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Stieltjes and inverse Stieltjes holomorphic families of linear relations and their representations

by

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Abstract. We study analytic and geometric properties of Stieltjes and inverse Stieltjes families defined on a separable Hilbert space and establish various minimal representations for them by means of compressed resolvents of various types of linear relations. Also, attention is paid to some new peculiar properties of Stieltjes and inverse Stieltjes families, including an analog for the notion of inner functions which will be characterized in an explicit manner. In addition, families which admit different types of scale invariance properties are described. Two transformers that naturally appear in Stieltjes and inverse Stieltjes classes are introduced and their fixed points are identified.

1. Introduction. The main objects in this paper are operator-valued Stieltjes and inverse Stieltjes functions, or more generally, Stieltjes and inverse Stieltjes families of linear relations. Elements in these classes are Nevanlinna functions [4, 15, 24, 27, 38, 39], or more generally, Nevanlinna families [19, 23, 31, 32, 36] which admit holomorphic continuation to the negative semiaxis $\mathbb{R}_-$. Such functions typically appear in boundary value problems when modeling physical phenomena and they offer an important analytic tool for the spectral analysis of associated nonnegative selfadjoint operators; see e.g. [1, 21, 22, 26, 27, 28] for some classical treatments in the case of scalar functions. In particular, functions belonging to the inverse Stieltjes class are relevant in the spectral analysis of the Friedrichs extension A_F of a nonnegative operator A (cf. [5, 21, 22, 27, 34, 35]).

In order to introduce a general definition of Stieltjes and inverse Stieltjes classes, first recall the definition of a Nevanlinna family (cf. e.g. [19, 20, 23, 36] and the references therein).

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Definition 1.1. A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, in a Hilbert space $\mathfrak{M}$ is called a *Nevanlinna family* if:

(i) $\mathcal{M}(\lambda)$ is maximal dissipative for every $\lambda \in \mathbb{C}_+$ (resp. accumulative for every $\lambda \in \mathbb{C}_-$);
(ii) $\mathcal{M}(\lambda)^* = \mathcal{M}(\bar{\lambda})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
(iii) for some (and hence for all) $\mu \in \mathbb{C}_+$ (resp. $\mu \in \mathbb{C}_-$) the operator family $(\mathcal{M}(\lambda)+\mu)^{-1}$ ($\in \mathbf{B}(\mathfrak{M})$) is holomorphic for all $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$).

Observe, that by (iii) the family $\mathcal{M}(\lambda)$ is holomorphic in the resolvent sense of T. Kato (see [30, Chapter VII, §1, in particular Theorem VII.1.3]). In this case $\mathcal{M}(\lambda)$ can be considered as the graph of a pair of bounded holomorphic functions in the half-planes $\mathbb{C}_\pm$:

$$\mathcal{M}(\lambda) = \bigl\{\{(\mathcal{M}(\lambda)+\mu)^{-1}h, (I-\mu(\mathcal{M}(\lambda)+\mu)^{-1})h\} : h \in \mathfrak{M}\ (\lambda,\mu \in \mathbb{C}_\pm)\bigr\}. \tag{1.1}$$

The class of all Nevanlinna families in a Hilbert space $\mathfrak{M}$ is denoted by $\widetilde{R}(\mathfrak{M})$. It is a consequence of the maximum modulus principle that the multi-valued part M_∞ of each $\mathcal{M} \in \widetilde{R}(\mathfrak{M})$ does not depend on $\lambda \in \mathbb{C}\setminus\mathbb{R}$; cf. (1.1), (2.11). It follows that $\mathcal{M}$ admits a unique decomposition

$$\mathcal{M}(\lambda) = M_{\rm op}(\lambda) \oplus M_\infty, \qquad M_\infty = \{0\}\times \operatorname{mul}\mathcal{M}(\lambda), \tag{1.2}$$

where $M_{\rm op}(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathfrak{M}\ominus\operatorname{mul}\mathcal{M}(\lambda)$ (see [31, §4.3], [36, Proposition 1.2]). If $\mathfrak{M}_\mathcal{M} := \operatorname{mul}\mathcal{M}(\lambda)$ (= const) and $\mathcal{M}_{\rm op}(\lambda)$ is the operator part of $\mathcal{M}(\lambda)$ (acting in the Hilbert space $\mathfrak{M}\ominus\mathfrak{M}_\mathcal{M}$), then due to (1.2), for the resolvent $(\mathcal{M}(\lambda)-\mu I)^{-1}$ one has (cf. e.g. [36, (1.6)])

$$(\mathcal{M}(\lambda)-\mu I)^{-1}\restriction\mathfrak{M}_\mathcal{M} = 0, \qquad (\mathcal{M}(\lambda)-\mu I)^{-1} = (\mathcal{M}_{\rm op}(\lambda)-\mu I)^{-1}P_{\mathfrak{M}\ominus\mathfrak{M}_\mathcal{M}}.$$

The main objective of the present paper is a study of *Stieltjes* and *inverse Stieltjes* holomorphic families of linear relations, which are defined as follows.

Definition 1.2. A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$, in a Hilbert space $\mathfrak{M}$ is said to be a *Stieltjes family* (respectively, *inverse Stieltjes family*) if it is a Nevanlinna family for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and moreover

(i) for all $x<0$ the linear relations $\mathcal{M}(x)$ are selfadjoint and nonnegative (respectively, nonpositive),
(ii) the family $\mathcal{M}(\lambda)$ is holomorphic on $\mathbb{R}_-$, i.e., for any $x<0$ and for some (and hence all) $\xi \in \rho(\mathcal{M}(x))$ the resolvent $(\mathcal{M}(\lambda)-\xi I)^{-1}$ exists and is holomorphic in λ from a neighborhood of x, depending on ξ.

The classes of all Stieltjes and inverse Stieltjes families in a Hilbert space $\mathfrak{M}$ are denoted by $\widetilde{\mathcal{S}}(\mathfrak{M})$ and $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$, respectively.

By definition, a Stieltjes family (resp. inverse Stieltjes family) is a Nevanlinna family which admits a holomorphic continuation to the negative semi-axis $(-\infty, 0)$ and whose values for $x \in (-\infty, 0)$ are nonnegative (resp. nonpositive) and selfadjoint. As in the case of scalar functions these classes are connected to each other via appropriate inversions: if $\mathcal{M}(\lambda)$ is a Stieltjes (resp. inverse Stieltjes) family, then

(a) $-\mathcal{M}(1/\lambda)$ is an inverse Stieltjes (resp. Stieltjes) family,
(b) $-\mathcal{M}^{-1}(\lambda)$ is an inverse Stieltjes (resp. Stieltjes) family.

Here (a) is clear. To see (b) apply the formula

$$\left(H^{-1} - \frac{1}{z}\right)^{-1} = -z - z^2(H - z)^{-1}, \qquad z \in \rho(H) \setminus \{0\}, \tag{1.3}$$

to $H = M(\lambda)$ and $z = -\mu$ (with $\lambda, \mu \in \mathbb{C}_+$ or $\lambda, \mu \in \mathbb{C}_-$). Hence, if $(\mathcal{M}(\lambda) + \mu)^{-1}$ ($\in \mathbf{B}(\mathfrak{M})$) admits an analytic continuation to $\lambda \in (-\infty, 0)$, the same is true for $(\mathcal{M}(\lambda)^{-1} + \mu^{-1})^{-1}$ ($\in \mathbf{B}(\mathfrak{M})$). This shows the equivalence in property (ii) of Definition 1.2; the equivalence in property (i) is clear from $M(x) \geq 0 \Leftrightarrow -M(x)^{-1} \leq 0$. The assertion in (b) becomes more visible when considering the graphs of $\mathcal{M}(\lambda)$ and $-\mathcal{M}(\lambda)^{-1}$ as in (1.1). In fact, another proof for (b) is contained in Lemma 3.2.

An important example of a Nevanlinna family is obtained by compressing the resolvent $(\widetilde{A} - \lambda)^{-1}$ of a selfadjoint relation $\widetilde{A}$ in a Hilbert space $\mathfrak{H}$ to some subspace $\mathfrak{M}$ of $\mathfrak{H}$:

$$P_{\mathfrak{M}}(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{M} \in \widetilde{R}(\mathfrak{M}), \tag{1.4}$$

which is an operator-valued Nevanlinna function. If, in addition, $\widetilde{A}$ is nonnegative, then $P_{\mathfrak{M}}(\widetilde{A} - \lambda)^{-1} \restriction \mathfrak{M}$ is a Stieltjes family of bounded operators.

A selfadjoint relation $\widetilde{A}$ in the orthogonal sum $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ is called *minimal with respect to* $\mathfrak{M}$, or $\mathfrak{M}$-*minimal* for short (see [19, p. 5366]), if

$$\mathfrak{H} = \overline{\operatorname{span}}\{\mathfrak{M} + (\widetilde{A} - \lambda I)^{-1}\mathfrak{M} : \lambda \in \mathbb{C} \setminus \mathbb{R}\}. \tag{1.5}$$

The set $\mathbb{C} \setminus \mathbb{R}$ in (1.5) can be replaced by a union of two open sets, with one open set from $\mathbb{C}_+$ and the other one from $\mathbb{C}_-$. Moreover, this definition of minimality can be extended to nonselfadjoint relations $\widetilde{A}$ in $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ with $\rho(\widetilde{A}) \neq \emptyset$ by replacing the set $\mathbb{C} \setminus \mathbb{R}$ in (1.5) by the resolvent set $\rho(\widetilde{A})$, or by a union of open sets, including one open set from each connected component of $\rho(\widetilde{A})$. In what follows, this minimality condition in this more general form is applied to nonnegative and, more generally, to maximal accretive relations, with λ in (1.5) taken from the left half-plane in $\mathbb{C}$.

Two selfadjoint relations $\widetilde{A}^{(1)}$ and $\widetilde{A}^{(2)}$ in the Hilbert spaces $\mathfrak{M} \oplus \mathfrak{K}^{(1)}$ and $\mathfrak{M} \oplus \mathfrak{K}^{(2)}$, respectively, are said to be *unitarily equivalent* if there exists a unitary operator $\mathcal{V}$ from $\mathfrak{K}^{(1)}$ onto $\mathfrak{K}^{(2)}$ such that

$$\widetilde{A}^{(2)} = \left\{ \left\{ \begin{bmatrix} \varphi \\ \mathcal{V}f \end{bmatrix}, \begin{bmatrix} \varphi' \\ \mathcal{V}f' \end{bmatrix} \right\} : \left\{ \begin{bmatrix} \varphi \\ f \end{bmatrix}, \begin{bmatrix} \varphi' \\ f' \end{bmatrix} \right\} \in \widetilde{A}^{(1)} \right\}, \quad \varphi, \varphi' \in \mathfrak{M},\ f, f' \in \mathfrak{K}.$$

In [19] in the context of the Weyl families of boundary relations it has been proved that for an arbitrary Nevanlinna family $\mathcal{M}$ in the Hilbert space $\mathfrak{M}$ there exists (up to unitary equivalence) a unique selfadjoint relation $\widetilde{A}$ in the Hilbert space $\mathfrak{M} \oplus \mathfrak{K}$ which is $\mathfrak{M}$-minimal such that

$$P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} {\upharpoonright} \mathfrak{M} = -(\mathcal{M}(\lambda) + \lambda I)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{1.6}$$

(see [19, proof of Theorem 3.9]). Inverting the formula (1.6) in the relation sense leads to an equivalent expression

$$\mathcal{M}(\lambda) = -\left(P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} {\upharpoonright} \mathfrak{M}\right)^{-1} - \lambda I_{\mathfrak{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.7}$$

If, in addition, $\widetilde{A}$ is nonnegative, then it follows from the general property (b) applied to the Stieltjes function (1.4) that the Nevanlinna family $\mathcal{M}(\cdot)$ in (1.7) is, in fact, an inverse Stieltjes family. As shown in Theorem 3.6, this representation describes all inverse Stieltjes families.

The formula (1.6) is closely related to description of generalized resolvents by A. V. Shtraus [40] (cf. [20, Theorem 5.2]). It should also be noted that representations of operator-valued Nevanlinna functions and Nevanlinna families as compressed resolvents of selfadjoint exit space extensions have been studied extensively and various related contributions can be found e.g. in [4, 6, 8, 10, 16, 23, 29, 31, 32, 33, 36, 37]. Some subclasses of Stieltjes and inverse Stieltjes matrix-valued functions have been considered in [41, 4], where the realizations of Nevanlinna matrix-valued functions as the impedance functions of singular L-systems are studied.

In the present paper the special attention is in characteristic properties as well as in various descriptions of Stieltjes and inverse Stieltjes families. Since these are special types of Nevanlinna families, it is of interest to characterize those selfadjoint relations which lead to their representations by means of compressed resolvents analogous to (1.6). A closer investigation of the properties of these families is obtained by using suitable linear fractional transformations of (the graphs of) selfadjoint relations. In particular, the following two transformations of $(\mathfrak{M} \oplus \mathfrak{K})^2$ will frequently appear in this paper:

$$\mathfrak{P}_{\mathfrak{M}} : \left\{ \begin{bmatrix} \varphi \\ f \end{bmatrix}, \begin{bmatrix} \varphi' \\ f' \end{bmatrix} \right\} \mapsto \left\{ \begin{bmatrix} \varphi' \\ f \end{bmatrix}, \begin{bmatrix} \varphi \\ f' \end{bmatrix} \right\}, \quad \varphi, \varphi' \in \mathfrak{M},\ f, f' \in \mathfrak{K}, \tag{1.8}$$

$$\mathfrak{J}_{\mathfrak{M}} : \left\{ \begin{bmatrix} \varphi \\ f \end{bmatrix}, \begin{bmatrix} \varphi' \\ f' \end{bmatrix} \right\} \mapsto \left\{ \begin{bmatrix} -i\varphi' \\ f \end{bmatrix}, \begin{bmatrix} i\varphi \\ f' \end{bmatrix} \right\}, \quad \varphi, \varphi' \in \mathfrak{M},\ f, f' \in \mathfrak{K}. \tag{1.9}$$

These mappings are involutions in $(\mathfrak{M} \oplus \mathfrak{K})^2$: $(\mathfrak{J}_{\mathfrak{M}})^2 = (\mathfrak{P}_{\mathfrak{M}})^2 = I_{(\mathfrak{M} \oplus \mathfrak{K})^2}$.

On the other hand, in establishing the main results of the present paper various relationships between selfadjoint contractions, nonnegative selfadjoint relations, resolvents of selfadjoint relations, and transfer functions of passive selfadjoint system (studied recently in [7], cf. the Appendix) will also be used.

The main results in this paper can be briefly described as follows:

- One-to-one correspondences between the classes of Stieltjes/inverse Stieltjes families in $\mathfrak{M}$ and the transfer functions from the combined Nevanlinna–Schur class $\mathcal{RS}(\mathfrak{M})$ (recently studied in [7]) of discrete-time passive selfadjoint systems are established (see Lemma 3.2).
- It is proved that: (a) inverse Stieltjes families in $\mathfrak{M}$ admit $\mathfrak{M}$-minimal representations of the form (1.6) by means of compressed resolvents of *nonnegative selfadjoint relations* $\widetilde{A}$ valid for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ (see Theorem 3.6); (b) for Stieltjes families there are $\mathfrak{M}$-minimal representations analogous to (1.6) either by means of compressed resolvents of *maximal accretive* relations $\widehat{B}$ valid for all λ in the open left half-plane, or by means of compressed resolvents of selfadjoint relations $\widehat{A}$ valid for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$; here $\widehat{A}$ and $\widehat{B}$ are connected with $\widetilde{A}$ via the transformations (1.8), (1.9): $\mathfrak{P}_{\mathfrak{M}}(\widehat{B}) = \mathfrak{J}_{\mathfrak{M}}(\widehat{A}) = \widetilde{A}$ (see Theorem 3.7); (c) because of close connections between Stieltjes and inverse Stieltjes families some further $\mathfrak{M}$-minimal representations for them by means of compressed resolvents of nonnegative selfadjoint relations $\widetilde{A}$ and $\widetilde{B}$ $(= \widetilde{A}^{-1})$ exist (see Theorem 3.9).
- In Section 4.1 *inner functions in the Stieltjes and inverse Stieltjes classes* are introduced and characterized.
- In Section 4.2 all those Stieltjes and inverse Stieltjes families $\mathcal{M}$ which admit the scaling property
$$\mathcal{M}(c\lambda) = c^p \mathcal{M}(\lambda) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$$
for some $c \in \mathbb{R}_+ \setminus \{1\}$ and for some $p \in \{0, 1, -1\}$ are described. We call such families *scale invariant.*
- In Section 4.3 two transformers Φ_+ and Φ_- appearing in the classes of Stieltjes and inverse Stieltjes families are briefly studied. In particular, we identify the fixed points of the mappings
$$\widetilde{\mathcal{S}}(\mathfrak{M}) \ni \mathcal{Q}(\lambda) \mapsto -\frac{\mathcal{Q}(\lambda)^{-1}}{\lambda} \in \widetilde{\mathcal{S}}(\mathfrak{M})$$
and
$$\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M}) \ni \mathcal{R}(\lambda) \mapsto -\lambda \mathcal{R}(\lambda)^{-1} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$$
and offer two appropriate realizations for them; these are also used as examples for the representation results given in Section 3.

Notations. We use the symbols $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$, and $\operatorname{mul} T$ for the domain, the range, the null-subspace, and the multi-valued part of a linear relation T. The closures of $\operatorname{dom} T$, $\operatorname{ran} T$ are denoted by $\overline{\operatorname{dom}}\, T$, $\overline{\operatorname{ran}}\, T$, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by I and sometimes by $I_{\mathfrak{H}}$. If $\mathfrak{L}$ is a subspace, i.e., a closed linear subset of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T{\restriction}\mathcal{N}$ means the restriction of a linear operator T to a set $\mathcal{N} \subset \operatorname{dom} T$. The resolvent set of T is denoted by $\rho(T)$.

The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathbf{B}(\mathfrak{H},\mathfrak{K})$, and the Banach algebra $\mathbf{B}(\mathfrak{H},\mathfrak{H})$ by $\mathbf{B}(\mathfrak{H})$.

$\mathbb{C}_+$ and $\mathbb{C}_-$ denote the open upper and lower half-planes of $\mathbb{C}$, $\mathbb{R}_+ := [0,\infty)$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk, $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ is the unit circle.

By $\mathbf{S}(\mathfrak{H}_1,\mathfrak{H}_2)$ we denote the *Schur class* (the set of all holomorphic and contractive $\mathbf{B}(\mathfrak{H}_1,\mathfrak{H}_2)$-valued functions on the unit disk), and $\mathbf{S}(\mathfrak{H}) := \mathbf{S}(\mathfrak{H},\mathfrak{H})$. For a contraction $T \in \mathbf{B}(\mathfrak{H},\mathfrak{K})$ the *defect operator* $(I - T^*T)^{1/2}$ is denoted by D_T, and $\mathfrak{D}_T := \overline{\operatorname{ran}}\, D_T$. For defect operators one has the commutation relations

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_TT^*.$$

A linear relation $\mathcal{A}$ in a Hilbert space $\mathfrak{H}$ is called *symmetric* if $\mathcal{A} \subset \mathcal{A}^*$, *selfadjoint* if $\mathcal{A} = \mathcal{A}^*$, *skew-symmetric* if $\mathcal{A} \subset -\mathcal{A}^*$, *skew-selfadjoint* if $\mathcal{A} = -\mathcal{A}^*$, and *nonnegative* if $(f', f) \geq 0$ for all $\{f, f'\} \in \mathcal{A}$.

Throughout this paper we consider separable Hilbert spaces over $\mathbb{C}$. For general treatments and various standard properties of linear relations we refer to [2, 17, 18, 23].

2. Transforms of linear relations, Nevanlinna families and the Schur class

2.1. Transforms of linear relations in orthogonally decomposed Hilbert spaces. Let $\mathfrak{H}$ be a Hilbert space, let $\mathfrak{M}$ be a subspace of $\mathfrak{H}$ and decompose $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$, where $\mathfrak{K} := \mathfrak{H} \ominus \mathfrak{M}$. Define a fundamental symmetry in $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ by

$$\widehat{J}_{\mathfrak{M}} = \begin{bmatrix} -I_{\mathfrak{M}} & 0 \\ 0 & I_{\mathfrak{K}} \end{bmatrix}. \tag{2.1}$$

The adjoint of T with respect to the indefinite inner product $(\widehat{J}_{\mathfrak{M}}h, k)_{\mathfrak{H}}$, $h, k \in \mathfrak{H}$, is denoted by $T^{[*]} := \widehat{J}_{\mathfrak{M}}T^*\widehat{J}_{\mathfrak{M}}$, where T^* stands for the Hilbert space adjoint of T in $\mathfrak{H}$ with respect to the original inner product $(h, k)_{\mathfrak{H}}$, $h, k \in \mathfrak{H}$. Then one can define the notions of $\widehat{J}_{\mathfrak{M}}$-*symmetric* ($\widehat{B} \subset \widehat{B}^{[*]}$), $\widehat{J}_{\mathfrak{M}}$-*selfadjoint* ($\widehat{B} = \widehat{B}^{[*]}$), and $\widehat{J}_{\mathfrak{M}}$-*dissipative* ($\operatorname{Im}(\widehat{J}_{\mathfrak{M}}u', u) \geq 0$, $\{u, u'\} \in \widehat{B}$) for linear relations $\widehat{B}$ in $\mathfrak{H}$.

The main properties of the transformation $\mathfrak{P}_{\mathfrak{M}}$ in (1.8) are described in the next proposition.

Proposition 2.1. *Let $\widetilde{A}$ be a linear relation in the Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and let $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ be defined by* (1.8)*. Then:*

(i) *The transformation $\mathfrak{P}_{\mathfrak{M}}$ preserves adjoints as follows:*

$$\mathfrak{P}_{\mathfrak{M}}(\widetilde{A}^*) = \widehat{B}^{[*]} \tag{2.2}$$

and it establishes a one-to-one correspondence between symmetric, selfadjoint, (*maximal*) *dissipative relations $\widetilde{A}$ in $\mathfrak{H}$ and $\widehat{J}_{\mathfrak{M}}$-symmetric, $\widehat{J}_{\mathfrak{M}}$-selfadjoint,* (*maximal*) *$\widehat{J}_{\mathfrak{M}}$-dissipative relations $\widehat{B}$ in $\mathfrak{H}$.*

(ii) *If $\widehat{h} = \{h, h'\}$, $\widehat{k} = \{k, k'\} \in \mathfrak{H}^2$ and $\{u, u'\} = \mathfrak{P}_{\mathfrak{M}}\widehat{h}$, $\{v, v'\} = \mathfrak{P}_{\mathfrak{M}}\widehat{k}$, then*

$$(h', k) + (h, k') = (u', v) + (u, v'), \tag{2.3}$$

in particular $\mathfrak{P}_{\mathfrak{M}}$ preserves the real parts, $\mathrm{Re}\,(f', f) = \mathrm{Re}\,(u', u)$, *and hence $\widetilde{A}$ is accretive* (*maximal accretive, skew-symmetric, skew-selfadjoint*) *precisely when the transform $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ is accretive* (*resp. maximal accretive, skew-symmetric, skew-selfadjoint*)*.*

(iii) *The transformation $\mathfrak{P}_{\mathfrak{M}}$ establishes a one-to-one correspondence between nonnegative* (*nonnegative selfadjoint, i.e. maximal nonnegative*) *relations $\widetilde{A}$ in $\mathfrak{H}$ and $\widehat{J}_{\mathfrak{M}}$-symmetric accretive* (*$\widehat{J}_{\mathfrak{M}}$-selfadjoint maximal accretive*) *relations $\widehat{B}$ in $\mathfrak{H}$.*

(iv) *The nonnegative selfadjoint relation $\widetilde{A}$ and its $\widehat{J}_{\mathfrak{M}}$-selfadjoint maximal accretive transform $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ are simultaneously $\mathfrak{M}$-minimal, and*

$$\begin{aligned} &\overline{\mathrm{span}}\{\mathfrak{M} + (\widetilde{A} - \lambda I)^{-1}\mathfrak{M} : \lambda \in \mathbb{C} \setminus \mathbb{R}_+\} \\ &\qquad = \overline{\mathrm{span}}\{\mathfrak{M} + (\widehat{B} - \lambda I)^{-1}\mathfrak{M} : \mathrm{Re}\,\lambda < 0\}. \end{aligned} \tag{2.4}$$

Proof. (i) Let $h = \begin{bmatrix} \varphi \\ f \end{bmatrix}$, $h' = \begin{bmatrix} \varphi' \\ f' \end{bmatrix} \in \mathfrak{H}$ and $k = \begin{bmatrix} \psi \\ g \end{bmatrix}$, $k' = \begin{bmatrix} \psi' \\ g' \end{bmatrix} \in \mathfrak{H}$ be decomposed according to $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$. Then

$$\begin{aligned} (h', k) - (h, k') &= (\varphi', \psi) + (f', g) - (\varphi, \psi') - (f, g') \\ &= \left(\widehat{J}_{\mathfrak{M}} \begin{bmatrix} \varphi \\ f' \end{bmatrix}, \begin{bmatrix} \psi' \\ g \end{bmatrix}\right) - \left(\widehat{J}_{\mathfrak{M}} \begin{bmatrix} \varphi' \\ f \end{bmatrix}, \begin{bmatrix} \psi \\ g' \end{bmatrix}\right). \end{aligned} \tag{2.5}$$

By applying this identity to $\widehat{h} = \{h, h'\} \in \widetilde{A}$ and $\mathfrak{P}_{\mathfrak{M}}\widehat{h} \in \mathfrak{P}_{\mathfrak{M}}(\widetilde{A}) = \widehat{B}$ one concludes that

$$\widehat{k} = \{k, k'\} \in \widetilde{A}^* \iff \mathfrak{P}_{\mathfrak{M}}\widehat{k} = \left\{ \begin{bmatrix} \psi' \\ g \end{bmatrix}, \begin{bmatrix} \psi \\ g' \end{bmatrix} \right\} \in \widehat{B}^{[*]}.$$

This proves (2.2). Hence, in particular,

$$\widetilde{A} \subset \widetilde{A}^* \iff \widehat{B} \subset \widehat{B}^{[*]}, \quad \widetilde{A} = \widetilde{A}^* \iff \widehat{B} = \widehat{B}^{[*]}.$$

Moreover, applying (2.5) with $\widehat{h}=\widehat{k}$ shows that

$$\operatorname{Im}(h',h)=\operatorname{Im}\left(\widehat{J}_{\mathfrak{M}}\begin{bmatrix}\varphi\\ f'\end{bmatrix},\begin{bmatrix}\varphi'\\ f\end{bmatrix}\right)$$

and hence $\widetilde{A}\geq 0$ is (maximal) dissipative precisely when $\widetilde{B}$ is (maximal) $\widehat{J}_{\mathfrak{M}}$-dissipative.

(ii) The formula (2.3) follows from

$$\begin{aligned}(h',k)+(h,k')&=(\varphi',\psi)+(f',g)+(\varphi,\psi')+(f,g')\\ &=\left(\begin{bmatrix}\varphi\\ f'\end{bmatrix},\begin{bmatrix}\psi'\\ g\end{bmatrix}\right)+\left(\begin{bmatrix}\varphi'\\ f\end{bmatrix},\begin{bmatrix}\psi\\ g'\end{bmatrix}\right).\end{aligned}$$

For $\widehat{h}=\widehat{k}$ and $\{u,u'\}=\mathfrak{P}_{\mathfrak{M}}\widehat{h}$ this shows that $2\operatorname{Re}(h',h)=2\operatorname{Re}(u',u)$. Hence $\operatorname{Re}(h',h)\geq 0$, $(=0)$ for all $\widehat{h}\in\widetilde{A}$ if and only if $\operatorname{Re}(u',u)\geq 0$ $(=0)$ for all $\widehat{u}\in\widehat{B}$, which proves the assertions.

(iii) is obtained by combining (i) and (ii).

(iv) By (iii), $\widetilde{A}$ is nonnegative and selfadjoint precisely when $\widehat{B}=\mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ is $\widehat{J}_{\mathfrak{M}}$-selfadjoint and maximal accretive. Hence, if $\operatorname{Re}\lambda<0$ then $\lambda\in\rho(\widetilde{A})\cap\rho(\widehat{B})$. Now let $\{h,h'\}\in\widetilde{A}$ and decompose $h,h'\in\mathfrak{H}$ as in the proof of (i). Then

$$\left\{\begin{bmatrix}\varphi'-\lambda\varphi\\ f'-\lambda f\end{bmatrix},\begin{bmatrix}\varphi\\ f\end{bmatrix}\right\}\in(\widetilde{A}-\lambda)^{-1}\iff\left\{\begin{bmatrix}\varphi-\lambda\varphi'\\ f'-\lambda f\end{bmatrix},\begin{bmatrix}\varphi'\\ f\end{bmatrix}\right\}\in(\widehat{B}-\lambda)^{-1}.$$

Since

$$\begin{bmatrix}\varphi'-\lambda\varphi\\ f'-\lambda f\end{bmatrix}\in\begin{matrix}\mathfrak{M}\\ \oplus\\ \{0\}\end{matrix}\iff f'=\lambda f\iff\begin{bmatrix}\varphi-\lambda\varphi'\\ f'-\lambda f\end{bmatrix}\in\begin{matrix}\mathfrak{M}\\ \oplus\\ \{0\}\end{matrix},$$

for every fixed $\lambda\in\mathbb{C}\setminus\mathbb{R}$ with $\operatorname{Re}\lambda<0$ one has

$$\begin{aligned}\mathfrak{M}+(\widetilde{A}-\lambda)^{-1}\mathfrak{M}&=\mathfrak{M}+\{f\in\mathfrak{K}:\widehat{h}=\{h,h'\}\in\widetilde{A},\ f'=\lambda f\}\\ &=\mathfrak{M}+\{f\in\mathfrak{K}:\mathfrak{P}_{\mathfrak{M}}\widehat{h}\in\mathfrak{P}_{\mathfrak{M}}(\widetilde{A})=\widehat{B},\ f'=\lambda f\}\\ &=\mathfrak{M}+(\widehat{B}-\lambda)^{-1}\mathfrak{M}.\end{aligned}$$

This implies (2.4), i.e., $\widetilde{A}$ and $\widehat{B}=\mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ are simultaneously $\mathfrak{M}$-minimal. ■

The main properties of the transformation $\mathfrak{J}_{\mathfrak{M}}$ in (1.9) are easier to describe. We state them for a slightly more general transformation $\mathfrak{J}_{\mathfrak{M}}^{(c)}$ involving a unimodular constant $|c|=1$:

(2.6) $$\mathfrak{J}_{\mathfrak{M}}^{(c)}:\left\{\begin{bmatrix}\varphi\\ f\end{bmatrix},\begin{bmatrix}\varphi'\\ f'\end{bmatrix}\right\}\mapsto\left\{\begin{bmatrix}-c\varphi'\\ f\end{bmatrix},\begin{bmatrix}c\varphi\\ f'\end{bmatrix}\right\},\quad \varphi,\varphi'\in\mathfrak{M},\ f,f'\in\mathfrak{K},$$

where the choice $c = i$ gives $\mathfrak{J}_{\mathfrak{M}}$ defined in (1.9); for simplicity the superscript (i) is dropped in this case.

Proposition 2.2. *Let $\widetilde{A}$ be a linear relation in the Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and let $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}^{(c)}(\widetilde{A})$, $|c| = 1$, be defined by (2.6). Then:*

(i) *The transformation $\mathfrak{J}_{\mathfrak{M}}^{(c)}$ satisfies the identity*

$$\mathfrak{J}_{\mathfrak{M}}^{(c)}(\widetilde{A}^*) = (\widehat{A})^*; \tag{2.7}$$

in particular, $\mathfrak{J}_{\mathfrak{M}}^{(c)}$ preserves the classes of symmetric, selfadjoint, and (maximal) dissipative relations in $\mathfrak{H}$.

(ii) *The selfadjoint relations $\widetilde{A}$ and $\widehat{A}$ are simultaneously $\mathfrak{M}$-minimal.*

Proof. (i) Again let $h = \left[\begin{smallmatrix}\varphi\\ f\end{smallmatrix}\right]$, $h' = \left[\begin{smallmatrix}\varphi'\\ f'\end{smallmatrix}\right] \in \mathfrak{H}$ and $k = \left[\begin{smallmatrix}\psi\\ g\end{smallmatrix}\right]$, $k' = \left[\begin{smallmatrix}\psi'\\ g'\end{smallmatrix}\right] \in \mathfrak{H}$ be decomposed according to $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and let $\mathfrak{J}_{\mathfrak{M}}^{(c)}\widehat{h}$ and $\mathfrak{J}_{\mathfrak{M}}^{(c)}\widehat{k}$ be the transforms of $\widehat{h} = \{h, h'\}$ and $\widehat{k} = \{k, k'\}$. Then

$$\begin{aligned}(h', k) - (h, k') &= (\varphi', \psi) + (f', g) - (\varphi, \psi') - (f, g') \\ &= \left(\begin{bmatrix}c\varphi\\ f'\end{bmatrix}, \begin{bmatrix}-c\psi'\\ g\end{bmatrix}\right) - \left(\begin{bmatrix}-c\varphi'\\ f\end{bmatrix}, \begin{bmatrix}c\psi\\ g'\end{bmatrix}\right).\end{aligned}$$

Therefore, taking $\widehat{h} \in \widetilde{A}$ and $\mathfrak{J}_{\mathfrak{M}}^{(c)}\widehat{h} \in \widehat{A}$ one concludes that

$$\widehat{k} \in \widetilde{A}^* \iff \mathfrak{J}_{\mathfrak{M}}^{(c)}\widehat{k} = \left\{\begin{bmatrix}-c\psi'\\ g\end{bmatrix}, \begin{bmatrix}c\psi\\ g'\end{bmatrix}\right\} \in (\widehat{A})^*.$$

This proves (2.7), and in particular

$$\widetilde{A} \subset (\widetilde{A})^* \iff \widehat{A} \subset (\widehat{A})^*, \quad \widetilde{A} = (\widetilde{A})^* \iff \widehat{A} = (\widehat{A})^*.$$

Moreover, the identity

$$\operatorname{Im}(h', h) = \operatorname{Im}\left(\begin{bmatrix}c\varphi\\ f'\end{bmatrix}, \begin{bmatrix}-c\varphi'\\ f\end{bmatrix}\right)$$

shows that $\widetilde{A}$ is (maximal) dissipative precisely when $\widehat{A}$ is.

(ii) Let $\widehat{h} \in \widetilde{A}$ and decompose $h, h' \in \mathfrak{H}$ as in the proof of (i). Then

$$\left\{\begin{bmatrix}\varphi' - \lambda\varphi\\ f' - \lambda f\end{bmatrix}, \begin{bmatrix}\varphi\\ f\end{bmatrix}\right\} \in (\widetilde{A} - \lambda)^{-1} \iff \left\{\begin{bmatrix}c(\varphi + \lambda\varphi')\\ f' - \lambda f\end{bmatrix}, \begin{bmatrix}-c\varphi'\\ f\end{bmatrix}\right\} \in (\widehat{A} - \lambda)^{-1}.$$

Since

$$\begin{bmatrix}\varphi' - \lambda\varphi\\ f' - \lambda f\end{bmatrix} \in \begin{matrix}\mathfrak{M}\\ \oplus\\ \{0\}\end{matrix} \iff f' = \lambda f \iff \begin{bmatrix}c(\varphi + \lambda\varphi')\\ f' - \lambda f\end{bmatrix} \in \begin{matrix}\mathfrak{M}\\ \oplus\\ \{0\}\end{matrix},$$

we see that for every fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned}\mathfrak{M} + (\widetilde{A} - \lambda)^{-1}\mathfrak{M} &= \mathfrak{M} + \{f \in \mathfrak{K} : \widehat{h} \in \widetilde{A},\ f' = \lambda f\} \\ &= \mathfrak{M} + \{f \in \mathfrak{K} : \mathfrak{J}^{(c)}_{\mathfrak{M}}\widehat{h} \in \widehat{A},\ f' = \lambda f\} = \mathfrak{M} + (\widehat{A} - \lambda)^{-1}\mathfrak{M}.\end{aligned}$$

Hence, $\widetilde{A}$ and $\widehat{A}$ are simultaneously $\mathfrak{M}$-minimal. ∎

Remark 2.3. If $\widetilde{A}$ is a nonnegative selfadjoint operator in $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and $\ker \widetilde{A} \cap \mathfrak{M} = \{0\}$, then $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A})$ and $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ are operators as well, and moreover

$$\begin{aligned}&\operatorname{dom} \widehat{A} = (-iP_{\mathfrak{M}}\widetilde{A} + P_{\mathfrak{K}}) \operatorname{dom} \widetilde{A}, \\ &\widehat{A}(-iP_{\mathfrak{M}}\widetilde{A}f + P_{\mathfrak{K}}f) = iP_{\mathfrak{M}}f + P_{\mathfrak{K}}\widetilde{A}f, \quad f \in \operatorname{dom} \widetilde{A}, \\ &\operatorname{dom} \widehat{B} = (P_{\mathfrak{M}}\widetilde{A} + P_{\mathfrak{K}}) \operatorname{dom} \widetilde{A}, \\ &\widehat{B}(P_{\mathfrak{M}}\widetilde{A}f + P_{\mathfrak{K}}f) = P_{\mathfrak{M}}f + P_{\mathfrak{K}}\widetilde{A}f, \quad f \in \operatorname{dom} \widetilde{A}.\end{aligned}$$

2.2. Compressed resolvents, Nevanlinna families, and the Schur class. As indicated in (1.6), (1.7) the resolvents and compressed resolvents are closely related to Nevanlinna families in $\mathfrak{M}$. Some further insight can be obtained by connecting selfadjoint relations $\widetilde{A}$ with unitary operators U, and Nevanlinna families in $\mathfrak{M}$ with operator-valued functions from the Schur class $\mathbf{S}(\mathfrak{M})$. In this subsection some basic connections between these objects are recalled and then augmented with some formulas that will be needed in later sections.

2.2.1. *Cayley transforms.* The basic connection between selfadjoint relations $\widetilde{A}$ and unitary operators U is obtained by the direct/inverse Cayley transform defined by

$$\begin{aligned}&\widetilde{A} \mapsto U = \mathcal{C}(\widetilde{A}) := \big\{\{(f' + if, f' - if\} : \{f, f'\} \in \widetilde{A}\big\}, \\ &U \mapsto \widetilde{A} = \mathcal{C}^{-1}(U) := \big\{\{(I - U)g, i(I + U)g\} : g \in \mathfrak{H}\big\},\end{aligned}$$

i.e., $\mathcal{C}(\widetilde{A}) = I - 2i(\widetilde{A} + iI)^{-1}$ and $\mathcal{C}^{-1}(U) = -iI + 2i(I - U)^{-1}$. These formulas establish a one-to-one correspondence between unitary operators U and selfadjoint relations $\widetilde{A}$ in a Hilbert space $\mathfrak{H}$ with $\operatorname{mul} \widetilde{A} = \ker(I - U)$. The resolvents of $\widetilde{A}$ and U are connected by

$$(2.8)\quad \begin{cases}(\widetilde{A} - \lambda I)^{-1} = -\frac{1}{\lambda+i}\, I - \frac{2i}{\lambda^2+1}\big(I - \frac{\lambda+i}{\lambda-i}U\big)^{-1}, & \lambda \in \rho(\widetilde{A}) \setminus \{\pm i\} \\ (\widetilde{A} + iI)^{-1} = \frac{1}{2i}(I - U), \quad (\widetilde{A} - iI)^{-1} = \frac{1}{2i}(U^{-1} - I), &\end{cases}$$

and

$$(I - zU)^{-1} = \frac{1}{1 - z}\, I - \frac{2iz}{(1 - z)^2}\left(\widetilde{A} + i\frac{1 + z}{1 - z} I\right)^{-1}, \quad z^{-1} \in \rho(U) \text{ or } z = 0.$$

2.2.2. *Connection between the Nevanlinna families and the Schur class.* A relationship between the class $\widetilde{R}(\mathfrak{M})$ of all Nevanlinna families in $\mathfrak{M}$ and the Schur class $\mathbf{S}(\mathfrak{M})$ can be given by fractional linear transformations of functions and their independent variables (cf. [12, 13]):

$$\widetilde{R}(\mathfrak{M}) \ni \mathcal{M} \mapsto \Psi(z) := I + 2i\left(\mathcal{M}\left(i\frac{z+1}{z-1}\right) - iI\right)^{-1} \in \mathbf{S}(\mathfrak{M}), \tag{2.9}$$

$$\begin{aligned} \mathbf{S}(\mathfrak{M}) \ni \Psi \mapsto \mathcal{M}(\lambda) \\ = \begin{cases} \{\{(I-\Psi(\tfrac{\lambda+i}{\lambda-i}))h, -i(I+\Psi(\tfrac{\lambda+i}{\lambda-i}))h\},\ h\in\mathfrak{M}\}, & \operatorname{Im}\lambda<0,\\ \{\{(I-\Psi(\tfrac{\bar\lambda+i}{\bar\lambda-i}))^*h, i(I+\Psi(\tfrac{\bar\lambda+i}{\bar\lambda-i}))^*h\},\ h\in\mathfrak{M}\}, & \operatorname{Im}\lambda>0 \end{cases} \\ \in \widetilde{R}(\mathfrak{M}). \end{aligned} \tag{2.10}$$

Note that from (2.9) and the maximum modulus principle one obtains

$$\operatorname{mul}\mathcal{M}(\lambda) = \text{const} = \ker(I-\Psi(0)), \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{2.11}$$

Moreover, $\operatorname{dom}\mathcal{M}(\lambda)=\mathfrak{M}$ if and only if $\operatorname{ran}(I-\Psi(0))=\mathfrak{M}$.

2.2.3. *Connections with compressed resolvents.* Let the selfadjoint relation $\widetilde{A}$ in $\mathfrak{H}$ and the unitary operator U be connected by the Cayley transform $U=\mathcal{C}(\widetilde{A})$. Let $\mathfrak{M}$ be a subspace of $\mathfrak{H}$ and decompose $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{K}$. Then the connection between the resolvents stated in (2.8) leads to useful connections between the compressed resolvents and the classes $\widetilde{R}(\mathfrak{M})$ and $\mathbf{S}(\mathfrak{M})$.

It follows from the Schur–Frobenius block formula (A.2) for the resolvent $(I-zU)^{-1}$ that

$$P_{\mathfrak{M}}(I_{\mathfrak{H}}-zU)^{-1}\upharpoonright\mathfrak{M} = (I_{\mathfrak{M}}-z\Psi(z))^{-1}, \quad z\in\mathbb{D}. \tag{2.12}$$

On the other hand, using (2.8) and (2.9) one gets

$$P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{M} = -(\mathcal{M}(\lambda)+\lambda I)^{-1}, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{2.13}$$

where $\mathcal{M}\in\widetilde{R}(\mathfrak{M})$ and $\Psi\in\mathbf{S}(\mathfrak{M})$ are connected by (2.9) and (2.10).

The connection between the resolvents of $\widetilde{A}$ and $U=\mathcal{C}(\widetilde{A})$ in (2.8) implies that there is also a direct connection between the minimality of $\widetilde{A}$ and U:

$$\begin{aligned} \mathfrak{H} &= \overline{\operatorname{span}}\{\mathfrak{M}+(\widetilde{A}-\lambda I)^{-1}\mathfrak{M} : \lambda\in\mathbb{C}\setminus\mathbb{R}\} \\ &= \overline{\operatorname{span}}\{(I-\xi U)^{-1}\mathfrak{M} : |\xi|\neq 1\}, \end{aligned} \tag{2.14}$$

i.e., $\widetilde{A}$ and U are simultaneously $\mathfrak{M}$-minimal. Since U is unitary, (2.14) is equivalent to

$$\overline{\operatorname{span}}\{U^n\mathfrak{M} : n\in\mathbb{Z}\} = \mathfrak{H}. \tag{2.15}$$

If, in addition, U is represented by a 2×2 block operator

$$U = \begin{bmatrix} D & C \\ B & F \end{bmatrix} : \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{K}\end{matrix} \to \begin{matrix}\mathfrak{M}\\ \oplus \\ \mathfrak{K}\end{matrix},$$

then (2.15) can be rewritten (cf. [3, Proposition 7.4.]) in one of the following forms:

$$
\begin{aligned}
(2.16)\qquad & \overline{\operatorname{span}}\{U^n\mathfrak{M} : n\in\mathbb{Z}\} = \mathfrak{H} \\
\iff\ & \overline{\operatorname{span}}\{F^nB\mathfrak{M}, F^{*n}C^*\mathfrak{M} :\ n\in\mathbb{N}_0\} = \mathfrak{H} \\
\iff\ & \Big(\bigcap_{n=0}^{\infty}\ker(B^*F^{*n})\Big)\cap\Big(\bigcap_{n=0}^{\infty}\ker(CF^n)\Big) = \{0\} \\
\iff\ & \Big(\bigcap_{z\in\mathcal{U}}\ker(B^*(I_{\mathfrak{K}}-zF^*)^{-1})\Big)\cap\Big(\bigcap_{z\in\mathcal{U}}\ker(C(I_{\mathfrak{K}}-zF)^{-1})\Big) = \{0\},
\end{aligned}
$$

where $\mathcal{U}$ can be taken to be some neighborhood of the origin. The first condition involving the block entries of U is often used as a definition for U to be a simple conservative realization of the function $\Psi(\cdot)\in\mathbf{S}(\mathfrak{M})$ (see Appendix).

The next result gives representations for the functions $\mathcal{M}(\lambda)\in\widetilde{R}(\mathfrak{M})$, $-\mathcal{M}(\lambda)^{-1}\in\widetilde{R}(\mathfrak{M})$, and $-\mathcal{M}(1/\lambda)\in\widetilde{R}(\mathfrak{M})$ as compressed resolvents of certain selfadjoint relations.

Theorem 2.4. *Let $\mathcal{M}(\cdot)$ be a Nevanlinna family in the Hilbert space $\mathfrak{M}$. Then, up to unitary equivalence, there exists a unique selfadjoint relation $\widetilde{A}$ in the Hilbert space $\mathfrak{M}\oplus\mathfrak{K}$ which is $\mathfrak{M}$-minimal and such that:*

(1) *The Nevanlinna family $\mathcal{M}(\lambda)$ has the representation*

$$(2.17)\qquad \mathcal{M}(\lambda) = -\big(P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\restriction\mathfrak{M}\big)^{-1}-\lambda I_{\mathfrak{M}},\qquad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

(2) *Moreover, if $\widehat{A}=\mathfrak{J}_{\mathfrak{M}}(\widetilde{A})$ is as defined in* (1.9)*, then*

$$(2.18)\qquad -\mathcal{M}^{-1}(\lambda) = -\big(P_{\mathfrak{M}}(\widehat{A}-\lambda I)^{-1}\restriction\mathfrak{M}\big)^{-1}-\lambda I_{\mathfrak{M}},\qquad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

(3) *If $\breve{A}=-\mathfrak{J}_{\mathfrak{K}}(\widetilde{A})$, then*

$$(2.19)\qquad -\mathcal{M}(1/\lambda) = -\big(P_{\mathfrak{M}}(\breve{A}-\lambda I)^{-1}\restriction\mathfrak{M}\big)^{-1}-\lambda I_{\mathfrak{M}},\qquad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

Proof. As indicated, the statement (1) is known [19] (cf. (1.7)), and an alternative proof via Cayley transforms is contained in [12, 13]. For later purposes it is convenient to derive the stated representations simultaneously by connecting each of the functions in (1)–(3) via Cayley transforms to functions from the Schur class $\mathbf{S}(\mathfrak{M})$.

(1) Let $\mathcal{M}(\lambda)\in\widetilde{R}(\mathfrak{M})$ and define

$$\Psi(z) = I + 2i\left(\mathcal{M}\left(i\frac{z+1}{z-1}\right)-iI\right)^{-1},\qquad z\in\mathbb{D}.$$

Then Ψ is an operator-valued function that belongs to $\mathbf{S}(\mathfrak{M})$. Hence one can represent $\Psi(z)$ as the transfer function of a unique (up to unitary similarity) simple conservative system

$$\tau = \left\{ \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{K} \right\}$$

with a state space $\mathfrak{K}$ (cf. Appendix). Thus,

$$\Psi(z) = U_{11} + zU_{12}(I - zU_{22})^{-1}U_{21}, \qquad |z| < 1,$$

where

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} : \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{K} \end{matrix} \to \begin{matrix} \mathfrak{M} \\ \oplus \\ \mathfrak{K} \end{matrix}$$

is a unitary operator. Then the inverse Cayley transform of U given by

$$\widetilde{A} = \left\{ \{(I - U)g, i(I + U)g\} : g \in \mathfrak{M} \oplus \mathfrak{K} \right\}$$

is selfadjoint. Using the equivalence of (2.12) and (2.13) we obtain (2.17) (cf. (1.7)). The uniqueness property of $\widetilde{A}$ holds by the $\mathfrak{M}$-minimality of $\widetilde{A}$ (see (2.14)).

(2) Here the following modifications of the unitary block operator U and the system τ from the proof of (1) are introduced:

$$\tau' = \left\{ U' = \begin{bmatrix} -U_{11} & -U_{12} \\ U_{21} & U_{22} \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{K} \right\}.$$

Clearly, U' is a unitary operator and hence τ' is a conservative system. Moreover, it is seen from (2.16) that τ' is simple precisely when τ is simple. The inverse Cayley transform of U',

$$\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A}) = \left\{ \{(I - U')g, i(I + U')g\} : g \in \mathfrak{M} \oplus \mathfrak{K} \right\},$$

is selfadjoint and (together with $\widetilde{A}$) also $\mathfrak{M}$-minimal (see Proposition 2.2). Moreover, the transfer function $\Psi_{\tau'}(z)$ is given by $\Psi_{\tau'}(z) = -\Psi(z)$. By applying (2.10) to $\Psi_{\tau'}(z)$ one obtains for $\mathcal{M}'(\lambda)$ the representation

$$\mathcal{M}'(\lambda) = \begin{cases} \left\{ \left\{ \left(I - \Psi_{\tau'}\left(\frac{\lambda+i}{\lambda-i}\right)\right)h, -i\left(I + \Psi_{\tau'}\left(\frac{\lambda+i}{\lambda-i}\right)\right)h \right\}, \, h \in \mathfrak{M} \right\}, & \operatorname{Im}\lambda < 0, \\ \left\{ \left\{ \left(I - \Psi_{\tau'}\left(\frac{\bar\lambda+i}{\bar\lambda-i}\right)\right)^* h, i\left(I + \Psi_{\tau'}\left(\frac{\bar\lambda+i}{\bar\lambda-i}\right)\right)^* h \right\}, \, h \in \mathfrak{M} \right\}, & \operatorname{Im}\lambda > 0. \end{cases}$$

Now substitute $\Psi_{\tau'}(z) = -\Psi(z)$ and compare the resulting formula with (2.10) to see that $\mathcal{M}'(\lambda) = -\mathcal{M}(\lambda)^{-1}$. It remains to replace $\widetilde{A}$ by $\widehat{A}$ and $\mathcal{M}(\lambda)$ by $-\mathcal{M}(\lambda)^{-1}$ in the representation (2.17) to get (2.18).

(3) Arguing as above introduce the conservative system

$$\tau'' = \left\{ U'' = \begin{bmatrix} U_{11} & U_{12} \\ -U_{21} & -U_{22} \end{bmatrix}; \mathfrak{M}, \mathfrak{M}, \mathfrak{K} \right\}$$

and its selfadjoint transform

$$\mathfrak{J}_{\mathfrak{K}}(\widetilde{A}) = \big\{\{(I-U'')g, i(I+U'')g\} : g \in \mathfrak{M}\oplus\mathfrak{K}\big\},$$

which is $\mathfrak{M}$-minimal since τ'' (together with τ) is simple, or equivalently $\widetilde{A}$ is $\mathfrak{M}$-minimal. The corresponding transfer function $\Psi_{\tau''}(z)$ is given by $\Psi_{\tau''}(z) = \Psi(-z)$, whose transform $\mathcal{M}''(z)$ has the expression

$$\mathcal{M}''(\lambda) = \begin{cases} \big\{\big\{\big(I-\Psi_{\tau''}\big(\frac{\lambda+i}{\lambda-i}\big)\big)h, -i\big(I+\Psi_{\tau''}\big(\frac{\lambda+i}{\lambda-i}\big)\big)h\big\}, \, h\in\mathfrak{M}\big\}, & \operatorname{Im}\lambda<0,\\ \big\{\big\{\big(I-\Psi_{\tau''}\big(\frac{\bar\lambda+i}{\bar\lambda-i}\big)\big)^*h, i\big(I+\Psi_{\tau''}\big(\frac{\bar\lambda+i}{\bar\lambda-i}\big)\big)^*h\big\}, \, h\in\mathfrak{M}\big\}, & \operatorname{Im}\lambda>0. \end{cases}$$

This means that $\mathcal{M}''(\lambda) = \mathcal{M}(-1/\lambda)$. Finally, replacing $\widetilde{A}$ by $\breve{A} = -\mathfrak{J}_{\mathfrak{K}}(\widetilde{A})$ and $\mathcal{M}(\lambda)$ by $-\mathcal{M}''(-\lambda)$ in (2.17) leads to (2.19). ∎

3. Representations of Stieltjes and inverse Stieltjes families. In Theorem 2.4 expressions for an arbitrary Nevanlinna family $\mathcal{M}(\lambda)$ and its transforms $-\mathcal{M}(\lambda)^{-1}$ and $-\mathcal{M}(1/\lambda)$ were given. In this section we assume in addition that $\mathcal{M}(\lambda)$ is a Stieltjes or an inverse Stieltjes family and construct various representations that take into account the additional properties of $\mathcal{M}(\lambda)$ implied by these further assumptions.

3.1. Stieltjes/inverse Stieltjes families & combined Nevanlinna–Schur class. It this subsection the classes $\widetilde{\mathcal{S}}(\mathfrak{M})$ and $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ of Stieltjes and inverse Stieltjes families are connected with a class of functions that has recently been studied in [7]. The definition reads as follows.

Definition 3.1. Let $\mathfrak{M}$ be a Hilbert space. A $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function Ω which is holomorphic on $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ is said to belong to the class $\mathcal{RS}(\mathfrak{M})$ if

$$-I \le \Omega(x) \le I, \quad x\in(-1,1).$$

It has been proved in [7] that $\mathcal{RS}(\mathfrak{M})$ is a subclass of the Schur functions $\mathbf{S}(\mathfrak{M})$. This means that $\mathcal{RS}(\mathfrak{M})$ consists of functions that are Nevanlinna functions in $\mathbb{C}\setminus\mathbb{R}$ and simultaneously Schur functions on the open unit disk. This class is called the *combined Nevanlinna–Schur class of* $\mathbf{B}(\mathfrak{M})$*-valued operator functions*, which explains the notation $\mathcal{RS}(\mathfrak{M})$, $\mathcal{R}$ standing for R-functions (Nevanlinna functions) and $\mathcal{S}$ for Schur functions. Some further characterizations for $\mathcal{RS}(\mathfrak{M})$ can be found in [7, Theorem 4.1] (see also Appendix).

The next lemma connects the classes $\widetilde{\mathcal{S}}(\mathfrak{M})$ and $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ with $\mathcal{RS}(\mathfrak{M})$. It will be used for some further analysis of Stieltjes and inverse Stieltjes families and it offers a tool for establishing some compressed resolvent formulas for these classes.

LEMMA 3.2. *Let $\Omega \in \mathcal{RS}(\mathfrak{M})$. Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,*

$$(3.1)\quad \mathcal{Q}(\lambda) = -I + 2\Big(I_{\mathfrak{M}} - \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)^{-1} = \Big\{\Big\{\Big(I_{\mathfrak{M}} - \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)h, \Big(I_{\mathfrak{M}} + \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)h\Big\} : h \in \mathfrak{M}\Big\}$$

is a Stieltjes family, and

$$(3.2)\quad \mathcal{R}(\lambda) = I - 2\Big(I_{\mathfrak{M}} + \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)^{-1} = \Big\{\Big\{\Big(I_{\mathfrak{M}} + \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)h, \Big(\Omega\Big(\frac{1+\lambda}{1-\lambda}\Big) - I_{\mathfrak{M}}\Big)h\Big\} : h \in \mathfrak{M}\Big\}$$

is an inverse Stieltjes family.

Conversely, if $\mathcal{Q}(\lambda)$ is a Stieltjes family (resp. $\mathcal{R}(\lambda)$ is an inverse Stieltjes family) in $\mathfrak{M}$, then there exists $\Omega \in \mathcal{RS}(\mathfrak{M})$ such that (3.1) *(resp.* (3.2)*) holds.*

Furthermore, the functions $\mathcal{Q}$ in (3.1) *and $\mathcal{R}$ in* (3.2) *are connected by $\mathcal{R} = -\mathcal{Q}^{-1}$, and thus $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ if and only if $-\mathcal{Q}^{-1} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$.*

Proof. Observe that

$$(3.3)\qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \iff z := \frac{1+\lambda}{1-\lambda} \in \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)),$$

with inverse transform for λ,

$$(3.4)\qquad \lambda = \frac{z-1}{z+1}, \quad \operatorname{Im} \lambda = \frac{2 \operatorname{Im} z}{|z+1|^2}.$$

Now assume that $\Omega(z) \in \mathcal{RS}(\mathfrak{M})$ and let $\mathcal{Q}(\lambda)$ be given by (3.1). Then $\Omega(z)$ is an operator-valued Nevanlinna function with $-I \le \Omega(x) \le I$ for $x \in (-1,1)$. Using (3.1) one obtains

$$(\mathcal{Q}(\lambda) + \mu I)^{-1} = -\frac{1}{1-\mu}\Big(I - \frac{2}{1-\mu}\Big(\frac{1+\mu}{1-\mu}I + \Omega\Big(\frac{1+\lambda}{1-\lambda}\Big)\Big)^{-1}\Big).$$

This shows that $(\mathcal{Q}(\lambda) + \mu I)^{-1}$ admits an analytic continuation to the negative semiaxis $(-\infty,0)$. On the other hand,

$$(3.5)\qquad -I \le \Omega(x) \le I \iff 2(I - \Omega(x))^{-1} \ge I,\ x \in (-1,1),$$

and hence $\mathcal{Q}(\lambda) \ge 0$ for $\lambda < 0$. By Definition 1.2 one concludes that $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$.

A comparison of (3.1) and (3.2) shows that $\mathcal{R}(\lambda) = -\mathcal{Q}^{-1}(\lambda)$. Therefore, $\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$.

Conversely, assume that $\mathcal{Q}(\lambda)$ is a Stieltjes family. Since $\mathcal{Q}(x)$ is a nonnegative selfadjoint relation for $x < 0$, the resolvent $(\mathcal{Q}(x) + I)^{-1} : \mathfrak{M} \to \mathfrak{M}$

is bounded for $x<0$. By assumption, $\mathcal{Q}(\lambda)$ is also a Nevanlinna family which admits a holomorphic continuation to the semiaxis $(-\infty,0)$ in the resolvent sense (see Definition 1.2). It follows that $(\mathcal{Q}(\lambda)+I)^{-1}$ is bounded when $\lambda=x+iy$ is sufficiently close to a real point $x<0$ [30, Chapter VII, §1]. Since $-(\mathcal{Q}(\lambda)+I)^{-1}$ is a Nevanlinna family, boundedness at a single point $\lambda_0\in\mathbb{C}\setminus\mathbb{R}$ implies boundedness of $-(\mathcal{Q}(\lambda)+I)^{-1}$ at every $\lambda\in\mathbb{C}\setminus\mathbb{R}$ (see e.g. [19, Proposition 4.18]). Now define

$$\Omega(z):=I-2\left(I+\mathcal{Q}\left(\frac{z-1}{1+z}\right)\right)^{-1},\qquad z\in\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty)).$$

Then $\Omega(z)$ is a $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function defined on $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ and

$$-I\le\Omega(x)\le I,\qquad x\in(-1,1)$$

(see (3.3)–(3.5)). Hence, $\Omega\in\mathcal{RS}(\mathfrak{M})$ and (3.1) holds.

In the case where $\mathcal{R}(\lambda)$ is from an inverse Stieltjes class one can use the result just proved for Stieltjes families by employing the identity $\mathcal{R}(\lambda)=-\mathcal{Q}^{-1}(\lambda)$, which is clear from (3.1) and (3.2). ■

3.2. Representations by means of compressed resolvents. In this subsection, representation theorems for general Stieltjes and inverse Stieltjes families are established as compressed resolvents along the lines of Theorem 2.4. These again involve transformations of a selfadjoint relation $\widetilde{A}$ which is in addition nonnegative. In this case it is convenient to introduce the following fractional linear transformation of $\widetilde{A}$:

$$T=-I+2(I+\widetilde{A})^{-1}. \tag{3.6}$$

We start with a lemma containing some simple, but useful, observations.

LEMMA 3.3. *Let $\widetilde{A}$ be a nonnegative selfadjoint relation in $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{K}$ and let $\widehat{B}=\mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ be defined by* (1.8). *Then the transforms*

$$T=-I+2(I+\widetilde{A})^{-1},\qquad \widehat{T}=-I+2(I+\widehat{B})^{-1}$$

are contractive, T is selfadjoint, $\widehat{T}$ is $\widehat{J}_{\mathfrak{M}}$-selfadjoint, i.e. $\widehat{J}_{\mathfrak{M}}\widehat{T}=\widehat{T}^*\widehat{J}_{\mathfrak{M}}$, where the fundamental symmetry $\widehat{J}_{\mathfrak{M}}$ is defined by (2.1), *and they have block representations*

$$T=\begin{bmatrix}D & C\\ C^* & F\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{K}\end{matrix}\to\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{K}\end{matrix} \tag{3.7}$$

and

$$\widehat{T}=\widehat{J}_{\mathfrak{M}}T=\begin{bmatrix}-D & -C\\ C^* & F\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{K}\end{matrix}\to\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{K}\end{matrix}. \tag{3.8}$$

Conversely, if T *is a selfadjoint contraction as in* (3.7) *and* $\widehat{T}$ *is given by* (3.8), *then*

$$\widetilde{A} = \big\{\{(I+T)h, (I-T)h\} : h \in \mathfrak{H}\big\} \tag{3.9}$$

is a nonnegative selfadjoint relation in $\mathfrak{H}$ *and the relation*

$$\widehat{B} = \big\{\{(I+\widehat{T})h, (I-\widehat{T})h\} : h \in \mathfrak{H}\big\} \tag{3.10}$$

is maximal accretive and $\widehat{J}_{\mathfrak{M}}$*-selfadjoint in* $\mathfrak{H}$, *and moreover* $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$.

Proof. Since (3.6) is an involution, $T = -I + 2(I+\widetilde{A})^{-1}$ and $\widetilde{A}$ are connected also by (3.9), and $\widehat{T} = -I + 2(I+\widehat{B})^{-1}$ and $\widehat{B}$ are connected by (3.10). It is well known and easy to check that $\widetilde{A}$ is selfadjoint and nonnegative precisely when T is a selfadjoint contraction. Moreover, $\widehat{B}$ is maximal accretive if and only if $\widehat{T}$ is a contraction in $\mathbf{B}(\mathfrak{H})$. On the other hand, by Proposition 2.1, $\widetilde{A}$ is selfadjoint and nonnegative if and only if $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ is $\widehat{J}_{\mathfrak{M}}$-selfadjoint and maximal accretive. If $\widehat{B}$ is $\widehat{J}_{\mathfrak{M}}$-selfadjoint then so is $\widehat{T}$, and conversely.

Now, using (3.7) and (3.9) one gets

$$\widetilde{A} = \left\{\left\{\begin{bmatrix}(I_{\mathfrak{M}}+D)\varphi + Cf\\ C^*\varphi + (I+F)f\end{bmatrix}, \begin{bmatrix}(I_{\mathfrak{M}}-D)\varphi - Cf\\ -C^*\varphi + (I-F)f\end{bmatrix}\right\} : \varphi \in \mathfrak{M},\ f \in \mathfrak{K}\right\},$$

while (3.8) and (3.10) lead to

$$\widehat{B} = \left\{\left\{\begin{bmatrix}(I_{\mathfrak{M}}-D)\varphi - Cf\\ C^*\varphi + (I+F)f\end{bmatrix}, \begin{bmatrix}(I_{\mathfrak{M}}+D)\varphi + Cf\\ -C^*\varphi + (I-F)f\end{bmatrix}\right\} : \varphi \in \mathfrak{M},\ f \in \mathfrak{K}\right\}.$$

One concludes that $\widehat{B}$ and $\widetilde{A}$ are connected by $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widehat{A})$, and conversely, if $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widehat{A})$, then their transforms $\widehat{T}$ and T are connected by $\widehat{T} = \widehat{J}_{\mathfrak{M}}T$ as in (3.8). ■

The contractive operators T and $\widehat{T}$ generate discrete-time passive linear systems which are briefly treated in the Appendix. The next lemma connects $\mathfrak{M}$-minimality of $\widetilde{A}$ and $\widehat{B}$ as in Lemma 3.3 with their simplicity.

LEMMA 3.4. *Let* $\widetilde{A}$, $\widehat{B}$, T, $\widetilde{T}$ *be as in Lemma* 3.3 *and let* $\tau = \{T, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$ *and* $\tau_- = \{\widehat{T}, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$ *be the discrete-time passive linear systems generated by* T *and* $\widehat{T}$, *respectively* (*see Appendix*). *Then the following statements are equivalent:*

(i) $\widetilde{A}$ *is* $\mathfrak{M}$*-minimal;*
(ii) $\widehat{B}$ *is* $\mathfrak{M}$*-minimal;*
(iii) *the system* τ *is simple;*
(iv) *the system* τ_- *is simple.*

Proof. The equivalence of (i) and (ii) was proved in Proposition 2.1(iv). To prove their equivalence to (iii) and (iv) decompose T and $\widehat{T}$ as in (3.7) and (3.8). Then using (3.6) it can be verified that the resolvents of T and $\widetilde{A}$ are connected by

$$(\widetilde{A}-\lambda I)^{-1}=\frac{1}{1-\lambda}(T+I)\left(I-\frac{1+\lambda}{1-\lambda}T\right)^{-1},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}_+. \tag{3.11}$$

Similarly the resolvents of $\widehat{T}$ and $\widehat{B}$ are connected by

$$(\widehat{B}-\lambda I)^{-1}=\frac{1}{1-\lambda}(\widehat{T}+I)\left(I-\frac{1+\lambda}{1-\lambda}\widehat{T}\right)^{-1},\quad \operatorname{Re}\lambda<0. \tag{3.12}$$

These relations combined with (A.4) imply

$$\begin{aligned}\overline{\operatorname{span}}\{\mathfrak{M}+(\widetilde{A}-\lambda I)^{-1}\mathfrak{M}:\lambda\in\mathbb{C}\setminus\mathbb{R}_+\}&=\overline{\operatorname{span}}\{(I-zT)^{-1}\mathfrak{M}:z\in\mathbb{D}\}\\&=\overline{\operatorname{span}}\{T^n\mathfrak{M}:n\in\mathbb{N}_0\}=\mathfrak{M}\oplus\overline{\operatorname{span}}\{F^nC^*\mathfrak{M}:n\in\mathbb{N}_0\}\end{aligned}$$

and

$$\begin{aligned}\overline{\operatorname{span}}\{\mathfrak{M}+(\widehat{B}-\lambda I)^{-1}\mathfrak{M}:\operatorname{Re}\lambda<0\}&=\overline{\operatorname{span}}\{(I-z\widehat{T})^{-1}\mathfrak{M}:z\in\mathbb{D}\}\\&=\overline{\operatorname{span}}\{\widehat{T}^n\mathfrak{M}:n\in\mathbb{N}_0\}=\mathfrak{M}\oplus\overline{\operatorname{span}}\{F^nC^*\mathfrak{M}:n\in\mathbb{N}_0\}.\end{aligned}$$

Since $T=T^*$ and $\widetilde{T}=\widehat{J}_{\mathfrak{M}}T$ has the simple expression (3.8), the equality (A.5) involving T^* and $\widehat{T}^*$ yields the same identities as stated above. Therefore, $\widehat{B}$ is $\mathfrak{M}$-minimal precisely when τ_- is simple, $\widetilde{A}$ is $\mathfrak{M}$-minimal precisely when τ is simple, and $\widehat{B}$ is $\mathfrak{M}$-minimal precisely when τ_- is simple. This proves the remaining equivalences. ∎

We now consider the compressed resolvents of the linear relations $\widetilde{A}$ and $\widehat{B}$ appearing in Lemma 3.3. While the discussion in Section 1 after (1.6) and (1.7) yields the first statement in the next theorem, it is convenient to use here the transformation (3.6) to keep connections visible with other forthcoming statements.

THEOREM 3.5. *Let $\mathfrak{M}$ be a subspace of the Hilbert space $\mathfrak{H}$ and let $\widetilde{A}$ be a nonnegative selfadjoint relation in $\mathfrak{H}=\mathfrak{M}\oplus\mathfrak{K}$. Then:*

(1) *The compressed resolvent $P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{M}$ admits the representation*

$$P_{\mathfrak{M}}(\widetilde{A}-\lambda I)^{-1}\upharpoonright\mathfrak{M}=-(\mathcal{R}(\lambda)+\lambda I_{\mathfrak{M}})^{-1},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}_+, \tag{3.13}$$

with $\mathcal{R}\in\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$.

(2) *If $\widehat{B}$ is a maximal accretive and $\widehat{J}_{\mathfrak{M}}$-selfadjoint relation with respect to the fundamental symmetry $\widehat{J}_{\mathfrak{M}}$ defined in* (2.1), *then the compressed resolvent $P_{\mathfrak{M}}(\widehat{B}-\lambda I)^{-1}\upharpoonright\mathfrak{M}$ admits the representation*

$$P_{\mathfrak{M}}(\widehat{B}-\lambda I)^{-1}\upharpoonright\mathfrak{M}=(\mathcal{Q}(\lambda)-\lambda I_{\mathfrak{M}})^{-1},\quad \operatorname{Re}\lambda<0, \tag{3.14}$$

with $\mathcal{Q}\in\widetilde{\mathcal{S}}(\mathfrak{M})$.

(3) *If* $\widehat{A}$ *is defined as follows (see* (1.8), (1.9)*):*

$$\widehat{A} = \mathfrak{J}_{\mathfrak{M}}\mathfrak{P}_{\mathfrak{M}}(\widehat{B}) = \left\{ \left\{ \begin{bmatrix} -ih \\ f \end{bmatrix}, \begin{bmatrix} ih' \\ f' \end{bmatrix} \right\} : \left\{ \begin{bmatrix} h \\ f \end{bmatrix}, \begin{bmatrix} h' \\ f' \end{bmatrix} \right\} \in \widehat{B} \right\}, \tag{3.15}$$

then $\widehat{A}$ *is a selfadjoint relation and*

$$P_{\mathfrak{M}}(\widehat{A} - \lambda I)^{-1} {\restriction} \mathfrak{M} = -(\mathcal{Q}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.16}$$

where $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ *is as in* (3.14).

(4) *If* $\breve{A} = -\mathfrak{J}_{\mathfrak{K}}(\widehat{A})$*, where* $\widehat{A}$ *is as defined in* (3.15)*, then* $\breve{A}$ *is a selfadjoint nonnegative relation in* $\mathfrak{H}$ *and*

$$P_{\mathfrak{M}}(\breve{A} - \lambda I)^{-1} {\restriction} \mathfrak{M} = -(-\mathcal{Q}(1/\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \tag{3.17}$$

where $-\mathcal{Q}(1/\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ *and* $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ *is as in* (3.14).

Moreover, if $\widetilde{A}$ *in* (3.13) *and* $\widehat{B}$ *in* (3.14) *are connected by* $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$ *then* $\mathcal{Q}(\lambda) = -\mathcal{R}^{-1}(\lambda)$*, and furthermore* $\breve{A} = \widetilde{A}^{-1}$.

Proof. (1) Since $\widetilde{A}$ is selfadjoint and nonnegative the transform $T = -I + 2(I + \widetilde{A})^{-1}$ is a selfadjoint contraction. Decompose T as in (3.7). Then the Schur–Frobenius formula (A.2) for the resolvent of T shows (cf. (A.3))

$$P_{\mathfrak{M}}(I - zT)^{-1} {\restriction} \mathfrak{M} = (I_{\mathfrak{M}} - z\Omega(z))^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)), \tag{3.18}$$

where

$$\Omega(z) = D + zC(I - zF)^{-1}C^*, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

According to [7] the function $\Omega(z)$ belongs to the class $\mathcal{RS}(\mathfrak{M})$; thus, by Lemma 3.2 the transform

$$\mathcal{R}(\lambda) = I - 2\left(I_{\mathfrak{M}} + \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+,$$

is an inverse Stieltjes family. Using (3.2) it is easy to check that

$$\begin{aligned} &(\mathcal{R}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} \\ &\quad = \frac{1}{\lambda - 1}\left(I_{\mathfrak{M}} + \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)\left(I_{\mathfrak{M}} - \frac{1+\lambda}{1-\lambda}\Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1}. \end{aligned} \tag{3.19}$$

On the other hand, it follows from (3.11) that

$$\begin{aligned} P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} {\restriction} \mathfrak{M} &= \frac{1}{1-\lambda} P_{\mathfrak{M}}(T + I)\left(I - \frac{1+\lambda}{1-\lambda}T\right)^{-1} {\restriction} \mathfrak{M} \\ &= \frac{1}{1-\lambda}\left(I_{\mathfrak{M}} + \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)\left(I_{\mathfrak{M}} - \frac{1+\lambda}{1-\lambda}\Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \end{aligned} \tag{3.20}$$

A comparison with (3.19) gives the stated formula (3.13).

(2) Consider the transform (3.6) of $\widehat{B}$, $\widehat{T} := -I + 2(I + \widehat{B})^{-1}$. By Lemma 3.3, $\widehat{T}$ is contractive and $\widehat{J}_{\mathfrak{M}}$-selfadjoint and the operator $T := \widehat{J}_{\mathfrak{M}}\widehat{T}$ is a selfadjoint contraction of the form (3.7), while $\widehat{T} = \widehat{J}_{\mathfrak{M}}T$ has the form (3.8). Define

$$\Omega(z) = D + zC(I - zF)^{-1}C^*, \qquad \Psi(z) = -\Omega(z), \qquad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)). \tag{3.21}$$

Notice that $\Omega \in \mathcal{RS}(\mathfrak{M})$ and Ψ are the transfer functions of the discrete-time passive linear systems $\tau = \{T, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$ and $\tau_- = \{\widehat{T}, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$ in Lemma 3.4 (see Appendix). Again by the Schur–Frobenius formula (A.2),

$$P_{\mathfrak{M}}(I - z\widehat{T})^{-1}{\restriction}\mathfrak{M} = (I_{\mathfrak{M}} - z\Psi(z))^{-1} = (I_{\mathfrak{M}} + z\Omega(z))^{-1}, \qquad z \in \mathbb{D}.$$

On the other hand, the resolvent formula (3.12) implies that for all $\operatorname{Re}\lambda < 0$ (cf. (3.20)),

$$\begin{aligned} &P_{\mathfrak{M}}(\widehat{B} - \lambda I)^{-1}{\restriction}\mathfrak{M} \\ &= \frac{1}{1-\lambda}P_{\mathfrak{M}}(\widehat{T} + I)\left(I - \frac{1+\lambda}{1-\lambda}\widehat{T}\right)^{-1}{\restriction}\mathfrak{M} \\ &= \frac{1}{1-\lambda}\left(I_{\mathfrak{M}} + \Psi\left(\frac{1+\lambda}{1-\lambda}\right)\right)\left(I_{\mathfrak{M}} - \frac{1+\lambda}{1-\lambda}\Psi\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1} \\ &= \frac{1}{1-\lambda}\left(I_{\mathfrak{M}} - \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)\left(I_{\mathfrak{M}} + \frac{1+\lambda}{1-\lambda}\Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1}. \end{aligned} \tag{3.22}$$

By Lemma 3.2 the function $\mathcal{Q}$ defined by

$$\mathcal{Q}(\lambda) = \left\{\left\{\left(I_{\mathfrak{M}} - \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)h, \left(I_{\mathfrak{M}} + \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)h\right\} : h \in \mathfrak{M}\right\}$$

for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ belongs to the Stieltjes class $\mathcal{S}(\mathfrak{M})$ and for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ one obtains

$$(\mathcal{Q}(\lambda) - \lambda I_{\mathfrak{M}})^{-1} = \frac{1}{1-\lambda}\left(I_{\mathfrak{M}} - \Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)\left(I_{\mathfrak{M}} + \frac{1+\lambda}{1-\lambda}\Omega\left(\frac{1+\lambda}{1-\lambda}\right)\right)^{-1}.$$

A comparison with (3.22) leads to (3.14).

(3) Let $\widehat{B}$, $\widehat{T}$ and T be as in the proof of (2) and define

$$\widetilde{A} = \{\{(I + T)h, (I - T)h\} : h \in \mathfrak{H}\}.$$

Then by Lemma 3.3, $\widetilde{A}$ is a nonnegative selfadjoint relation, which is connected with $\widehat{B}$ by $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$, i.e., $\widetilde{A} = \mathfrak{P}_{\mathfrak{M}}(\widehat{B})$ and $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A})$. As was proved in (1) the compressed resolvent $P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1}{\restriction}\mathfrak{M}$ admits the representation (3.13), where the function $\mathcal{R}$ is given by (3.2). On the other hand, the proof of (2) shows that $P_{\mathfrak{M}}(\widehat{B} - \lambda I)^{-1}{\restriction}\mathfrak{M}$ admits the representation

(3.14), where the function $\mathcal{Q}$ is given by (3.1). Hence, $\mathcal{Q}(\lambda) = -\mathcal{R}^{-1}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Since $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A})$, it follows from Theorem 2.4 that

$$(\mathcal{Q}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} = -P_{\mathfrak{M}}(\mathfrak{J}_{\mathfrak{M}}(\widetilde{A}) - \lambda I)^{-1} \upharpoonright \mathfrak{M}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

(4) By assumption $\breve{A} = -\mathfrak{J}_{\mathfrak{K}}(\widehat{A})$, and the proof of (3) shows that $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A})$, where $\widetilde{A} = \mathfrak{P}_{\mathfrak{M}}(\widehat{B})$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. Thus,

$$\breve{A} = -\mathfrak{J}_{\mathfrak{K}}(\widehat{A}) = -\mathfrak{J}_{\mathfrak{K}}\mathfrak{J}_{\mathfrak{M}}(\widetilde{A}) = \widetilde{A}^{-1},$$

and in particular $\breve{A}$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. Moreover, (3.17) follows from Theorem 2.4 and (3.16).

The last assertion is clear from the arguments used above to prove (3) and (4). ■

The next theorem shows that all inverse Stieltjes families $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ can be characterized by the statement (1) in Theorem 3.5.

THEOREM 3.6. *Let $\mathcal{R}$ belong to the inverse Stieltjes class in $\mathfrak{M}$. Then there exists up to unitary equivalence a unique nonnegative selfadjoint relation $\widetilde{A}$ in the Hilbert space $\mathfrak{M} \oplus \mathfrak{K}$ such that $\widetilde{A}$ is $\mathfrak{M}$-minimal and*

$$P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} \upharpoonright \mathfrak{M} = -(\mathcal{R}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{3.23}$$

Proof. Let $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$. Then according to Lemma 3.2 the operator-valued function

$$\Omega(z) := -I + 2\left(I_{\mathfrak{M}} - \mathcal{R}\left(\frac{z-1}{1+z}\right)\right)^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)),$$

is in $\mathcal{RS}(\mathfrak{M})$. By [6, Theorem 4.3] there exists up to unitary equivalence a unique simple passive selfadjoint system $\tau = \{T, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$, where T is a selfadjoint contraction, such that the transfer function of τ coincides with $\Omega(z)$. Then the fractional linear transformation

$$\widetilde{A} = -I + 2(I+T)^{-1} = \big\{\{(I+T)h, (I-T)h\} : h \in \mathfrak{M} \oplus \mathfrak{K}\big\}$$

is a nonnegative selfadjoint relation in $\mathfrak{M} \oplus \mathfrak{K}$. Since the system τ is simple, $\widetilde{A}$ is $\mathfrak{M}$-minimal by Lemma 3.4. Now it is clear that (3.18)–(3.20) hold and thus (3.23) is obtained from Theorem 3.5. For the uniqueness of $\widetilde{A}$ see the discussion in Section 1. ■

For Stieltjes families $\mathcal{R} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ we have the following characterizations.

THEOREM 3.7. *Let $\mathcal{Q}$ belong to the Stieltjes class in $\mathfrak{M}$. Then the following statements hold:*

(1) *There exists up to unitary equivalence a unique maximal accretive linear relation $\widehat{B}$ in the Hilbert space $\mathfrak{M} \oplus \mathfrak{K}$ such that $\widehat{B}$ is $\widehat{J}_{\mathfrak{M}}$-selfadjoint with respect to the fundamental symmetry $\widehat{J}_{\mathfrak{M}}$ in* (2.1), *$\widehat{B}$ is $\mathfrak{M}$-minimal and*

$$P_{\mathfrak{M}}(\widehat{B} - \lambda I)^{-1} \upharpoonright \mathfrak{M} = (\mathcal{Q}(\lambda) - \lambda I_{\mathfrak{M}})^{-1} \tag{3.24}$$

for all λ with $\operatorname{Re} \lambda < 0$.

(2) *There exists up to unitary equivalence a unique selfadjoint relation* $\widehat{A}$ *in* $\mathfrak{M} \oplus \mathfrak{K}$ *such that* $\widehat{A}$ *is* $\mathfrak{M}$*-minimal and its transform* $\mathfrak{J}_{\mathfrak{M}}(\widehat{A})$ *is nonnegative, and*

$$P_{\mathfrak{M}}(\widehat{A} - \lambda I)^{-1}{\upharpoonright}\mathfrak{M} = -(\mathcal{Q}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} \quad \textit{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Furthermore, one can choose $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}\mathfrak{P}_{\mathfrak{M}}(\widehat{B})$*.*

Proof. Let $\mathcal{R} \in \widetilde{\mathcal{S}}(\mathfrak{M})$. Then according to Lemma 3.2 the operator-valued function

$$\Omega(z) := I - 2\left(I_{\mathfrak{M}} + \mathcal{Q}\left(\frac{z-1}{1+z}\right)\right)^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)),$$

belongs to $\mathcal{RS}(\mathfrak{M})$. Again by [6, Theorem 4.3] there exists up to unitary equivalence a unique simple passive selfadjoint system $\tau = \{T, \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\}$ whose transfer function coincides with $\Omega(z)$. Here T is a selfadjoint contraction of the form (3.7) and the associated operator $\widehat{T} := \widehat{J}_{\mathfrak{M}}T$ is a $\widehat{J}_{\mathfrak{M}}$-selfadjoint contraction in $\mathfrak{H}$ of the form (3.8). The corresponding discrete-time linear system τ_- in Lemma 3.4 is also simple and has the transfer function

$$\Psi_{\tau_-}(z) = -\Omega(z), \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$$

(cf. (3.21)). By Lemma 3.3 the transform $\widetilde{A}$ of T defined by (3.9) is nonnegative and selfadjoint and the transform $\widehat{B}$ defined by (3.10) is maximal accretive and $\widehat{J}_{\mathfrak{M}}$-selfadjoint, and $\widetilde{A}$ and $\widehat{B}$ are connected by $\widehat{B} = \mathfrak{P}_{\mathfrak{M}}(\widetilde{A})$. Moreover, by Lemma 3.4, $\widetilde{A}$ and $\widehat{B}$ are $\mathfrak{M}$-minimal. Since we are now in the setting used to prove Theorem 3.5(2), the representation (3.24) is obtained from (3.14). The stated uniqueness property of $\widehat{B}$ is a consequence of its $\mathfrak{M}$-minimality. This completes the proof of (1).

To prove (2) consider the transform $\widehat{A} = \mathfrak{J}_{\mathfrak{M}}(\widetilde{A}) = \mathfrak{J}_{\mathfrak{M}}\mathfrak{P}_{\mathfrak{M}}(\widehat{B})$. Then we may apply Theorem 3.5(3); the stated resolvent formula is immediate from (3.16). According to Proposition 2.2, $\widehat{A}$ is $\mathfrak{M}$-minimal, since $\widetilde{A}$ is $\mathfrak{M}$-minimal, and this implies the uniqueness property of $\widehat{A}$. ∎

Next some further representations for Stieltjes and inverse Stieltjes families will be established by means of specific transformation properties of these families. The basic properties of scalar Stieltjes and inverse Stieltjes functions can be found in [27]. The next lemma is an extension of [27, Lemma S1.5.2, Theorem S1.5.3] from the scalar case to our present general setting.

Lemma 3.8. *With* $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ *the following assertions are equivalent:*

(i) $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$;
(ii) $-\mathcal{Q}^{-1}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$;
(iii) $\lambda\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$.

Proof. The equivalence of (i) and (ii) was proved in Lemma 3.2 (see also Section 1). We now use the same approach to prove the equivalence of (i) and (iii).

Let $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ and $\Omega \in \mathcal{RS}(\mathfrak{M})$ be connected by (3.1). Then a straightforward calculation shows that for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,

$$\lambda \mathcal{Q}(\lambda) = \left\{ \left\{ \left(I + \Upsilon\left(\frac{1+\lambda}{1-\lambda}\right)\right) h, \left(\Upsilon\left(\frac{1+\lambda}{1-\lambda}\right) - I\right) h \right\} : h \in \mathfrak{M} \right\},$$

where $\Upsilon(z)$ is given by the fractional linear transformation

$$\Upsilon(z) = (zI - \Omega(z))(I - z\Omega(z))^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

As shown in [7, Theorem 4.1] the function $\Upsilon(z)$ (together with $\Omega(z)$) belongs to $\mathcal{RS}(\mathfrak{M})$ (see also (A.6)). Therefore, by Lemma 3.2, $\lambda\mathcal{Q}(\lambda)$ is an inverse Stieltjes family. Hence, (i)⇒(iii).

Conversely, let $\lambda\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$. By applying the already proven implications we get

$$\begin{aligned} -(\lambda\mathcal{Q}(\lambda))^{-1} &= -\frac{\mathcal{Q}^{-1}(\lambda)}{\lambda} \in \widetilde{\mathcal{S}}(\mathfrak{M}) \\ &\Longrightarrow \lambda(-\lambda\mathcal{Q}(\lambda))^{-1} = -\mathcal{Q}^{-1}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M}) \implies \mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M}), \end{aligned}$$

which proves (iii)⇒(i). ■

Notice that Lemma 3.8 also gives the equivalence

$$\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M}) \iff \frac{\mathcal{R}(\lambda)}{\lambda} \in \widetilde{\mathcal{S}}(\mathfrak{M}).$$

Therefore, every function $\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ is of the form $\mathcal{R}(\lambda) = \lambda\mathcal{Q}(\lambda)$ for some $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$. Similarly, every function $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$ is of the form $\mathcal{Q}(\lambda) = \mathcal{R}(\lambda)/\lambda$ for some $\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$.

We are now ready to state the following further characterizations for Stieltjes and inverse Stieltjes families.

THEOREM 3.9. *The following representations hold:*

(1) *Let $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$. Then there is a Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and up to unitary equivalence a unique $\mathfrak{M}$-minimal nonnegative selfadjoint relation $\widetilde{A}$ in $\mathfrak{H}$ such that*

$$\mathcal{Q}(\lambda) = -\frac{1}{\lambda}\left(P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} \restriction \mathfrak{M}\right)^{-1} - I_{\mathfrak{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{3.25}$$

(2) *Let $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$. Then there is a Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$ and up to unitary equivalence a unique $\mathfrak{M}$-minimal nonnegative selfadjoint relation $\widetilde{B}$ in $\mathfrak{H}$ such that*

$$\mathcal{R}(\lambda) = I_{\mathfrak{M}} - \left(P_{\mathfrak{M}}(I - \lambda\widetilde{B})^{-1} \restriction \mathfrak{M}\right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{3.26}$$

Moreover, if $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$ *is represented by means of* $\widetilde{A}$ *in* (3.25), *then* $-\mathcal{Q}(\lambda)^{-1}$ *admits the representation* (3.26) *by means of* $\widetilde{B} = \widetilde{A}^{-1}$.

Proof. (1) Let $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$. As indicated, it follows from Lemma 3.8 that $\mathcal{Q}(\lambda) = \mathcal{R}(\lambda)/\lambda$ for some $\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$. Hence, from Theorem 3.6 we obtain the following representation for $\mathcal{R}(\lambda)$:

$$\mathcal{R}(\lambda) = -\big(P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1}{\upharpoonright}\mathfrak{M}\big)^{-1} - \lambda I_{\mathfrak{M}}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+.$$

Dividing this expression by λ yields the representation (3.25) for $\mathcal{Q}(\lambda)$.

(2) The resolvent formula (1.3) gives $\lambda(\widetilde{A} - \lambda I)^{-1} = (I - \lambda\widetilde{A}^{-1})^{-1} - I$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Hence, (3.25) can be rewritten as

$$\mathcal{Q}(\lambda) = P_{\mathfrak{M}}(I - \lambda\widetilde{A}^{-1})^{-1}{\upharpoonright}\mathfrak{M}\big(I_{\mathfrak{M}} - P_{\mathfrak{M}}(I - \lambda\widetilde{A}^{-1})^{-1}{\upharpoonright}\mathfrak{M}\big)^{-1}$$

and thus

$$-\mathcal{Q}(\lambda)^{-1} = \big(P_{\mathfrak{M}}(I - \lambda\widetilde{A}^{-1})^{-1}{\upharpoonright}\mathfrak{M} - I_{\mathfrak{M}}\big)\big(P_{\mathfrak{M}}(I - \lambda\widetilde{A}^{-1})^{-1}{\upharpoonright}\mathfrak{M}\big)^{-1}.$$

This leads to (3.26) with the choices $\mathcal{R}(\lambda) = -\mathcal{Q}(\lambda)^{-1}$ and $\widetilde{B} = \widetilde{A}^{-1}$. To complete the proof observe that $\widetilde{A}^{-1}$ is $\mathfrak{M}$-minimal if and only if $\widetilde{A}$ is $\mathfrak{M}$-minimal. Indeed, if $T = -I + 2(I + \widetilde{A})^{-1}$ then $-T = -I + 2(I + \widetilde{A}^{-1})^{-1}$ (cf. (3.6), (3.9)). Now the claim follows from Lemma 3.4, since $-T$ is simple precisely when T is simple. ∎

3.3. Nevanlinna families as Weyl families of boundary relations. Let $\mathfrak{K}$ be a Hilbert space and define

$$J_{\mathfrak{K}} = \begin{bmatrix} 0 & -iI_{\mathfrak{K}} \\ iI_{\mathfrak{K}} & 0 \end{bmatrix} : \begin{array}{c} \mathfrak{K} \\ \oplus \\ \mathfrak{K} \end{array} \to \begin{array}{c} \mathfrak{K} \\ \oplus \\ \mathfrak{K} \end{array}.$$

The operator $J_{\mathfrak{K}}$ is a *fundamental symmetry* ($J_{\mathfrak{K}} = J_{\mathfrak{K}}^* = J_{\mathfrak{K}}^{-1}$) in the Hilbert space $\mathfrak{K}^2 := \mathfrak{K}\oplus\mathfrak{K}$. Define a linear transformation from $\mathfrak{M}^2\oplus\mathfrak{K}^2$ into $(\mathfrak{M}\oplus\mathfrak{K})^2$ by

$$\mathcal{J} : \left\{ \begin{bmatrix} \varphi \\ \varphi' \end{bmatrix}, \begin{bmatrix} f \\ f' \end{bmatrix} \right\} \mapsto \left\{ \begin{bmatrix} \varphi \\ f \end{bmatrix}, \begin{bmatrix} -\varphi' \\ f' \end{bmatrix} \right\}, \qquad \varphi, \varphi' \in \mathfrak{M},\ f, f' \in \mathfrak{K};$$

for this and for further results and details in this connection we refer to [19]. The formula $\widetilde{A} = \mathcal{J}(\mathit{\Gamma})$ for the so-called *main transform* establishes a one-to-one correspondence between all unitary relations $\mathit{\Gamma}$ from the Kreĭn space $\langle\mathfrak{K}^2, J_{\mathfrak{K}}\rangle$ into the Kreĭn space $\langle\mathfrak{M}^2, J_{\mathfrak{M}}\rangle$ and all selfadjoint relations $\widetilde{A}$ in $(\mathfrak{M} \oplus \mathfrak{K})^2$. If $\mathit{\Gamma}$ is a unitary relation from $\langle\mathfrak{K}^2, J_{\mathfrak{K}}\rangle$ into $\langle\mathfrak{M}^2, J_{\mathfrak{M}}\rangle$, then $\mathit{\Gamma}$ is called a *boundary relation* of S^*, where $S = \ker \mathit{\Gamma}$. A boundary relation $\mathit{\Gamma}$ is called *minimal* if $\widetilde{A} = \mathcal{J}(\mathit{\Gamma})$ is $\mathfrak{M}$-minimal.

Let $\mathcal{T} = \operatorname{dom} \Gamma$. Then $\mathcal{T} \subset \mathfrak{K}^2$ is a linear relation in $\mathfrak{K}$. Define $\mathfrak{N}_\lambda(\mathcal{T}) := \ker(\mathcal{T} - \lambda I)$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and denote

$$\widehat{\mathfrak{N}}_\lambda(\mathcal{T}) = \{\{g_\lambda, \lambda g_\lambda\} : g_\lambda \in \mathfrak{N}_\lambda(\mathcal{T})\}.$$

The *Weyl family* $\mathcal{W}(\lambda)$ associated with Γ is defined as follows [19]:

$$\mathcal{M}(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(\mathcal{T})), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

According to [19, Theorem 3.9] there is (up to unitary equivalence) a unique minimal boundary relation $\Gamma : \mathfrak{K}^2 \to \mathfrak{M}^2$ (where $\mathfrak{K}$ is some Hilbert space) whose Weyl family coincides with the given Nevanlinna family $\mathcal{M}(\lambda)$ in $\mathfrak{M}$. In terms of compressed resolvent this means that there is a unique (up to unitary equivalence) $\mathfrak{M}$-minimal selfadjoint relation $\widetilde{A}$ in the Hilbert space $\mathfrak{M} \oplus \mathfrak{K}$ (which is the main transform $\mathcal{J}(\Gamma)$) such that (see [19])

$$P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} {\restriction} \mathfrak{M} = -(\mathcal{M}(\lambda) + \lambda I)^{-1}, \qquad \operatorname{Im} \lambda \neq 0.$$

The latter is equivalent to (2.17).

Let $\widetilde{A}$ be a selfadjoint relation in the Hilbert space $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{K}$. For $\lambda \in \mathbb{C}$ define

$$\mathcal{M}(\lambda) := \left\{ \{\varphi_\lambda, -\varphi'_\lambda\} : \left\{ \begin{bmatrix} \varphi_\lambda \\ f_\lambda \end{bmatrix}, \begin{bmatrix} \varphi'_\lambda \\ \lambda f_\lambda \end{bmatrix} \right\} \in \widetilde{A},\ f_\lambda \in \mathfrak{K},\ \varphi_\lambda, \varphi'_\lambda \in \mathfrak{M} \right\}.$$

Then for $\lambda \in \rho(\widetilde{A})$ one has

$$\mathcal{M}(\lambda) = \left\{ \{P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} m, -\lambda P_{\mathfrak{M}}(\widetilde{A} - \lambda I)^{-1} m - m\} : m \in \mathfrak{M} \right\}.$$

Since the transformation

$$\Gamma^T = \begin{bmatrix} 0 & iI \\ -iI & 0 \end{bmatrix} \Gamma$$

defines a unitary relation as well, its main transform $\widetilde{A}' = \mathcal{J}(\Gamma^T)$ is a selfadjoint relation. The connection between $\widetilde{A}'$ and $\widetilde{A}$ is given by

$$\left\{ \begin{bmatrix} h \\ f \end{bmatrix}, \begin{bmatrix} h' \\ f' \end{bmatrix} \right\} \in \widetilde{A} \iff \left\{ \begin{bmatrix} -ih' \\ f \end{bmatrix}, \begin{bmatrix} ih \\ f' \end{bmatrix} \right\} \in \widetilde{A}'.$$

The Weyl family of Γ^T coincides with $-\mathcal{M}^{-1}(\lambda)$ (see [19]). These facts and formulas lead to various interpretations for the results on compressed resolvents appearing in the present paper. We will discuss these connections in more detail elsewhere (cf. also [5]).

The notion of boundary relation is a generalization of the notion of a space of boundary values or a boundary triplet. We will now recall some basic notions, since they will be used in some examples.

Let S be a closed densely defined symmetric operator with equal defect numbers in $\mathfrak{K}$. Let $\mathfrak{M}$ be some Hilbert space, Γ_0 and Γ_1 be linear mappings of $\operatorname{dom} S^*$ into $\mathfrak{M}$. A triplet $\{\mathfrak{M}, \Gamma_0, \Gamma_1\}$ is called a *space of boundary values* (s.b.v.) or an *ordinary boundary triplet* for S^* (see [19, 21, 22, 25]) if

(a) for all $x, y \in \operatorname{dom} S^*$ the *Green identity*

$$(S^*x, y) - (x, S^*y) = (\Gamma_1 x, \Gamma_0 y)_{\mathfrak{M}} - (\Gamma_0 x, \Gamma_1 y)_{\mathfrak{M}}$$

holds, and

(b) the mapping

$$\operatorname{dom} S^* \ni x \mapsto \Gamma x = \{\Gamma_0 x, \Gamma_1 x\} \in \mathfrak{M} \times \mathfrak{M}$$

is surjective.

From this definition it follows that $\ker \Gamma_k \supset \mathcal{D}(S)$, $k = 0, 1$, the operators

$$A_0 = S^* {\upharpoonright} \ker \Gamma_0, \qquad A_1 = S^* {\upharpoonright} \ker \Gamma_1$$

are selfadjoint extensions of S, and moreover they are transversal:

$$\operatorname{dom} S^* = \operatorname{dom} A_0 + \operatorname{dom} A_1.$$

The function

$$M(\lambda)(\Gamma_0 x_\lambda) = \Gamma_1 x_\lambda, \qquad x_\lambda \in \mathfrak{N}_\lambda,$$

where $\mathfrak{N}_\lambda$ is a defect subspace of S, is called the *Weyl function* of the boundary triplet (cf. [19, 21, 22]). If

$$\gamma(\lambda) := (\Gamma_0 {\upharpoonright} \mathfrak{N}_\lambda)^{-1},$$

then $M(\lambda) = \Gamma_1 \gamma(\lambda)$. The main transform of an ordinary boundary triplet is a selfadjoint operator in $\mathfrak{M} \oplus \mathfrak{H}$ and it is determined by (cf. [5])

$$\widetilde{A}\begin{bmatrix} \Gamma_0 f \\ f \end{bmatrix} = \begin{bmatrix} -\Gamma_1 f \\ S^* f \end{bmatrix}.$$

4. Further properties of Stieltjes and inverse Stieltjes families. In this section a couple of special types of Stieltjes and inverse Stieltjes families are studied. First an analog for the notion of an inner function is introduced in the setting of such families, and then inner families in the classes $\widetilde{\mathcal{S}}(\mathfrak{M})$ and $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ are characterized. Then we study Stieltjes and inverse Stieltjes families which admit certain scaling invariance properties. We also investigate some qualitative properties of the following two mappings arising from Lemma 3.8:

$$\Phi_+ : \mathcal{Q}(\lambda) \mapsto -\frac{\mathcal{Q}(\lambda)^{-1}}{\lambda} \quad \text{and} \quad \Phi_- : \mathcal{R}(\lambda) \mapsto -\lambda \mathcal{R}(\lambda)^{-1}.$$

Notice that by Lemma 3.8, Φ_+ maps $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ back into $\widetilde{\mathcal{S}}(\mathfrak{M})$, while Φ_- maps $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ back into $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$. The fixed points of Φ_+ in $\widetilde{\mathcal{S}}(\mathfrak{M})$ and of Φ_- in $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ will be described.

4.1. Inner functions in Stieltjes and inverse Stieltjes classes. Recall that an operator-valued Schur function is said to be *inner/co-inner/bi-inner* if almost everywhere on the unit disk the nontangential limit values to the unit circle $\mathbb{T}$ are, respectively, isometric/co-isometric/unitary. It is proved in [7] that a function $\Omega \in \mathcal{RS}(\mathfrak{M})$ is inner if and only if it admits the representation

$$\Omega(z) = (zI + \widetilde{D})(I + z\widetilde{D})^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)), \tag{4.1}$$

where $\widetilde{D}$ is a selfadjoint contraction in $\mathfrak{M}$.

The Stieltjes class $\mathcal{S}(\mathfrak{M})$ and the inverse Stieltjes class $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ are connected with the class $\mathcal{RS}(\mathfrak{M})$ as described in Lemma 3.2. Notice that (cf. (3.4))

$$\mathrm{Re}\,\lambda = \frac{|z|^2 - 1}{|z+1|^2}, \quad \lambda = \frac{z-1}{z+1}.$$

In particular, the transform $z \mapsto \frac{z-1}{z+1}$ maps the nonreal part $\mathbb{T} \setminus \{1, -1\}$ of the unit circle bijectively onto $\{iy : y \in \mathbb{R},\, y \neq 0\}$, i.e. the imaginary axis excluding the origin. This motivates the following definition.

Definition 4.1. A family S in $\widetilde{\mathcal{S}}(\mathfrak{M})$ (or $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$) is said to be *inner* if

$$\mathrm{Re}\,(S(iy)f, f) = 0, \quad f \in \mathrm{dom}\, S(iy),$$

for all $y \in \mathbb{R} \setminus \{0\}$.

Theorem 4.2. *All inner families in Stieltjes and inverse Stieltjes classes are described as follows:*

(1) *The inner families in* $\widetilde{\mathcal{S}}(\mathfrak{M})$ *are of the form*

$$\mathcal{Q}(\lambda) = -\lambda^{-1}\mathcal{B}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \tag{4.2}$$

where $\mathcal{B}$ *runs through the set of all nonnegative selfadjoint relations in* $\mathfrak{M}$.

(2) *The inner families in* $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ *are of the form*

$$\mathcal{R}(\lambda) = \lambda\mathcal{C}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \tag{4.3}$$

where $\mathcal{C}$ *runs through the set of all nonnegative selfadjoint relations in* $\mathfrak{M}$.

Proof. Let $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$. Then by Lemma 3.2 the function

$$\Omega(z) := I - 2\left(I + \mathcal{Q}\left(\frac{z-1}{1+z}\right)\right)^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)),$$

belongs to $\mathcal{RS}(\mathfrak{M})$ and satisfies $\mathcal{Q}(\lambda) = (I + \Omega(z))(I - \Omega(z))^{-1}$ with $z = \frac{1+\lambda}{1-\lambda}$. Since

$$\mathrm{Re}\,(\mathcal{Q}(\lambda)f, f) = \big([I - \Omega(z)^*\Omega(z)](I - \Omega(z))^{-1}f, (I - \Omega(z))^{-1}f\big), \quad f \in \mathrm{dom}\,\mathcal{Q}(\lambda), \tag{4.4}$$

we conclude that $\mathcal{Q}$ is inner if and only if $\varOmega$ is inner. Therefore, $\varOmega$ has the expression (4.1) and hence its transform $\mathcal{Q}$ takes the form (4.2) with $\mathcal{B} = (I + \widetilde{D})(I - \widetilde{D})^{-1}$.

Similarly, using the transform (3.2) from Lemma 3.2 one derives from (4.1) the formula (4.3) with $\mathcal{C} = (I - \widetilde{D})(I + \widetilde{D})^{-1}$. ∎

REMARK 4.3. Let P be an orthogonal projection in $\mathfrak{M}$. Then the constant family

$$\mathcal{M}(\lambda) = \big\{\{Pf, (I-P)f\} : f \in \mathfrak{M}\big\}$$

is an inner Stieltjes family and an inner inverse Stieltjes family simultaneously.

4.2. Scale invariant Stieltjes and inverse Stieltjes families

DEFINITION 4.4. A Nevanlinna family $\mathcal{M}$ in the Hilbert space $\mathfrak{M}$ is said to be *scale invariant* if for some $c \in \mathbb{R}_+ \setminus \{1\}$ and some $p \in \{0, 1, -1\}$,

$$\mathcal{M}(c\lambda) = c^p \mathcal{M}(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

As shown in the next theorem, scale invariant Stieltjes and inverse Stieltjes families admit specific characterizations for each of the choices $p = 0$, $p = 1$, and $p = -1$.

THEOREM 4.5. *Let $\mathfrak{M}$ be a Hilbert space and let $c \in \mathbb{R}_+ \setminus \{1\}$.*

(1) *Each Stieltjes family in $\mathfrak{M}$ satisfying $\mathcal{Q}(c\lambda) = \mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is of the form $\mathcal{Q}(\lambda) \equiv \mathcal{A}$, where $\mathcal{A}$ is a nonnegative selfadjoint relation.*
(2) *Each inverse Stieltjes family in $\mathfrak{M}$ satisfying $\mathcal{R}(c\lambda) = \mathcal{R}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is of the form $\mathcal{R}(\lambda) \equiv -\mathcal{A}$, where $\mathcal{A}$ is a nonnegative selfadjoint relation.*
(3) *Each Stieltjes family in $\mathfrak{M}$ satisfying $\mathcal{Q}(c\lambda) = c^{-1}\mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$ is of the form*

$$\mathcal{Q}(\lambda) = -\lambda^{-1}\mathcal{B},$$

where $\mathcal{B}$ is a nonnegative selfadjoint relation.
(4) *Each inverse Stieltjes family in $\mathfrak{M}$ satisfying $\mathcal{R}(c\lambda) = c\mathcal{R}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ is of the form*

$$\mathcal{R}(\lambda) = \lambda\,\mathcal{C},$$

where $\mathcal{C}$ is a nonnegative selfadjoint relation.
(5) *Each Stieltjes family $\mathcal{Q}$ in $\mathfrak{M}$ satisfying $\mathcal{Q}(c\lambda) = c\mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}_+$, and each inverse Stieltjes family $\mathcal{R}$ in $\mathfrak{M}$ satisfying $\mathcal{R}(c\lambda) = c^{-1}\mathcal{R}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, is of the form*

$$\mathcal{Q}(\lambda) \equiv \{\{Pf, (I-P)f\}\},$$

where P is an orthogonal projection in $\mathfrak{M}$.

Proof. First notice that if in $z = \frac{1+\lambda}{1-\lambda}$ one replaces λ by $c\lambda$, then

$$\frac{1+c\lambda}{1-c\lambda} = \frac{z+a}{1+za}, \quad \text{where} \quad a = \frac{1-c}{1+c} \in (-1,1), \ c > 0. \tag{4.5}$$

Hence, if $\mathcal{Q}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$ or $\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ is represented by (3.1) or (3.2) with $\Omega(z) \in \mathcal{RS}(\mathfrak{M})$, then $\mathcal{Q}(c\lambda)$ and $\mathcal{R}(c\lambda)$ are represented by means of $\Omega\left(\frac{z+a}{1+za}\right)$. Clearly the condition $c \neq 1$ is equivalent to $a \neq 0$.

On the other hand, if $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ and $\Omega(z)$ (see Lemma 3.2) is defined by

$$\Omega(z) = I - 2\left(I + \mathcal{Q}\left(\frac{z-1}{z+1}\right)\right)^{-1} \in \mathcal{RS}(\mathfrak{M}), \tag{4.6}$$

then the function

$$\Psi(z) = I - 2\left(I + c\mathcal{Q}\left(\frac{z-1}{z+1}\right)\right)^{-1}, \quad z \in \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)),$$

can be expressed in the form

$$\Psi(z) = (\Omega(z) - aI_{\mathfrak{M}})(I_{\mathfrak{M}} - a\Omega(z))^{-1},$$

where a is given by (4.5). Similarly,

$$I - 2\left(I + \frac{1}{c}\mathcal{Q}\left(\frac{z-1}{z+1}\right)\right)^{-1} = (\Omega(z) + aI_{\mathfrak{M}})(I_{\mathfrak{M}} + a\Omega(z))^{-1}.$$

We now prove the assertions (1)–(5) in three parts.

Verification of (1) & (2). By the transformations in Lemma 3.2 and the observations just made above, the equalities $\mathcal{Q}(c\lambda) = \mathcal{Q}(\lambda)$ with $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ and $\mathcal{R}(c\lambda) = \mathcal{R}(\lambda)$ with $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, are equivalent to the equality

$$\Omega\left(\frac{z+a}{1+za}\right) = \Omega(z) \quad \forall z \in \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty))$$

for $\Omega \in \mathcal{RS}(\mathfrak{M})$. According to [7, Proposition 6.13] this equality with $a \neq 0$ has only constant solutions. Hence, $\Omega(z) \equiv \Omega(0)$ is a selfadjoint contraction in $\mathfrak{M}$. Here

$$-I \leq \Omega(0) \leq I \iff 2(I - \Omega(0))^{-1} \geq I,$$

and now applying Lemma 3.2 once again, we deduce (1) and (2).

Verification of (3) & (4). By the above observations, the equality $\mathcal{Q}(c\lambda) = c^{-1}\mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ with $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ is equivalent to

$$\Omega\left(\frac{z+a}{1+za}\right) = (\Omega(z) + aI_{\mathfrak{M}})(I_{\mathfrak{M}} + a\Omega(z))^{-1}, \tag{4.7}$$

$$z \in \mathbb{C} \setminus ((-\infty,-1] \cup [1,\infty)),$$

where $\Omega(z)$ is given by (4.6). It is proved in [7, Theorem 6.18] that the solutions to (4.7) with $a \neq 0$ consist of the inner functions from $\mathcal{RS}(\mathfrak{M})$. Since $\mathcal{Q}$ is inner if and only if Ω is inner (see (4.4)), we conclude from Theorem 4.2 that $\mathcal{Q}(\lambda) = -\lambda^{-1}\mathcal{B}$, where $\mathcal{B}$ is a nonnegative selfadjoint realtion in $\mathfrak{M}$. On the other hand, the equality $\mathcal{R}(c\lambda) = c\mathcal{R}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ with $\mathcal{R} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ is equivalent to

$$-\mathcal{R}^{-1}(c\lambda) = c^{-1}(-\mathcal{R}^{-1}(\lambda)) \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}_+,$$

where $-\mathcal{R}^{-1} \in \widetilde{\mathcal{S}}(\mathfrak{M})$. This implies that $\mathcal{R}(\lambda) = \lambda\mathcal{C}$, where $\mathcal{C} = \mathcal{B}^{-1}$ is a nonnegative selfadjoint relation.

Verification of (5). Suppose that a Stieltjes family $\mathcal{Q}$ satisfies $\mathcal{Q}(c\lambda) = c\mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ with some $c > 0$, $c \neq 1$. Recall that $\mathcal{Q}(\lambda)$ is a Nevanlinna family with a holomorphic continuation to the negative semiaxis. Then $\mathcal{Q}(x)$ is a monotonically nondecreasing function of the negative semiaxis and by assumption $\mathcal{Q}(x) \geq 0$ for all $x < 0$. Now fix $x < 0$. By assumption $cQ(x) = Q(cx)$ with $c \neq 1$. In particular, $\operatorname{dom} \mathcal{Q}(x) = \operatorname{dom} \mathcal{Q}(cx)$ and moreover $\mathcal{Q}(x)$ and $\mathcal{Q}(cx)$ generate closed nonnegative forms with the same form domain. If, for instance, $c > 1$ then $cx < x$ and using monotonicity we conclude that

$$0 \leq c\mathcal{Q}(x) = \mathcal{Q}(cx) \leq \mathcal{Q}(x),$$

and hence $0 \leq (c-1)\mathcal{Q}(x) \leq 0$ (for a proper meaning of monotonicity in this general setting we refer to [14]). This implies that for all $f \in \operatorname{dom} \mathcal{Q}(x)$ one has $\mathcal{Q}(x)f = 0$. Thus $\operatorname{dom} \mathcal{Q}(x) = \ker \mathcal{Q}(x)$, i.e., $\mathcal{Q}(x)$ is a singular relation. Since $\mathcal{Q}(x)$ is selfadjoint, $\operatorname{mul} \mathcal{Q}(x) = \operatorname{dom} \mathcal{Q}(x)^{\perp}$ and we conclude that

$$\mathcal{Q}(x) = \left\{ \{Pf, (I_{\mathfrak{M}} - P)f\} : f \in \mathfrak{M} \right\},$$

where P stands for the orthogonal projection onto $\overline{\operatorname{dom}}\, \mathcal{Q}(x) = \operatorname{dom} \mathcal{Q}(x)$. However, $\mathcal{Q}(x)$ is continuous (by assumption even holomorphic) as a function of x and therefore the projector P cannot depend on $x < 0$ (by a general principle concerning continuous paths of projectors on connected sets). Furthermore, by holomorphy we conclude that $\mathcal{Q}(\lambda) \equiv \left\{ \{Pf, (I - P)f\} \right\}$.

In the same way one treats the case $0 < c < 1$. Therefore, the statement for Stieltjes families is proven. The statement concerning inverse Stieltjes families is obtained by passing to inverses. ■

Remark 4.6. (i) There is a connection between assertions (1) and (4) and assertions (2) and (3) in Theorem 4.5, which can be seen by means of Lemma 3.8. For instance, assume that $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ satisfies $\mathcal{Q}(c\lambda) = \mathcal{Q}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Then by Lemma 3.8 the function $\mathcal{R}(\lambda) := \lambda\mathcal{Q}(\lambda)$ belongs to $\widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ and satisfies

$$\mathcal{R}(c\lambda) = c\lambda\mathcal{Q}(c\lambda) = c\lambda\mathcal{Q}(\lambda) = c\mathcal{R}(\lambda).$$

Conversely, if $\mathcal{R}(c\lambda) = c\mathcal{R}(\lambda)$ then the function $\mathcal{Q}(\lambda) := \lambda^{-1}\mathcal{R}(\lambda) \in \widetilde{\mathcal{S}}(\mathfrak{M})$ satisfies $\mathcal{Q}(c\lambda) = \mathcal{Q}(\lambda)$. Hence, if e.g. (1) holds, then $\mathcal{Q}(\lambda) \equiv \mathcal{A}$, where $\mathcal{A}$ is a nonnegative selfadjoint relation. This means that $\mathcal{R}(\lambda) = \lambda\mathcal{Q}(\lambda) = \lambda\mathcal{A}$. Similarly, we get the connection between assertions (2) and (3) in Theorem 4.5.

(ii) The intersection of the Stieltjes and inverse Stieltjes classes in $\mathfrak{M}$ coincides with the class of constant functions of the form appearing in part (5) of Theorem 4.5:

$$\mathcal{Q}(\lambda) = \big\{\{Pf, (I_{\mathfrak{M}} - P)f\} : f \in \mathfrak{M}\big\}.$$

This is clear from the conditions $\mathcal{Q}(x) \geq 0$ for Stieltjes families and $\mathcal{Q}(x) \leq 0$ for inverse Stieltjes families, which imply that $(\varphi', \varphi) = 0$ for all $\{\varphi, \varphi'\} \in \mathcal{Q}(x)$. Hence, $\operatorname{dom}\mathcal{Q}(x) = \ker\mathcal{Q}(x)$ and the claim follows (cf. the proof of Theorem 4.5(5)).

(iii) Theorem 4.5(5) can also be obtained by making a connection with the class $\mathcal{RS}(\mathfrak{M})$. Namely, the equality $\mathcal{Q}(c\lambda) = c\mathcal{Q}(\lambda)$ for the Stieltjes family $\mathcal{Q}$ in $\mathfrak{M}$ with $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ and $c \neq 1$ is equivalent to

$$(\varOmega(z) - aI_{\mathfrak{M}})(I_{\mathfrak{M}} - a\varOmega(z))^{-1} = \varOmega\left(\frac{z+a}{1+za}\right) \qquad \forall z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)),$$

where $\varOmega(z)$ is given by (4.6) and $a \neq 0$. It is proved in [7, Theorem 6.19] that the only solutions to this equation are the constant functions $\varOmega(z) \equiv D$, $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$, where D is a fundamental symmetry in $\mathfrak{M}$. Because $D = I - 2P$, where P is an orthogonal projection in $\mathfrak{M}$, this yields

$$\mathcal{Q}(\lambda) = \big\{\{Pf, (I_{\mathfrak{M}} - P)f\} : f \in \mathfrak{M}\big\}.$$

4.3. The mappings $\varPhi_+$, $\varPhi_-$ and their fixed points. Recall that by Lemma 3.8 the transformation

$$\widetilde{\mathcal{S}}(\mathfrak{M}) \ni \mathcal{Q}(\lambda) \overset{\varPhi_+}{\longmapsto} \widetilde{\mathcal{Q}}(\lambda) := -\mathcal{Q}(\lambda)^{-1}/\lambda \in \widetilde{\mathcal{S}}(\mathfrak{M})$$

is well defined. In fact, $\varPhi_+$ is an automorphism of $\widetilde{\mathcal{S}}(\mathfrak{M})$. Analogously, the transformation

$$\widetilde{\mathcal{S}}^{-1}(\mathfrak{M}) \ni \mathcal{R}(\lambda) \overset{\varPhi_-}{\longmapsto} \widetilde{\mathcal{R}}(\lambda) := -\lambda\mathcal{R}(\lambda)^{-1} \in \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$$

is an automorphism of $\widetilde{\mathcal{S}}^{-1}(\mathfrak{M})$. Here the main purpose is to find the fixed points of these two mappings.

Proposition 4.7. *Let $\varPhi_+ : \widetilde{\mathcal{S}}(\mathfrak{M}) \to \widetilde{\mathcal{S}}(\mathfrak{M})$ and $\varPhi_- : \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M}) \to \widetilde{\mathcal{S}}^{(-1)}(\mathfrak{M})$ be as defined above. Then with $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$:*

(1) *the mapping $\varPhi_+$ has a unique fixed point*

$$\mathcal{Q}_0(\lambda) = \frac{i}{\sqrt{\lambda}} I_{\mathfrak{M}}, \qquad \mathcal{Q}_0(-1) = I_{\mathfrak{M}};$$

(2) *the mapping* Φ_- *has a unique fixed point*

$$\mathcal{R}_0(\lambda) = i\sqrt{\lambda}\, I_{\mathfrak{M}}, \qquad \mathcal{R}_0(-1) = -I_{\mathfrak{M}}.$$

Proof. (1) Let $\mathcal{Q} \in \widetilde{\mathcal{S}}(\mathfrak{M})$ and consider the equation

$$\mathcal{Q}(\lambda) = -\mathcal{Q}(\lambda)^{-1}/\lambda, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{4.8}$$

Then $\ker \mathcal{Q}(\lambda) = \operatorname{mul} \mathcal{Q}(\lambda)$ and $\operatorname{dom} \mathcal{Q}(\lambda) = \operatorname{ran} \mathcal{Q}(\lambda)$. In particular, since $\mathcal{Q}(x)$ is selfadjoint for $x < 0$, one has $\ker \mathcal{Q}(\lambda) \perp \operatorname{mul} \mathcal{Q}(\lambda)$ and hence $\ker \mathcal{Q}(\lambda) = \operatorname{mul} \mathcal{Q}(\lambda) = \{0\}$. Moreover, $\operatorname{dom} \mathcal{Q}(x) = \operatorname{ran} \mathcal{Q}(x)$ implies that $\mathcal{Q}(x)$ is a bounded selafadjoint operator. Then by holomorphy $\mathcal{Q}(\lambda)$ is bounded when $\operatorname{Im} \lambda$ is sufficiently small, and thus by [19, Proposition 4.18], $\mathcal{Q}(\lambda) \in \mathbf{B}(\mathfrak{M})$ for all $\mathbb{C} \setminus \mathbb{R}_+$. Moreover, $\ker \mathcal{Q}(\lambda) = \operatorname{mul} \mathcal{Q}(\lambda) = \{0\}$ for all $\mathbb{C} \setminus \mathbb{R}_+$. This implies that (4.8) is equivalent to

$$\mathcal{Q}(\lambda)^2 = -1/\lambda. \tag{4.9}$$

Since $\mathcal{Q}(\lambda)$ is a Nevanlinna function and is holomorphic on the simply connected set $\mathcal{D} \setminus \mathbb{R}_+$, the unique solution to (4.9) is

$$\mathcal{Q}(\lambda) = (i/\sqrt{\lambda}) I_{\mathfrak{M}}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+.$$

(2) This is obtained from (1) by passing to inverses. ∎

Remark 4.8. The equation (4.8) is equivalent to

$$\lambda \mathcal{Q}(\lambda) = -\mathcal{Q}(\lambda)^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \tag{4.10}$$

The functions on both sides of (4.10) are inverse Stieltjes families by Lemma 3.8. Applying (3.2) to (4.10) leads to the equivalent condition

$$(zI_{\mathfrak{M}} - \Omega(z))(I_{\mathfrak{M}} - z\Omega(z))^{-1} = \Omega(z).$$

It is shown in [7, Proposition 6.6] that the unique solution to this last equation in $\mathcal{RS}(\mathfrak{M})$ is the function

$$\Omega_0(z) = \frac{zI_{\mathfrak{M}}}{1 + \sqrt{1 - z^2}}, \qquad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

A straightforward calculation shows that $\Omega_0(z)$ and $\mathcal{Q}_0(\lambda)$ in Proposition 4.7 are connected by (3.1).

To complete this section we construct realizations for the functions $\mathcal{Q}_0$ and $\mathcal{R}_0$ in Proposition 4.7. In this way we simultaneously illustrate the general results obtained in this paper.

(A) We illustrate Theorems 3.5–3.7 by treating an L_2-model for the functions $\mathcal{Q}_0$ and $\mathcal{R}_0$. Let $\mathfrak{M}$ be a Hilbert space and consider the weighted Hilbert space $\mathfrak{H}_0 = L_2(\mathfrak{M}, \mathbb{R}_+, \rho_0(t))$, where

$$\rho_0(t) = \frac{2}{\pi} \frac{1}{1 + t^2}, \qquad t \in \mathbb{R}_+.$$

The inner product is given by

$$(f,g)_{\mathfrak{H}_0}=\int\limits_{\mathbb{R}_+}(f(t),g(t))_{\mathfrak{M}}\rho_0(t)\,dt.$$

The Hilbert space $\mathfrak{M}$ can be identified with the subspace of $\mathfrak{H}_0$ consisting of constant functions. It is easy to see that

$$P_{\mathfrak{M}}f(t)=\int\limits_{\mathbb{R}_+}f(t)\rho_0(t)\,dt.$$

Let $\widetilde{A}_0$ be multiplication by the squared independent variable:

$$\widetilde{A}_0f(t)=t^2f(t),\qquad \operatorname{dom}\widetilde{A}_0=\Big\{f\in\mathfrak{H}_0:\int\limits_{\mathbb{R}_+}t^4\|f(t)\|^2_{\mathfrak{M}}\rho_0(t)\,dt<\infty\Big\}.$$

Then $\widetilde{A}_0$ is selfadjoint, nonnegative, and

$$(\widetilde{A}_0-\lambda)^{-1}g(t)=\frac{g(t)}{t^2-\lambda},\qquad g\in\mathfrak{H}_0,\ \lambda\in\mathbb{C}\setminus\mathbb{R}_+.$$

Let $g_0(t)=g_0$, $t\in\mathbb{R}_+$, $g_0\in\mathfrak{M}$. Then for $\lambda\in\mathbb{C}\setminus\mathbb{R}_+$, using residues, one obtains

$$\begin{aligned}P_{\mathfrak{M}}(\widetilde{A}_0-\lambda I)^{-1}g_0(t)&=\int\limits_{\mathbb{R}_+}\frac{g_0}{t^2-\lambda}\rho_0(t)\,dt\\&=\frac{2g_0}{\pi}\int\limits_{\mathbb{R}_+}\frac{dt}{(t^2-\lambda)(1+t^2)}=\frac{-\,g_0}{\lambda+i\sqrt{\lambda}},\end{aligned}$$

where the branch $\operatorname{Im}\lambda>0\Rightarrow\operatorname{Im}\sqrt{\lambda}>0$ is chosen. Therefore,

$$P_{\mathfrak{M}}(\widetilde{A}_0-\lambda)^{-1}\restriction\mathfrak{M}=-(\mathcal{R}_0(\lambda)+\lambda I_{\mathfrak{M}})^{-1},$$

where $\mathcal{R}_0(\lambda)=i\sqrt{\lambda}\,I_{\mathfrak{M}}$.

Clearly, $\ker\widetilde{A}_0=\{0\}$. Then according to Remark 2.3 the operators $\widehat{A}_0=\mathfrak{J}_{\mathfrak{M}}(\widetilde{A}_0)$ and $\widehat{B}_0=\mathfrak{P}_{\mathfrak{M}}(\widetilde{A}_0)$ are of the form

$$\widehat{A}_0\bigg(f(t)-\frac{2}{\pi}\int\limits_{\mathbb{R}_+}\frac{it^2+1}{t^2+1}f(t)\,dt\bigg)=t^2f(t)+\frac{2}{\pi}\int\limits_{\mathbb{R}_+}\frac{i-t^2}{t^2+1}f(t)\,dt,$$

$$\widehat{B}_0\bigg(f(t)+\frac{2}{\pi}\int\limits_{\mathbb{R}_+}\frac{t^2-1}{t^2+1}f(t)\,dt\bigg)=t^2f(t)-\frac{2}{\pi}\int\limits_{\mathbb{R}_+}\frac{t^2-1}{t^2+1}f(t)\,dt,$$

where $f(t)\in\operatorname{dom}\widetilde{A}_0$. Now by Theorem 3.5 we have

$$P_{\mathfrak{M}}(\widehat{A}_0-\lambda)^{-1}\restriction\mathfrak{M}=-\left(\frac{i}{\sqrt{\lambda}}+\lambda\right)^{-1}I_{\mathfrak{M}},\qquad\lambda\in\mathbb{C}\setminus\mathbb{R},$$

$$P_{\mathfrak{M}}(\widehat{B}_0-\lambda)^{-1}\restriction\mathfrak{M}=\left(\frac{i}{\sqrt{\lambda}}-\lambda\right)^{-1}I_{\mathfrak{M}},\qquad\operatorname{Re}\lambda<0.$$

(B) Now we treat a realization by means of a second order differential operator on the semiaxis $\mathbb{R}_+$. Let $\mathfrak{H}_0 = \mathfrak{M} \oplus L_2(\mathfrak{M}, \mathbb{R}_+, dt)$ and let $H^2(\mathfrak{M}, \mathbb{R}_+)$ be the Sobolev space. Define

$$\widetilde{\mathcal{A}}_0 \begin{bmatrix} u(0) \\ u(x) \end{bmatrix} = \begin{bmatrix} -u'(0) \\ -u''(x) \end{bmatrix}, \quad u \in H^2(\mathfrak{M}, \mathbb{R}_+). \tag{4.11}$$

Consider a closed symmetric nonnegative operator S_0 in $L_2(\mathfrak{M}, \mathbb{R}_+, dt)$:

$$S_0 u = -u'', \quad \operatorname{dom} S_0 = \{u \in H^2(\mathfrak{M}, \mathbb{R}_+) : u(0) = u'(0) = 0\}.$$

The adjoint operator S_0^* is given by

$$S_0^* u = -u'', \quad \operatorname{dom} S_0^* = H^2(\mathfrak{M}, \mathbb{R}_+).$$

Define a pair of boundary mappings by

$$\mathit{\Gamma}_0 u = u(0), \quad \mathit{\Gamma}_1(0) = u'(0), \quad u \in H^2(\mathfrak{M}, \mathbb{R}_+).$$

Then $\{\mathfrak{M}, \mathit{\Gamma}_0, \mathit{\Gamma}_1\}$ is a boundary triplet for S_0^* (see Subsection 3.3), and

$$(S^* u, u) - (\mathit{\Gamma}_1 u, \mathit{\Gamma}_0 u)_{\mathfrak{M}} = \int\limits_{\mathbb{R}_+} \|u'(x)\|^2_{\mathfrak{M}}\, dx \geq 0 \quad \forall u \in H^2(\mathfrak{M}, \mathbb{R}_+).$$

The operator $\widetilde{A}_0$ defined in (4.11) is the main transform of the boundary triplet $\{\mathfrak{M}, \mathit{\Gamma}_0, \mathit{\Gamma}_1\}$. It is selfadjoint and nonnegative in $\mathfrak{H}_0$.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. The system of equations

$$\begin{cases} -u'(0) - \lambda u(0) = h, \\ -u''(x) - \lambda u(x) = 0 \end{cases}$$

has a unique solution

$$\begin{bmatrix} \dfrac{-\,h}{i\sqrt{\lambda} + \lambda} \\ \dfrac{-\,e^{i\sqrt{\lambda}} h}{i\sqrt{\lambda} + \lambda} \end{bmatrix}, \quad \text{where } h \in \mathfrak{M}.$$

Hence,

$$P_{\mathfrak{M}}(\widetilde{\mathcal{A}}_0 - \lambda I)^{-1} h = -\frac{h}{i\sqrt{\lambda} + \lambda}$$

(see Theorem 3.5). Now the transforms $\widehat{\mathcal{A}}_0 = \mathfrak{J}_{\mathfrak{M}}(\widetilde{\mathcal{A}}_0)$ and $\widehat{\mathcal{B}}_0 = \mathfrak{P}_{\mathfrak{M}}(\widetilde{\mathcal{A}}_0)$ are of the form

$$\widehat{\mathcal{A}}_0 \begin{bmatrix} iu'(0) \\ u(x) \end{bmatrix} = \begin{bmatrix} iu(0) \\ -u''(x) \end{bmatrix}, \quad \widehat{\mathcal{B}}_0 \begin{bmatrix} -u'(0) \\ u(x) \end{bmatrix} = \begin{bmatrix} u(0) \\ -u''(x) \end{bmatrix}, \quad u(x) \in H^2(\mathfrak{M}, \mathbb{R}_+).$$

It remains to note that (see Theorems 3.5–3.7)

$$P_{\mathfrak{M}}(\widehat{\mathcal{A}}_0-\lambda I)^{-1}\upharpoonright\mathfrak{M}=-\frac{1}{\dfrac{i}{\sqrt{\lambda}}+\lambda}I_{\mathfrak{M}},\qquad \lambda\in\mathbb{C}\setminus\mathbb{R},$$

$$P_{\mathfrak{M}}(\widehat{\mathcal{B}}_0-\lambda I)^{-1}\upharpoonright\mathfrak{M}=\frac{1}{\dfrac{i}{\sqrt{\lambda}}-\lambda}I_{\mathfrak{M}},\qquad \operatorname{Re}\lambda<0.$$

Appendix. Discrete-time systems and their transfer functions. Let $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{H}$ be separable Hilbert spaces. A linear system

$$\tau=\left\{\begin{bmatrix}D & C\\ B & F\end{bmatrix};\mathfrak{M},\mathfrak{N},\mathfrak{K}\right\}$$

with bounded linear operators F,B,C,D of the form

$$\begin{cases}\sigma_k=Ch_k+D\xi_k,\\ h_{k+1}=Fh_k+B\xi_k,\end{cases}\qquad k\in\mathbb{N}_0,$$

where $\{\xi_k\}\subset\mathfrak{M}$, $\{\sigma_k\}\subset\mathfrak{N}$, $\{h_k\}\subset\mathfrak{K}$, is called a *discrete-time invariant system* (cf. [11]). The Hilbert spaces $\mathfrak{M}$ and $\mathfrak{N}$ are called the *input* and the *output spaces*, respectively, and the Hilbert space $\mathfrak{K}$ is called the *state space.* The transfer function of the system τ is defined by

$$\Omega(z):=D+zC(I_{\mathfrak{K}}-zF)^{-1}B \tag{A.1}$$

and it is holomorphic in a neighborhood of the origin.

Associate with τ the block operator matrix

$$T=\begin{bmatrix}D & C\\ B & F\end{bmatrix}:\begin{matrix}\mathfrak{M}\\ \oplus\\ \mathfrak{K}\end{matrix}\to\begin{matrix}\mathfrak{N}\\ \oplus\\ \mathfrak{K}\end{matrix}.$$

If T is contractive, then the corresponding discrete-time system is said to be *passive* [11]. If T is unitary, then the system is *conservative.* The transfer function of a passive system τ belongs to the *Schur class* $\mathcal{S}(\mathfrak{M},\mathfrak{N})$ of all holomorphic and contractive $\mathbf{B}(\mathfrak{M},\mathfrak{N})$-valued functions on the unit disk $\mathbb{D}$.

The subspaces

$$\mathfrak{K}^c:=\overline{\operatorname{span}}\{F^nB\mathfrak{M}:n\in\mathbb{N}\cup\{0\}\},\quad \mathfrak{K}^o:=\overline{\operatorname{span}}\{F^{*n}C^*\mathfrak{N}:n\in\mathbb{N}\cup\{0\}\}$$

are called the *controllable* and *observable subspaces* of τ, respectively. If $\mathfrak{K}^c=\mathfrak{K}$ (respectively, $\mathfrak{K}^o=\mathfrak{K}$), then the system τ is called *controllable* (resp. *observable*). If

$$\operatorname{clos}\{\mathfrak{K}^c+\mathfrak{K}^o\}=\mathfrak{K},$$

then τ is called *simple*, and if $\mathfrak{K}^c=\mathfrak{K}^o=\mathfrak{K}$, i.e., τ is controllable and observable, then it is called *minimal.*

The resolvent $\mathcal{R}_T(z) = (I - zT)^{-1}$ of the block operator T is of the form (the Schur–Frobenius formula for the resolvent):

$$\text{(A.2)}\quad \mathcal{R}_T(z) = \begin{bmatrix} (I - z\Omega(z))^{-1} & z(I - z\Omega(z))^{-1}C\mathcal{R}_F(z) \\ z\mathcal{R}_F(z)B(I - z\Omega(z))^{-1} & \mathcal{R}_F(z)(I + zB(I - z\Omega(z))^{-1}C\mathcal{R}_F(z)) \end{bmatrix},$$
$$z^{-1} \in \rho(T) \cap \rho(F) \text{ or } z = 0,$$

where Ω is given by (A.1). In particular, (A.2) shows that

$$\text{(A.3)}\qquad P_{\mathfrak{M}}(I - zT)^{-1}\upharpoonright\mathfrak{M} = (I_{\mathfrak{M}} - z\Omega(z))^{-1}, \qquad z \in \mathbb{D}.$$

Moreover, the resolvent formula (A.2) yields

$$\text{(A.4)}\quad \overline{\operatorname{span}}\{T^n\mathfrak{M} : n \in \mathbb{N}_0\} = \overline{\operatorname{span}}\{(I - zT)^{-1}\mathfrak{M} : z \in \mathcal{U}\}$$
$$= \mathfrak{M} \oplus \overline{\operatorname{span}}\{F^nB\mathfrak{M} : n \in \mathbb{N} \cup \{0\}\},$$
$$\text{(A.5)}\quad \overline{\operatorname{span}}\{T^{*n}\mathfrak{M} : n \in \mathbb{N}_0\} = \overline{\operatorname{span}}\{(I - zT^*)^{-1}\mathfrak{M} : z \in \mathcal{U}\}$$
$$= \mathfrak{M} \oplus \overline{\operatorname{span}}\{F^{*n}C^*\mathfrak{M} : n \in \mathbb{N} \cup \{0\}\}$$

for any small neighborhood $\mathcal{U}$ of the origin.

For a passive selfadjoint system $\tau = \left\{\left[\begin{smallmatrix} D & C \\ C^* & F \end{smallmatrix}\right]; \mathfrak{M}, \mathfrak{M}, \mathfrak{K}\right\}$ the controllable and observable subspaces coincide.

Definition A.1 ([7]). Let $\mathfrak{M}$ be a Hilbert space. A $\mathbf{B}(\mathfrak{M})$-valued Nevanlinna function Ω holomorphic on $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ is said to belong to $\mathcal{RS}(\mathfrak{M})$ if $-I \le \Omega(x) \le I$ for $x \in (-1, 1)$.

Let $\Omega \in \mathcal{RS}(\mathfrak{M})$. By [9, Theorem 5.1, Proposition 5.6] there exists up to unitary equivalence a unique minimal passive selfadjoint system τ as above whose transfer function coincides with $\Omega(z)$, i.e.,

$$\Omega(z) = D + zC(I - zF)^{-1}C^*, \qquad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

The Schur–Frobenius formula (A.2) yields

$$P_{\mathfrak{M}}(I - zT)^{-1}\upharpoonright\mathfrak{M} = (I_{\mathfrak{M}} - z\Omega(z))^{-1}, \qquad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)).$$

Theorem A.2 ([7]). *Let Ω be an operator-valued Herglotz–Nevanlinna function defined in the region $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Then the following statements are equivalent:*

(i) $\Omega \in \mathcal{RS}(\mathfrak{M})$*;*

(ii) Ω *satisfies*

$$I - \Omega^*(z)\Omega(z) - (1 - |z|^2)\frac{\operatorname{Im} \Omega(z)}{\operatorname{Im} z} \ge 0, \qquad \operatorname{Im} z \ne 0;$$

(iii) *the function*

$$K(z, w) := I - \Omega^*(w)\Omega(z) - \frac{1 - \bar{w}z}{z - \bar{w}}(\Omega(z) - \Omega^*(w)$$

is a nonnegative kernel on the domains $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ *for* $\operatorname{Im} z > 0$ *and for* $\operatorname{Im} z < 0$;

(iv) *the following transform of* Ω:

$$\Upsilon(z) = (zI - \Omega(z))(I - z\Omega(z))^{-1}, \qquad z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty)), \tag{A.6}$$

belongs to $\mathcal{RS}(\mathfrak{M})$.

Theorem A.3 ([7]). *Let* $\mathfrak{M}$ *be a Hilbert space and let* $\Omega \in \mathcal{RS}(\mathfrak{M})$. *Then there exist a Hilbert space* $\widetilde{\mathfrak{M}}$ *containing* $\mathfrak{M}$ *and a selfadjoint contraction* $\widetilde{T}$ *in* $\widetilde{\mathfrak{M}}$ *such that for all* $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$,

$$\Omega(z) = P_{\mathfrak{M}}(zI_{\widetilde{\mathfrak{M}}} + \widetilde{T})(I_{\widetilde{\mathfrak{M}}} + z\widetilde{T})^{-1}\upharpoonright\mathfrak{M}. \tag{A.7}$$

Moreover, the pair $\{\widetilde{\mathfrak{M}}, \widetilde{T}\}$ *can be chosen such that* $\widetilde{T}$ *is* $\mathfrak{M}$*-minimal, i.e.,*

$$\overline{\operatorname{span}}\{\widetilde{T}^n\mathfrak{M} : n \in \mathbb{N}_0\} = \widetilde{\mathfrak{M}}. \tag{A.8}$$

The function Ω *is inner if and only if* $\widetilde{\mathfrak{M}} = \mathfrak{M}$ *in* (A.8).

If there are two representations of the form (A.7) *with pairs* $\{\widetilde{\mathfrak{M}}_1, \widetilde{T}_1\}$ *and* $\{\widetilde{\mathfrak{M}}_2, \widetilde{T}_2\}$ *that are* $\mathfrak{M}$*-simple, then there exists a unitary operator* $\widetilde{U} \in \mathbf{B}(\widetilde{\mathfrak{M}}_1, \widetilde{\mathfrak{M}}_2)$ *such that*

$$\widetilde{U}\upharpoonright\mathfrak{M} = I_{\mathfrak{M}}, \qquad \widetilde{T}_2\widetilde{U} = \widetilde{U}\widetilde{T}_1.$$

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