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Yet another note on the arithmetic-geometric mean inequality

by

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Abstract. It was shown by E. Gluskin and V. D. Milman in [GAFA Lecture Notes in Math. 1807, 2003] that the classical arithmetic-geometric mean inequality can be reversed (up to a multiplicative constant) with high probability, when applied to coordinates of a point chosen with respect to the surface unit measure on a high-dimensional Euclidean sphere. We present two asymptotic refinements of this phenomenon in the more general setting of the surface probability measure on a high-dimensional ℓ_p-sphere, and also show that sampling the point according to either the cone probability measure on ℓ_p or the uniform distribution on the ball enclosed by ℓ_p yields the same results. First, we prove a central limit theorem, which allows us to identify the precise constants in the reverse inequality. Second, we prove the large deviations counterpart to the central limit theorem, thereby describing the asymptotic behaviour beyond the Gaussian scale, and identify the rate function.

1. Introduction and main results. The classical inequality of arithmetic and geometric means states that the arithmetic mean of a finite sequence of non-negative real numbers is greater than or equal to the geometric mean of the sequence, i.e., for any $n \in \mathbb{N}$ and $x_1, \dots, x_n \in [0, \infty)$,

$$\Big(\prod_{i=1}^n x_i\Big)^{1/n} \le \frac{1}{n}\sum_{i=1}^n x_i$$

with equality if and only if $x_1 = \dots = x_n$. This inequality may be written in the form

$$\Big(\prod_{i=1}^n y_i\Big)^{1/n} \le \sqrt{\frac{1}{n}\sum_{i=1}^n y_i^2},$$

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where $y_1, \dots, y_n > 0$. This means that if $y = (y_1, \dots, y_n)$ is an element of the Euclidean unit sphere $\mathbb{S}^{n-1}$, then

$$\Big(\prod_{i=1}^n |y_i|\Big)^{1/n} \le \frac{1}{\sqrt{n}}.$$

A natural question is how sharp this inequality is for a typical point on the sphere. In [6, Proposition 1], Gluskin and Milman showed that for large $n \in \mathbb{N}$ the arithmetic and geometric means are actually equivalent (up to multiplicative constants) with very high probability. More precisely, if we denote by σ the rotationally invariant surface probability measure on $\mathbb{S}^{n-1}$, then for any $t \in (0, \infty)$,

$$\sigma\bigg(\Big\{x \in \mathbb{S}^{n-1} : \Big(\prod_{i=1}^n |x_i|\Big)^{1/n} \ge t \cdot \frac{1}{\sqrt{n}}\Big\}\bigg) \ge 1 - (1.6\sqrt{t}\,)^n. \tag{1.1}$$

In other words, if we sample a point uniformly at random on the unit Euclidean sphere, then it will satisfy a reverse (up to a constant) arithmetic-geometric mean inequality with high probability. Alternatively, we can sample a point uniformly at random from the unit Euclidean ball, and put it into the sphere by dividing by its norm.

The problem has been revisited by Aldaz [2, Theorem 2.8], who showed that for any $k, \varepsilon > 0$ there exists an $N = N(k, \varepsilon) \in \mathbb{N}$ such that for every $n \ge N$, the set

$$B_n = \Big\{x \in \mathbb{S}^{n-1} : \frac{(1-\varepsilon)e^{-\frac{1}{2}(\gamma + \log 2)}}{\sqrt{n}} < \Big(\prod_{i=1}^n |x_i|\Big)^{1/n} < \frac{(1+\varepsilon)e^{-\frac{1}{2}(\gamma + \log 2)}}{\sqrt{n}}\Big\}$$

satisfies

$$\sigma(B_n) \ge 1 - 1/n^k, \tag{1.2}$$

where γ denotes Euler's constant. This identifies the exact constant $e^{-\frac{1}{2}(\gamma+\log 2)}$ around which the ratio of geometric and arithmetic means concentrates. Aldaz also obtained a similar result for points chosen on the ℓ_1^n-sphere (with concentration around the constant $e^{-\gamma}$) and studied weighted versions of the arithmetic-geometric mean inequality. We also refer the reader to [1].

In this note we complement the inequalities (1.1) and (1.2) by finding the (logarithmic) asymptotics of their left-hand sides. Our first observation is the following central limit theorem. We state and prove it in the setting of the inequality between geometric and p-generalized means, which says that for all $p > 0$, $n \in \mathbb{N}$, and $x_1, \dots, x_n \in \mathbb{R}$,

$$\Big(\prod_{i=1}^n |x_i|\Big)^{1/n} \le \Big(\frac{1}{n}\sum_{i=1}^n |x_i|^p\Big)^{1/p}.$$

Since both sides of this inequality scale linearly, we can assume that the right-hand side equals (or does not exceed) 1. If $1 \le p < \infty$, this means that $(x_1, \dots, x_n)$ belongs to the unit ℓ_p^n-sphere (or the ℓ_p^n-ball), for which we use the standard notation

$$\mathbb{S}_p^{n-1} = \{x \in \mathbb{R}^n : \|x\|_p = 1\} \quad \text{and} \quad \mathbb{B}_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$$

with the $\|\cdot\|_p$-norm of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ given by

$$\|x\|_p = \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}.$$

Then it is natural to study the behaviour of the geometric mean for typical x_i's, which corresponds to choosing x at random. There are several natural probability measures on $\mathbb{B}_p^n$ and $\mathbb{S}_p^{n-1}$. Restricting the Lebesgue measure to $\mathbb{B}_p^n$ and normalizing it, we obtain the uniform probability distribution on $\mathbb{B}_p^n$. We shall also consider the surface probability measure and cone probability measure on $\mathbb{S}_p^{n-1}$, which we denote respectively by σ_p and μ_p; see Subsection 2.2 for the definition of μ_p.

Recall that for $x > 0$, the digamma function ψ is defined via

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)},$$

where Γ denotes the gamma function and Γ' its derivative.

THEOREM 1.1. *Let $n \in \mathbb{N}$ and $p \in [1, \infty)$. Suppose $\mathbf{X}_n = (X_1^{(n)}, \dots, X_n^{(n)})$ is a random vector that is either uniformly distributed over $\mathbb{B}_p^n$ or distributed according to μ_p or σ_p on $\mathbb{S}_p^{n-1}$. Then, for every $a \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\Big[\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} \ge e^{m_p}\Big(1 + \frac{a}{\sqrt{n}}\Big) \cdot \Big(\frac{1}{n}\sum_{i=1}^n |X_i^{(n)}|^p\Big)^{1/p}\Big] = 1 - \Phi\Big(\frac{pa}{\sqrt{\psi'(1/p) - p}}\Big),$$

where Φ denotes the distribution function of a standard normal random variable and

$$m_p := \frac{\psi(1/p) + \log p}{p}$$

is a negative constant only depending on p.

In other words, the sequence of (random) ratios of geometric and p-generalized means, given by

$$\mathcal{R}_n := \frac{(\prod_{i=1}^n |X_i^{(n)}|)^{1/n}}{(n^{-1}\sum_{i=1}^n |X_i^{(n)}|^p)^{1/p}}, \qquad n \in \mathbb{N}, \tag{1.3}$$

satisfies a central limit theorem with the normalization $\sqrt{n}(e^{-m_p}\mathcal{R}_n - 1)$.

Remark 1.2. Some particular values of e^{m_p} for $p \in \{1,2,4\}$ and $p \to \infty$ are

$$e^{m_p} = \begin{cases} e^{-\gamma} \approx 0.561, & p = 1, \\ \exp\left(-\frac{\gamma + \log 2}{2}\right) \approx 0.529, & p = 2, \\ \exp\left(-\frac{2\gamma + \pi + 2\log 2}{8}\right) \approx 0.491, & p = 4, \\ e^{-1}, & p \to \infty, \end{cases}$$

and $\psi'(1) = \pi^2/6$, $\psi'(1/2) = \pi^2/2$ (see [7, Section 8.366]). Let us also note that, setting $a = 0$ in Theorem 1.1, we obtain

$$\lim_{n\to\infty} \mathbb{P}\left[\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} \le e^{m_p} \cdot \Big(\frac{1}{n}\sum_{i=1}^n |X_i^{(n)}|^p\Big)^{1/p}\right] = \frac{1}{2},$$

which means that with probability approaching $1/2$ the inequality between geometric and p-generalized means can be improved with the multiplicative constant $e^{m_p} < 1$. Similarly, using Theorem 1.1 with $a \to +\infty$ and $a \to -\infty$, we see that with probability approaching 1, the inequality holds true with any constant $c > e^{m_p}$ but ceases to hold with any constant $c < e^{m_p}$. The precise rate of convergence of these probabilities will be identified in our large deviations result, Theorem 1.4.

Remark 1.3. As will be shown at the end of the proof of Theorem 1.1, we also have

$$\lim_{n\to\infty} \mathbb{P}\left[\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} \ge e^{m_p}\Big(1 + \frac{a}{\sqrt{n}}\Big) \cdot n^{-1/p}\right] = 1 - \Phi\Big(\frac{pa}{\sqrt{\psi'(1/p) - p}}\Big),$$

which of course trivially follows from the theorem if the distribution of $\mathbf{X}_n$ is μ_p or σ_p.

Our second observation concerns large deviations of the ratio $\mathcal{R}_n$ given in (1.3). While large deviations are extensively studied in probability theory (see, e.g., [4, 8] and the references cited therein), they have not been considered—in contrast to central limit theorems—in geometric functional analysis until the very recent paper by Gantert, Kim, and Ramanan [5]. Already shortly after, this work was extended and complemented in [3, 10, 9, 12, 13]. In contrast to the universality in central limit theorems, the probabilities of (large) deviations on the scale of laws of large numbers are non-universal, being sensitive to the distribution of the random variables considered. This non-universality is reflected in the so-called rate function, which essentially defines the large deviations probabilities.

Theorem 1.4. *Let $n \in \mathbb{N}$ and $p \in [1, \infty)$. Suppose $\mathbf{X}_n = (X_1^{(n)}, \dots, X_n^{(n)})$ is a random vector that is either uniformly distributed over $\mathbb{B}_p^n$ or distributed* *according to* μ_p *or* σ_p *on* $\mathbb{S}_p^{n-1}$. *Then for* $\theta \in [e^{m_p}, 1)$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\bigg[\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} \geq \theta \cdot \Big(\frac{1}{n}\sum_{i=1}^n |X_i^{(n)}|^p\Big)^{1/p}\bigg] = -\mathcal{J}_p(\theta),$$

and for $\theta \in (0, e^{m_p}]$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\bigg[\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} \leq \theta \cdot \Big(\frac{1}{n}\sum_{i=1}^n |X_i^{(n)}|^p\Big)^{1/p}\bigg] = -\mathcal{J}_p(\theta),$$

where

$$\begin{aligned}\mathcal{J}_p(\theta) := {}& [pG_p(\theta) - 1]\log\theta + G_p(\theta)[\log G_p(\theta) - 1] \\ & - \log\Gamma(G_p(\theta)) + \frac{1}{p} + \frac{1}{p}\log p + \log\Gamma\Big(\frac{1}{p}\Big), \qquad \theta \in (0,1),\end{aligned}$$

with $G_p(\theta) := H^{-1}(p\log\theta)$ *and* $H : (0,\infty) \to (-\infty, 0)$ *being an increasing bijection given by*

$$H(x) := \psi(x) - \log x.$$

The function $\mathcal{J}_p$ *is non-negative, satisfies* $\mathcal{J}_p(e^{m_p}) = 0$, *and* $\mathcal{J}_p(0+) = \mathcal{J}_p(1-) = +\infty$.

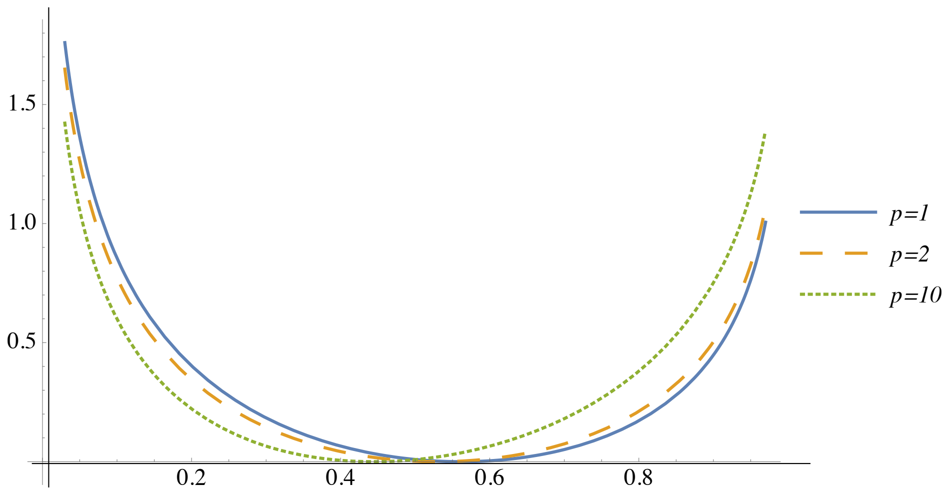

Fig. 1. The rate function $\mathcal{J}_p$ for $p = 1, 2, 10$

Remark 1.5. (a) Note that the equality $\mathcal{J}_p(e^{m_p}) = 0$ agrees with Theorem 1.1. This equality also follows directly from the identity $m_p = \frac{1}{p}H\big(\frac{1}{p}\big)$.

(b) In fact, we shall prove the following large deviations principle for $\mathcal{R}_n$ with rate function $\mathcal{J}_p$. Put $\mathcal{J}_p \equiv +\infty$ on $\mathbb{R} \setminus (0,1)$. Then for all Borel measurable sets $A \subseteq \mathbb{R}$,

$$-\inf_{\theta\in A^\circ} \mathcal{J}_p(\theta) \le \liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}[\mathcal{R}_n \in A] \le \limsup_{n\to\infty} \frac{1}{n}\log \mathbb{P}[\mathcal{R}_n \in A] \le -\inf_{\theta\in \bar{A}} \mathcal{J}_p(\theta),$$

where A° and $\bar{A}$ are the interior and closure of A, respectively.

(c) In the setting of the uniform distribution on the sphere, Theorems 1.1 and 1.4 use results for the Radon–Nikodym density of μ_p and σ_p (see Proposition 2.1 and Lemma 2.2). Their extensions to the regime $0 < p < 1$ are not fully known (see, e.g., [15, Section 5, Comment (5)]), so for the uniform distribution on the sphere we truly need $1 \le p < \infty$. For the other two distributions one can easily see that the results continue to hold in the regime $0 < p < 1$ by following verbatim the proofs presented.

2. Preliminaries. We shall present here the notation and background material used throughout the text. Having in mind a broad readership from both probability theory and geometric functional analysis, we introduce the material on large deviations in slightly more detail.

2.1. Notation. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and equip it with the standard inner product $\langle \cdot,\cdot\rangle$. For a subset A of $\mathbb{R}^n$, we denote by A° the interior of A and by $\bar{A}$ its closure. For a Borel measurable set $A \subseteq \mathbb{R}^n$, we denote by $|A|$ its n-dimensional Lebesgue measure. The asymptotic notation $\sim$ means that the ratio of functions or sequences tends to 1.

2.2. The ℓ_p^n-balls. For any $p \in [1,\infty)$ the ℓ_p^n-norm of $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ is given by

$$\|x\|_p := \Big(\sum_{i=1}^n |x_i|^p\Big)^{1/p}.$$

For any n and p denote by $\mathbb{B}_p^n := \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$ the unit ball and by $\mathbb{S}_p^{n-1} := \{x \in \mathbb{R}^n : \|x\|_p = 1\}$ the unit sphere with respect to this norm. The restriction of the Lebesgue measure to $\mathbb{B}_p^n$ provides a natural volume measure on $\mathbb{B}_p^n$. We will supply $\mathbb{S}_p^{n-1}$ with the cone probability measure μ_p defined as follows: for a Borel set $A \subseteq \mathbb{S}_p^{n-1}$,

$$\mu_p(A) := \frac{|\{rx : x \in A,\, r \in [0,1]\}|}{|\mathbb{B}_p^n|}. \tag{2.1}$$

Let σ_p be the $(n-1)$-dimensional Hausdorff probability measure or, equivalently, the $(n-1)$-dimensional normalized Riemannian volume measure on $\mathbb{S}_p^{n-1}$, $p \in [1,\infty)$. We remark that the cone measure μ_p coincides with σ_p if and only if $p = 1$ or $p = 2$. In particular, μ_2 is the normalized spherical Lebesgue measure.

We shall use the following result on the Radon–Nikodym density of the cone and surface measures, proved in [15, Lemma 2].

PROPOSITION 2.1. *Let $n \in \mathbb{N}$ and $1 \le p < \infty$. Then, for all $x = (x_1, \dots, x_n) \in \mathbb{S}_p^{n-1}$,*

$$h_{n,p}(x) := \frac{\mathrm{d}\sigma_p}{\mathrm{d}\mu_p}(x) = C_{n,p} \cdot \Big(\sum_{i=1}^{n} |x_i|^{2p-2}\Big)^{1/2},$$

where

$$C_{n,p} := \Big(\int_{\mathbb{S}_p^{n-1}} \Big(\sum_{i=1}^{n} |x_i|^{2p-2}\Big)^{1/2} \mu_p(\mathrm{d}x)\Big)^{-1}.$$

We refer to [14, 15] for more details on the relation between these two measures. We shall also use the following result.

LEMMA 2.2. *Let $1 \le p < \infty$. There is a constant $C = C(p) \in (0,\infty)$ such that, for all $n \in \mathbb{N}$ and every $x \in \mathbb{S}_p^{n-1}$,*

$$n^{-C} \le h_{n,p}(x) \le n^{C}.$$

Proof. It suffices to show that there is a constant $A > 0$ such that $n^{-2A} \le \sum_{i=1}^n |x_i|^{2p-2} \le n^{2A}$ for all $x \in \mathbb{S}_p^{n-1}$. Indeed, from this it follows that $n^{-A} \le C_{n,p} \le n^A$ by the definition of $C_{n,p}$, which yields the claim. Write $y_i := |x_i|^p \ge 0$, so that $\sum_{i=1}^n y_i = 1$. It is an easy consequence of Hölder's inequality that under this constraint we always have $\sum_{i=1}^n y_i^\alpha \le \max\{n^{1-\alpha}, 1\}$ and $\sum_{i=1}^n y_i^\alpha \ge \min\{n^{1-\alpha}, 1\}$ for all $\alpha \ge 0$. Taking $\alpha = (2p-2)/p$, we obtain the required bounds on $\sum_{i=1}^n y_i^\alpha = \sum_{i=1}^n |x_i|^{2p-2}$. ∎

The proofs of our results rely on the following probabilistic representation for the cone probability measure on $\mathbb{S}_p^{n-1}$ for $p \in [1,\infty)$ (and for the uniform distribution over $\mathbb{B}_p^n$), which is due to Schechtman and Zinn [17] and was independently obtained by Rachev and Rüschendorf [16].

PROPOSITION 2.3. *Let $n \in \mathbb{N}$ and $p \in [1,\infty)$. Suppose that $Z_1, \dots, Z_n$ are independent p-generalized Gaussian random variables whose distribution has density*

$$f_p(x) := \frac{1}{2p^{1/p}\Gamma(1+1/p)} e^{-|x|^p/p}$$

with respect to the Lebesgue measure on $\mathbb{R}$. Then, for $Z := (Z_1, \dots, Z_n) \in \mathbb{R}^n$, we have:

(i) *The random vector $Z/\|Z\|_p \in \mathbb{S}_p^{n-1}$ is independent of $\|Z\|_p$ and its distribution is μ_p.*
(ii) *If U is a random variable uniformly distributed on $[0,1]$ and independent of Z, then the random vector $U^{1/n} Z/\|Z\|_p$ is uniformly distributed on $\mathbb{B}_p^n$.*

2.3. Large deviations principles. We start with the definition of a large deviations principle. In this subsection we denote for clarity the space dimension by d instead of n in order to distinguish it from our index parameter n. Finally, we make the assumption that all random objects we are dealing with are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For thorough introductions to the theory of large deviations, we refer the reader to the monographs [4, 8] or the book [11].

DEFINITION 2.4. Let $\mathbf{X} := (X^{(n)})_{n\in\mathbb{N}}$ be a sequence of random vectors taking values in $\mathbb{R}^d$. Further, let $s : \mathbb{N} \to (0, \infty]$ be a positive sequence and $\mathcal{J} : \mathbb{R}^d \to [0, \infty]$ be a lower semicontinuous function. We say that $\mathbf{X}$ satisfies a *large deviations principle* (*LDP*) with *speed* $s(n)$ and *rate function* $\mathcal{J}$ if

$$\begin{aligned}-\inf_{x\in A^\circ} \mathcal{J}(x) &\le \liminf_{n\to\infty} \frac{1}{s(n)} \log \mathbb{P}[X^{(n)} \in A] \\ &\le \limsup_{n\to\infty} \frac{1}{s(n)} \log \mathbb{P}[X^{(n)} \in A] \le -\inf_{x\in\bar{A}} \mathcal{J}(x)\end{aligned}$$

for all Borel sets $A \subseteq \mathbb{R}^d$. If $\mathcal{J}$ has compact level sets $\{x \in \mathbb{R}^d : \mathcal{J}(x) \le \alpha\}$, $\alpha \in \mathbb{R}$, then $\mathcal{J}$ is called a *good* rate function.

We notice that on the class of all $\mathcal{J}$-continuity sets, that is, on the class of Borel sets $A \subseteq \mathbb{R}^d$ for which $\mathcal{J}(A^\circ) = \mathcal{J}(\bar{A})$ with $\mathcal{J}(A) := \inf\{\mathcal{J}(x) : x \in A\}$, one has the exact limit relation

$$\lim_{n\to\infty} \frac{1}{s(n)} \log \mathbb{P}[X^{(n)} \in A] = -\mathcal{J}(A).$$

Let $d \ge 1$ be a fixed integer and let X be an $\mathbb{R}^d$-valued random vector. We write

$$\Lambda(u) = \Lambda_X(u) := \log \mathbb{E} e^{\langle X,u\rangle}, \qquad u \in \mathbb{R}^d,$$

for the cumulant generating function of X. Moreover, we define the (effective) domain of Λ to be the set $D_\Lambda := \{u \in \mathbb{R}^d : \Lambda(u) < \infty\} \subseteq \mathbb{R}^d$.

DEFINITION 2.5. The *Legendre–Fenchel transform* of a convex function $\Lambda : \mathbb{R}^d \to (-\infty, +\infty]$ is defined as

$$\Lambda^*(x) := \sup_{u\in\mathbb{R}^d} [\langle u, x\rangle - \Lambda(u)], \qquad x \in \mathbb{R}^d.$$

The Legendre–Fenchel transform of the cumulant generating function plays a crucial rôle in the following result, usually referred to as Cramér's theorem, (see, e.g., [4, Theorem 2.2.30, Theorem 6.1.3, Corollary 6.1.6] or [11, Theorem 27.5]).

PROPOSITION 2.6 (Cramér's theorem). *Let $X, X_1, X_2, \ldots$ be independent and identically distributed random vectors taking values in $\mathbb{R}^d$. Assume that $0 \in D_\Lambda^\circ$. Then the partial sums $n^{-1}\sum_{i=1}^n X_i$, $n \in \mathbb{N}$, satisfy an LDP with speed n and good rate function Λ^*.*

It will be important for us to deduce from an already existing large deviations principle a new one by applying a suitable transformation. The next result allows such a 'transport' by means of a continuous function. This device is known as the contraction principle and we refer to [4, Theorem 4.2.1] or [11, Theorem 27.11(i)].

PROPOSITION 2.7 (Contraction principle). *Let* $d_1, d_2 \in \mathbb{N}$, *and let* $F : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ *be a continuous function. Further, let* $\mathbf{X} = (X^{(n)})_{n\in\mathbb{N}}$ *be a sequence of* $\mathbb{R}^{d_1}$*-valued random vectors that satisfies an LDP with speed* $s(n)$ *and a good rate function* $\mathcal{J}_{\mathbf{X}}$. *Then the sequence* $\mathbf{Y} := (F(X^{(n)}))_{n\in\mathbb{N}}$ *of* $\mathbb{R}^{d_2}$*-valued random vectors satisfies an LDP with the same speed and with the good rate function* $\mathcal{J}_{\mathbf{Y}} = \mathcal{J}_{\mathbf{X}} \circ F^{-1}$, *i.e.,* $\mathcal{J}_{\mathbf{Y}}(y) := \inf\{\mathcal{J}_{\mathbf{X}}(x) : F(x) = y\}$, $y \in \mathbb{R}^{d_2}$, *with the convention that* $\mathcal{J}_{\mathbf{Y}}(y) = +\infty$ *if* $F^{-1}(\{y\}) = \emptyset$.

3. Arithmetic-geometric mean CLT for p-balls

We shall now present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $(X_1^{(n)}, \ldots, X_n^{(n)})$ be chosen uniformly at random from $\mathbb{B}_p^n$. Consider independent p-generalized Gaussians $Z_1, \ldots, Z_n$ and a random variable U uniformly distributed on $[0,1]$ and independent of the Z_i's. We know from the Schechtman–Zinn result (Proposition 2.3) that

$$(X_1^{(n)}, \ldots, X_n^{(n)}) \stackrel{d}{=} U^{1/n} \frac{(Z_1, \ldots, Z_n)}{\|(Z_1, \ldots, Z_n)\|_p}.$$

With this representation, we have

$$\begin{aligned} \mathcal{R}_n &= \frac{(\prod_{i=1}^n |X_i^{(n)}|)^{1/n}}{(n^{-1}\sum_{i=1}^n |X_i^{(n)}|^p)^{1/p}} \stackrel{d}{=} \frac{(\prod_{i=1}^n |Z_i|)^{1/n}}{(n^{-1}\sum_{i=1}^n |Z_i|^p)^{1/p}} \\ &= \frac{\exp(n^{-1}\sum_{i=1}^n \log|Z_i|)}{(n^{-1}\sum_{i=1}^n |Z_i|^p)^{1/p}}, \end{aligned} \tag{3.1}$$

where $\stackrel{d}{=}$ denotes equality of distributions. If $(X_1^{(n)}, \ldots, X_n^{(n)})$ is chosen at random with respect to the cone measure μ_p on $\mathbb{S}_p^{n-1}$, then the factor $U^{1/n}$ does not appear in the Schechtman–Zinn representation (see Proposition 2.3(i)) and the formula for $\mathcal{R}_n$ above does not change as the corresponding factor cancels out. For any $a \in \mathbb{R}$, the tail probability for $\mathcal{R}_n$ reads

$$\begin{aligned} &\mathbb{P}[\mathcal{R}_n \geq e^{m_p + a/\sqrt{n}}] \\ &\quad= \mathbb{P}\bigg[\exp\bigg(-m_p - \frac{a}{\sqrt{n}} + \frac{1}{n}\sum_{i=1}^n \log|Z_i|\bigg) \geq \bigg(\frac{1}{n}\sum_{i=1}^n |Z_i|^p\bigg)^{1/p}\bigg] \\ &\quad= \mathbb{P}\bigg[\exp\bigg(-a + \frac{1}{\sqrt{n}}\sum_{i=1}^n (\log|Z_i| - m_p)\bigg) \geq \bigg(1 + \frac{1}{n}\sum_{i=1}^n (|Z_i|^p - 1)\bigg)^{\sqrt{n}/p}\bigg]. \end{aligned}$$

Taking the logarithm, we can further write this as

$$\begin{aligned}
(3.2)\qquad & \mathbb{P}[\mathcal{R}_n \geq e^{m_p + a/\sqrt{n}}] \\
&= \mathbb{P}\Big[\frac{1}{\sqrt{n}}\sum_{i=1}^n (\log|Z_i| - m_p) - \frac{\sqrt{n}}{p}\log\Big(1 + \frac{1}{n}\sum_{i=1}^n (|Z_i|^p - 1)\Big) \geq a\Big] \\
&= \mathbb{P}\Big[\frac{1}{\sqrt{n}}\sum_{i=1}^n (\log|Z_i| - m_p) - \frac{1}{p\sqrt{n}}\sum_{i=1}^n (|Z_i|^p - 1) \\
&\qquad\qquad - \frac{\sqrt{n}}{p}\alpha\Big(\frac{1}{n}\sum_{i=1}^n (|Z_i|^p - 1)\Big) \geq a\Big],
\end{aligned}$$

where the function $\alpha(x)$ is defined by $\log(1+x) = x + \alpha(x)$ for $x > -1$, so that $\alpha(x) = o(x)$ as $x \to 0$. Using Mathematica, we can find that $\mathbb{E}\log|Z_1| = m_p$, $\mathbb{E}|Z_1|^p = 1$, and $\mathbb{E}|Z_1|^p\log|Z_1| = (1/p)(\log p + \psi(1+1/p)) = m_p + 1$, where in the last equality we have used the fact that the digamma function satisfies $\psi(x+1) = \psi(x) + 1/x$, $x > 0$. Let (N_1, N_2) be a bivariate centered normal random vector with the same covariance matrix as that of $(\log|Z_1|, |Z_1|^p)$, i.e., with the diagonal elements $\operatorname{Var}(\log|Z_1|) = (1/p^2)\psi'(1/p)$ and $\operatorname{Var}(|Z_1|^p) = p$ and the covariance terms equal 1. Then the bivariate central limit theorem states that

$$\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n (\log|Z_i| - m_p), \frac{1}{p\sqrt{n}}\sum_{i=1}^n (|Z_i|^p - 1)\Big) \xrightarrow[n\to\infty]{d} (N_1, p^{-1}N_2),$$

where $\xrightarrow{d}$ denotes distributional convergence. Further, by the strong law of large numbers, $n^{-1}\sum_{i=1}^n (|Z_i|^p - 1)$ converges to 0 a.s. Using the relation $\alpha(x) = o(x)$ as $x \to 0$ together with Slutsky's theorem, we arrive at

$$\begin{aligned}
&\frac{\sqrt{n}}{p}\alpha\Big(\frac{1}{n}\sum_{i=1}^n (|Z_i|^p - 1)\Big) \\
&\qquad = \frac{1}{p\sqrt{n}}\sum_{i=1}^n (|Z_i|^p - 1) \cdot \frac{\alpha(n^{-1}\sum_{i=1}^n (|Z_i|^p - 1))}{n^{-1}\sum_{i=1}^n (|Z_i|^p - 1)} \xrightarrow[n\to\infty]{d} 0
\end{aligned}$$

since the first factor converges to $p^{-1}N_2$ in distribution, whereas the second one converges to 0 a.s. Taking everything together and using the continuous mapping theorem together with Slutsky's theorem, we arrive at

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^n (\log|Z_i| - m_p) - \frac{1}{p\sqrt{n}}\sum_{i=1}^n (|Z_i|^p - 1) \\
&\qquad - \frac{\sqrt{n}}{p}\alpha\Big(\frac{1}{n}\sum_{i=1}^n (|Z_i|^p - 1)\Big) \xrightarrow[n\to\infty]{d} N_1 - p^{-1}N_2.
\end{aligned}$$

Recalling (3.2), we obtain

$$\lim_{n\to\infty} \mathbb{P}[\mathcal{R}_n \geq e^{m_p + a/\sqrt{n}}] = \mathbb{P}[N_1 - p^{-1}N_2 \geq a] = 1 - \Phi\Big(\frac{pa}{\sqrt{\psi'(1/p) - p}}\Big), \tag{3.3}$$

where we have used the fact that $\mathrm{Var}(N_1 - p^{-1}N_2) = \frac{1}{p^2}\psi'\big(\frac{1}{p}\big) - 2\frac{1}{p} + \frac{1}{p} = \frac{1}{p^2}\psi'\big(\frac{1}{p}\big) - \frac{1}{p}$.

To prove that $e^{m_p + a/\sqrt{n}}$ can be replaced with $e^{m_p}(1 + a/\sqrt{n})$ in (3.3), fix some $\varepsilon > 0$ and note that $1 + a/\sqrt{n}$ is sandwiched between $e^{(a-\varepsilon)/\sqrt{n}}$ and $e^{(a+\varepsilon)/\sqrt{n}}$ provided n is sufficiently large. Thus, for all such n,

$$\mathbb{P}[\mathcal{R}_n \geq e^{m_p + (a+\varepsilon)/\sqrt{n}}] \leq \mathbb{P}[\mathcal{R}_n \geq e^{m_p}(1 + a/\sqrt{n})] \leq \mathbb{P}[\mathcal{R}_n \geq e^{m_p + (a-\varepsilon)/\sqrt{n}}].$$

Taking first the limit as $n \to \infty$ (which is given by (3.3)) and then letting $\varepsilon \downarrow 0$ and using the continuity of Φ, we arrive at

$$\lim_{n\to\infty} \mathbb{P}\Big[\mathcal{R}_n \geq e^{m_p}\Big(1 + \frac{a}{\sqrt{n}}\Big)\Big] = 1 - \Phi\Big(\frac{pa}{\sqrt{\psi'(1/p) - p}}\Big).$$

This proves the claim of Theorem 1.1 for the uniform distribution on $\mathbb{B}_p^n$ and the cone probability measure μ_p.

Consider now the case where $(X_1^{(n)}, \ldots, X_n^{(n)})$ is chosen with respect to the probability measure σ_p on $\mathbb{S}_p^{n-1}$. It was proved in [15, Theorem 2] that the total variation distance between μ_p and σ_p, written $\mathrm{d}_{\mathrm{TV}}(\mu_p, \sigma_p)$, is bounded above by a constant $c_p \in (0, \infty)$ (only depending on p) times $n^{-1/2}$. Let us consider the sets

$$A_n := \Big\{x = (x_1, \ldots, x_n) \in \mathbb{S}_p^{n-1} : \frac{(\prod_{i=1}^n |x_i|)^{1/n}}{(n^{-1}\sum_{i=1}^n |x_i|^p)^{1/p}} \geq e^{m_p}\Big(1 + \frac{a}{\sqrt{n}}\Big)\Big\}$$

for $n \in \mathbb{N}$. As shown above,

$$\lim_{n\to\infty} \mu_p(A_n) = 1 - \Phi\Big(\frac{pa}{\sqrt{\psi'(1/p) - p}}\Big). \tag{3.4}$$

Since by [15, Theorem 2], $|\sigma_p(A_n) - \mu_p(A_n)| \leq \mathrm{d}_{\mathrm{TV}}(\sigma_p, \mu_p) \xrightarrow{n\to\infty} 0$, we can replace μ_p by σ_p in (3.4).

We shall now briefly give the argument for Remark 1.3. Since the formula there trivially holds when $\mathbf{X}_n \in \mathbb{S}_p^{n-1}$, we consider only the case of the uniform distribution on the ball $\mathbb{B}_p^n$. Define the quantity

$$\widetilde{\mathcal{R}}_n := n^{1/p}\Big(\prod_{i=1}^n |X_i^{(n)}|\Big)^{1/n} = U^{1/n} \cdot \mathcal{R}_n,$$

where U is uniformly distributed on $[0,1]$ and independent of $Z_1, \ldots, Z_n$ as in Proposition 2.3. Then

$$\sqrt{n}(\log \widetilde{\mathcal{R}}_n - m_p) = \sqrt{n}(\log \mathcal{R}_n - m_p) + n^{-1/2}\log U.$$

Since $n^{-1/2}\log U \to 0$ in probability, it follows from Slutsky's theorem that $\sqrt{n}(\log \widetilde{\mathcal{R}}_n - m_p)$ satisfies the same central limit theorem as $\sqrt{n}(\log \mathcal{R}_n - m_p)$. Hence, the analogue of (3.3) with $\mathcal{R}_n$ replaced by $\widetilde{\mathcal{R}}_n$ holds. ∎

4. Arithmetic-geometric mean LDP for p-balls. We shall present here the proof of Theorem 1.4. Essentially, the result is a consequence of Cramér's theorem, the contraction principle and the probabilistic representation of Schechtman and Zinn. However, most of the work is a careful analysis required to obtain the rate function, which is represented in terms of the inverse function of $H(x) = \psi(x) - \log x$.

Proof of Theorem 1.4. We want to study the large deviations behaviour of the ratios $\mathcal{R}_n$ given in (3.1). As we shall see later, $\mathcal{R}_n$ can be written as a function of the partial sums

$$\frac{1}{n}\sum_{i=1}^{n}(\log|Z_i|, |Z_i|^p)$$

with i.i.d. increments $(\log|Z_i|, |Z_i|^p)$, $i \in \mathbb{N}$, and where the Z_i's are i.i.d. p-generalized Gaussians. The goal is to prove a large deviations principle via Cramér's theorem and then to apply the contraction principle. In order to do that, we first need to check that $0 \in D_\Lambda^\circ$. We have

$$\begin{aligned}
\Lambda(s,t) &= \log \mathbb{E} e^{\langle(\log|Z_i|,|Z_i|^p),(s,t)\rangle} \\
&= \log\bigg(\frac{1}{p^{1/p}\Gamma(1+1/p)}\int_0^\infty e^{s\log x-(1/p-t)x^p}\,\mathrm{d}x\bigg) \\
&= \log\bigg(\frac{1}{p(1-pt)^{1/p}\Gamma(1+1/p)}\Big(\frac{p}{1-pt}\Big)^{s/p}\Gamma\Big(\frac{s+1}{p}\Big)\bigg) \\
&= -\frac{1}{p}\log(1-pt) + \frac{s}{p}[\log p - \log(1-pt)] \\
&\quad + \log\Gamma\Big(\frac{s+1}{p}\Big) - \log\Gamma\Big(\frac{1}{p}\Big),
\end{aligned}$$

where $t < 1/p$ and $s > -1$. Otherwise, we have $\Lambda(s,t) = +\infty$ as can be seen from the second line. In particular, $D_\Lambda = (-1,\infty) \times (-\infty, 1/p)$ and thus $0 \in D_\Lambda^\circ$. This means that we can apply Cramér's theorem (Proposition 2.6) and obtain an LDP for the partial sums

$$\frac{1}{n}\sum_{i=1}^{n}(\log|Z_i|, |Z_i|^p) \tag{4.1}$$

with speed n and the rate function given by the Legendre–Fenchel transform of Λ, which is

$$\begin{aligned}\Lambda^*(\alpha,\beta) &= \sup_{(s,t)\in\mathbb{R}^2}[\langle(s,t),(\alpha,\beta)\rangle - \Lambda(s,t)]\\ &= \sup_{(s,t)\in(-1,\infty)\times(-\infty,1/p)}[\langle(s,t),(\alpha,\beta)\rangle - \Lambda(s,t)].\end{aligned}$$

First, we observe that the function $(s,t)\mapsto\langle(s,t),(\alpha,\beta)\rangle-\Lambda(s,t)$ is concave on $(-1,\infty)\times(-\infty,1/p)$. We shall show that if $\beta > e^{p\alpha}$, then the gradient of this function vanishes at a unique point $(s^*,t^*)\in(-1,\infty)\times(-\infty,1/p)$. This implies that the supremum is attained at this point.

Computing the partial derivatives to find the point (s^*,t^*), we obtain the following two conditions:

$$\frac{\partial}{\partial s}\Lambda(s^*,t^*) = \frac{1}{p}\left[\frac{\Gamma'\big(\frac{s^*+1}{p}\big)}{\Gamma\big(\frac{s^*+1}{p}\big)} + \log\left(\frac{p}{1-pt^*}\right)\right] = \alpha, \tag{4.2}$$

$$\frac{\partial}{\partial t}\Lambda(s^*,t^*) = \frac{s^*+1}{1-pt^*} = \beta. \tag{4.3}$$

Let $\beta > e^{p\alpha}$. To prove that the system (4.2), (4.3) has indeed a unique solution (s^*,t^*), we define $v^*\in(0,\infty)$ as the unique solution to the equation

$$H(v^*) = p\alpha - \log\beta,$$

where we recall that $H(x)=\psi(x)-\log x$ is negative on $(0,\infty)$. To see that this equation has a unique solution $v^*\in(0,\infty)$, we use the representation (see, e.g., [7, Eq. 8.361.8, p. 903])

$$H(x) = \psi(x) - \log x = -\int_0^\infty e^{-tx}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) dt.$$

The integrand is positive and decreases in $x>0$ for every $t>0$. Therefore, H is increasing, and it follows from the monotone convergence theorem that $H(0+)=-\infty$ and $H(+\infty)=0$ (for the former equality, we use the fact that the integral diverges at $x=0$). Hence, H is an increasing bijection between $(0,\infty)$ and $(-\infty,0)$, which also shows that m_p defined in Theorem 1.1 is negative.

Let $s^*>-1$ and $t^*<1/p$ be defined by

$$v^* = \frac{s^*+1}{p} \quad\text{and}\quad t^* = \frac{1}{p} - \frac{s^*+1}{\beta p}.$$

Then it can be easily checked that (4.2) and (4.3) hold. The uniqueness follows from the fact that (4.2) and (4.3) imply

$$H\left(\frac{s^*+1}{p}\right) = p\alpha - \log\beta < 0, \tag{4.4}$$

which has a unique solution $s^* > -1$ as seen above. From this we can determine the unique solution t^* to (4.3).

The contraction principle (Proposition 2.7) will be applied to the random vectors in (4.1) with the continuous function

$$F : (x,y) \mapsto \frac{e^x}{y^{1/p}}, \qquad x \in \mathbb{R},\, y \ge e^{px},$$

and $F(x,y) = 1$ otherwise. This means that the sequence of random variables

$$F\bigg(\frac{1}{n}\sum_{i=1}^{n}(\log|Z_i|, |Z_i|^p)\bigg) = \frac{(\prod_{i=1}^n |Z_i|)^{1/n}}{(n^{-1}\sum_{i=1}^n |Z_i|^p)^{1/p}} \overset{d}{=} \mathcal{R}_n$$

satisfies an LDP with speed n and a rate function $\mathcal{J}_p$ to be determined. For $\theta \in (0,1)$, the rate function is given by

$$\mathcal{J}_p(\theta) = \inf_{(\alpha,\beta):\, F(\alpha,\beta)=\theta} \Lambda^*(\alpha,\beta) = \inf_{\beta>0,\, \alpha=\log\theta+\frac{1}{p}\log\beta} [\alpha s^* + \beta t^* - \Lambda(s^*,t^*)],$$

and $\mathcal{J}_p \equiv +\infty$ on $\mathbb{R}\setminus(0,1]$. Let $\theta\in(0,1)$. Note that (4.3) implies

$$\frac{\beta}{(s^*+1)/p} = \frac{p}{1-pt^*} \quad \text{and} \quad \frac{\beta}{p} - \frac{s^*+1}{p} = \beta t^*. \tag{4.5}$$

Now, using (4.5) to exclude t^* from $\Lambda(s^*,t^*)$, we obtain

$$\Lambda(s^*,t^*) = \frac{s^*+1}{p}\log\bigg(\frac{\beta}{s^*+1}\bigg) + \frac{s^*}{p}\log p + \log\Gamma\bigg(\frac{s^*+1}{p}\bigg) - \log\Gamma\bigg(\frac{1}{p}\bigg).$$

Hence, excluding α and βt^* (using (4.5) for the latter) yields

$$\begin{aligned} \mathcal{J}_p(\theta) &= \inf_{\beta>0}\bigg[s^*\log\theta + \frac{s^*}{p}\log\beta + \frac{\beta}{p} - \frac{s^*+1}{p} - \Lambda(s^*,t^*)\bigg] \\ &= \inf_{\beta>0}\bigg[s^*\log\theta + \frac{\beta}{p} - \frac{s^*+1}{p} - \frac{1}{p}\log\beta \\ &\quad + \frac{s^*+1}{p}\log(s^*+1) - \frac{s^*}{p}\log p - \log\Gamma\bigg(\frac{s^*+1}{p}\bigg) + \log\Gamma\bigg(\frac{1}{p}\bigg)\bigg]. \end{aligned} \tag{4.6}$$

Note that the equalities $F(\alpha,\beta)=\theta$ and (4.4) imply

$$p\log\theta = H\bigg(\frac{s^*+1}{p}\bigg), \tag{4.7}$$

Since s^* given by (4.7) is a function of θ only and is independent of β (under $F(\alpha,\beta)=\theta$), it is clear that the infimum in (4.6) with $\theta\in(0,1)$ is attained at $\beta=1$ (which minimizes $p^{-1}(\beta-\log\beta)$). Thus,

$$\begin{aligned} \mathcal{J}_p(\theta) &= (\log\theta)s^* + \frac{s^*+1}{p}\bigg[\log\bigg(\frac{s^*+1}{p}\bigg) - 1\bigg] - \log\Gamma\bigg(\frac{s^*+1}{p}\bigg) \\ &\quad + \frac{1}{p} + \frac{1}{p}\log p + \log\Gamma\bigg(\frac{1}{p}\bigg). \end{aligned}$$

Finally, using (4.7) to exclude s^* and recalling that $G_p(\theta) = H^{-1}(p\log\theta)$, we obtain

$$\mathcal{J}_p(\theta) = [pG_p(\theta) - 1]\log\theta + G_p(\theta)[\log G_p(\theta) - 1] - \log\Gamma(G_p(\theta)) \\ + \frac{1}{p} + \frac{1}{p}\log p + \log\Gamma\left(\frac{1}{p}\right), \qquad \theta\in(0,1).$$

We shall now prove that $\mathcal{J}_p(1) = +\infty$, where by the contraction principle

$$\mathcal{J}_p(1) = \inf_{F(\alpha,\beta)=1} \Lambda^*(\alpha,\beta) = \inf_{(\alpha,\beta):\, \beta\le e^{p\alpha}} \Lambda^*(\alpha,\beta).$$

We claim that for all pairs (α,β) with $\beta \le e^{p\alpha}$, we have

$$\Lambda^*(\alpha,\beta) \equiv \sup_{s>-1,\, t<1/p} [\alpha s + \beta t - \Lambda(s,t)] = +\infty.$$

To prove this it is enough to consider a sequence of pairs (s_k, t_k) such that $s_k \to +\infty$ (as $k\to\infty$) and

$$t_k := \frac{1}{p} - \frac{s_k+1}{\beta p}, \qquad k\in\mathbb{N},$$

and to show that $\alpha s_k + \beta t_k - \Lambda(s_k,t_k) \to \infty$ as $k\to\infty$. It follows from the definition of t_k that if $v_k = (s_k+1)/p$, then

$$1 - pt_k = \frac{v_k p}{\beta} \quad\text{and}\quad \frac{p}{1-pt_k} = \frac{\beta}{v_k}.$$

Using the expression for $\Lambda(s_k,t_k)$ and excluding s_k and t_k, we get

$$\alpha s_k + \beta t_k - \Lambda(s_k,t_k) = v_k(p\alpha - \log\beta) \\ + (v_k\log v_k - v_k) - \log\Gamma(v_k) + c(\alpha,\beta,p)$$

where $c(\alpha,\beta,p)$ is a term independent of the sequence v_k. Note that $v_k \to +\infty$ as $k\to\infty$. Hence, by Stirling's formula, we have

$$v_k\log v_k - v_k - \log\Gamma(v_k) = \tfrac{1}{2}\log v_k - \tfrac{1}{2}\log 2\pi + o(1).$$

Since $p\alpha - \log\beta \ge 0$, we have, as $k\to\infty$,

$$\alpha s_k + \beta t_k - \Lambda(s_k,t_k) \to +\infty.$$

This proves that $\mathcal{J}_p(1) = +\infty$.

Therefore, we have proved an LDP for $\mathcal{R}_n$ with speed n and rate function $\mathcal{J}_p$ as stated in Remark 1.5. Since the function H is continuous, so is G_p on $(0,1)$, and consequently the same holds for $\mathcal{J}_p$ on $(0,1)$. Thus, the LDP for $\mathcal{R}_n$ yields the limiting behaviour as in the statement of Theorem 1.4 in the case of the uniform distribution on $\mathbb{B}_p^n$ or the cone measure on $\mathbb{S}_p^{n-1}$. We will now prove that the same LDP holds for the uniform distribution on $\mathbb{S}_p^{n-1}$. Let $A\subseteq\mathbb{R}$ be a Borel set and define

$$D_n := \left\{x = (x_1,\dots,x_n)\in\mathbb{S}_p^{n-1} : \frac{(\prod_{i=1}^n |x_i|)^{1/n}}{(n^{-1}\sum_{i=1}^n |x_i|^p)^{1/p}} \in A\right\}, \qquad n\in\mathbb{N}.$$

Then

$$\frac{1}{n}\log\sigma_p(D_n) = \frac{1}{n}\log\int\limits_{D_n} h_{n,p}(x)\,\mu_p(\mathrm{d}x),$$

and it follows from Lemma 2.2 that, as $n\to\infty$,

$$\left|\frac{1}{n}\log\mu_p(D_n) - \frac{1}{n}\log\sigma_p(D_n)\right| \to 0.$$

This shows the LDP for $\mathcal{R}_n$ in the case of the surface probability measure σ_p on $\mathbb{S}_p^{n-1}$ with the same rate function $\mathcal{J}_p$.

The limit relations $\mathcal{J}_p(0+) = \mathcal{J}_p(1-) = +\infty$ immediately follow from lower semicontinuity of $\mathcal{J}_p$ and the identities $\mathcal{J}_p(0) = \mathcal{J}_p(1) = +\infty$. ∎

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