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The ∂-complex on the Segal–Bargmann space

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To the memory of Józef Siciak

Abstract. We study certain densely defined unbounded operators on the Segal–Bargmann space. These are the annihilation and creation operators of quantum mechanics. In several complex variables we have the ∂-operator and its adjoint ∂^* acting on $(p,0)$-forms with coefficients in the Segal–Bargmann space. We consider the corresponding ∂-complex and study the spectral properties of the corresponding complex Laplacian $\tilde{\Box} = \partial\partial^* + \partial^*\partial$. Finally, we study a more general complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, where D is a differential operator of polynomial type, to find the canonical solutions to the inhomogeneous equations $Du = \alpha$ and $D^*v = \beta$.

1. Introduction. The purpose of this paper is to consider the ∂-complex and to use the powerful classical methods of the $\overline{\partial}$-complex based on the theory of unbounded densely defined operators on Hilbert spaces (see [8], [16]). The main difference from the classical theory is that the underlying Hilbert space is now not an L^2-space but a closed subspace of an L^2-space—the Segal–Bargmann space $A^2(\mathbb{C}^n, e^{-|z|^2})$ of entire functions. It is well known that differentiation with respect to z_j defines an unbounded operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$. We consider the operator

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j,$$

which is densely defined on $A^2(\mathbb{C}^n, e^{-|z|^2})$ and maps to $A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$, the space of $(1,0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$. In general, we get

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the ∂-complex

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}),$$

where $1 \leq p \leq n-1$ and ∂^* denotes the adjoint operator of ∂.

We will choose the domain $\mathrm{dom}(\partial)$ in such a way that ∂ becomes a closed operator on $A^2(\mathbb{C}^n, e^{-|z|^2})$. In addition, the corresponding complex Laplacian

$$\tilde{\Box}_p = \partial^*\partial + \partial\partial^*$$

with $\mathrm{dom}(\tilde{\Box}_p) = \{f \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*) : \partial f \in \mathrm{dom}(\partial^*) \text{ and } \partial f^* \in \mathrm{dom}(\partial)\}$ acts as an unbounded self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. We point out that in this case the complex Laplacian is a differential operator of order one. Nevertheless, for these differential operators of order one we can use the general features of a Laplacian.

Using an estimate which is analogous to the basic estimate for the $\overline{\partial}$-complex, we find that the map $\tilde{\Box}_p$ has a bounded inverse $\tilde{N}_p$: $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$, and we show that $\tilde{N}_p$ is even compact. In addition we compute the spectrum of $\tilde{\Box}_p$.

The inspiration for this comes from quantum mechanics, where the annihilation operator a_j can be represented by differentiation with respect to z_j on $A^2(\mathbb{C}^n, e^{-|z|^2})$, and its adjoint, the creation operator a_j^*, is represented by multiplication by z_j, both operators being unbounded densely defined (see [5], [4]). One can show that $A^2(\mathbb{C}^n, e^{-|z|^2})$ with this action of the a_j and a_j^* is an irreducible representation M of the Heisenberg group; by the Stone–von Neumann theorem it is the only one up to unitary equivalence. Physically, M can be thought of as the Hilbert space of a harmonic oscillator with n degrees of freedom and Hamiltonian operator

$$H = \sum_{j=1}^n \frac{1}{2}(P_j^2 + Q_j^2) = \sum_{j=1}^n \frac{1}{2}(a_j^* a_j + a_j a_j^*).$$

In the second part we consider a general $\mathcal{C}^2$ weight function $\varphi : \mathbb{C}^n \to \mathbb{R}$ and the corresponding weighted space $A^2(\mathbb{C}^n, e^{-\varphi})$ of entire functions. The ∂-complex now has the form

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-\varphi}) \underset{\partial^*_\varphi}{\overset{\partial}{\leftrightarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) \underset{\partial^*_\varphi}{\overset{\partial}{\leftrightarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}),$$

where $1 \leq p \leq n-1$. Furthermore, we prove a formula which is analogous to the Kohn–Morrey formula for the classical $\overline{\partial}$-complex (see [8], [16] or [1]):

$$\|\partial u\|^2_\varphi + \|\partial^*_\varphi u\|^2_\varphi = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda + T$$

for $u \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*_\varphi)$, where the term T is non-positive.

Finally, we investigate operators of the form $Du = \sum_{j=1}^n p_j(u)dz_j$, where $u \in A^2(\mathbb{C}^n, e^{-|z|^2})$ and $p_j\big(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\big)$ are polynomial differential operators with constant coefficients. Differential operators of polynomial type on the Segal–Bargmann space were also investigated by J. D. Newman and H. Shapiro [12], [13] and by H. Render [14] in the real analytic setting. Replacing ∂ by D one gets the corresponding complex Laplacian $\tilde{\Box}_D = DD^* + D^*D$, for which one can use duality and the machinery of the $\overline{\partial}$-Neumann operator [9], [10] in order to prove existence and boundedness of the inverse to $\tilde{\Box}_D$ and to find the canonical solutions to the inhomogeneous equations $Du = \alpha$ and $D^*v = \beta$.

2. The Segal–Bargmann space. We consider the *Segal–Bargmann space* $A^2(\mathbb{C}^n, e^{-|z|^2})$ consisting of all entire functions f such that

$$\|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \infty.$$

It is clear that it is a Hilbert space with the inner product

$$(f, g) = \int_{\mathbb{C}^n} f(z)\, \overline{g(z)}\, e^{-|z|^2}\, d\lambda(z).$$

Setting $n = 1$, we see for $f \in A^2(\mathbb{C}, e^{-|z|^2})$ that

$$\begin{aligned}|f(z)| &\le \frac{1}{\pi r^2} \int_{D_r(z)} e^{|w|^2/2} |f(w)| e^{-|w|^2/2}\, d\lambda(w) \\ &\le \frac{1}{\pi r^2} \Big(\int_{D_r(z)} e^{|w|^2}\, d\lambda(w)\Big)^{1/2} \Big(\int_{D_r(z)} |f(w)|^2 e^{-|w|^2}\, d\lambda(w)\Big)^{1/2} \\ &\le C\Big(\int_{\mathbb{C}} |f(w)|^2 e^{-|w|^2}\, d\lambda(w)\Big)^{1/2} = C\|f\|,\end{aligned}$$

where C is a constant only depending on z. In addition, for each compact subset L of $\mathbb{C}$ there exists a constant $C_L > 0$ such that

$$\sup_{z\in L} |f(z)| \le C_L \|f\| \quad \text{for all } f \in A^2(\mathbb{C}, e^{-|z|^2}). \tag{2.1}$$

For several variables one immediately gets an analogous estimate. This implies that the Segal–Bargmann space $A^2(\mathbb{C}^n, e^{-|z|^2})$ has the reproducing property. The monomials $\{z^\alpha\}$ constitute an orthogonal basis, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, and the norms of the monomials are

$$\|z^\alpha\|^2 = \int_{\mathbb{C}} |z_1|^{2\alpha_1} e^{-|z_1|^2}\, d\lambda(z_1) \cdots \int_{\mathbb{C}} |z_n|^{2\alpha_n} e^{-|z_n|^2}\, d\lambda(z_n)$$

$$= (2\pi)^n \int_0^\infty r^{2\alpha_1+1} e^{-r^2}\, dr \cdots \int_0^\infty r^{2\alpha_n+1} e^{-r^2}\, dr$$
$$= \pi^n \alpha_1! \dots \alpha_n!.$$

It follows that each function $f \in A^2(\mathbb{C}^n, e^{-|z|^2})$ can be written in the form

$$f = \sum_\alpha f_\alpha \varphi_\alpha,$$

where

$$\varphi_\alpha(z) = \frac{z^\alpha}{\sqrt{\pi^n \alpha!}} \quad \text{and} \quad \sum_\alpha |f_\alpha|^2 < \infty \tag{2.2}$$

and $\alpha! = \alpha_1! \dots \alpha_n!$.

Hence the Bergman kernel of $A^2(\mathbb{C}^n, e^{-|z|^2})$ is

$$K(z,w) = \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|^2} = \frac{1}{\pi^n} \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{z^\alpha \overline{w}^\alpha}{\alpha_1! \dots \alpha_n!} \tag{2.3}$$
$$= \frac{1}{\pi^n} \exp(z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n).$$

See [19] for an extensive study of the Segal–Bargmann space.

We point out that the space $A^2(\mathbb{C}, e^{-|z|^2})$ serves for representation of states in quantum mechanics (see [4]), where

$$a(f) = \frac{df}{dz}$$

is the annihilation operator and

$$a^*(f) = zf$$

is the creation operator, both being densely defined unbounded operators on $A^2(\mathbb{C}, e^{-|z|^2})$. Indeed, the span of the finite linear combinations of the basis functions φ_α is dense in $A^2(\mathbb{C}, e^{-|z|^2})$. Moreover

$$F(z) = \sum_{k=2}^\infty \frac{\varphi_k(z)}{\sqrt{k(k-1)}} \in A^2(\mathbb{C}, e^{-|z|^2})$$

but

$$F'(z) = \sum_{k=1}^\infty \frac{\varphi_k(z)}{\sqrt{k}} \notin A^2(\mathbb{C}, e^{-|z|^2}),$$

and

$$G(z) = \sum_{k=0}^\infty \frac{\varphi_k(z)}{k+1} \in A^2(\mathbb{C}, e^{-|z|^2})$$

but

$$zG(z) = \sum_{k=1}^{\infty} \frac{\varphi_k(z)}{\sqrt{k}} \notin A^2(\mathbb{C}, e^{-|z|^2}),$$

hence both a and a^* are unbounded operators on $A^2(\mathbb{C}, e^{-|z|^2})$.

On the other hand, taking a primitive of a function $f \in A^2(\mathbb{C}, e^{-|z|^2})$ yields a bounded operator

$$T : A^2(\mathbb{C}, e^{-|z|^2}) \to A^2(\mathbb{C}, e^{-|z|^2}).$$

Indeed, let

$$f(z) = \sum_{k=0}^{\infty} f_k \frac{z^k}{\sqrt{\pi}\sqrt{k!}}, \qquad \sum_{k=0}^{\infty} |f_k|^2 < \infty.$$

Then the function

$$h(z) = \sum_{k=0}^{\infty} f_k \frac{z^{k+1}}{\sqrt{\pi}\sqrt{k+1}\sqrt{(k+1)!}}$$

defines a primitive of f and we can write

$$T(f) = h = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k!}} (f, \tilde{\varphi}_k)\varphi_k,$$

where $\tilde{\varphi}_k = \varphi_{k-1}$ and the constant term in the Fourier series expansion is 0. This implies immediately that T is even a compact operator.

This is also a special case in the theory of Volterra-type integration operators on the Segal–Bargmann space of the form

$$T_g f(z) = \int_0^z f g' \, d\zeta$$

(see [2]).

Next, we define the domain of the operator a to be

$$\mathrm{dom}(a) = \{f \in A^2(\mathbb{C}, e^{-|z|^2}) : f' \in A^2(\mathbb{C}, e^{-|z|^2})\}.$$

Then $\mathrm{dom}(a^*)$ consists of all functions $g \in A^2(\mathbb{C}, e^{-|z|^2})$ such that the densely defined linear functional $L(f) = (a(f), g)$ is continuous on $\mathrm{dom}(a)$. This implies that there exists a function $h \in A^2(\mathbb{C}, e^{-|z|^2})$ such that $L(f) = (a(f), g) = (f, h)$.

Next, we show that

$$\mathrm{dom}(a^*) = \{g \in A^2(\mathbb{C}, e^{-|z|^2}) : zg \in A^2(\mathbb{C}, e^{-|z|^2})\}.$$

Let $f \in \mathrm{dom}(a)$ and $g \in \mathrm{dom}(a^*)$. Then

$$(a(f), g) = \int_{\mathbb{C}} \frac{df(z)}{dz} \overline{g(z)}\, e^{-|z|^2}\, d\lambda(z) = -\int_{\mathbb{C}} f(z) \frac{d}{dz}(\overline{g(z)}\, e^{-|z|^2})\, d\lambda(z)$$
$$= \int_{\mathbb{C}} f(z)\, \overline{zg(z)}\, e^{-|z|^2}\, d\lambda(z) = (f, a^*(g)),$$

where we have used integration by parts in the first step (see (3.3) for a detailed proof) and the fact that

$$\frac{d}{dz}(\overline{g(z)}\, e^{-|z|^2}) = \overline{g(z)}(-\overline{z} e^{-|z|^2}).$$

An alternative proof uses Fourier series expansion: Let

$$f(z) = \sum_{k=0}^{\infty} f_k \varphi_k(z) \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} g_k \varphi_k(z),$$

where $(f_k)_k, (g_k)_k \in l^2$. Then

$$(a(f), g) = \sum_{k=0}^{\infty} \sqrt{k+1}\, f_{k+1}\, \overline{g_k} = (f, zg) = (f, a^*(g)).$$

REMARK 2.1. (a) We point out that the commutator satisfies $[a, a^*] = I$, which is of importance in quantum mechanics (see [4]).

(b) For $f \in \mathrm{dom}(a) \cap \mathrm{dom}(a^*)$ we deduce from the last results that

$$\|a(f)\|^2 + \|a^*(f)\|^2 = 2\|a(f)\|^2 + \|f\|^2, \tag{2.4}$$

which implies that $\mathrm{dom}(a) = \mathrm{dom}(a^*)$.

LEMMA 2.2. *The operators a and a^* are densely defined operators on $A^2(\mathbb{C}, e^{-|z|^2})$ with closed graphs.*

Proof. By general properties of unbounded operators, it suffices to prove the assertion for a (see [8] or [18]). Let $(f_j)_j$ be a sequence in $\mathrm{dom}(a)$ such that $\lim_{j\to\infty} f_j = f$ and $\lim_{j\to\infty} a(f_j) = g$ in $A^2(\mathbb{C}, e^{-|z|^2})$. We have to show that $f \in \mathrm{dom}(a)$ and $a(f) = g$. By (2.1) it follows that $(f_j)_j$ converges uniformly on each compact subset of $\mathbb{C}$ to f and the same is true for the derivatives $\lim_{j\to\infty} f_j' = f'$. This implies that $f' = a(f) = g$. Since $g \in A^2(\mathbb{C}, e^{-|z|^2})$, and taking primitives we do not leave $A^2(\mathbb{C}, e^{-|z|^2})$, it follows that $f \in \mathrm{dom}(a)$, which proves the assertion. ∎

For the rest of this section we consider the Segal–Bargmann space in several variables with the weight $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$. We will denote the derivative with respect to z by ∂ and in the following we will consider the ∂-complex for the Segal–Bargmann space in several variables,

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}),$$

where $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ denotes the Hilbert space of $(p,0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$, and

$$\partial f = \sum_{|J|=p}{}' \sum_{j=1}^{n} \frac{\partial f_J}{\partial z_j}\, dz_j \wedge dz_J$$

for a $(p,0)$-form

$$f = \sum_{|J|=p}{}' f_J\, dz_J$$

with summation over increasing multiindices $J = (j_1, \dots, j_p)$, $1 \le p \le n-1$; and we take

$$\operatorname{dom}(\partial) = \{f \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : \partial f \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2})\}.$$

Now let

$$\tilde{\Box}_p = \partial^* \partial + \partial \partial^*,$$

with $\operatorname{dom}(\tilde{\Box}_p) = \{f \in \operatorname{dom}(\partial) \cap \operatorname{dom}(\partial^*) : \partial f \in \operatorname{dom}(\partial^*) \text{ and } \partial^* f \in \operatorname{dom}(\partial)\}$. Then $\tilde{\Box}_p$ acts as an unbounded self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ (see [8, p. 77]).

Remark 2.3. (a) Let $g = \sum_{j=1}^n g_j dz_j$ be a $(1,0)$-form with holomorphic coefficients. The pull back of g by a holomorphic map $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ is again a $(1,0)$-form with holomorphic coefficients: we have

$$F^* g = \sum_{l=1}^n g_l\, dF_l = \sum_{j=1}^n \left(\sum_{l=1}^n g_l \frac{\partial F_l}{\partial z_j} \right) dz_j,$$

where we have used the fact that

$$dF_l = \partial F_l + \overline{\partial}\, F_l = \sum_{j=1}^n \frac{\partial F_l}{\partial z_j}\, dz_j + \sum_{j=1}^n \frac{\partial F_l}{\partial \overline{z}_j}\, d\overline{z}_j = \sum_{j=1}^n \frac{\partial F_l}{\partial z_j}\, dz_j.$$

The expressions $\frac{\partial F_l}{\partial z_j}$ are holomorphic.

(b) For $p = 0$ and $f \in \operatorname{dom}(\tilde{\Box}_0)$ we have

$$\tilde{\Box}_0 f = \partial^* \partial f = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}.$$

If $1 \le p \le n-1$ and $f = \sum'_{|J|=p} f_J\, dz_J \in \operatorname{dom}(\tilde{\Box}_p)$ is a $(p,0)$-form, we have

$$\tilde{\Box}_p f = \sum_{|J|=p}{}' \left(\sum_{k=1}^n z_k \frac{\partial f_J}{\partial z_k} + p f_J \right) dz_J. \tag{2.5}$$

For $p = n$ and an $(n,0)$-form $F \in \mathrm{dom}(\tilde{\Box}_n)$ (here we identify the $(n,0)$-form with a function), we have

$$\tilde{\Box}_n F = \partial\partial^* F = \sum_{j=1}^n z_j \frac{\partial F}{\partial z_j} + nF.$$

Before we continue the study of the box operator $\tilde{\Box}_p$, we collect some facts about Segal–Bargmann spaces with more general weights.

3. Generalized Segal–Bargmann spaces. Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic $\mathcal{C}^\infty$ function. Let

$$A^2(\mathbb{C}^n, e^{-\varphi}) = \Big\{ f : \mathbb{C}^n \to \mathbb{C} \text{ entire} : \|f\|_\varphi^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi}\, d\lambda < \infty \Big\},$$

with inner product

$$(f,g)_\varphi = \int_{\mathbb{C}^n} f\overline{g} e^{-\varphi}\, d\lambda.$$

It is easily seen that $A^2(\mathbb{C}^n, e^{-\varphi})$ is a Hilbert space with the reproducing property. Hence it has a reproducing kernel $K_\varphi(z,w)$ (Bergman kernel) which has the following properties: $K_\varphi(w,z) = \overline{K_\varphi(z,w)}$, the function $z \mapsto K_\varphi(z,w)$ belongs to $A^2(\mathbb{C}^n, e^{-\varphi})$ and

$$f(z) = \int_{\mathbb{C}^n} K_\varphi(z,w) f(w) e^{-\varphi(w)}\, d\lambda(w) \quad \text{for } f \in A^2(\mathbb{C}^n, e^{-\varphi}).$$

The Bergman projection $P_\varphi : L^2(\mathbb{C}^n, e^{-\varphi}) \to A^2(\mathbb{C}^n, e^{-\varphi})$ can be written in the form

$$P_\varphi F(z) = \int_{\mathbb{C}^n} K_\varphi(z,w) F(w) e^{-\varphi(w)}\, d\lambda(w) \text{ for } F \in L^2(\mathbb{C}^n, e^{-\varphi}).$$

REMARK 3.1. We mention that $A^2(\mathbb{C}^n, e^{-\varphi})$ is infinite-dimensional if the lowest eigenvalue μ_φ of the Levi matrix

$$\left(\frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j} \right)_{j,k=1}^n$$

satisfies $\lim_{|z|\to\infty} |z|^2 \mu_\varphi(z) = +\infty$ (see [15] or [8]).

We study the ∂-complex

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-\varphi}) \underset{\partial^*_\varphi}{\overset{\partial}{\leftrightarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) \underset{\partial^*_\varphi}{\overset{\partial}{\leftrightarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}),$$

where $A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi})$ denotes the Hilbert space of $(p,0)$-forms with coefficients in $A^2(\mathbb{C}^n, e^{-\varphi})$, and

$$\partial f = {\sum_{|J|=p}}' \sum_{j=1}^{n} \frac{\partial f_J}{\partial z_j}\, dz_j \wedge dz_J$$

for a $(p,0)$-form $f = \sum'_{|J|=p} f_J\, dz_J$ with summation over increasing multi-indices $J = (j_1, \dots, j_p)$, $1 \le p \le n-1$; and we take

$$\mathrm{dom}(\partial) = \{f \in A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi}) : \partial f \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi})\}.$$

Now we suppose that $\mathrm{dom}(\partial)$ is dense in $A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi})$, which is true if the $(p,0)$-forms with polynomial components are dense in $A^2_{(p,0)}(\mathbb{C}^n, e^{-\varphi})$. If φ is a convex function such that its conjugate convex function

$$\varphi^*(w) = \sup\{\Re\langle z, w\rangle - \varphi(z) : z \in \mathbb{C}^n\}$$

is finite on a neighborhood of the origin in $\mathbb{C}^n$, then the polynomials are dense in $A^2(\mathbb{C}^n, e^{-\varphi})$ (see [17], or [6] for other conditions on φ implying density of the polynomials).

The adjoint operator to ∂ depends on the weight:

$$\mathrm{dom}(\partial^*_\varphi) = \{g \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi}) : f \mapsto (\partial f, g)_\varphi \text{ is continuous on } \mathrm{dom}(\partial)\}.$$

We use the Gauß–Green Theorem in order to compute the adjoint ∂^*_φ.

Let $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$, where r is a real valued $\mathcal{C}^1$-function with

$$\nabla_z r := \left(\frac{\partial r}{\partial z_1}, \dots, \frac{\partial r}{\partial z_n}\right) \neq 0$$

on $b\Omega = \{z : r(z) = 0\}$. Without loss of generality we can suppose that $|\nabla_z r| = |\nabla r| = 1$ on $b\Omega$. For $u, v \in \mathcal{C}^\infty(\overline{\Omega})$ and

$$(u, v) = \int_\Omega u(z)\, \overline{v(z)}\, d\lambda(z),$$

the Gauß–Green Theorem implies that

$$\left(\frac{\partial u}{\partial z_k}, v\right) = -\left(u, \frac{\partial v}{\partial \overline{z}_k}\right) + \int_{b\Omega} u(z)\, \overline{v(z)}\, \frac{\partial r}{\partial z_k}(z)\, d\sigma(z), \tag{3.1}$$

where $d\sigma$ is the surface measure on $b\Omega$.

In our case we have holomorphic components f_J and g_{jJ} and the inner product

$$\left(\frac{\partial f_J}{\partial z_j}, g_{jJ}\right)_\varphi = \int_{\mathbb{C}^n} \frac{\partial f_J}{\partial z_j}\, \overline{g_{jJ}}\, e^{-\varphi}\, d\lambda.$$

Now let $\Omega = \{z : |z| < R\}$, take $r(z) = \frac{|z|^2 - R^2}{R}$ and apply (3.1) to get

$$\int_{|z|\le R} \frac{\partial f_J}{\partial z_j}\, \overline{g_{jJ}}\, e^{-\varphi}\, d\lambda - \int_{|z|\le R} f_J\, \overline{\frac{\partial \varphi}{\partial z_j} g_{jJ}}\, e^{-\varphi}\, d\lambda = \int_{|z|=R} f_J\, \overline{g_{jJ}}\, \frac{\overline{z_j}}{R}\, e^{-\varphi}\, d\sigma. \tag{3.2}$$

By Cauchy–Schwarz we get

$$\Big| \int\limits_{|z|=R} f_J\, \overline{g_{jJ}}\, \frac{\overline{z_j}}{R}\, e^{-\varphi}\, d\sigma \Big|^2 \leq \int\limits_{|z|=R} |f_J|^2 e^{-\varphi}\, d\sigma \int\limits_{|z|=R} |g_{jJ}|^2 e^{-\varphi}\, d\sigma,$$

and as

$$\|f_J\|_\varphi^2 = \int\limits_{\mathbb{C}^n} |f_J|^2 e^{-\varphi}\, d\lambda = \int\limits_0^\infty R^{2n-1} \int\limits_{|z|=R} |f_J|^2 e^{-\varphi}\, d\sigma\, dR < \infty,$$

the right-hand side of (3.2) tends to zero as $R \to \infty$. If we suppose that $\frac{\partial\varphi}{\partial\overline{z}_j}\, g_{jJ} \in L^2(\mathbb{C}^n, e^{-\varphi})$, we obtain, for the components of the $(p,0)$-form f and the $(p+1,0)$-form g,

$$\begin{aligned} \Big(\frac{\partial f_J}{\partial z_j}, g_{jJ}\Big)_\varphi &= \Big(f_J, \frac{\partial\varphi}{\partial\overline{z}_j}\, g_{jJ}\Big)_\varphi = \Big(P_\varphi(f_J), \frac{\partial\varphi}{\partial\overline{z}_j}\, g_{jJ}\Big)_\varphi \\ &= \Big(f_J, P_\varphi\Big(\frac{\partial\varphi}{\partial\overline{z}_j}\, g_{jJ}\Big)\Big)_\varphi, \end{aligned}$$

where we have used the fact that the components f_J are holomorphic. Hence we obtain

$$\partial_\varphi^* u = {\sum_{|K|=p-1}}' \sum_{j=1}^n P_\varphi\Big(\frac{\partial\varphi}{\partial\overline{z}_j}\, u_{jK}\Big)\, dz_K \tag{3.3}$$

for a $(p,0)$-form $u \in \operatorname{dom}(\partial_\varphi^*)$.

Similar to Lemma 2.2 one shows that $\partial : \operatorname{dom}(\partial) \to A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi})$ has closed graph. It follows that $\operatorname{dom}(\partial_\varphi^*)$ is dense in $A^2_{(p+1,0)}(\mathbb{C}^n, e^{-\varphi})$ (see [8, p. 52] or [18, p. 90]).

In the following we prove an identity which is analogous to the Kohn–Morrey formula for the $\overline{\partial}$-complex (see [16], [8]). Now an additional non-positive term appears, which vanishes for the weighted $\overline{\partial}$-complex.

THEOREM 3.2. *Let $u = \sum_{j=1}^n u_j\, dz_j \in A^2_{(1,0)}(\mathbb{C}^n, e^{-\varphi})$ and suppose that $u \in \operatorname{dom}(\partial) \cap \operatorname{dom}(\partial_\varphi^*)$ and $\frac{\partial\varphi}{\partial\overline{z}_j} u_j \in L^2(\mathbb{C}^n, e^{-\varphi})$, $j = 1, \ldots, n$. Then*

$$\begin{aligned} \|\partial u\|_\varphi^2 + \|\partial_\varphi^* u\|_\varphi^2 &= \sum_{j,k=1}^n \int\limits_{\mathbb{C}^n} \Big|\frac{\partial u_j}{\partial z_k}\Big|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^n \int\limits_{\mathbb{C}^n} \frac{\partial^2\varphi}{\partial z_k \partial\overline{z}_j}\, u_j \overline{u}_k e^{-\varphi}\, d\lambda \\ &\quad - \sum_{j,k=1}^n \Big(\frac{\partial\varphi}{\partial\overline{z}_j} u_j - P_\varphi\Big(\frac{\partial\varphi}{\partial\overline{z}_j} u_j\Big), \frac{\partial\varphi}{\partial\overline{z}_k} u_k\Big)_\varphi. \end{aligned}$$

Proof. Since

$$\partial u = \sum_{j<k} \Big(\frac{\partial u_j}{\partial z_k} - \frac{\partial u_k}{\partial z_j}\Big)\, dz_j \wedge dz_k \quad \text{and} \quad \partial_\varphi^* u = \sum_{j=1}^n P_\varphi\Big(\frac{\partial\varphi}{\partial\overline{z}_j} u_j\Big),$$

we get

$$
\begin{aligned}
&\|\overline{\partial}u\|_\varphi^2 + \|\partial_\varphi^* u\|_\varphi^2 \\
&\quad = \int_{\mathbb{C}^n} \sum_{j<k} \left| \frac{\partial u_j}{\partial z_k} - \frac{\partial u_k}{\partial z_j} \right|^2 e^{-\varphi}\, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j\Big) \overline{P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_k} u_k\Big)}\, e^{-\varphi}\, d\lambda \\
&\quad = \sum_{j,k=1}^n \left[\int_{\mathbb{C}^n} \left(\left| \frac{\partial u_j}{\partial z_k} \right|^2 + P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j\Big) \overline{P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_k} u_k\Big)} - \frac{\partial u_j}{\partial z_k} \overline{\frac{\partial u_k}{\partial z_j}} \right) e^{-\varphi}\, d\lambda \right] \\
&\quad = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial z_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left[\frac{\partial}{\partial z_k}, P_\circ \frac{\partial \varphi}{\partial \overline{z}_j} \right] u_j \overline{u}_k\, e^{-\varphi}\, d\lambda,
\end{aligned}
$$

where we have used the fact that for $f, g \in A^2(\mathbb{C}^n, e^{-\varphi})$ we have

$$
\Big(\frac{\partial f}{\partial z_k}, g \Big)_\varphi = \Big(f, P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_k} g \Big) \Big)_\varphi
$$

and hence

$$
\Big(\Big[\frac{\partial}{\partial z_k}, P_\varphi \circ \frac{\partial \varphi}{\partial \overline{z}_j} \Big] u_j, u_k \Big)_\varphi = \Big(P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big), P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_k} u_k \Big) \Big)_\varphi - \Big(\frac{\partial u_j}{\partial z_k}, \frac{\partial u_k}{\partial z_j} \Big)_\varphi .
$$

Since

$$
\begin{aligned}
\Big(\Big[\frac{\partial}{\partial z_k}, P_\varphi \circ \frac{\partial \varphi}{\partial \overline{z}_j} \Big] u_j, u_k \Big)_\varphi &= \Big(\Big[\frac{\partial}{\partial z_k}, P_\varphi \Big] \Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big), u_k \Big)_\varphi \\
&\quad + \Big(\frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j} u_j, u_k \Big)_\varphi
\end{aligned}
$$

and we have

$$
\begin{aligned}
\Big(\Big[\frac{\partial}{\partial z_k}, P_\varphi \Big] \Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big), u_k \Big)_\varphi &= \Big(P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big), P_\varphi\big(\frac{\partial \varphi}{\partial \overline{z}_k} u_k \big) \Big)_\varphi \\
&\quad - \Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j, \frac{\partial \varphi}{\partial \overline{z}_k} u_k \Big)_\varphi \\
&= \Big(P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big) - \frac{\partial \varphi}{\partial \overline{z}_j} u_j, \frac{\partial \varphi}{\partial \overline{z}_k} u_k \Big)_\varphi ,
\end{aligned}
$$

we get the desired result. ■

REMARK 3.3. The last term

$$
\sum_{j,k=1}^n \Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j - P_\varphi\Big(\frac{\partial \varphi}{\partial \overline{z}_j} u_j \Big), \frac{\partial \varphi}{\partial \overline{z}_k} u_k \Big)_\varphi
$$

vanishes for $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$, and we obtain

$$\|\partial u\|_\varphi^2 + \|\partial_\varphi^* u\|_\varphi^2 = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left|\frac{\partial u_j}{\partial z_k}\right|^2 e^{-|z|^2}\, d\lambda + \sum_{j=1}^n \int_{\mathbb{C}^n} |u_j|^2 e^{-|z|^2}\, d\lambda. \tag{3.4}$$

If $n = 1$ and u is a $(1,0)$-form, we have $\partial u = 0$ and

$$\|\partial u\|_\varphi^2 + \|\partial_\varphi^* u\|_\varphi^2 = \|\partial_\varphi^* u\|_\varphi^2 = \|u\|_\varphi^2 + \|u'\|_\varphi^2.$$

THEOREM 3.4. *The last term in Theorem* 3.2*,*

$$\sum_{j,k=1}^n \left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j - P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j\right), \frac{\partial\varphi}{\partial\overline{z}_k}u_k\right)_\varphi,$$

is always non-negative:

$$\sum_{j,k=1}^n \left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j - P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j\right), \frac{\partial\varphi}{\partial\overline{z}_k}u_k\right)_\varphi$$
$$= \sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, \frac{\partial\varphi}{\partial\overline{z}_j}\right]u_j, u_k\right)_\varphi - \sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, P_\varphi\circ\frac{\partial\varphi}{\partial\overline{z}_j}\right]u_j, u_k\right)_\varphi$$
$$= \|R_\varphi v_1 + \cdots + R_\varphi v_n\|_\varphi^2 = \|V\|_\varphi^2 - \|P_\varphi V\|_\varphi^2,$$

where R_φ denotes the orthogonal projection $R_\varphi = I - P_\varphi$ and

$$V = \sum_{j=1}^n v_j = \sum_{j=1}^n \frac{\partial\varphi}{\partial\overline{z}_j}u_j.$$

Proof. Since

$$\sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, \frac{\partial\varphi}{\partial\overline{z}_j}\right]u_j, u_k\right)_\varphi = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_j}u_j\overline{u}_k e^{-\varphi}\, d\lambda,$$

from the proof of Theorem 3.2 we get

$$\sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, \frac{\partial\varphi}{\partial\overline{z}_j}\right]u_j, u_k\right)_\varphi - \sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, P_\varphi\circ\frac{\partial\varphi}{\partial\overline{z}_j}\right]u_j, u_k\right)_\varphi$$
$$= -\sum_{j,k=1}^n \left(\left[\frac{\partial}{\partial z_k}, P_\varphi\right]\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j\right), u_k\right)_\varphi,$$

which equals

$$\sum_{j,k=1}^n \left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j - P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_j\right), \frac{\partial\varphi}{\partial\overline{z}_k}u_k\right)_\varphi,$$

by the last computation in the proof of Theorem 3.2. This term can be written in the form

$$\sum_{j,k=1}^{n}(R_\varphi v_j, v_k)_\varphi = \sum_{j,k=1}^{n}(R_\varphi v_j, R_\varphi v_k)_\varphi$$
$$= (R_\varphi v_1 + \cdots + R_\varphi v_n, R_\varphi v_1 + \cdots + R_\varphi v_n)_\varphi$$
$$= \|R_\varphi v_1 + \cdots + R_\varphi v_n\|_\varphi^2 = \|V\|_\varphi^2 - \|P_\varphi V\|_\varphi^2,$$

and we are done. ∎

REMARK 3.5. Notice that for $u = \sum_{j=1}^{n} u_j\, dz_j \in \mathrm{dom}(\partial)\cap\mathrm{dom}(\partial_\varphi^*)$ we have

$$\left\|\frac{\partial u_j}{\partial z_k}\right\|_\varphi^2 = \left\|\frac{\partial \varphi}{\partial \overline{z}_k}\,u_j\right\|_\varphi^2 - \int_{\mathbb{C}^n}\frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_k}|u_j|^2 e^{-\varphi}\,d\lambda. \tag{3.5}$$

This follows from

$$\left\|\frac{\partial u_j}{\partial z_k}\right\|_\varphi^2 = \left(P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_k}\frac{\partial u_j}{\partial z_k}\right), u_j\right)_\varphi = \left(\frac{\partial\varphi}{\partial\overline{z}_k}\frac{\partial u_j}{\partial z_k}, u_j\right)_\varphi = \left(\frac{\partial u_j}{\partial z_k}, \frac{\partial\varphi}{\partial z_k}u_j\right)_\varphi$$
$$= -\left(u_j, \frac{\partial}{\partial\overline{z}_k}\left(\frac{\partial\varphi}{\partial z_k}u_j e^{-\varphi}\right)\right)$$
$$= -\left(u_j, \frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_k}u_j\right)_\varphi + \left(u_j, \frac{\partial\varphi}{\partial z_k}\frac{\partial\varphi}{\partial\overline{z}_k}u_j\right)_\varphi$$
$$= \left\|\frac{\partial\varphi}{\partial\overline{z}_k}u_j\right\|_\varphi^2 - \int_{\mathbb{C}^n}\frac{\partial^2\varphi}{\partial z_k\partial\overline{z}_k}|u_j|^2 e^{-\varphi}\,d\lambda,$$

where we have again used the fact that the components u_j are holomorphic.

Now we generalize Theorem 3.2 for $(p,0)$-forms $u = \sum_{|J|=p}' u_J\, dz_J$ with coefficients in $A^2(\mathbb{C}^n, e^{-\varphi})$ where $1 \le p \le n-1$. We notice that

$$\partial u = {\sum_{|J|=p}}'\sum_{j=1}^{n}\frac{\partial u_J}{\partial z_j}\,dz_j\wedge dz_J,$$
$$\partial_\varphi^* u = {\sum_{|K|=p-1}}'\sum_{j=1}^{n}P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_{jK}\right)dz_K.$$

We obtain

$$\|\partial u\|_\varphi^2 + \|\partial_\varphi^* u\|_\varphi^2 = {\sum_{|J|=|M|=p}}'\sum_{j,k=1}^{n}\epsilon_{jJ}^{kM}\int_{\mathbb{C}^n}\frac{\partial u_J}{\partial z_j}\overline{\frac{\partial u_M}{\partial z_k}}e^{-\varphi}\,d\lambda$$
$$+ {\sum_{|K|=p-1}}'\sum_{j,k=1}^{n}\int_{\mathbb{C}^n}P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_j}u_{jK}\right)\overline{P_\varphi\left(\frac{\partial\varphi}{\partial\overline{z}_k}u_{kK}\right)}e^{-\varphi}\,d\lambda,$$

where $\epsilon^{kM}_{jJ} = 0$ if $j \in J$ or $k \in M$ or $k \cup M \neq j \cup J$, and equals the sign of the permutation $\binom{kM}{jJ}$ otherwise. The right-hand side of the last formula can be rewritten as

$$\sum_{|J|=p}{}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial z_j} \right\|_{\varphi}^{2} + \sum_{|K|=p-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left(P_\varphi\left(\frac{\partial\varphi}{\partial \overline{z}_j} u_{jK}\right) \overline{P_\varphi\left(\frac{\partial\varphi}{\partial \overline{z}_k} u_{kK}\right)} - \frac{\partial u_{jK}}{\partial z_k} \overline{\frac{\partial u_{kK}}{\partial z_j}} \right) e^{-\varphi}\, d\lambda. \tag{3.6}$$

In order to prove this we first consider the (non-zero) terms where $j = k$ (and hence $M = J$). These terms result in the portion of the first sum in (3.6) where $j \notin J$. On the other hand, if $j \neq k$, then $j \in M$ and $k \in J$, and deletion of j from M and k from J results in the strictly increasing multi-index K of length $p-1$. Consequently, these terms can be collected into the second sum in (3.6) (in the part with the minus sign, we have interchanged the summation indices j and k). In this sum, the terms where $j = k$ compensate for the terms in the first sum where $j \in J$.

Now one can use the same reasoning as in the last proof to get

$$\begin{aligned} \|\partial u\|_\varphi^2 + \|\partial^* u\|_\varphi^2 &= \sum_{|J|=p}{}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial z_j} \right\|_{\varphi}^{2} \\ &\quad + \sum_{|K|=p-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j} u_{jK} \overline{u_{kK}}\, e^{-\varphi}\, d\lambda \\ &\quad - \sum_{|K|=p-1}{}' \sum_{j,k=1}^{n} \left(\frac{\partial\varphi}{\partial \overline{z}_j} u_{jK} - P_\varphi\left(\frac{\partial\varphi}{\partial \overline{z}_j} u_{jK}\right), \frac{\partial\varphi}{\partial \overline{z}_k} u_{kK} \right)_\varphi . \end{aligned} \tag{3.7}$$

REMARK 3.6. For $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$ we obtain

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \sum_{|J|=p}{}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial z_j} \right\|^{2} + p \sum_{|J|=p}{}' \int_{\mathbb{C}^n} |u_J|^2 e^{-|z|^2}\, d\lambda. \tag{3.8}$$

4. The ∂-Neumann operator on the Segal–Bargmann space. As an immediate consequence of (3.4) and (3.8) we get what is called *the basic estimates.*

LEMMA 4.1. *Let $1 \le p \le n-1$ and $u = \sum_{|J|=p}' u_J\, dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ and suppose that $u \in \operatorname{dom}(\partial) \cap \operatorname{dom}(\partial^*)$. Then*

$$\|u\|^2 \le \frac{1}{p}(\|\partial u\|^2 + \|\partial^* u\|^2). \tag{4.1}$$

This follows easily from the corresponding results for general Segal–Bargmann spaces (see Theorem 3.2 and (3.8)).

Now we can use the machinery of the classical $\overline{\partial}$-Neumann operator to show the following results.

LEMMA 4.2. *Both operators ∂ and ∂^* have closed range. If we endow* $\mathrm{dom}(\partial)\cap\mathrm{dom}(\partial^*)$ *with the graph norm* $(\|f\|^2+\|\partial f\|^2+\|\partial^* f\|^2)^{1/2}$, *the dense subspace* $\mathrm{dom}(\partial)\cap\mathrm{dom}(\partial^*)$ *of* $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ *becomes a Hilbert space.*

Proof. We use general results about unbounded operators on Hilbert spaces (see for instance [8, p. 81]) to show that $\mathrm{im}\,\partial$ and $\mathrm{im}\,\partial^*$ are closed. The last assertion follows again by (4.1) (see [8, Chapter 4]). ∎

The next result describes the implication of the basic estimates (4.1) for the $\tilde{\Box}$-operator.

THEOREM 4.3. *The operator* $\tilde{\Box} : \mathrm{dom}(\tilde{\Box}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ *is bijective and has a bounded inverse*

$$\tilde{N} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \mathrm{dom}(\tilde{\Box}).$$

In addition

$$\|\tilde{N}u\| \leq \frac{1}{p}\|u\| \quad \text{for } u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}). \tag{4.2}$$

Proof. The proof follows the proof of [8, Proposition 4.58] very closely (see also [1, Section 4.4]). ∎

Following the classical $\overline{\partial}$-Neumann calculus we obtain

THEOREM 4.4. *Let $\tilde{N}_p$ denote the inverse of $\tilde{\Box}$ on* $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$. *Then*

$$\tilde{N}_{p+1}\partial = \partial\tilde{N}_p \quad \text{on } \mathrm{dom}(\partial), \tag{4.3}$$

$$\tilde{N}_{p-1}\partial^* = \partial^*\tilde{N}_p \quad \text{on } \mathrm{dom}(\partial^*). \tag{4.4}$$

In addition, $\partial^*\tilde{N}_p$ is zero on $(\ker\partial)^\perp$.

Proof. See [8, proof of Proposition 4.61] (and [1, Section 4.4]). ∎

Now we can also prove a solution formula for the equation $\partial u = \alpha$, where α is a given $(p,0)$-form in $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ with $\partial\alpha = 0$.

THEOREM 4.5. *Let* $\alpha \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ *with $\partial\alpha = 0$. Then $u_0 = \partial^*\tilde{N}_p\alpha$ is the canonical solution of $\partial u = \alpha$, that is, $\partial u_0 = \alpha$ and $u_0 \in (\ker\partial)^\perp = \mathrm{im}\,\partial^*$, and*

$$\|\partial^*\tilde{N}_p\alpha\| \leq p^{-1/2}\,\|\alpha\|. \tag{4.5}$$

Proof. See [8, proof of Proposition 4.62]. ∎

Now we discuss a different approach to the operator $\tilde{N}$ which is related to the quadratic form

$$Q(u,v) = (\partial u, \partial v) + (\partial^* u, \partial^* v).$$

For this purpose we consider the embedding

$$\iota : \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}),$$

where $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ is endowed with the graph norm

$$u \mapsto (\|\partial u\|^2 + \|\partial^* u\|^2)^{1/2}.$$

The graph norm comes from the inner product

$$Q(u, v) = (u, v)_Q = (\tilde{\Box} u, v) = (\partial u, \partial v) + (\partial^* u, \partial^* v).$$

The basic estimates (4.1) imply that ι is a bounded operator with operator norm

$$\|\iota\| \le 1/\sqrt{p}.$$

By (4.1) it follows in addition that $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ endowed with the graph norm $u \mapsto (\|\partial u\|^2 + \|\partial^* u\|^2)^{1/2}$ is a Hilbert space (see Lemma 4.2).

Since $(u, v) = (u, \iota v)$, we have $(u, v) = (\iota^* u, v)_Q$.

For $u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ and $v \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ we get

$$(u, v) = (\tilde{\Box}\tilde{N} u, v) = ((\partial\partial^* + \partial^*\partial)\tilde{N} u, v) = (\partial^* \tilde{N} u, \partial^* v) + (\partial \tilde{N} u, \partial v). \tag{4.6}$$

This suggests that as an operator to $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$, $\tilde{N}$ coincides with ι^*, and as an operator to $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$, $\tilde{N}$ is equal to $\iota \circ \iota^*$ (see [8, p. 82] or [16, p. 30] for the details).

Hence $\tilde{N}$ is compact if and only if the embedding

$$\iota : \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$$

is compact. This will be used to prove the following theorem.

THEOREM 4.6. *The operator* $\tilde{N} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$, $1 \le p \le n$, *is compact.*

Proof. First we consider the case when $p = 1$. We use (3.4) for the graph norm on $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ and note that it suffices to consider one component u_j of the $(1,0)$-form u. So denote u_j by f. We have to handle

$$\sum_{k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial f}{\partial z_k} \right|^2 e^{-|z|^2}\, d\lambda + \int_{\mathbb{C}^n} |f|^2\, e^{-|z|^2}\, d\lambda$$

for the graph norm. We use the complete orthonormal system $(\varphi_\alpha)_\alpha$ of $A^2(\mathbb{C}^n, e^{-|z|^2})$ (see (2.2)). Let $f = \sum_\alpha f_\alpha \varphi_\alpha \in \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$. We have $\iota(f) = f$ and hence

$$\iota(f) = \sum_\alpha (f, \varphi_\alpha)\varphi_\alpha$$

in $A^2(\mathbb{C}^n, e^{-|z|^2})$. The basis elements φ_α are normalized in $A^2(\mathbb{C}^n, e^{-|z|^2})$. First we have to compute the graph norm of φ_α. Notice that

$$\frac{\partial \varphi_\alpha}{\partial z_k} = \frac{1}{\sqrt{\pi^n}} \frac{z_1^{\alpha_1}}{\sqrt{\alpha_1!}} \cdots \frac{z_{k-1}^{\alpha_{k-1}}}{\sqrt{\alpha_{k-1}!}} \frac{\alpha_k z_k^{\alpha_k - 1}}{\sqrt{\alpha_k!}} \frac{z_{k+1}^{\alpha_{k+1}}}{\sqrt{\alpha_{k+1}!}} \cdots \frac{z_n^{\alpha_n}}{\sqrt{\alpha_n!}} = \sqrt{\alpha_k}\, \varphi_{(\alpha k-1)},$$

where $(\alpha k - 1) = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \dots, \alpha_n)$. Hence the graph norm of φ_α equals

$$\|\varphi_\alpha\|_Q = \left(\|\varphi_\alpha\|^2 + \sum_{k=1}^n \left\| \frac{\partial \varphi_\alpha}{\partial z_k} \right\|^2 \right)^{1/2} = \sqrt{1 + |\alpha|},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Now let

$$\psi_\alpha = \frac{\varphi_\alpha}{\sqrt{1 + |\alpha|}}.$$

Then $(\psi_\alpha)_\alpha$ constitutes a complete orthonormal system in the Hilbert space $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ endowed with the graph norm; notice that

$$(f, \psi_\alpha)_Q = \sum_{k=1}^n \left(\frac{\partial f}{\partial z_k}, \frac{\partial \psi_\alpha}{\partial z_k} \right) + (f, \psi_\alpha) = \sum_{k=1}^n \frac{\alpha_k}{\sqrt{1 + |\alpha|}} f_\alpha + \frac{1}{\sqrt{1 + |\alpha|}} f_\alpha$$
$$= \sqrt{1 + |\alpha|}\, f_\alpha,$$

and we have

$$\iota(f) = f = \sum_\alpha (f, \psi_\alpha)_Q \psi_\alpha.$$

For the norm of $A^2(\mathbb{C}^n, e^{-|z|^2})$ we have

$$\begin{aligned} \Big\| \iota(f) - \sum_{|\alpha| \le N} (f, \psi_\alpha)_Q \psi_\alpha \Big\|^2 &= \Big\| \sum_{|\alpha| \ge N+1} (f, \psi_\alpha)_Q \psi_\alpha \Big\|^2 \\ &= \Big\| \sum_{|\alpha| \ge N+1} \frac{1}{\sqrt{1 + |\alpha|}} (f, \psi_\alpha)_Q \varphi_\alpha \Big\|^2 \\ &= \sum_{|\alpha| \ge N+1} \left| \frac{1}{\sqrt{1 + |\alpha|}} (f, \psi_\alpha)_Q \right|^2 \\ &\le \frac{\|f\|_Q^2}{N + 2}, \end{aligned}$$

where we have used Bessel's inequality for the Hilbert space $\mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*)$ endowed with the graph norm. This proves that

$$\iota : \mathrm{dom}(\partial) \cap \mathrm{dom}(\partial^*) \to A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2})$$

is a compact operator, and the same is true for

$$\tilde{N} : A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}) \to A^2_{(1,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

For arbitrary p between 1 and n we can use (3.8) and the same reasoning as before to get the desired conclusion. ∎

Compare with the $\overline{\partial}$-Neumann operator N on $L^2(\mathbb{C}^n, e^{-|z|^2})$: in this case N fails to be compact (see [8]). This is related to the fact that the kernel of $\overline{\partial}$ is large (it is the Segal–Bargmann space) in the case of the $\overline{\partial}$-complex, but the kernel of ∂ consists just of the constant functions in case of the ∂-complex on the Segal–Bargmann space.

To compute the spectrum of the operator $\tilde{\Box}_p$ we will use

LEMMA 4.7 (see [3] or [8]). *Let A be a symmetric operator on a Hilbert space H with domain* $\mathrm{dom}(A)$, *and suppose that $(x_k)_k$ is a complete orthonormal system in H. If each x_k lies in* $\mathrm{dom}(A)$, *and there exist $\lambda_k \in \mathbb{R}$ such that*

$$Ax_k = \lambda_k x_k$$

for every $k \in \mathbb{N}$, then A is essentially self-adjoint and the spectrum of $\overline{A}$ is the closure in $\mathbb{R}$ of the set of all λ_k.

THEOREM 4.8. *The spectrum of $\tilde{\Box}_p$, where $0 \le p \le n$, consists of all numbers $m+p$ for $m = 0, 1, \ldots,$ where $m+p$ has multiplicity $\binom{n+m-1}{n-1}\binom{n}{p}$.*

Proof. Recall that the monomials $(\varphi_\alpha)_\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, constitute a complete orthonormal system in $A^2(\mathbb{C}^n, e^{-|z|^2})$. We use (2.5) to compute

$$\left(\sum_{k=1}^n z_k \frac{\partial \varphi_\alpha}{\partial z_k}\right) + p\varphi_\alpha = (|\alpha| + p)\varphi_\alpha.$$

We use Lemma 4.7 and note that there are $\binom{n+|\alpha|-1}{n-1}$ monomials of degree $|\alpha|$. Hence we get the assertion about the multiplicity from the fact that $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ is the direct sum of $\binom{n}{p}$ copies of $A^2(\mathbb{C}^n, e^{-|z|^2})$. ∎

The last result implies also that the inverse $\tilde{N}_p$ of $\tilde{\Box}_p$ is a compact operator with eigenvalues $1/(m+p)$.

Note that the complex Laplacian $\Box_q$ of the $\overline{\partial}$-complex on $L^2(\mathbb{C}^n, e^{-|z|^2})$ also has the eigenvalues $q+m$ for $m = 0, 1, \ldots,$ but each of them has infinite multiplicity (see [11], [7], [8]).

5. The generalized ∂-complex. Now we return to the classical Segal–Bargmann space but replace a single derivative with respect to z_j by a differential operator of the form $p_j\big(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\big)$, where p_j is a complex polynomial on $\mathbb{C}^n$ (see [12], [13]). We consider the densely defined operators

$$Du = \sum_{j=1}^n p_j(u)\, dz_j, \tag{5.1}$$

where $u \in A^2(\mathbb{C}^n, e^{-|z|^2})$ and $p_j\big(\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\big)$ are polynomial differential operators with constant coefficients.

More generally we define

$$Du = \sum_{|J|=p}{}' \sum_{k=1}^{n} p_k(u_J) dz_k \wedge dz_J, \tag{5.2}$$

where $u = \sum_{|J|=p}' u_J dz_J$ is a $(p,0)$-form with coefficients in $A^2(\mathbb{C}^n, e^{-|z|^2})$.

It is clear that $D^2 = 0$ and

$$(Du, v) = (u, D^* v), \tag{5.3}$$

where $u \in \operatorname{dom}(D) = \{u \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) : Du \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2})\}$ and

$$D^* v = \sum_{|K|=p-1}{}' \sum_{j=1}^{n} p_j^* v_{jK} dz_K$$

for $v = \sum_{|J|=p}' v_J dz_J$ and where $p_j^*(z_1, \dots, z_n)$ is the polynomial p_j with complex conjugate coefficients, taken as a multiplication operator.

Now the corresponding D-complex has the form

$$A^2_{(p-1,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{D^*}{\overset{D}{\leftrightarrows}} A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \underset{D^*}{\overset{D}{\leftrightarrows}} A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2}).$$

In what follows we consider the generalized box operator $\tilde{\Box}_{D,p} := D^*D + DD^*$ as a densely defined self-adjoint operator on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ with

$$\begin{aligned} &\operatorname{dom}(\tilde{\Box}_{D,p}) \\ &\quad = \{f \in \operatorname{dom}(D) \cap \operatorname{dom}(D^*) : Df \in \operatorname{dom}(D^*) \text{ and } D^* f \in \operatorname{dom}(D)\}. \end{aligned}$$

We want to find conditions under which $\tilde{\Box}_{D,1}$ has a bounded inverse. For this purpose we have to consider the graph norm $(\|u\|^2 + \|Du\|^2 + \|D^*u\|^2)^{1/2}$ on $\operatorname{dom}(D) \cap \operatorname{dom}(D^*)$.

THEOREM 5.1. *Let $u = \sum_{j=1}^n u_j dz_j \in \operatorname{dom}(\tilde{\Box}_{D,1})$ and suppose that there exists a constant $C > 0$ such that*

$$\|u\|^2 \le C \sum_{j,k=1}^{n} ([p_k, p_j^*] u_j, u_k). \tag{5.4}$$

Then

$$\|u\|^2 \le C(\|Du\|^2 + \|D^*u\|^2). \tag{5.5}$$

Proof. First we have

$$Du = \sum_{j<k} (p_j(u_k) - p_k(u_j)) dz_j \wedge dz_k \quad \text{and} \quad D^* u = \sum_{j=1}^{n} p_j^* u_j,$$

hence

$$\|Du\|^2 + \|D^*u\|^2$$
$$= \int_{\mathbb{C}^n} \sum_{j<k} |p_k(u_j) - p_j(u_k)|^2 e^{-|z|^2}\, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n p_j^* u_j\, \overline{p_k^* u_k}\, e^{-|z|^2}\, d\lambda$$
$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k(u_j)|^2 e^{-|z|^2}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} (p_j^* u_j\, \overline{p_k^* u_k} - p_k(u_j)\, \overline{p_j(u_k)})\, e^{-|z|^2}\, d\lambda$$
$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} |p_k(u_j)|^2 e^{-|z|^2}\, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} [p_k, p_j^*] u_j\, \overline{u_k}\, e^{-|z|^2}\, d\lambda,$$

where we have used (5.3). Now the assumption (5.4) implies the desired result. ∎

Let $1 \le p \le n-1$ and let

$$u = {\sum_{|J|=p}}' u_J\, dz_J \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$$

and suppose that $u \in \operatorname{dom}(\tilde{\Box}_{D,p})$. In a similar way to (3.7) we get

$$\|Du\|^2 + \|D^*u\|^2 = {\sum_{|J|=p}}' \sum_{k=1}^n \|p_k(u_J)\|^2 + {\sum_{|K|=p-1}}' \sum_{j,k=1}^n \int_{\mathbb{C}^n} [p_k, p_j^*] u_{jK}\, \overline{u_{kK}}\, e^{-|z|^2}\, d\lambda, \tag{5.6}$$

and if we suppose that

$$\|u\|^2 \le C {\sum_{|K|=p-1}}' \sum_{j,k=1}^n ([p_k, p_j^*] u_{jK}, u_{kK}), \tag{5.7}$$

we get the basic estimate (5.5), which also implies that both $\operatorname{im} D$ and $\operatorname{im} D^*$ are closed (see for instance [8, Chapter 4]). With the basic estimate (5.5) we are now able to use the machinery of the corresponding Neumann operator, the bounded inverse of $\tilde{\Box}_{D,p}$ (see Theorems 4.3 and 4.5) to get

THEOREM 5.2. *Let D be as in* (5.2) *and $u \in \operatorname{dom}(\tilde{\Box}_{D,p})$. Suppose that there exists a constant $C > 0$ such that*

$$\|u\|^2 \le C {\sum_{|K|=p-1}}' \sum_{j,k=1}^n ([p_k, p_j^*] u_{jK}, u_{kK}).$$

Then $\tilde{\Box}_{D,p}$ has a bounded inverse

$$\tilde{N}_{D,p} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \operatorname{dom}(\tilde{\Box}_{D,p}).$$

If $\alpha \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ satisfies $D\alpha = 0$, then $u_0 = D^*\tilde{N}_{D,p}\alpha$ is the canonical solution of $Du = \alpha$, that is, $Du_0 = \alpha$ and $u_0 \in (\ker D)^\perp = \operatorname{im} D^*$, and $\|D^*\tilde{N}_{D,p}\alpha\| \le C\|\alpha\|$ for some constant $C > 0$ independent of α.

EXAMPLE 5.3. (a) Let $p_k = \frac{\partial^2}{\partial z_k^2}$. Then $p_j^*(z) = z_j^2$ and we have

$$\sum_{j,k=1}^n ([p_k, p_j^*]u_j, u_k) = \sum_{j,k=1}^n (2\delta_{j,k}u_j, u_k) + \sum_{j,k=1}^n \left(4\delta_{jk} z_j \frac{\partial u_j}{\partial z_k}, u_k\right)$$
$$= 2\|u\|^2 + 4\sum_{j=1}^n \left\|\frac{\partial u_j}{\partial z_j}\right\|^2$$

for $u = \sum_{j=1}^n u_j dz_j \in \operatorname{dom}(\tilde{\Box}_{D,1})$. Hence (5.4) is satisfied.

(b) Let $n = 2$ and take $p_1 = \frac{\partial^2}{\partial z_1 \partial z_2}$ and $p_2 = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2}$. Then $p_1^*(z) = z_1 z_2$ and $p_2^*(z) = z_1^2 + z_2^2$ and we have

$$([p_1, p_1^*]u_1, u_1) = (u_1, u_1) + \left(\frac{\partial u_1}{\partial z_1}, \frac{\partial u_1}{\partial z_1}\right) + \left(\frac{\partial u_1}{\partial z_2}, \frac{\partial u_1}{\partial z_2}\right),$$
$$([p_1, p_2^*]u_2, u_1) = 2\left(\frac{\partial u_2}{\partial z_1}, \frac{\partial u_1}{\partial z_2}\right) + 2\left(\frac{\partial u_2}{\partial z_2}, \frac{\partial u_1}{\partial z_1}\right),$$
$$([p_2, p_1^*]u_1, u_2) = 2\left(\frac{\partial u_1}{\partial z_1}, \frac{\partial u_2}{\partial z_2}\right) + 2\left(\frac{\partial u_1}{\partial z_2}, \frac{\partial u_2}{\partial z_1}\right),$$
$$([p_2, p_2^*]u_2, u_2) = 4(u_2, u_2) + 4\left(\frac{\partial u_2}{\partial z_1}, \frac{\partial u_2}{\partial z_1}\right) + 4\left(\frac{\partial u_2}{\partial z_2}, \frac{\partial u_2}{\partial z_2}\right).$$

So we obtain

$$\sum_{j,k=1}^2 ([p_k, p_j^*]u_j, u_k)$$
$$= \int_{\mathbb{C}^2} \left(|u_1|^2 + 4|u_2|^2 + \left|\frac{\partial u_1}{\partial z_1} + 2\frac{\partial u_2}{\partial z_2}\right|^2 + \left|\frac{\partial u_1}{\partial z_2} + 2\frac{\partial u_2}{\partial z_1}\right|^2\right) e^{-|z|^2}\, d\lambda$$

for $u = \sum_{j=1}^2 u_j dz_j \in \operatorname{dom}(\tilde{\Box}_{D,1})$. Again, (5.4) is satisfied.

We remark that we can interchange the roles of D and D^* and obtain

THEOREM 5.4. *Suppose that $n > 1$ and $1 \le p \le n-1$. Let D be as in (5.2) and suppose that*

$$\|u\|^2 \le C {\sum_{|K|=p-1}}' \sum_{j,k=1}^n ([p_k, p_j^*]u_{jK}, u_{kK})$$

for all $u \in \operatorname{dom}(\tilde{\Box}_{D,p})$. If $\beta \in A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ satisfies $D^*\beta = 0$, then $v_0 = D\tilde{N}_{D,p}\beta \in A^2_{(p+1,0)}(\mathbb{C}^n, e^{-|z|^2})$ is the canonical solution of $D^*v = \beta$, *that is,* $D^* v_0 = \beta$ *and* $v_0 \in (\ker D^*)^\perp = \operatorname{im} D$*, and* $\|D\tilde{N}_{D,p}\beta\| \leq C\|\beta\|$ *for some constant* $C > 0$ *independent of* β*.*

Proof. As in Theorem 5.2 we find that $\tilde{\Box}_{D,p}$ has a bounded inverse

$$\tilde{N}_{D,p} : A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2}) \to \operatorname{dom}(\tilde{\Box}_{D,p}).$$

Now, using the $\overline{\partial}$-Neumann calculus, we obtain

$$0 = D^*\beta = D^*(D^*D + DD^*)\tilde{N}_{D,p}\beta = D^*DD^*\tilde{N}_{D,p}\beta,$$

hence

$$0 = (D^*DD^*\tilde{N}_{D,p}\beta, D^*\tilde{N}_{D,p}\beta) = (DD^*\tilde{N}_{D,p}\beta, DD^*\tilde{N}_{D,p}\beta).$$

This implies that $DD^*\tilde{N}_{D,p}\beta = 0$ and we get

$$D^*v_0 = D^*D\tilde{N}_{D,p}\beta = (DD^* + D^*D)\tilde{N}_{D,p}\beta = \beta,$$

and $(v_0, f) = (D\tilde{N}_{D,p}\beta, f) = (\tilde{N}_{D,p}\beta, D^*f) = 0$ for all $f \in \ker D^*$. ■

Example 5.5. We take Example 5.3(b) and consider $f = f_1\, dz_1 + f_2\, dz_2 \in A^2_{(1,0)}(\mathbb{C}^2, e^{-|z|^2})$ such that $D^*f = p_1^* f_1 + p_2^* f_2 = 0$. By Theorem 5.4 we get

$$g = g\, dz_1 \wedge dz_2 = D\tilde{N}_{D,1}f \in A^2_{(2,0)}(\mathbb{C}^2, e^{-|z|^2})$$

such that $D^*g = -p_2^* g\, dz_1 + p_1^* g\, dz_2 = f$ and $\|D\tilde{N}_{D,1}f\| \leq C\|f\|$ for some constant $C > 0$.

Remark 5.6. Finally, we point out that the ∂-Neumann operator $\tilde{N}_{D,p}$ exists and is bounded on $A^2_{(p,0)}(\mathbb{C}^n, e^{-|z|^2})$ if and only if the basic estimate (5.5) holds (see for instance [8, Remark 9.12] for the details).

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