in Sections 5 and 6.

(2) Variables denoted x and y will generally refer to coordinates of graphs of dilated floor functions and/or to graphs of commutator functions $[f_\alpha, f_\beta](x)$. These coordinates are (positive or negative) real-valued and we refer to them as “function coordinates.”

3. Mixed sign dilations and preorder theorem. In this section we consider the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$ when the dilation factors α, β differ in sign.

3.1. Floor function basics. The floor function $\lfloor x \rfloor$ is defined to be the greatest integer which is no larger than x , while the ceiling function $\lceil x \rceil$ is defined to be the least integer which is no smaller than x . In other words,

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}, \quad \lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}.$$

We let $f_\alpha(x) = \lfloor \alpha x \rfloor$ and $g_\alpha(x) = \lceil \alpha x \rceil$.

The following properties of the floor and ceiling functions are immediate:

$$(3.1) \quad x - 1 < \lfloor x \rfloor \leq x, \quad x \leq \lceil x \rceil < x + 1;$$

$$(3.2) \quad x \leq y \quad \text{implies} \quad \lfloor x \rfloor \leq \lfloor y \rfloor \quad \text{and} \quad \lceil x \rceil \leq \lceil y \rceil.$$

The floor and ceiling functions are conjugate under negation, i.e.

$$(3.3) \quad \lceil x \rceil = -\lfloor -x \rfloor.$$

Thus $\lfloor -x \rfloor = -\lceil x \rceil$.

3.2. Preorder property: proof of Theorem 1.5. We let \mathbb{R}^* denote the nonzero real numbers, and let $(\mathbb{R}^*, \preceq_C)$ denote the binary relation on \mathbb{R}^* defined by

$$\alpha \preceq_C \beta \quad \text{if} \quad [f_\alpha, f_\beta] \geq 0.$$

This relation is reflexive, i.e. $\alpha \preceq_C \alpha$ for all α . By Theorem 1.1, we have $\alpha \preceq_C \beta$ whenever $\alpha < 0 < \beta$.

We prove Theorem 1.5, which asserts that this relation is transitive, and so defines a preorder on the nonzero reals \mathbb{R}^* . We restate the theorem in terms of \preceq_C for the reader's convenience.

THEOREM 1.5 (Nonnegative commutator preorder). *The nonnegative commutator relation $(\mathbb{R}^*, \preceq_C)$ is transitive, meaning that for nonzero dilation factors α, β, γ ,*

$$[f_\alpha, f_\beta] \geq 0 \quad \text{and} \quad [f_\beta, f_\gamma] \geq 0 \quad \text{imply that} \quad [f_\alpha, f_\gamma] \geq 0.$$

Proof. Suppose α, β are nonzero. We observe that after the change of variable $x = \frac{y}{\alpha\beta}$, the nonnegative commutator relation $[f_\alpha, f_\beta](x) \geq 0$ can be rewritten as

$$(3.4) \quad \left| \alpha \left\lfloor \frac{1}{\alpha} y \right\rfloor \right| \geq \left| \beta \left\lfloor \frac{1}{\beta} y \right\rfloor \right| \quad \text{for all } y \in \mathbb{R}.$$

This equivalent version of the nonnegative commutator relation (1.1) has the important feature that the α and β parameters are *separated* on the two sides of the inequality.

Now suppose that α, β , and γ are nonzero such that $[f_\alpha, f_\beta] \geq 0$ and $[f_\beta, f_\gamma] \geq 0$. Then by the criterion (3.4), for all $y \in \mathbb{R}$ we have $\left\lfloor \alpha \left\lfloor \frac{1}{\alpha} y \right\rfloor \right\rfloor \geq \left\lfloor \beta \left\lfloor \frac{1}{\beta} y \right\rfloor \right\rfloor$ and $\left\lfloor \beta \left\lfloor \frac{1}{\beta} y \right\rfloor \right\rfloor \geq \left\lfloor \gamma \left\lfloor \frac{1}{\gamma} y \right\rfloor \right\rfloor$. Thus $\left\lfloor \alpha \left\lfloor \frac{1}{\alpha} y \right\rfloor \right\rfloor \geq \left\lfloor \gamma \left\lfloor \frac{1}{\gamma} y \right\rfloor \right\rfloor$ holds for all $y \in \mathbb{R}$, whence $[f_\alpha, f_\gamma] \geq 0$. ■

REMARK 3.1. This proof of Theorem 1.5 establishes the preorder without determining anything about its structure. The fact that for $\alpha > 0$ the partial order \preceq_C contains the divisibility relation follows from the symmetries of S described in Theorem 2.1(i).

3.3. Mixed sign dilations: proof of Theorem 1.1. We restate Theorem 1.1 for the convenience of the reader.

THEOREM 1.1 (Mixed sign dilations classification). *Suppose dilation factors α and β have opposite signs.*

- (a) *If $\alpha < 0$ and $\beta > 0$, then the commutator relation $[f_\alpha, f_\beta] \geq 0$ is satisfied.*
- (b) *If $\alpha > 0$ and $\beta < 0$, then the commutator relation $[f_\alpha, f_\beta] \geq 0$ is not satisfied.*

Proof. We set $\alpha' = |\alpha|$ and $\beta' = |\beta|$, so that $\alpha', \beta' > 0$.

(a) (Case $\alpha < 0$, $\beta > 0$) By setting $x = \frac{1}{\alpha\beta}y$, it suffices to show that $\lfloor \alpha \lfloor \frac{1}{\alpha}y \rfloor \rfloor \geq \lfloor \beta \lfloor \frac{1}{\beta}y \rfloor \rfloor$ for all $y \in \mathbb{R}$. Note that $(\alpha, \beta) = (-\alpha', \beta')$ with $\alpha', \beta' > 0$. We have

$$\begin{aligned} -\alpha' \left\lfloor \frac{1}{-\alpha'}y \right\rfloor &= \alpha' \left\lfloor \frac{1}{\alpha'}y \right\rfloor && \text{by (3.3)} \\ &\geq y && \text{by (3.1)} \\ &\geq \beta' \left\lfloor \frac{1}{\beta'}y \right\rfloor && \text{by (3.1)}. \end{aligned}$$

By (3.2), this implies $\lfloor -\alpha' \lfloor \frac{1}{-\alpha'}y \rfloor \rfloor \geq \lfloor \beta' \lfloor \frac{1}{\beta'}y \rfloor \rfloor$, so the commutator relation $\lfloor f_{-\alpha'}, f_{\beta'} \rfloor \geq 0$ holds.

(b) (Case $\alpha > 0$, $\beta < 0$) Now we have $(\alpha, \beta) = (\alpha', -\beta')$. By the above argument, for any y ,

$$\left\lfloor \alpha' \left\lfloor \frac{1}{\alpha'}y \right\rfloor \right\rfloor \leq \left\lfloor -\beta' \left\lfloor \frac{1}{-\beta'}y \right\rfloor \right\rfloor.$$

Moreover, this inequality is strict for $y = -\epsilon$ a negative number sufficiently close to zero, since

$$\left\lfloor \alpha' \left\lfloor -\frac{1}{\alpha'}\epsilon \right\rfloor \right\rfloor = \lfloor -\alpha' \rfloor \leq -1 < 0 = \left\lfloor -\beta' \left\lfloor \frac{1}{\beta'}\epsilon \right\rfloor \right\rfloor.$$

In particular $\lfloor f_{\alpha'}, f_{-\beta'} \rfloor(x) < 0$ for $x = \frac{\epsilon}{\alpha'\beta'}$, so $\lfloor f_{\alpha'}, f_{-\beta'} \rfloor$ is not always nonnegative. ■

4. Rounding functions. To deal with the nonnegative commutator relation for positive dilations, we introduce a new family of functions, which we call rounding functions.

The dilated floor function $f_\alpha(x) = \lfloor \alpha x \rfloor$ is a variation of the floor function which sends $\mathbb{R} \rightarrow \mathbb{Z}$ but is modified to have average slope α . Rounding functions, on the other hand, are variations of the floor function which all have average slope 1, but instead map $\mathbb{R} \rightarrow \alpha\mathbb{Z}$ for some real parameter α .

DEFINITION 4.1. Given parameter $\alpha \neq 0$, let $\lfloor x \rfloor_\alpha$ and $\lceil x \rceil_\alpha$ denote the (*lower*) rounding function defined by

$$\lfloor x \rfloor_\alpha := \alpha \left\lfloor \frac{1}{\alpha}x \right\rfloor$$

and the (*upper*) rounding function defined by

$$\lceil x \rceil_\alpha := \alpha \left\lceil \frac{1}{\alpha}x \right\rceil.$$

The usual floor and ceiling functions are contained in this family as $\lfloor x \rfloor = \lfloor x \rfloor_1$ and $\lceil x \rceil = \lceil x \rceil_1$, respectively. The identity (3.3) says that $\lceil x \rceil = \lfloor x \rfloor_{-1}$;

since we allow α to be negative, the above definition gives a single family of rounding functions, namely

$$(4.1) \quad \lceil x \rceil_\alpha = \lfloor x \rfloor_{-\alpha}.$$

Furthermore it is natural to extend the definition to $\alpha = 0$ by setting $\lfloor x \rfloor_0 = x$, observing that x is the limit of $\lfloor x \rfloor_\alpha$ as α approaches zero from either direction. This family of functions seems of some interest in its own right. We also use the notation $r_\alpha(x) = \lfloor x \rfloor_\alpha$ for the rounding function when it is more convenient.

Rounding functions appear in the separation of variables criterion (3.4) for a nonnegative commutator, for all nonzero α, β . That criterion can be rewritten in terms of rounding functions as

$$(4.2) \quad \lfloor \lfloor y \rfloor_\alpha \rfloor_1 \geq \lfloor \lfloor y \rfloor_\beta \rfloor_1 \quad \text{for all } y \in \mathbb{R}.$$

A related family of functions, the *strict rounding functions*, will be required in the negative dilations case treated in Part II [15].

4.1. Rounding functions: ordering inequalities. We classify when two rounding functions are comparable in the sense that the graph of one lies (weakly) below the other. These relations will be used to analyze symmetries of the set S of dilation factors (α, β) which satisfy $\lfloor f_\alpha, f_\beta \rfloor \geq 0$.

PROPOSITION 4.2 (Rounding function ordering inequalities). *Suppose dilation factors α, β are positive.*

- (a) *The inequality $\lfloor x \rfloor_\alpha \leq \lfloor x \rfloor_\beta$ holds for all $x \in \mathbb{R}$ if and only if $\lceil x \rceil_\alpha \geq \lceil x \rceil_\beta$ holds for all $x \in \mathbb{R}$.*
- (b) *We have $\lfloor x \rfloor_\alpha \leq \lfloor x \rfloor_\beta$ for all $x \in \mathbb{R}$ if and only if $\alpha = m\beta$ for some integer $m \geq 1$.*
- (c) *We have $\lceil x \rceil_\alpha \leq \lceil x \rceil_\beta$ for all $x \in \mathbb{R}$ if and only if $\beta = m\alpha$ for some integer $m \geq 1$.*

Proof. (a) This equivalence follows from “rotating the graph by 180° .” Namely, since $\lceil x \rceil_\alpha = -\lfloor -x \rfloor_\alpha$ the condition $\lceil x \rceil_\alpha \geq \lceil x \rceil_\beta$ is equivalent to

$$-\lfloor -x \rfloor_\alpha \geq -\lfloor -x \rfloor_\beta \quad \text{for all } -x \in \mathbb{R}.$$

Multiplying by -1 switches the direction of the inequality.

(b) By definition of the rounding functions, after multiplication by $\frac{1}{\beta}$ the inequality $\lfloor x \rfloor_\alpha \leq \lfloor x \rfloor_\beta$ is equivalent to the inequality $\lfloor \frac{1}{\beta}x \rfloor_{\alpha/\beta} \leq \lfloor \frac{1}{\beta}x \rfloor_1$. Thus it suffices to show that the condition

$$\lfloor x \rfloor_\alpha \leq \lfloor x \rfloor_1 = \lfloor x \rfloor \quad \text{for all } x \in \mathbb{R}$$

holds if and only if α is a positive integer. If $\alpha = m$ is a positive integer, then $\lfloor x \rfloor_m$ is an integer no larger than x so $\lfloor x \rfloor_m \leq \lfloor x \rfloor$ by definition of the

floor function. Conversely, if α is not an integer, then at $x = \alpha$ we have $\lfloor \alpha \rfloor_\alpha = \alpha > \lfloor \alpha \rfloor$.

(c) This follows directly from (a) and (b). ■

4.2. Rounding function criterion: positive dilations. We next give conditions in terms of rounding functions equivalent to the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$ on dilated floor functions $f_\alpha(x) = \lfloor \alpha x \rfloor$. While conditions (R1) and (R3) use the floor function, condition (R2) uses the ceiling function. The ceiling function in some sense behaves as the “functional inverse” of the floor function, and the steps (4.7) at the heart of the proof combine both floor and ceiling functions.

PROPOSITION 4.3 (Nonnegative commutator relation: rounding function criteria). *For $\alpha, \beta > 0$, the following properties are equivalent.*

(R1) *The nonnegative commutator relation holds:*

$$[f_\alpha, f_\beta](x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

(R2) (Upper rounding function) *We have*

$$\lceil n \rceil_\alpha \leq \lceil n \rceil_\beta \quad \text{for all } n \in \mathbb{Z},$$

where $\lceil x \rceil_\alpha = \alpha \lceil \frac{1}{\alpha} x \rceil$.

(R3) (Lower rounding function) *We have*

$$\lfloor n \rfloor_\alpha \geq \lfloor n \rfloor_\beta \quad \text{for all } n \in \mathbb{Z}.$$

Proof. We first show (R1) \Leftrightarrow (R2). The *upper level set* $U[f](y)$ of a real-valued function f at level y is

$$U[f](y) := \{x \in \mathbb{R} : f(x) \geq y\}.$$

For real-valued functions f and g on \mathbb{R} , to prove that $f(x) \geq g(x)$ holds for all x , it suffices to show the set inclusions for corresponding upper level sets:

$$(4.3) \quad U[g](y) \subset U[f](y) \quad \text{for all } y \text{ in the range of } g.$$

For positive α, β the nonnegative commutator condition $[f_\alpha, f_\beta] \geq 0$ is equivalent to the condition

$$(4.4) \quad \left\lceil \alpha \left\lfloor \frac{1}{\alpha} x \right\rfloor \right\rceil \geq \left\lceil \beta \left\lfloor \frac{1}{\beta} x \right\rfloor \right\rceil \quad \text{for all } x \in \mathbb{R}.$$

The functions $r_1 \circ r_\alpha(x) = \lfloor \alpha \lfloor \frac{1}{\alpha} x \rfloor \rfloor$ and $r_1 \circ r_\beta(x) = \lfloor \beta \lfloor \frac{1}{\beta} x \rfloor \rfloor$ both have integer values. Applying the level set criterion above to $f = r_1 \circ r_\alpha$ and $g = r_1 \circ r_\beta$, we conclude that (4.4) holds if and only if

$$(4.5) \quad U[r_1 \circ r_\beta](n) \subset U[r_1 \circ r_\alpha](n) \quad \text{for all } n \in \mathbb{Z}.$$

The equivalence (R1) \Leftrightarrow (R2) is a consequence of the following formula for the upper level sets: for all $\alpha > 0$,

$$(4.6) \quad U[r_1 \circ r_\alpha](n) = \{x : x \geq \lceil n \rceil_\alpha\}.$$

This formula follows by the chain of equivalences:

$$\begin{aligned}
 \left\lceil \alpha \left\lfloor \frac{1}{\alpha} x \right\rfloor \right\rceil \geq n &\Leftrightarrow \alpha \left\lfloor \frac{1}{\alpha} x \right\rfloor \geq n && \text{(the right side is in } \mathbb{Z} \text{)} \\
 &\Leftrightarrow \left\lfloor \frac{1}{\alpha} x \right\rfloor \geq \frac{1}{\alpha} n && \text{(since } \alpha > 0 \text{)} \\
 (4.7) \quad &\Leftrightarrow \left\lfloor \frac{1}{\alpha} x \right\rfloor \geq \left\lfloor \frac{1}{\alpha} n \right\rfloor && \text{(the left side is in } \mathbb{Z} \text{)} \\
 &\Leftrightarrow \frac{1}{\alpha} x \geq \left\lfloor \frac{1}{\alpha} n \right\rfloor && \text{(the right side is in } \mathbb{Z} \text{)} \\
 &\Leftrightarrow x \geq \alpha \left\lfloor \frac{1}{\alpha} n \right\rfloor = \lceil n \rceil_{\alpha} && \text{(since } \alpha > 0 \text{)}.
 \end{aligned}$$

The level set inclusion (4.5) is equivalent by (4.6) to the condition

$$\lceil n \rceil_{\alpha} \leq \lceil n \rceil_{\beta} \quad \text{for all } n \in \mathbb{Z},$$

as asserted.

We next show (R2) \Leftrightarrow (R3). Using (4.1) we find (R2) is equivalent to

$$(4.8) \quad \lfloor n \rfloor_{-\alpha} \leq \lfloor n \rfloor_{-\beta} \quad \text{for all } n \in \mathbb{Z}.$$

Since $\lfloor n \rfloor_{-\alpha} = -\lceil -n \rceil_{\alpha}$, this inequality transforms to

$$\lceil -n \rceil_{\alpha} \geq \lceil -n \rceil_{\beta} \quad \text{for all } n \in \mathbb{Z},$$

which is (R3) because \mathbb{Z} is invariant under negation. ■

4.3. Symmetries of S for positive dilations: proof of Theorem 2.1.

We deduce symmetries of the set S for positive dilations $\alpha, \beta > 0$ from symmetries of the integer rounding criterion given in Proposition 4.3.

Proof of Theorem 2.1. We suppose $\alpha, \beta > 0$ and are to show:

- (i) for any integer $m \geq 1$, if $(\alpha, \beta) \in S$ then $(\alpha, m\beta) \in S$,
- (ii) for any integer $m \geq 1$, if $(\alpha, \beta) \in S$ then $(\frac{1}{m}\alpha, \frac{1}{m}\beta) \in S$.

By Proposition 4.3 we have $(\alpha, \beta) \in S$ if and only if these parameters satisfy the rounding function inequalities

$$(4.9) \quad \lceil n \rceil_{\alpha} \leq \lceil n \rceil_{\beta} \quad \text{for all } n \in \mathbb{Z}.$$

Therefore, given that (α, β) satisfies (4.9) it suffices to show that

- (i) for any integer $m \geq 1$, the dilation factors $(\alpha, m\beta)$ also satisfy (4.9),
- (ii) for any integer $m \geq 1$, the dilation factors $(\frac{1}{m}\alpha, \frac{1}{m}\beta)$ also satisfy (4.9).

To show these, note that for any integer $m \geq 1$:

- (i) Proposition 4.2 implies $\lceil x \rceil_{\beta} \leq \lceil x \rceil_{m\beta}$ for all $x \in \mathbb{R}$, so in particular $\lceil n \rceil_{\beta} \leq \lceil n \rceil_{m\beta}$.

(ii) By definition of the rounding functions, $\lceil n \rceil_{\alpha/m} \leq \lceil n \rceil_{\beta/m}$ is equivalent to $\lceil mn \rceil_{\alpha} \leq \lceil mn \rceil_{\beta}$. ■

5. Positive dilations classification: sufficiency. In this section we prove the sufficiency of the criterion of Theorem 1.2 for membership $(\alpha, \beta) \in S$ of a pair of positive dilations. Namely, we show that if positive dilation parameters α, β satisfy the condition

$$(2.1) \quad m\alpha + n\frac{\alpha}{\beta} = 1 \quad \text{for some integers } m, n \geq 0$$

then they also satisfy the nonnegative commutator relation $[f_{\alpha}, f_{\beta}] \geq 0$.

To do so it is convenient to make a birational change of coordinates of the parameter space describing the two dilations. We map (α, β) -coordinates of parameter space to (μ, ν) -coordinates, given by

$$(5.1) \quad (\mu, \nu) := \left(\frac{1}{\alpha}, \frac{\beta}{\alpha} \right).$$

The map $(\alpha, \beta) \mapsto (1/\alpha, \beta/\alpha)$ is an involution sending the open first quadrant to itself, as in the other direction we have $(\mu, \nu) \mapsto (1/\mu, \nu/\mu) = (\alpha, \beta)$.

The positive dilation part of the set S is pictured in (μ, ν) -coordinates in Figure 5.1.

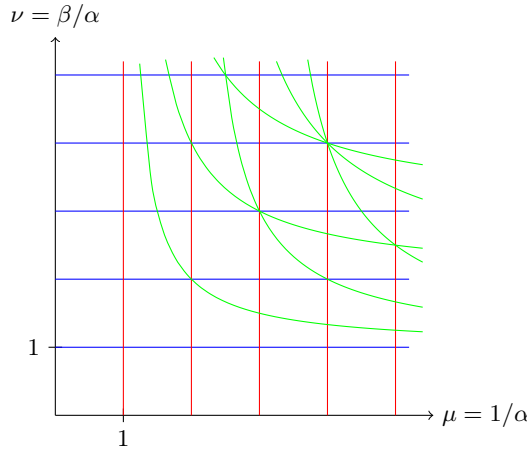


Fig. 5.1. Positive dilation solutions in S in (μ, ν) -coordinates: $\mu = 1/\alpha, \nu = \beta/\alpha$

In the new coordinates, Theorem 2.1 says that solutions to $[f_{1/\mu}, f_{\nu/\mu}] \geq 0$ are preserved under the maps

$$(\mu, \nu) \mapsto (m\mu, \nu) \quad \text{and} \quad (\mu, \nu) \mapsto (\mu, m\nu)$$

for any integer $m \geq 1$. These symmetries of the nonnegative commutator condition (1.1) are visually apparent after this coordinate change.

5.1. Lattice disjointness criterion: positive dilations. The non-negative commutator relation has a convenient reformulation in terms of the new parameters (μ, ν) as follows. It is a geometric criterion which involves a disjointness property of the rectangular lattice $\Lambda_{\mu, \nu} = \mu\mathbb{Z} \times \nu\mathbb{Z}$ in \mathbb{R}^2 from an “enlarged diagonal set” \mathcal{D} ; see (P2) of Proposition 5.2 below.

DEFINITION 5.1. The *enlarged diagonal set* \mathcal{D} in \mathbb{R}^2 is the region

$$(5.2) \quad \begin{aligned} \mathcal{D} &:= \bigcup_{n \in \mathbb{Z}} \{(x, y) : n < x < n + 1 \text{ and } n < y < n + 1\} \\ &= \bigcup_{n \in \mathbb{Z}} \{x \in (n, n + 1)\} \times \{y \in (n, n + 1)\}. \end{aligned}$$

The region \mathcal{D} is a disjoint union of open unit squares. Each open unit square is symmetric by reflection across the diagonal, $(x, y) \mapsto (y, x)$. We can also characterize this set in terms of rounding functions, by

$$(5.3) \quad \mathcal{D} = \{(x, y) : \lfloor x \rfloor < y < \lceil x \rceil\} = \{(x, y) : \lfloor y \rfloor < x < \lceil y \rceil\}.$$

The set \mathcal{D} is pictured in Figure 5.2 below.

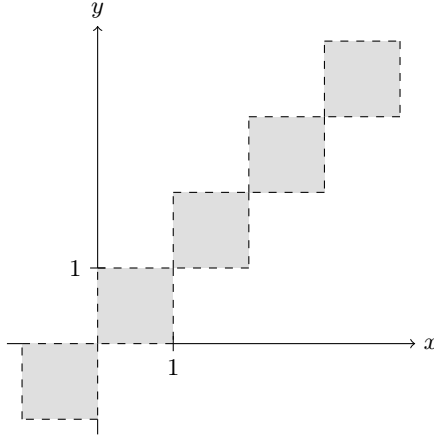


Fig. 5.2. Region $\mathcal{D} = \{\lfloor y \rfloor < x < \lceil y \rceil\}$ of \mathbb{R}^2 , in gray

We also relate this geometric criterion to a disjoint reduced Beatty sequence criterion, given in (P3) below. Recall from Section 2.3 that for $u > 0$ the reduced Beatty sequence $\mathcal{B}_0(u)$ in \mathbb{Z} is given by

$$\mathcal{B}_0(u) := \{\lfloor x \rfloor : x \in u\mathbb{Z} \setminus \mathbb{Z}\}.$$

PROPOSITION 5.2 (Nonnegative commutator relation: lattice disjointness criterion). *For $\mu, \nu > 0$, the following three properties are equivalent.*

(P1) *The nonnegative commutator relation holds:*

$$[f_{1/\mu}, f_{\nu/\mu}](x) \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

(P2) *The two-dimensional rectangular lattice $\Lambda_{\mu,\nu} := \mathbb{Z}\mu \times \mathbb{Z}\nu = \{(m\mu, n\nu) : m, n \in \mathbb{Z}\}$ in \mathbb{R}^2 is disjoint from the enlarged diagonal region $\mathcal{D} = \{(x, y) : \lfloor y \rfloor < x < \lceil y \rceil\}$:*

$$(5.4) \quad \Lambda_{\mu,\nu} \cap \mathcal{D} = \emptyset.$$

(P3) *The reduced Beatty sequences $\mathcal{B}_0(\mu)$ and $\mathcal{B}_0(\nu)$ in \mathbb{Z} are disjoint:*

$$(5.5) \quad \mathcal{B}_0(\mu) \cap \mathcal{B}_0(\nu) = \emptyset.$$

Proof. To establish (P1) \Leftrightarrow (P2) we prove the contrapositive. By Proposition 4.3, the commutator inequality $[f_{1/\mu}, f_{\nu/\mu}] \geq 0$ fails if and only if

$$\lceil n \rceil_{1/\mu} > \lceil n \rceil_{\nu/\mu} \quad \text{for some } n \in \mathbb{Z}.$$

By multiplying this inequality by μ and expanding $\lceil n \rceil_{1/\mu} = \frac{1}{\mu} \lceil \mu n \rceil_1$, we get the equivalent condition

$$(5.6) \quad \lceil n\mu \rceil_1 > \lceil n\mu \rceil_\nu \quad \text{for some } n \in \mathbb{Z}.$$

If true, this means that the ν -multiple $\lceil n\mu \rceil_\nu$ lies in the half-open interval $\{x \in \mathbb{R} : n\mu \leq x < \lceil n\mu \rceil\}$, or in other words,

$$(5.7) \quad \text{there exist } m, n \in \mathbb{Z} \text{ such that } n\mu \leq m\nu < \lceil n\mu \rceil$$

where we pick the integer m to satisfy $\lceil n\mu \rceil_\nu = m\nu$. Conversely, (5.7) implies (5.6) so these conditions are equivalent.

Now consider the lattice $\Lambda_{\mu,\nu} = \{(m\mu, n\nu) : m, n \in \mathbb{Z}\} \subset \mathbb{R}^2$. Condition (5.7) says:

$$\lceil f_{1/\mu}, f_{\nu/\mu} \rceil \geq 0 \text{ fails} \Leftrightarrow \Lambda_{\mu,\nu} \text{ intersects } \mathcal{D}_0 = \{(x, y) : x \leq y < \lceil x \rceil\} \subset \mathbb{R}^2.$$

Note that \mathcal{D}_0 is the ‘‘upper half’’ of $\mathcal{D} = \{(x, y) : \lfloor x \rfloor < y < \lceil x \rceil\}$. The ‘‘lower half’’ is the image of \mathcal{D}_0 under the 180° rotation $(x, y) \mapsto (-x, -y)$. If we let $-\mathcal{D}_0$ denote the image of \mathcal{D}_0 under this rotation, then $\mathcal{D} = \mathcal{D}_0 \cup (-\mathcal{D}_0)$. Since the lattice $\Lambda_{\mu,\nu}$ is fixed by the rotation $(x, y) \mapsto (-x, -y)$, we have

$$\begin{aligned} \Lambda_{\mu,\nu} \text{ intersects } \mathcal{D}_0 &\Leftrightarrow -\Lambda_{\mu,\nu} = \Lambda_{\mu,\nu} \text{ intersects } -\mathcal{D}_0 \\ &\Leftrightarrow \Lambda_{\mu,\nu} \text{ intersects } (-\mathcal{D}_0) \cup \mathcal{D}_0 = \mathcal{D}. \end{aligned}$$

The contrapositive is proved.

To establish (P2) \Leftrightarrow (P3), we observe that

$$\begin{aligned} \Lambda_{\mu,\nu} \cap \mathcal{D} = \emptyset &\Leftrightarrow \text{for all } (k, \ell) \in \mathbb{Z}^2, (k\mu, \ell\nu) \notin \mathcal{D} \\ &\Leftrightarrow \text{if } \lfloor k\mu \rfloor = m \text{ and } \lfloor \ell\nu \rfloor = m, \text{ then } k\mu \in \mathbb{Z} \text{ or } \ell\nu \in \mathbb{Z} \\ &\Leftrightarrow \text{for all } m \in \mathbb{Z}, m \notin \mathcal{B}_0(\mu) \cap \mathcal{B}_0(\nu) \\ &\Leftrightarrow \mathcal{B}_0(\mu) \cap \mathcal{B}_0(\nu) = \emptyset, \end{aligned}$$

as required. ■

5.2. Birational symmetry property for positive dilations: Proof of Theorem 2.2. Given the criterion (P2) of Proposition 5.2, the symmetries of the nonnegative commutator relation may be deduced from the symmetries of \mathcal{D} . Recall that the region \mathcal{D} is symmetric by reflection across the diagonal, $(x, y) \mapsto (y, x)$. Combining this observation with Proposition 5.2 yields an extra symmetry of the commutator relation for positive dilation factors.

LEMMA 5.3 (Birational symmetry property of S). *Suppose $\mu, \nu > 0$.*

- (i) *The nonnegative commutator relation $[f_{1/\mu}, f_{\nu/\mu}] \geq 0$ holds if and only if $[f_{1/\nu}, f_{\mu/\nu}] \geq 0$ holds.*
- (ii) *The positive dilation part of the set S viewed in (μ, ν) -coordinates is invariant under the interchange of μ and ν .*

Proof. (i) Interchanging μ and ν has the effect of reflecting the lattice $\Lambda_{\mu, \nu}$ across the line of slope 1 through the origin. The enlarged diagonal region \mathcal{D} is preserved under this reflection, so disjointness of $\Lambda_{\mu, \nu}$ and \mathcal{D} is preserved under interchanging μ and ν . By Proposition 5.2, this disjointness condition (P2) is equivalent to membership in S , i.e. to (P1).

(ii) This restates (i), using the definition of the nonnegative commutator set S . ■

Proof of Theorem 2.2. We are to show that if $\alpha, \beta > 0$ and $(\alpha, \beta) \in S$, then $(1/\beta, \alpha/\beta) \in S$.

Written in (α, β) -coordinates, Lemma 5.3(i) gives the result. Namely, for $\alpha, \beta > 0$ set $\alpha = 1/\mu$ and $\beta = \nu/\mu$, whence $1/\nu = \alpha/\beta$ and $\mu/\nu = 1/\beta$. Lemma 5.3(i) implies that if $(\alpha, \beta) = (1/\mu, \nu/\mu) \in S$, then $(\alpha/\beta, 1/\beta) = (1/\nu, \mu/\nu) \in S$. ■

5.3. Proof of sufficiency in Theorem 1.2. The sufficiency direction of Theorem 1.2 for positive dilations $\alpha, \beta > 0$ asserts that $(\alpha, \beta) \in S$ whenever $m\alpha + n\frac{\alpha}{\beta} = 1$ for some integers $m, n \geq 0$. Restated in (μ, ν) -coordinates, this is equivalent to the following proposition.

THEOREM 5.4 (Sufficiency condition in (μ, ν) -coordinates). *Given parameters $\mu, \nu > 0$, suppose there are integers $m, n \geq 0$ such that*

$$\frac{m}{\mu} + \frac{n}{\nu} = 1.$$

Then the nonnegative commutator relation $[f_{1/\mu}, f_{\nu/\mu}] \geq 0$ is satisfied.

This proposition will follow using the following lemma, which is the special case $(m, n) = (1, 1)$.

LEMMA 5.5 (Rectangular hyperbola sufficiency). *If (μ, ν) lies on the rectangular hyperbola $1/x + 1/y = 1$, then the lattice $\Lambda_{\mu, \nu}$ is disjoint from the set \mathcal{D} .*

Proof. To show that the lattice $\Lambda_{\mu,\nu} = \{(m\mu, n\nu) : m, n \in \mathbb{Z}\}$ is disjoint from \mathcal{D} , it suffices to show they are disjoint in the positive quadrant since both sets are preserved by $(x, y) \mapsto (-x, -y)$ and \mathcal{D} does not intersect the second or fourth quadrants. For any integers $m, n \geq 1$, the point $(m\mu, n\nu)$ lies on the positive branch of the hyperbola $m/x + n/y = 1$. This curve is monotonically decreasing and intersects the diagonal at the integer point $(m+n, m+n)$. This implies the hyperbola is disjoint from \mathcal{D} , so $(m\mu, n\nu)$ is not in \mathcal{D} . ■

Proof of Theorem 5.4. There are three cases, listed in Section 2.1 and below. In each case, it suffices to show that condition (P2) of Proposition 5.2 holds, as (P2) is equivalent to the desired nonnegative commutator condition (P1).

Case (i-c). Suppose μ, ν satisfy $m/\mu + n/\nu = 1$ with both $m, n \geq 1$. Setting $\mu_0 = \frac{1}{m}\mu$ and $\nu_0 = \frac{1}{n}\nu$, we see that the lattice $\Lambda_{\mu,\nu}$ is contained in the lattice Λ_{μ_0,ν_0} . By Lemma 5.5 the lattice Λ_{μ_0,ν_0} does not intersect \mathcal{D} , so neither does $\Lambda_{\mu,\nu}$.

Case (i-b). If $n = 0$, then $m/\mu = 1$ implies $\mu = m$ is an integer. The region \mathcal{D} does not contain any points with integer x -coordinate, so $\Lambda_{\mu,\nu}$ is disjoint from \mathcal{D} .

Case (i-a). If $m = 0$, the result follows by interchange of μ and ν from the preceding case, using Lemma 5.3. ■

Converting Theorem 5.4 from (μ, ν) -coordinates to (α, β) -coordinates proves one direction of Theorem 1.2, classifying positive dilations $\alpha, \beta > 0$ satisfying the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$.

6. Positive dilations classification: necessity. In this section we prove the necessity of the criterion of Theorem 1.2 for membership in \mathcal{S} . That is, if positive dilations α, β satisfy $[f_\alpha, f_\beta] \geq 0$ then they necessarily satisfy

$$(2.1) \quad m\alpha + n\frac{\alpha}{\beta} = 1 \quad \text{for some integers } m, n \geq 0.$$

For this direction of the proof we birationally transform the parameter space to a third set of coordinates, (σ, τ) -coordinates, given by

$$(6.1) \quad (\sigma, \tau) := \left(\frac{1}{\mu}, \frac{1}{\nu} \right) = \left(\alpha, \frac{\alpha}{\beta} \right).$$

Note that although $\sigma = \alpha$, we use a different variable name as a reminder that we are using a different coordinate system. This map takes the open first quadrant of (α, β) -coordinates onto the open first quadrant in (σ, τ) -coordinates, and its inverse map is $(\sigma, \tau) \mapsto (\sigma, \sigma/\tau) = (\alpha, \beta)$.

Figure 6.1 pictures the positive dilation solutions of S in the (σ, τ) -coordinate system.

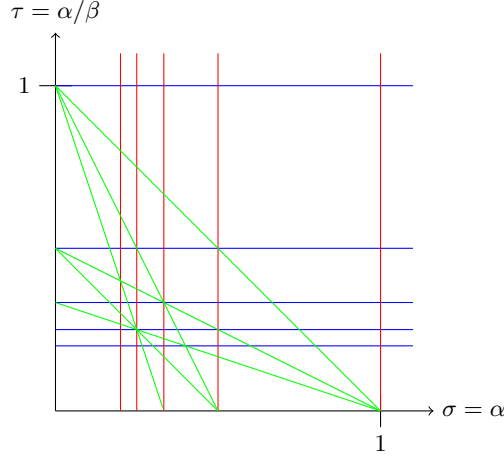


Fig. 6.1. Positive dilation solutions of S in (σ, τ) -coordinates: $\sigma = \alpha$, $\tau = \alpha/\beta$

In (σ, τ) -coordinates the set S consists of straight lines: either vertical or horizontal half-infinite lines, or oblique line segments connecting the two axes.

6.1. Torus subgroup criterion: positive dilations. To prove necessity of the classification given in Theorem 1.2, we establish a criterion for nonnegative commutator, given in terms of (σ, τ) -coordinates. It is expressed in terms of a cyclic subgroup of the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ avoiding a set $\tilde{\mathcal{C}}_{\sigma, \tau}$ which also varies with the parameters. The set to be avoided, viewed in \mathbb{R}^2 , is an open rectangular region with one corner at the origin.

DEFINITION 6.1. We define the *corner rectangle* $\mathcal{C}_{\sigma, \tau}$ to be the open region

$$(6.2) \quad \mathcal{C}_{\sigma, \tau} := \{(x, y) : 0 < x < \sigma, 0 < y < \tau\} \subset \mathbb{R}^2,$$

and let $\mathcal{C} = \mathcal{C}_{1,1}$, i.e.

$$(6.3) \quad \mathcal{C} := \{(x, y) : 0 < x, y < 1\} \subset \mathbb{R}^2.$$

Given a real number x , we let \tilde{x} denote its image under the quotient map $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. We project $\mathcal{C}_{\sigma, \tau}$ to the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$, and let $\tilde{\mathcal{C}}_{\sigma, \tau}$ denote the image of $\mathcal{C}_{\sigma, \tau}$ via coordinatewise projection $\mathbb{R}^2 \rightarrow \mathbb{T}$. The region $\tilde{\mathcal{C}}_{\sigma, \tau}$ in \mathbb{T} is either an open rectangle (if $\sigma, \tau \leq 1$), or an open annulus (if either $\sigma > 1$ or $\tau > 1$), or the whole torus (if both $\sigma, \tau > 1$). See Figure 6.2.

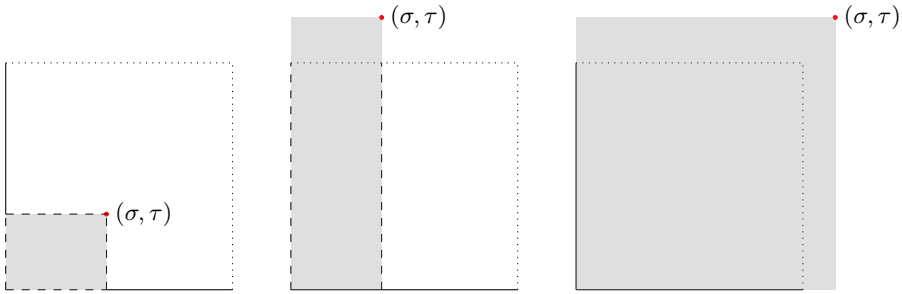


Fig. 6.2. Corner rectangles $\tilde{\mathcal{C}}_{\sigma,\tau}$ in the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ for varying parameters (σ, τ)

PROPOSITION 6.2 (Nonnegative commutator relation: torus subgroup criterion). *For $\sigma, \tau > 0$, the following conditions are equivalent.*

(Q1) *The nonnegative commutator relation holds:*

$$[f_\sigma, f_{\sigma/\tau}] \geq 0.$$

(Q2) *The cyclic torus subgroup*

$$\langle (\sigma, \tau) \rangle_{\mathbb{T}} = \{(\tilde{n}\sigma, \tilde{n}\tau) : n \in \mathbb{Z}\} \subset \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$$

is disjoint from the corner rectangle $\tilde{\mathcal{C}}_{\sigma,\tau} = \{(\tilde{x}, \tilde{y}) : 0 < x < \sigma, 0 < y < \tau\}$ in \mathbb{T} .

Proof. Let $\mu = \sigma^{-1}$ and $\nu = \tau^{-1}$. By Proposition 5.2 the commutator relation $[f_\sigma, f_{\sigma/\tau}] \geq 0$ is equivalent to the lattice $\Lambda_{\sigma^{-1}, \tau^{-1}}$ being disjoint from \mathcal{D} as subsets of \mathbb{R}^2 . The lattice $\Lambda_{\sigma^{-1}, \tau^{-1}}$ is the image of the map

$$\varphi_{\sigma,\tau} : \mathbb{Z}^2 \rightarrow \mathbb{R}^2, \quad (m, n) \mapsto (m\sigma^{-1}, n\tau^{-1}).$$

Note that any point in \mathcal{D} may be uniquely expressed as the sum of a point (k, k) for some integer k and a point inside the open unit square $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < 1\}$. Therefore the image of $\varphi_{\sigma,\tau}$ is disjoint from \mathcal{D} if and only if the image of $\varphi_{\sigma,\tau,1}$ is disjoint from \mathcal{C} , where we let

$$\varphi_{\sigma,\tau,1} : \mathbb{Z}^3 \rightarrow \mathbb{R}^2, \quad (m, n, k) \mapsto (m\sigma^{-1} + k, n\tau^{-1} + k).$$

(Recall, by Proposition 5.2, that this happens if and only if $[f_{1/\mu}, f_{\nu/\mu}] = [f_\sigma, f_{\sigma/\tau}] \geq 0$.)

After a rescaling of \mathbb{R}^2 by $(x, y) \mapsto (\sigma x, \tau y)$, the condition $[f_\sigma, f_{\sigma/\tau}] \geq 0$ is equivalent to the disjointness of the image of

$$\phi_{\sigma,\tau,1} : \mathbb{Z}^3 \rightarrow \mathbb{R}^2, \quad (m, n, k) \mapsto (m + k\sigma, n + k\tau),$$

from the open corner rectangle

$$\mathcal{C}_{\sigma,\tau} = \{(x, y) \in \mathbb{R}^2 : 0 < x < \sigma, 0 < y < \tau\}.$$

The image of $\phi_{\sigma,\tau,1}$ inside \mathbb{R}^2 is disjoint from $\mathcal{C}_{\sigma,\tau}$ if and only if their projections under $\mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ remain disjoint, since the image $\phi_{\sigma,\tau,1}(\mathbb{Z}^3) \subset \mathbb{R}^2$ is

clearly invariant under \mathbb{Z}^2 -translations. The projection of $\phi_{\sigma,\tau,1}(\mathbb{Z}^3)$ to the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ is the cyclic torus subgroup $\langle(\tilde{\sigma}, \tilde{\tau})\rangle_{\mathbb{T}}$ generated by the projection of (σ, τ) . The projection of $\mathcal{C}_{\sigma,\tau}$ to the torus is $\tilde{\mathcal{C}}_{\sigma,\tau}$. This proves the equivalence of the disjointness criterion (Q2) with the nonnegative commutator relation (Q1). ■

6.2. Proof of necessity in Theorem 1.2. The necessity condition in Theorem 1.2 states that if positive α, β have $(\alpha, \beta) \in S$, then necessarily $m\alpha + n\frac{\alpha}{\beta} = 1$ for some integers $m, n \geq 0$. Restated in the (σ, τ) -coordinates, this is equivalent to the following theorem.

THEOREM 6.3 (Necessity in (σ, τ) -coordinates). *Suppose parameters $\sigma, \tau > 0$ satisfy the nonnegative commutator relation $[f_{\sigma}, f_{\sigma/\tau}] \geq 0$. Then there are integers $m, n \geq 0$ such that*

$$m\sigma + n\tau = 1.$$

We prove Theorem 6.3 in the remainder of Section 6.2. The analysis concerns cyclic subgroups of the torus, using the criterion in Proposition 6.2. In the next subsection we recall two useful facts for the analysis.

6.2.1. Technical tools. We state two technical results used in the following proofs.

THEOREM 6.4 (Closed subgroup theorem). *Given a Lie group G and a subgroup $H \subset G$, the topological closure \bar{H} of the subspace H is a Lie subgroup $\bar{H} \subset G$.*

Proof. See Lee [16, Theorem 20.12, p. 523]. Under the given hypotheses, \bar{H} is a closed subgroup of G . ■

THEOREM 6.5 (Sylvester duality theorem). *Given a, b coprime positive integers, let*

$$\mathcal{S} := \mathcal{S}(a, b) = a\mathbb{N} + b\mathbb{N}$$

denote the semigroup generated by a and b inside $\mathbb{N} = \{0, 1, 2, \dots\}$. Then $n \in \mathcal{S}$ if and only if $ab - a - b - n \notin \mathcal{S}$.

Proof. See Beck and Robins [6, Theorem 1.3, p. 6] and the discussion thereafter, on pp. 12–14. ■

The name Sylvester is associated to this result from his problem [22] posed in 1884.

REMARK 6.6. Given positive integers a_1, \dots, a_n with $\gcd(a_1, \dots, a_n) = 1$, let $\mathcal{S} = \mathcal{S}(a_1, \dots, a_n)$ denote the nonnegative integer semigroup generated by a_1, \dots, a_n . The gcd condition implies that \mathcal{S} includes all sufficiently large integers. We call

$$\text{NR}(a_1, \dots, a_n) := \mathbb{N} \setminus \mathcal{S}(a_1, \dots, a_n)$$

the *Frobenius nonrealizing set* of the generators a_1, \dots, a_n , and its largest member is called the *Frobenius number* of \mathcal{S} . The *Diophantine Frobenius problem* is the problem of determining the Frobenius number as a function of (a_1, \dots, a_n) . This problem has been extensively studied: see Ramírez Alfonsín [19]. In the two-generator case $\mathcal{S} = \mathcal{S}(a, b)$ the Frobenius number is $ab - a - b$, a result implied by Sylvester duality. In 2006 Tuenter [23] gave a characterization of the complete set $\text{NR}(a, b)$ in terms of its moments.

6.2.2. Proof of necessity

Proof of Theorem 6.3. We divide the proof into two cases, depending on whether or not both the parameters σ, τ are rational. In each case, we use the torus disjointness condition (Q2) of Proposition 6.2 in place of the nonnegative commutator condition (Q1). Namely, it suffices to show that for (Q2) to hold, it is necessary that $m\sigma + n\tau = 1$ for some integers $m, n \geq 0$.

Case 1: At least one of σ, τ is irrational. In this case the map

$$\tilde{\phi}_{\sigma, \tau} : \mathbb{Z} \rightarrow \mathbb{T}, \quad k \mapsto (\tilde{k}\sigma, \tilde{k}\tau),$$

is injective and $H = \langle (\tilde{\sigma}, \tilde{\tau}) \rangle_{\mathbb{T}} = \text{im}(\tilde{\phi}_{\sigma, \tau})$ is an infinite cyclic subgroup of the torus \mathbb{T} . Since \mathbb{T} is compact, H cannot be a discrete subset of points. Thus by Theorem 6.4 the closure of this subspace is a Lie subgroup \bar{H} of dimension 1 or 2. If \bar{H} is dimension 2 then H is dense in \mathbb{T} and will intersect the non-empty open rectangle $\tilde{\mathcal{C}}_{\sigma, \tau}$, so condition (Q2) fails.

If \bar{H} is dimension 1, then the subgroup \bar{H} is the projection to \mathbb{T} of the lines $\{(x, y) \in \mathbb{R}^2 : mx + ny \in \mathbb{Z}\}$ for some integers m, n . (The subgroup \bar{H} determines the pair (m, n) almost uniquely, up to a sign $\pm(m, n)$; the ratio $-m/n$ is the slope of the lines in \bar{H} , and $\text{gcd}(m, n)$ is the number of connected components of \bar{H} .) The parameters σ, τ must satisfy some integer relation

$$m\sigma + n\tau = k, \quad m, n, k \in \mathbb{Z}.$$

We must have $\text{gcd}(m, n, k) = 1$ since the cyclic subgroup $H = \langle (\tilde{\sigma}, \tilde{\tau}) \rangle_{\mathbb{T}}$ is assumed to be dense in $\{(\tilde{x}, \tilde{y}) \in \mathbb{T} : m\tilde{x} + n\tilde{y} \in \mathbb{Z}\}$; otherwise, $\text{gcd}(m, n, k)$ would give the index of H inside $\{(\tilde{x}, \tilde{y}) \in \mathbb{T} : m\tilde{x} + n\tilde{y} \in \mathbb{Z}\}$.

If the integers m, n have opposite sign, then the lines in \bar{H} will have positive slope and H will intersect $\tilde{\mathcal{C}}_{\sigma, \tau}$ in a neighborhood of $(0, 0)$. See Figure 6.3.

Otherwise m, n have the same sign (including the case where m or n is zero) and we may assume that m, n, k are all nonnegative. If $k \geq 2$, then the point $(\frac{1}{k}\sigma, \frac{1}{k}\tau)$ lies in the closed subgroup \bar{H} and in the open rectangle $\tilde{\mathcal{C}}_{\sigma, \tau}$. See Figure 6.4. This implies H must also intersect the open region $\tilde{\mathcal{C}}_{\sigma, \tau}$, so condition (Q2) fails. Thus $k = 1$ is necessary for (Q2) to hold in this case.

Case 2: The parameters σ, τ are both rational. We may express $(\sigma, \tau) = (s\frac{a}{b}, t\frac{a}{b})$ where s, t are coprime positive integers and a, b are coprime positive

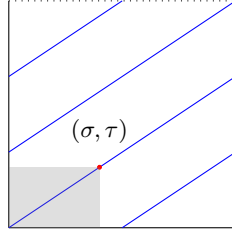


Fig. 6.3. Solutions to $2x - 3y \in \mathbb{Z}$ in the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$

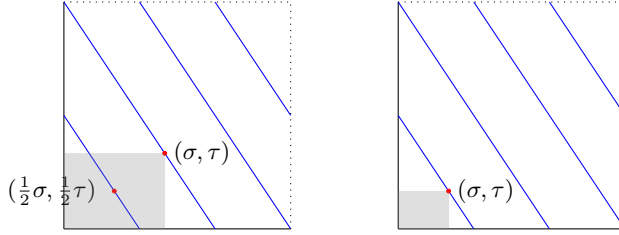


Fig. 6.4. Solutions to $3x + 2y \in \mathbb{Z}$ in the torus \mathbb{T} with $k = 2$ and $k = 1$

integers. In this case the subgroup

$$H = \langle (\tilde{\sigma}, \tilde{\tau}) \rangle_{\mathbb{T}} = \left\{ \left(\frac{\widetilde{nsa}}{b}, \frac{\widetilde{nta}}{b} \right) \in \mathbb{T} : n \in \mathbb{Z} \right\}$$

is a discrete subspace of the torus, containing precisely b points. By the assumption that $\gcd(a, b) = 1$, we have

$$H = \left\langle \left(\frac{\tilde{s}}{b}, \frac{\tilde{t}}{b} \right) \right\rangle_{\mathbb{T}} = \left\{ \left(\frac{\widetilde{ns}}{b}, \frac{\widetilde{nt}}{b} \right) : 0 \leq n \leq b - 1 \right\}.$$

Case 2-a: $a \geq 2$. In this case $(\frac{\tilde{s}}{b}, \frac{\tilde{t}}{b})$ belongs to H and lies inside the open corner rectangle $\tilde{\mathcal{C}}_{\sigma, \tau}$. Thus condition (Q2) does not hold.

Case 2-b: $a = 1$. In this case, we have $(\sigma, \tau) = (s/b, t/b)$. We are to show that $(\sigma, \tau) = (s/b, t/b)$ satisfies the nonnegative commutator relation $[f_{\sigma}, f_{\sigma/\tau}] \geq 0$ only if there exist nonnegative integers m, n such that

$$m \frac{s}{b} + n \frac{t}{b} = 1,$$

or equivalently, only if $b \in s\mathbb{N} + t\mathbb{N}$, the semigroup generated by s and t inside $\mathbb{N} = \{0, 1, 2, \dots\}$. To prove this we consider its contrapositive.

To show the contrapositive, suppose $b \notin s\mathbb{N} + t\mathbb{N}$. Then by the Sylvester Duality Theorem 6.5 we have

$$st - s - t - b \in s\mathbb{N} + t\mathbb{N}.$$

In consequence $1 \leq b < st$ and we can express

$$st - b = ms + nt$$

for some positive integers $m, n \geq 1$. These integers necessarily satisfy $1 \leq m < t$ and $1 \leq n < s$, since $1 \leq st - b < st$.

Since $\gcd(s, t) = 1$ there exist (possibly negative) integers m_0, n_0 satisfying

$$1 = m_0s + n_0t.$$

Note that m_0 is a multiplicative inverse of s modulo t , and similarly n_0 is inverse to t modulo s . We now assert the following claim:

CLAIM. *The point $(\widetilde{\frac{s-n}{b}}, \widetilde{\frac{m}{b}})$ is in the torus subgroup $H = \langle (\widetilde{\frac{s}{b}}, \widetilde{\frac{t}{b}}) \rangle_{\mathbb{T}}$. Specifically, in the torus \mathbb{T} , the point $(\widetilde{\frac{s-n}{b}}, \widetilde{\frac{m}{b}})$ is equal to the multiple $(\widetilde{N\frac{s}{b}}, \widetilde{N\frac{t}{b}})$ where*

$$N = m_0(s - n) + n_0m.$$

First we verify that N satisfies the following relations to b : since

$$b = st - ms - nt = (s - n)t - ms$$

we have

$$\begin{aligned} m_0b &= m_0(s - n)t - mm_0s \\ &= m_0(s - n)t + m(n_0t - 1) \\ &= (m_0(s - n) + n_0m)t - m = Nt - m \end{aligned}$$

and

$$\begin{aligned} -n_0b &= -(s - n)n_0t + n_0ms \\ &= (s - n)(m_0s - 1) + n_0ms \\ &= (m_0(s - n) + n_0m)s - (s - n) = Ns - (s - n). \end{aligned}$$

This shows that

$$N = \frac{1}{t}(m_0b + m) = \frac{1}{s}(-n_0b + s - n).$$

Second, consider the multiple $(N\sigma, N\tau) = (N\frac{s}{b}, N\frac{t}{b})$. We have

$$N\frac{s}{b} = \left(\frac{-n_0b + s - n}{s} \right) \frac{s}{b} = -n_0 + \frac{s - n}{b}$$

so $N\frac{s}{b} \equiv \frac{s-n}{b}$ modulo 1, and

$$N\frac{t}{b} = \left(\frac{m_0b + m}{t} \right) \frac{t}{b} = m_0 + \frac{m}{b}$$

so $N\frac{t}{b} \equiv \frac{m}{b}$ modulo 1. This shows that $(\widetilde{\frac{s-n}{b}}, \widetilde{\frac{m}{b}}) = (\widetilde{N\frac{s}{b}}, \widetilde{N\frac{t}{b}})$ is in the torus subgroup $H = \langle (\widetilde{\frac{s}{b}}, \widetilde{\frac{t}{b}}) \rangle_{\mathbb{T}}$, proving the claim.

Now the claim certifies that $(\sigma, \tau) = (s/b, t/b)$ fails to satisfy property (Q2) in Proposition 6.2, since $(\frac{s-n}{b}, \frac{\tilde{m}}{b}) \in H$, while the bounds

$$0 < s - n < s \quad \text{and} \quad 0 < m < t$$

show that $(\frac{s-n}{b}, \frac{\tilde{m}}{b})$ lies in the open rectangle $\tilde{C}_{\sigma, \tau}$. We conclude property (Q1) does not hold. The contrapositive is proved, completing Case 2-b. ■

By converting Theorem 6.3 from (σ, τ) -coordinates to (α, β) -coordinates, we have the proof of the other direction of Theorem 1.2. Thus Theorem 1.2 is now proved, by combining the statements of Theorems 5.4 and 6.3.

7. Proof of Closure Theorem 1.4 for positive and mixed sign dilations. In this section we prove the closure property of Theorem 1.4 restricted to the closed set of nonnegative dilations and mixed sign dilations, on the closed region \mathbb{R}_{nn}^2 where at least one coordinate is nonnegative.

This closure property does not seem obvious a priori, since although the composition $\lfloor \alpha \lfloor \beta x \rfloor \rfloor$ is an upper semicontinuous function of α, β and x , the commutator $[f_\alpha, f_\beta](x)$ does not share this property. Moreover for a fixed x_0 the set of (α, β) satisfying $[f_\alpha, f_\beta](x_0) \geq 0$ is in general not closed. For example, when $x_0 = 1$ the set

$$\{(\alpha, \beta) : \lfloor \alpha \lfloor \beta \rfloor \rfloor - \lfloor \beta \lfloor \alpha \rfloor \rfloor \geq 0\}$$

does not contain $(2, 1/2)$ but does contain $(2 - \epsilon, 1/2)$ for any $\epsilon > 0$.

THEOREM 7.1. *Let S be the set of all dilation factors (α, β) which satisfy the nonnegative commutator relation $[f_\alpha, f_\beta] \geq 0$, where $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let*

$$\mathbb{R}_{\text{nn}}^2 = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq 0 \text{ or } \beta \geq 0\}.$$

Then $S \cap \mathbb{R}_{\text{nn}}^2$ is a closed subset of \mathbb{R}_{nn}^2 .

Proof. To show that $S \subset \mathbb{R}_{\text{nn}}^2$ is closed, it suffices to check that its intersection with each closed quadrant is closed. The set \mathbb{R}_{nn}^2 is the union of three closed quadrants. Since the two coordinate axes are entirely in S , it suffices to check that S is closed in each open quadrant. For the second and fourth quadrants, this is immediate by Theorem 1.1.

In the open first quadrant \mathbb{R}_+^2 , let $S_+ = \{(\alpha, \beta) > 0 : [f_\alpha, f_\beta] \geq 0\}$. Since the change of coordinates $\mu = 1/\alpha, \nu = \beta/\alpha$ is a homeomorphism on the open first quadrant, it suffices to show that $S_+^{\mu, \nu} = \{(\mu, \nu) > 0 : [f_{1/\mu}, f_{\nu/\mu}] \geq 0\}$ is closed in \mathbb{R}_+^2 . The classification of Theorem 2.6 states that $S_+^{\mu, \nu}$ is a countably infinite union of closed arcs,

$$S_+^{\mu, \nu} = \bigcup_{m, n \geq 0} \{(\mu, \nu) : m/\mu + n/\nu = 1\}.$$

If U is any bounded open region of \mathbb{R}_+^2 then U intersects only finitely many of these arcs, hence $S_+^{\mu, \nu} \cap U$ is closed in U . It follows that $S_+^{\mu, \nu}$ is closed in \mathbb{R}_+^2 , since the bounded open subsets of \mathbb{R}_+^2 form an open cover. ■

REMARK 7.2. The fact that $S_+^{\mu,\nu}$ is closed may also be deduced from the lattice disjointness criterion in Proposition 5.2, without reference to the full classification theorem. Namely, the lattice disjointness criterion (P2) allows us to express the complement of $S_+^{\mu,\nu}$ in \mathbb{R}_+^2 as the union of the open diagonal region \mathcal{D} and a collection of its rescalings, which are also open. We leave the details to the reader.

Acknowledgements. We are indebted to David Speyer for his observation that our results implied the preorder property stated in Theorem 1.5, and to the anonymous referee for several clarifying suggestions. The second author thanks Yilin Yang for helpful conversations.

Work of the first author was partially supported by NSF grants DMS-1401224 and DMS-1701576, and work of the second author was partially supported by NSF grant DMS-1600223.

References

- [1] V. Baldoni, N. Berline, J. De Loera, M. Köppe and M. Vergne, *Computation of the highest coefficients of weighted Ehrhart quasi-polynomials of rational polyhedra*, Found. Comp. Math. 12 (2012), 435–469.
- [2] V. Baldoni, N. Berline, J. De Loera, M. Köppe and M. Vergne, *Intermediate sums on polyhedra II: bidegree and Poisson formula*, Mathematika 62 (2016), 653–684.
- [3] V. Baldoni, N. Berline, M. Köppe and M. Vergne, *Intermediate sums on polyhedra: computation and real Ehrhart theory*, Mathematika 59 (2013), 1–22.
- [4] Th. Bang, *On the sequences $[n\alpha]$, $n = 1, 2, \dots$. Supplementary note to the preceding paper by Th. Skolem*, Math. Scand. 5 (1957), 69–76.
- [5] S. Beatty, *Problem 3173*, Amer. Math. Monthly 33 (1926), 159 (solution: Amer. Math. Monthly 34 (1927), 159–160).
- [6] M. Beck and S. Robins, *Computing the Continuous Discretely. Integer-Point Enumeration in Polyhedra*, Springer, New York, 2007.
- [7] V. Bergelson and A. Leibman, *Distribution of values of bounded generalized polynomials*, Acta Math. 198 (2007), 155–230.
- [8] J.-P. Cardinal, *Symmetric matrices related to the Mertens function*, Linear Algebra Appl. 432 (2010), 161–172.
- [9] A. S. Fraenkel, *The bracket function and complementary sets of integers*, Canad. J. Math. 21 (1969), 6–27.
- [10] A. S. Fraenkel, *Iterated floor function, algebraic numbers, discrete chaos, Beatty subsequences, semigroups*, Trans. Amer. Math. Soc. 341 (1994), 639–664.
- [11] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, Reading, MA, 1994.
- [12] R. Graham and K. O’Bryant, *Can you hear the shape of a Beatty sequence?* in: *Additive Number Theory, Festschrift in Honor of the Sixtieth Birthday of Melvyn B. Nathanson* (D. Chudnovsky and G. Chudnovsky, eds.), Springer, New York, 2010, 39–52.
- [13] I. J. Høaland and D. E. Knuth, *Polynomials involving the floor function*, Math. Scand. 76 (1995), 194–200.

- [14] J. C. Lagarias, T. Murayama and D. H. Richman, *Dilated floor functions that commute*, Amer. Math. Monthly 123 (2016), 1033–1038.
- [15] J. C. Lagarias and D. H. Richman, *Dilated floor functions having nonnegative commutator: II. Negative dilations*, in preparation.
- [16] J. M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Grad. Texts in Math. 218, Springer, New York, 2013.
- [17] I. Niven, *Diophantine Approximations*, Interscience Tracts in Pure Math. 14, Interscience Publ., Wiley, New York, 1963.
- [18] K. O’Byrant, *Fraenkel’s partition and Brown’s decomposition*, Integers 3 (2003), A11, 17 pp.
- [19] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford Univ. Press, Oxford, 2005.
- [20] J. W. S. Rayleigh, *The Theory of Sound*, 2nd ed., Vol. 1, MacMillan, London, 1894.
- [21] T. Skolem, *On certain distributions of integers in pairs with given differences*, Math. Scand. 5 (1957), 57–68.
- [22] J. J. Sylvester, *Problem 7382*, Educational Times 37 (1884), 26; reprinted in: *Mathematical Questions with Their Solution, from the Educational Times*, W. J. C. Miller (ed.), Vol. 41, 1884, 21.
- [23] H. J. H. Tuentler, *The Frobenius problem, sums of powers of integers, and recurrences for the Bernoulli numbers*, J. Number Theory 117 (2006), 376–386.
- [24] J. V. Uspensky, *On a problem arising out of the theory of a certain game*, Amer. Math. Monthly 34 (1927), 516–521.

Jeffrey C. Lagarias, David Harry Richman
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043, U.S.A.
E-mail: lagarias@umich.edu
hrichman@umich.edu