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On the Erdős–Fuchs theorem

by

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1. Introduction. The Gauss circle conjecture states that

$$|\{(a,b) : a,b \in \mathbb{N},\ a^2+b^2 \le n\}| = \pi n + O(n^{1/4+\epsilon}) \tag{1.1}$$

for any $\epsilon > 0$. The best known result due to Huxley has $O(n^{1/4+\epsilon})$ replaced by $O(n^{\frac{131}{416}}(\log n)^{2.26})$. In general, for two non-empty subsets A, B of $\mathbb{N}$ and $n \in \mathbb{N}$, define

$$r_{A,B}(n) := |\{(a,b) : a \in A,\ b \in B,\ a+b=n\}|.$$

Also, define

$$R_{A,B}(n) := \sum_{j\le n} r_{A,B}(j),$$

i.e.,

$$R_{A,B}(n) = |\{(a,b) : a \in A,\ b \in B,\ a+b \le n\}|.$$

Clearly (1.1) can be rewritten as

$$R_{\mathbb{N}^2,\mathbb{N}^2}(n) = \frac{\pi}{4}n + \sqrt{n} + O(n^{1/4+\epsilon}),$$

where $\mathbb{N} = \{0,1,2,\dots\}$ and $\mathbb{N}^2 = \{a^2 : a \in \mathbb{N}\}$.

On the other hand, with the aid of Fourier analysis, Hardy found that the remainder $O(n^{1/4+\epsilon})$ in (1.1) cannot be replaced by $O(n^{1/4}(\log n)^{1/4})$. In 1956, for any non-empty infinite subset A of $\mathbb{N}$, Erdős and Fuchs [4] proved that as $n \to \infty$,

$$R_{A,A}(n) = cn + o(n^{1/4}(\log n)^{-1/2}) \tag{1.2}$$

cannot hold for any constant $c > 0$. Subsequently, Jurkat (unpublished) and later Montgomery and Vaughan [9] showed that the $(\log n)^{-1/2}$ in the

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remainder term of (1.2) can be removed, i.e., it is impossible that

$$R_{A,A}(n) = cn + o(n^{1/4}), \quad n \to \infty, \tag{1.3}$$

for any constant $c > 0$.

In [10], Sarközy considered the extension of the Erdős–Fuchs theorem for the sum of two different subsets of $\mathbb{N}$. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be two infinite subsets of $\mathbb{N}$. Suppose that for each $i \geq 1$, a_i is not very far from b_i, explicitly,

$$a_i - b_i = o(a_i^{1/2}(\log a_i)^{-1}). \tag{S}$$

Then Sarközy proved that

$$R_{A,B}(n) = cn + o(n^{1/4}(\log n)^{-1/2}), \quad n \to \infty, \tag{1.4}$$

cannot hold for any constant $c > 0$.

In [7], Horváth tried to remove the term $(\log n)^{-1/2}$ on the right side of (1.4). Define

$$A(n) := |\{a \in A : a \leq n\}|.$$

Under the assumptions

$$a_i - b_i = o(a_i^{1/2}), \qquad i \to \infty, \tag{H1}$$

$$A(n) - B(n) = O(1), \quad n \geq 1, \tag{H2}$$

Horváth proved that

$$R_{A,B}(n) = cn + o(n^{1/4}), \quad n \to \infty, \tag{1.5}$$

cannot happen.

Notice that the assumption (H2), which says $A(n)$ and $B(n)$ are almost equal, seems a little too strong. So we wish to weaken the requirement for $A(n) - B(n)$, under the assumption that the difference $a_i - b_i$ is much smaller than $o(a_i^{1/2})$. In this paper, we shall give such a generalization of Horváth's result.

Theorem 1.1. *Suppose that $0 \leq \alpha \leq 1/4$. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be infinite subsets of $\mathbb{N}$ such that*

(1) $a_i - b_i = o(a_i^{1/2-\alpha})$ *as* $i \to \infty$;
(2) $A(n) - B(n) = O(n^\alpha)$ *for each* $n \in \mathbb{N}$.

Then

$$R_{A,B}(n) = cn + o(n^{1/4}), \quad n \to \infty, \tag{1.6}$$

cannot hold for any constant $c > 0$.

Note that $a_i - b_i = o(a_i^{1/2-\alpha})$ implies $A(n) - B(n) = o(n^{1/2-\alpha})$. Hence setting $\alpha = 1/4$, we get

Corollary 1.1.

$$R_{A,B}(n) = cn + o(n^{1/4}), \qquad n \to \infty,$$

cannot hold for any constant $c > 0$ *under the unique assumption*

$$a_i - b_i = o(a_i^{1/4}).$$

It is natural to consider generalizations of the Erdős–Fuchs theorem for sums of more than two subsets of $\mathbb{N}$. Suppose that $A_1, \ldots, A_k$ are non-empty subsets of $\mathbb{N}$. Define

$$r_{A_1,\ldots,A_k}(n) = |\{(a_1, \ldots, a_k) : a_1 + \cdots + a_k = n,\ a_1 \in A_1, \ldots, a_k \in A_k\}|,$$

and let

$$R_{A_1,\ldots,A_k}(n) = \sum_{j \le n} r_{A_1,\ldots,A_k}(j).$$

Horváth [6] proved that for any $A \subseteq \mathbb{N}$,

$$R_{\underbrace{A,\ldots,A}_{k \text{ times}}}(n) = cn + o(n^{1/4}(\log n)^{-1/2}) \tag{1.7}$$

cannot hold for any constant $c > 0$. Subsequently, Tang [11] obtained an extension of (1.3) for the sum of k A's, i.e., it is impossible that

$$R_{\underbrace{A,\ldots,A}_{k \text{ times}}}(n) = cn + o(n^{1/4}). \tag{1.8}$$

Chen and Tang [2] also proved a quantitative version of (1.8).

In [5, 6], Horváth considered $R_{A_1,\ldots,A_k}(n)$. Assume $A_1 = \{a_{1,1}, a_{1,2}, \ldots\}$ and $A_2 = \{a_{2,1}, a_{2,2}, \ldots\}$. Let $f = \Theta(g)$ mean $g \ll f \ll g$, i.e., there exist constants $c_1, c_2 > 0$ such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for any sufficiently large n. Assume that

$$a_{1,i} - a_{2,i} = o(a_{1,i}^{1/2}(\log a_{1,i})^{-k/2}), \qquad i \to \infty, \tag{h1}$$

$$A_j(n) = \Theta(A_1(n)), \qquad j = 3, \ldots, k. \tag{h2}$$

Horváth [6] showed that for any constant $c > 0$,

$$R_{A_1,\ldots,A_k}(n) = cn + o(n^{1/4}(\log n)^{1-3k/4}), \qquad n \to \infty, \tag{1.9}$$

is impossible. Recently, under some additional assumptions, Tang [12] improved Horváth's result and showed that the remainder term can be reduced to $o(n^{1/4}(\log n)^{-1/2})$ or $o(n^{1/4}(\log n)^{-\frac{k+1}{2(k-1)}})$ according to whether k is even or odd.

Here we shall give an extension of Theorem 1.1 concerning $R_{A_1,\ldots,A_k}(n)$.

Theorem 1.2. *Suppose that* $0 < \beta \le 1/2$ *and* $0 \le \alpha \le \beta/2$. *Let* $A_1, \ldots, A_k$ *be non-empty subsets of* $\mathbb{N}$ *such that*

(1) $a_{1,i} - a_{2,i} = o(a_{1,i}^{\beta-\alpha})$ *as* $i \to \infty$, *where* $a_{1,i}$ (*resp.* $a_{2,i}$) *is the* i*th element of* A_1 (*resp.* A_2);
(2) $A_1(n) - A_2(n) = O(n^\alpha)$ *for each* $n \in \mathbb{N}$;
(3) $A_1(n) = \Theta(n^\beta)$.

Then

$$R_{A_1,\dots,A_k}(n) = cn + o(n^{1/4}), \quad n \to \infty, \tag{1.10}$$

cannot hold for any constant $c > 0$.

Clearly the assumptions (2) and (3) of Theorem 1.2 also imply $A_2(n) = \Theta(n^\beta)$. Furthermore, if $A_j(n) = \Theta(A_1(n))$ for $j = 2, \dots, k$ and $R_{A_1,\dots,A_k}(n) = \Theta(n)$, then it is easy to verify that $A_1(n) = \Theta(n^{1/k})$, i.e., (3) of Theorem 1.2 is valid for $\beta = 1/k$ under Horváth's assumption (h2).

In [1], Bateman showed that

$$\sum_{j\le n} (R_{A,A}(j) - cj)^2 = o(n^{3/2}(\log n)^{-1}), \quad n \to \infty, \tag{1.11}$$

cannot hold for any constant $c > 0$. Clearly the result of Bateman also implies the Erdős–Fuchs theorem. In [3], Chen and Tang showed that for any constant $c > 0$, it is impossible that

$$\sum_{j\le n} (\underbrace{R_{A,\dots,A}}_{k \text{ times}}(j) - cj)^2 = o(n^{3/2}), \quad n \to \infty. \tag{1.12}$$

Now we can prove

Theorem 1.3.

(i) *Under the assumptions of Theorem* 1.1,

$$\sum_{j\le n} (R_{A,B}(j) - cj)^2 = o(n^{3/2}), \quad n \to \infty, \tag{1.13}$$

cannot hold for any constant $c > 0$.

(ii) *Under the assumptions of Theorem* 1.2,

$$\sum_{j\le n} (R_{A_1,\dots,A_k}(j) - cj)^2 = o(n^{3/2}), \quad n \to \infty, \tag{1.14}$$

cannot hold for any constant $c > 0$.

In Section 2, we shall give the proof of Theorem 1.1. In particular, the key ingredient is an auxiliary lemma (Lemma 2.2), which is also necessary for the proof of Theorem 1.2 given in Section 3. Finally, we prove Theorem 1.3 in Section 4.

2. Proof of Theorem 1.1

Lemma 2.1. *Let* $\{a_0, a_1, \dots\}$ *be a sequence of non-negative real numbers and* $1/2 \le \rho < 1$.

(i) *Suppose that*

$$\sum_{n\le x} a_n = o(x^c), \qquad x\to\infty,$$

where $c>1$ is a constant. Then

$$\sum_{n=0}^{\infty} a_n\rho^n = o\biggl(\frac{1}{(1-\rho)^c}\biggr), \qquad \rho\to 1. \tag{2.1}$$

(ii) *Suppose that*

$$\sum_{n\le x} a_n = O(x^c), \qquad x\to\infty,$$

for some constant $c>1$. Then

$$\sum_{n=0}^{\infty} a_n\rho^n = O\biggl(\frac{1}{(1-\rho)^c}\biggr), \tag{2.2}$$

where the implied constant only depends on c.

Proof. We only prove (i), since the proof of (ii) is very similar. Write

$$\psi(x) = \sum_{n\le x} a_n.$$

Then

$$\sum_{n=0}^{\infty} a_n\rho^n = \sum_{n=1}^{\infty}\bigl(\psi(n)-\psi(n-1)\bigr)\rho^n + \psi(0) = (1-\rho)\sum_{n=1}^{\infty}\psi(n)\rho^n.$$

So if $|\psi(n)|<\epsilon n^c$ for $n>n_0$, then

$$\biggl|\sum_{n=0}^{\infty} a_n\rho^n\biggr| < \epsilon(1-\rho)\sum_{n=n_0}^{\infty} n^c\rho^n + O_\epsilon(1) = \epsilon(1-\rho)\sum_{n=1}^{\infty} n^c\rho^n + O_\epsilon(1).$$

Since $(-1)^n\binom{-c-1}{n} = \binom{n+c}{n} \gg n^c$, it not difficult to get the desired result. ∎

Lemma 2.2. *Suppose that $0<\beta\le 1/2$ and $0\le\alpha\le\beta/2$. Let $A=\{a_1,a_2,\ldots\}$ and $B=\{b_1,b_2,\ldots\}$ be infinite subsets of $\mathbb{N}$ such that*

(1) $a_i - b_i = o(a_i^{\beta-\alpha})$ *for each $i\ge 1$;*
(2) $A(n)-B(n) = O(n^\alpha)$*;*
(3) $A(n),B(n) = O(n^\beta)$.

Then as $N\to\infty$, we have

$$\sum_{n=0}^{\infty}\Bigl(1-\frac{1}{N}\Bigr)^{2n}\cdot\Bigl(\sum_{\substack{a\in A\\ a\le n}} a - \sum_{\substack{b\in B\\ b\le n}} b\Bigr)^2 = o(N^{2+2\beta}), \tag{2.3}$$

$$\sum_{n=0}^{\infty}\Bigl(1-\frac{1}{N}\Bigr)^{2n}\cdot(A(n)-B(n))^2 = o(N^{2\beta}). \tag{2.4}$$

Proof. For each $j \geq 1$, let

$$\mathcal{I}_j = \big\{n \in \mathbb{N} : \min\{a_j, b_j\} \leq n \leq \max\{a_j, b_j\} - 1\big\}.$$

Evidently $|\mathcal{I}_j| = o({a_j}^{\beta-\alpha})$. For each $n \in \mathbb{N}$, define

$$\lambda(n) = |\{j : n \in \mathcal{I}_j\}|.$$

In particular, $\lambda(n) = 0$ if $n \notin \mathcal{I}_j$ for any $j \geq 1$. Note that $n \in \mathcal{I}_j$ if and only if either $a_j \leq n < b_j$ or $b_j \leq n < a_j$. Hence

$$\lambda(n) = |A(n) - B(n)| = O(n^\alpha).$$

Thus

$$\begin{aligned}\Big|\sum_{\substack{a\in A\\ a\leq n}} a - \sum_{\substack{b\in B\\ b\leq n}} b\Big| &\leq \sum_{\substack{j\geq 1\\ n\in\mathcal{I}_j}} \min\{a_j, b_j\} + \sum_{\substack{j\geq 1\\ a_j, b_j\leq n}} |a_j - b_j| \\ &\leq \lambda(n)\cdot n + A(n)\cdot o(n^{\beta-\alpha}) = \lambda(n)\cdot n + o(n^{2\beta-\alpha}),\end{aligned}$$

where in the last step we have used the assumption $A(n) \ll n^\beta$. It follows that

$$\begin{aligned}\sum_{n\leq x}\Big(\sum_{\substack{a\in A\\ a\leq n}} a - \sum_{\substack{b\in B\\ b\leq n}} b\Big)^2 &\ll \sum_{n\leq x}\big(\lambda(n)^2 n^2 + o(n^{4\beta-2\alpha})\big) \\ &\ll x^2\sum_{n\leq x}\lambda(n)^2 + o(x^{1+4\beta-2\alpha})\end{aligned}$$

as $x \to \infty$. Define

$$\mathfrak{J}(x) = \max\{j \geq 1 : a_j \leq x \text{ or } b_j \leq x\}.$$

Clearly $\mathfrak{J}(x) \ll x^\beta$. We have

$$\sum_{n\leq x}\lambda(n)^2 \ll x^\alpha \sum_{n\leq x}\lambda(n) \leq x^\alpha \sum_{j\leq\mathfrak{J}(x)} |\mathcal{I}_j| = x^\alpha\cdot x^\beta\cdot o(x^{\beta-\alpha}) = o(x^{2\beta})$$

as $x \to \infty$, i.e.,

$$\sum_{n\leq x}\Big(\sum_{\substack{a\in A\\ a\leq n}} a - \sum_{\substack{b\in B\\ b\leq n}} b\Big)^2 = o(x^{2+2\beta}).$$

Thus applying Lemma 2.1(i) with $\rho = (1 - N^{-1})^2$, we get (2.3).

Similarly, we have

$$\sum_{n\leq x}(A(n) - B(n))^2 = \sum_{n\leq x}\lambda(n)^2 \ll x^\alpha\sum_{n\leq x}\lambda(n) \leq x^\alpha \sum_{j\leq\mathfrak{J}(x)}|\mathcal{I}_j| = o(x^{2\beta})$$

as $x \to \infty$. Then (2.4) follows, too. ∎

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume on the contrary that (1.6) is true. Let

$$\vartheta(n) = R_{A,B}(n) - cn.$$

Then $\vartheta(n) = o(n^{1/4})$. Furthermore, since

$$A(n)B(n) \le R_{A,B}(2n) = 2cn + o(n^{1/4}),$$

we also have $A(n), B(n) \le 2c^{1/2}n^{1/2}$ for sufficiently large n.

For $|z| < 1$, let

$$F(z) = \sum_{a\in A} z^a, \qquad G(z) = \sum_{b\in B} z^b.$$

Clearly

$$F(z)G(z) = \Big(\sum_{a\in A} z^a\Big)\cdot\Big(\sum_{b\in B} z^b\Big) = \sum_{n=0}^{\infty} z^n \sum_{\substack{a\in A,\, b\in B\\ a+b=n}} 1 = \sum_{n=0}^{\infty} r_{A,B}(n)z^n.$$

It follows that

$$\begin{aligned}\frac{F(z)G(z)}{1-z} &= \Big(\sum_{n=0}^{\infty} r_{A,B}(n)z^n\Big)\cdot\Big(\sum_{n=0}^{\infty} z^n\Big)\\ &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} r_{A,B}(j) = \sum_{n=0}^{\infty} R_{A,B}(n)z^n.\end{aligned}$$

Thus

$$\begin{aligned}\frac{(F(z)+G(z))^2}{4(1-z)} &= \frac{F(z)G(z)}{1-z} + \frac{(F(z)-G(z))^2}{4(1-z)}\\ &= c\sum_{n=0}^{\infty} nz^n + \sum_{n=0}^{n} \vartheta(n)z^n + \frac{(F(z)-G(z))^2}{4(1-z)},\end{aligned}$$

i.e.,

$$\frac{(F(z)+G(z))^2}{2} = \frac{2cz}{1-z} + 2(1-z)\sum_{n=0}^{n}\vartheta(n)z^n + \frac{(F(z)-G(z))^2}{2}.$$

Taking the derivative in z of both sides of the above equation, we get

$$\begin{aligned}(2.5)\qquad &(F'(z)+G'(z))(F(z)+G(z))\\ &\quad= \frac{2c}{(1-z)^2} + (F'(z)-G'(z))(F(z)-G(z))\\ &\qquad + 2(1-z)\sum_{n=1}^{n} n\vartheta(n)z^{n-1} - 2\sum_{n=0}^{n}\vartheta(n)z^n.\end{aligned}$$

Let m be a large integer to be chosen later. Let $\rho = 1 - 1/N$ and $z(\theta) = \rho e^{2\pi\sqrt{-1}\,\theta}$. For convenience, we abbreviate $z(\theta)$ as z. Clearly for any $n_1, n_2 \in \mathbb{N}$,

$$\int_0^1 z^{n_1} \cdot \overline{z}^{n_2}\, d\theta = \begin{cases} \rho^{2n_1} & \text{if } n_1 = n_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$J = \int_0^1 |(F'(z) + G'(z))(F(z) + G(z))| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \tag{2.6}$$

$$J_1 = \int_0^1 \left| \frac{2c}{(1 - z)^2} \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \tag{2.7}$$

$$J_2 = \int_0^1 \left| 2 \sum_{n=0}^{\infty} \vartheta(n) z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \tag{2.8}$$

$$J_3 = \int_0^1 \left| 2(1 - z) \sum_{n=0}^{\infty} (n + 1)\vartheta(n + 1) z^n \right| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta, \tag{2.9}$$

$$J_4 = \int_0^1 |(F'(z) - G'(z))(F(z) - G(z))| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta. \tag{2.10}$$

Evidently by (2.5), we have

$$J \le J_1 + J_2 + J_3 + J_4.$$

In [7], Horváth has shown that

$$J \gg mN^{3/2}, \qquad J_1, J_2 \ll m^2 N, \qquad J_3 = o(m^{1/2} N^{7/4}). \tag{2.11}$$

It suffices to give an upper bound for J_4. By the Cauchy–Schwarz inequality,

$$\begin{aligned} J_4 &\le 4 \int_0^1 \left| \frac{F'(z) - G'(z)}{1 - z} \right| \cdot \left| \frac{F(z) - G(z)}{1 - z} \right| d\theta \\ &\le \frac{4}{\rho} \left(\int_0^1 \left| \frac{zF'(z) - zG'(z)}{1 - z} \right|^2 d\theta \right)^{1/2} \left(\int_0^1 \left| \frac{F(z) - G(z)}{1 - z} \right|^2 d\theta \right)^{1/2}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{zF'(z) - zG'(z)}{1 - z} &= \frac{1}{1 - z} \Big(\sum_{a \in A} a z^a - \sum_{b \in B} b z^b \Big) \\ &= \sum_{n=0}^{\infty} z^n \Big(\sum_{\substack{a \in A \\ a \le n}} a - \sum_{\substack{b \in B \\ b \le n}} b \Big). \end{aligned}$$

Applying (2.3) with $\beta = 1/2$, we have

$$(2.12)\qquad \int_0^1 \left|\frac{zF'(z) - zG'(z)}{1-z}\right|^2 d\theta$$
$$= \int_0^1 \Big(\sum_{n=0}^{\infty} z^n \Big(\sum_{\substack{a\in A\\ a\le n}} a - \sum_{\substack{b\in B\\ b\le n}} b\Big)\Big)\Big(\sum_{n=0}^{\infty} \bar{z}^n \Big(\sum_{\substack{a\in A\\ a\le n}} a - \sum_{\substack{b\in B\\ b\le n}} b\Big)\Big)\, d\theta$$
$$= \sum_{n=0}^{\infty} \rho^{2n} \Big(\sum_{\substack{a\in A\\ a\le n}} a - \sum_{\substack{b\in B\\ b\le n}} b\Big)^2 = o(N^3).$$

Similarly, by (2.4),

$$\int_0^1 \left|\frac{F(z) - G(z)}{1-z}\right|^2 d\theta = \sum_{n=0}^{\infty} \rho^{2n} \Big(\sum_{\substack{a\in A\\ a\le n}} 1 - \sum_{\substack{b\in B\\ b\le n}} 1\Big)^2 = o(N).$$

Thus $J_4 = o(N^2)$.

Since $J \le J_1 + J_2 + J_3 + J_4$, in view of (2.11) there exists a constant $C > 1$ such that

$$mN^{3/2} \le Cm^2 N + o(m^{1/2} N^{7/4}) + o(N^2), \qquad N \to \infty.$$

By letting $m = [C^{-2} N^{1/2}]$, we get an evident contradiction when N is sufficiently large. ∎

3. Proof of Theorem 1.2

LEMMA 3.1. *Suppose that* $0 < \beta \le 1/2$ *and* $A_1, \ldots, A_k$ *are non-empty subsets of* $\mathbb{N}$. *Assume that* $A_1(n), A_2(n) = \Theta(n^\beta)$ *and* $R_{A_1,\ldots,A_k}(n) = \Theta(n)$ *for each* $n \ge 1$. *Then*

$$R_{A_3,\ldots,A_k}(n) = \Theta(n^{1-2\beta}).$$

Proof. Evidently

$$R_{A_1,\ldots,A_k}(n) = \sum_{u=0}^{n} r_{A_1,\ldots,A_k}(u) = \sum_{\substack{0\le v,w\le n\\ v+w=n}} r_{A_1,A_2}(v) r_{A_3,\ldots,A_k}(w)$$
$$\le \sum_{v=0}^{n} r_{A_1,A_2}(v) \sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w) \le A_1(n)A_2(n) \sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w).$$

Since $R_{A_1,\ldots,A_k}(n) \gg n$ and $A_1(n), A_2(n) \ll n^\beta$, we get

$$\sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w) \ge \frac{R_{A_1,\ldots,A_k}(n)}{A_1(n)A_2(n)} \gg n^{1-2\beta}.$$

On the other hand, we also have

$$R_{A_1,\ldots,A_k}(3n) = \sum_{\substack{0\le v,w\le 3n\\ v+w=3n}} r_{A_1,A_2}(v) r_{A_3,\ldots,A_k}(w) \ge \sum_{v=0}^{2n} r_{A_1,A_2}(v) \sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w)$$
$$\ge A_1(n)A_2(n) \sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w) \gg n^{2\beta} \sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w).$$

It follows from $R_{A_1,\ldots,A_k}(3n) \ll n$ that

$$\sum_{w=0}^{n} r_{A_3,\ldots,A_k}(w) \ll n^{1-2\beta}. \blacksquare$$

Assume on the contrary that (1.10) holds. Let $\vartheta(n) = R_{A_1,\ldots,A_k}(n) - cn$. Let $F_i(z) = \sum_{a\in A_i} z^a$ for each $1 \le i \le k$. Then

$$\frac{F_1(z)F_2(z)\cdots F_k(z)}{1-z} = \sum_{n=0}^{\infty} R_{A_1,\ldots,A_k}(n) z^n = \frac{cz}{(1-z)^2} + \sum_{n=0}^{\infty} \vartheta(n) z^n,$$

i.e.,

$$(F_1(z)+F_2(z))^2 F_3(z)\cdots F_k(z)$$
$$= \frac{4cz}{1-z} + 4(1-z)\sum_{n=0}^{\infty} \vartheta(n)z^n + (F_1(z)-F_2(z))^2 F_3(z)\cdots F_k(z).$$

Taking the derivative in z, we obtain

$$2(F_1'(z)+F_2'(z))(F_1(z)+F_2(z))\prod_{j=3}^{k} F_j(z) + (F_1(z)+F_2(z))^2 \sum_{j=3}^{k} F_j'(z) \prod_{\substack{3\le i\le k\\ i\ne j}} F_i(z)$$
$$= 2(F_1'(z)-F_2'(z))(F_1(z)-F_2(z)) \prod_{j=3}^{k} F_j(z)$$
$$+ (F_1(z)-F_2(z))^2 \sum_{j=3}^{k} F_j'(z) \prod_{\substack{3\le i\le k\\ i\ne j}} F_i(z)$$
$$+ \frac{4c}{(1-z)^2} + 4(1-z)\sum_{n=0}^{\infty}(n+1)\vartheta(n+1)z^n - 4\sum_{n=0}^{\infty}\vartheta(n)z^n.$$

Let $\rho = 1 - 1/N$, $z = \rho e^{2\pi\sqrt{-1}\theta}$ and let m be a large integer to be chosen later. Let

$$J = \int_0^1 \Big| 2({F_1}'(z) + {F_2}'(z)) \prod_{j=3}^{k} F_j(z) + (F_1(z)+F_2(z)) \sum_{j=3}^{k} F_j'(z) \prod_{\substack{3\le i\le k\\ i\ne j}} F_i(z) \Big|$$
$$\cdot |F_1(z)+F_2(z)| \cdot \left| \frac{1-z^m}{1-z} \right|^2 d\theta$$

and

$$J_4 = \int_0^1 \left| 2({F_1}'(z) - {F_2}'(z)) \prod_{j=3}^k F_j(z) + (F_1(z) - F_2(z)) \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \le i \le k \\ i \ne j}} F_i(z) \right| \cdot |F_1(z) - F_2(z)| \cdot \left| \frac{1 - z^m}{1 - z} \right|^2 d\theta.$$

And let J_1, J_2, J_3 be as in (2.7)–(2.9) respectively. Then

$$|J| \le 2|J_1| + 2|J_2| + 2|J_3| + |J_4|.$$

First, we shall give a lower bound of J. Since $A_1(n) = \Theta(n^\beta)$ and $A_1(n) - A_2(n) = O(n^\alpha)$, we may assume that

$$c_1 n^\beta \le A_1(n), A_2(n) \le c_2 n^\beta$$

for some constants $c_1, c_2 > 0$, whenever n is sufficiently large. Let

$$G(z) := 2({F_1}'(z) + {F_2}'(z)) \prod_{j=3}^k F_j(z) + (F_1(z) + F_2(z)) \sum_{j=3}^k F_j'(z) \prod_{\substack{3 \le i \le k \\ i \ne j}} F_i(z).$$

Write

$$G(z) = \sum_{n=0}^\infty g_n z^n, \qquad (F_1'(z) + F_2'(z)) \prod_{j=3}^k F_j(z) = \sum_{n=0}^\infty h_n z^n.$$

Clearly $g_n \ge 2h_n \ge 0$ for each $n \ge 0$. Let $\mathcal{A}$ denote the multiset $A_1 \cup A_2$, i.e., the common elements of A_1 and A_2 have multiplicity 2 in $\mathcal{A}$. Then

$$F_1(z) + F_2(z) = \sum_{a \in A_1} z^a + \sum_{a \in A_2} z^a = \sum_{a \in \mathcal{A}} z^a.$$

Thus

$$\begin{aligned} J &\ge \left| \int_0^1 \overline{G(z)} \cdot (F_1(z) + F_2(z)) \cdot \frac{1 - z^m}{1 - z} \cdot \frac{1 - \overline{z}^m}{1 - \overline{z}} \, d\theta \right| \\ &= \left| \int_0^1 \Big(\sum_{n=0}^\infty g_n \overline{z}^n \Big) \cdot \Big(\sum_{a \in \mathcal{A}} z^a \Big) \cdot \Big(\sum_{n=0}^{m-1} z^n \Big) \cdot \Big(\sum_{n=0}^{m-1} \overline{z}^n \Big) \, d\theta \right| \\ &= \sum_{\substack{a \in \mathcal{A},\, u \ge 0 \\ 0 \le v, w \le m-1 \\ a+v=u+w}} \rho^{a+u+v+w} g_u \ge 2 \sum_{\substack{a \in \mathcal{A},\, u \ge 0 \\ 0 \le v, w \le m-1 \\ a+v=u+w}} \rho^{a+u+v+w} h_u. \end{aligned}$$

Note that

$$(F_1{}'(z)+F_2{}'(z))\prod_{j=3}^{k}F_j(z)=\sum_{a\in\mathcal{A}}az^{a-1}\cdot\sum_{n=0}^{\infty}r_{A_3,\dots,A_k}(n)z^n$$
$$=\sum_{n=0}^{\infty}z^n\sum_{\substack{a\in\mathcal{A}\\ a\le n+1}}ar_{A_3,\dots,A_k}(n-a+1).$$

It follows that

$$\sum_{\substack{a\in\mathcal{A},\,u\ge0\\0\le v,w\le m-1\\a+v=u+w}}\rho^{a+u+v+w}h_u=\sum_{\substack{a\in\mathcal{A},\,u\ge0\\0\le v,w\le m-1\\a+v=u+w}}\rho^{a+u+v+w}\sum_{\substack{b\in\mathcal{A}\\b\le u+1}}br_{A_3,\dots,A_k}(u+1-b)$$
$$\ge\sum_{\substack{a,b\in\mathcal{A},\,b\le u\\0\le v,w\le m-1\\a+v=u+w}}\rho^{a+u+v+w}\cdot br_{A_3,\dots,A_k}(u+1-b).$$

We may restrict the above summation to those a,b,u,v,w satisfying the following conditions:

(1) $c_3N\le a\le N$, where $c_3=(c_1/(4c_2))^{1/\beta}$;
(2) $b=a$, $a\le u\le N+m/2$;
(3) $1\le w<m/2$, $w\le v\le w+m/2$.

Thus

$$J\ge2\sum_{\substack{a\in\mathcal{A},\,c_3N\le a\le N\\0\le w<m/2,\ w\le v\le w+m/2\\a\le u\le N+m/2,\ u-a=v-w}}\rho^{a+u+v+w}\cdot ar_{A_3,\dots,A_k}(u+1-a)$$
$$\ge2\sum_{\substack{a\in\mathcal{A}\\c_3N\le a\le N}}a\sum_{\substack{0\le w<m/2\\w\le v\le w+m/2}}\rho^{a+(v-w+a)+v+w}r_{A_3,\dots,A_k}(v-w+1)$$
$$\ge2c_3N\cdot(\mathcal{A}(N)-\mathcal{A}(c_3N))\cdot\frac{\rho^{2N+2m}m}{2}\sum_{j=0}^{m/2}r_{A_3,\dots,A_k}(j+1).$$

We have $\mathcal{A}(N)\ge A_1(N)\ge c_1N^\beta$ and

$$\mathcal{A}(c_3N)\le A_1(c_3N)+A_2(c_3N)\le2c_2c_3^\beta N^\beta\le\frac{c_1}{2}N^\beta.$$

Furthermore, since $m\le N$,

$$\rho^{2N+2m}\ge\left(1-\frac{1}{N}\right)^{4N}\ge\frac{1}{2e^4}.$$

Hence by Lemma 3.1,

$$J \gg N^{1+\beta} m \sum_{j=0}^{m/2} r_{A_3,\ldots,A_k}(j+1) \gg m^{2-2\beta} N^{1+\beta}.$$

Next, let us find an upper bound for J_4. Clearly

$$J_4 \le 8 \int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right| \cdot \left| \frac{F_1(z) - F_2(z)}{1-z} \right| \cdot \prod_{j=3}^{k} |F_j(z)| \, d\theta$$
$$+ 4 \sum_{j=3}^{k} \int_0^1 \left| \frac{F_1(z) - F_2(z)}{1-z} \right|^2 \cdot |F_j'(z)| \prod_{\substack{3 \le i \le k \\ i \ne j}} |F_i(z)| \, d\theta.$$

Note that

$$|F_3(z) \cdots F_k(z)| \le |F_3(\rho) \cdots F_k(\rho)| = \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3,\ldots,A_k}(n). \tag{3.1}$$

By Lemma 3.1,

$$\sum_{n \le x} r_{A_3,\ldots,A_k}(n) = O(x^{1-2\beta}).$$

It follows from Lemma 2.1(ii) that

$$\sum_{n=0}^{\infty} \rho^n \cdot r_{A_3,\ldots,A_k}(n) = O(N^{1-2\beta}).$$

Thus by the Cauchy–Schwarz inequality,

$$\int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right| \cdot \left| \frac{F_1(z) - F_2(z)}{1-z} \right| \cdot \prod_{j=3}^{k} |F_j(z)| \, d\theta$$
$$\ll N^{1-2\beta} \left(\int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right|^2 d\theta \right)^{1/2} \cdot \left(\int_0^1 \left| \frac{F_1(z) - F_2(z)}{1-z} \right|^2 d\theta \right)^{1/2}.$$

Applying Lemma 2.2, we obtain

$$\int_0^1 \left| \frac{F_1'(z) - F_2'(z)}{1-z} \right|^2 d\theta \le \frac{1}{\rho^2} \int_0^1 \left| \frac{zF_1'(z) - zF_2'(z)}{1-z} \right|^2 d\theta$$
$$= \frac{1}{\rho^2} \sum_{n=0}^{\infty} \rho^{2n} \Big(\sum_{\substack{a_1 \in A_1 \\ a_1 \le n}} a_1 - \sum_{\substack{a_2 \in A_2 \\ a_2 \le n}} a_2 \Big)^2 = o(N^{2+2\beta})$$

and

$$\int_0^1 \left|\frac{F_1(z)-F_2(z)}{1-z}\right|^2 d\theta = \sum_{n=0}^{\infty} \rho^{2n}\Big(\sum_{\substack{a_1\in A_1\\ a_1\le n}} 1 - \sum_{\substack{a_2\in A_2\\ a_2\le n}} 1\Big)^2 = o(N^{2\beta}).$$

So as $N\to\infty$,

$$\int_0^1 \left|\frac{F_1'(z)-F_2'(z)}{1-z}\right| \cdot \left|\frac{F_1(z)-F_2(z)}{1-z}\right| \cdot \prod_{j=3}^{k} |F_j(z)|\, d\theta \ll N^{1-2\beta}\cdot o(N^{1+2\beta}) = o(N^2).$$

Similarly, for each $3\le j\le k$,

$$|F_j'(z)| \prod_{\substack{3\le i\le k\\ i\ne j}} |F_i(z)| \le |F_j'(\rho)| \prod_{\substack{3\le i\le k\\ i\ne j}} |F_i(\rho)| = \sum_{a_3\in A_3,\dots,a_k\in A_k} a_j \rho^{a_3+\cdots+a_k}$$

$$= \sum_{n=0}^{\infty} \rho^n \sum_{\substack{a_3\in A_3,\dots,a_k\in A_k\\ a_3+\cdots+a_k=n}} a_j \le \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3,\dots,A_k}(n) n.$$

Note that

$$\sum_{n\le x} r_{A_3,\dots,A_k}(n) n \le x \sum_{n\le x} r_{A_3,\dots,A_k}(n) = O(x^{2-2\beta}).$$

Using Lemma 2.1(ii), we get

$$|F_j'(z)| \prod_{\substack{3\le i\le k\\ i\ne j}} |F_i(z)| \le \sum_{n=0}^{\infty} \rho^n \cdot r_{A_3,\dots,A_k}(n) n = O(N^{2-2\beta}).$$

It follows that

$$\int_0^1 \left|\frac{F_1(z)-F_2(z)}{1-z}\right|^2 \cdot |F_j'(z)| \prod_{\substack{3\le i\le k\\ i\ne j}} |F_i(z)|\, d\theta \ll N^{2-2\beta} \int_0^1 \left|\frac{F_1(z)-F_2(z)}{1-z}\right|^2 d\theta$$

$$= N^{2-2\beta}\cdot o(N^{2\beta}) = o(N^2).$$

Thus we get

$$J_4 = o(N^2).$$

Recall that $J \le 2J_1 + 2J_2 + 2J_3 + J_4$ and

$$J_1, J_2 \ll m^2 N, \qquad J_3 = o(m^{1/2} N^{7/4}).$$

We may choose a large constant $C>1$ such that

$$m^{2-2\beta} N^{1+\beta} \le C m^2 N + o(m^{1/2} N^{7/4}) + o(N^2).$$

This immediately leads to a contradiction by setting $m = [C^{-1/\beta} N^{1/2}]$.

4. Proof of Theorem 1.3. Here we only give the proof of (i) of Theorem 1.3, since the proof of (ii) is the same.

Suppose that N is sufficiently large and $\rho = 1 - 1/N$. Let $\vartheta(n) = R_{A,B}(n) - cn$ and let J, J_1, J_2, J_3, J_4 be given by (2.6)–(2.10). We shall give upper bounds of J_2, J_3 under the assumption

$$\sum_{n\le x} \vartheta(n)^2 = o(x^{3/2}), \qquad x \to \infty. \tag{4.1}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} J_2 &= 2\int_0^1 \Big|\sum_{n=0}^{\infty} \vartheta(n) z^n\Big| \cdot \Big|\frac{1-z^m}{1-z}\Big|^2 \, d\theta \\ &\le 2\Big(\int_0^1 \Big|\sum_{n=0}^{\infty} \vartheta(n) z^n\Big|^2 \, d\theta\Big)^{1/2} \cdot \Big(\int_0^1 \Big|\frac{1-z^m}{1-z}\Big|^4 \, d\theta\Big)^{1/2}, \end{aligned}$$

where $z = \rho e^{2\pi\sqrt{-1}\,\theta}$. Clearly

$$\int_0^1 \Big|\frac{1-z^m}{1-z}\Big|^4 \, d\theta \le \sum_{\substack{0\le a,b,c,d\le m-1 \\ a+b=c+d}} 1 \le m^3.$$

And

$$\int_0^1 \Big|\sum_{n=0}^{\infty} \vartheta(n) z^n\Big|^2 \, d\theta = \sum_{n=0}^{\infty} \vartheta(n)^2 \rho^{2n}.$$

By (4.1) and Lemma 2.1(i), we have

$$\sum_{n=0}^{\infty} \vartheta(n)^2 \rho^{2n} = o(N^{3/2}), \qquad N \to \infty,$$

i.e.,

$$J_2 = o(m^{3/2} N^{3/4}).$$

On the other hand, since $|1 - z^m| \le 2$, we have

$$\begin{aligned} J_3 &= 2\int_0^1 \Big|(1-z)\sum_{n=0}^{\infty} (n+1)\vartheta(n+1) z^n\Big| \cdot \Big|\frac{1-z^m}{1-z}\Big|^2 \, d\theta \\ &\le 4\int_0^1 \Big|\sum_{n=0}^{\infty} (n+1)\vartheta(n+1) z^n\Big| \cdot \Big|\frac{1-z^m}{1-z}\Big| \, d\theta \\ &\le 4\Big(\int_0^1 \Big|\sum_{n=0}^{\infty} (n+1)\vartheta(n+1) z^n\Big|^2 \, d\theta\Big)^{1/2} \cdot \Big(\int_0^1 \Big|\frac{1-z^m}{1-z}\Big|^2 \, d\theta\Big)^{1/2}. \end{aligned}$$

Clearly

$$\int_0^1 \left|\frac{1-z^m}{1-z}\right|^2 d\theta \le m.$$

And

$$\int_0^1 \left|\sum_{n=0}^{\infty}(n+1)\vartheta(n+1)z^n\right|^2 d\theta = \sum_{n=0}^{\infty}(n+1)^2\vartheta(n+1)^2\rho^{2n} \le 4\sum_{n=0}^{\infty} n^2\vartheta(n)^2\rho^{2n}.$$

Since

$$\sum_{n\le x} n^2\vartheta(n)^2 \le x^2\sum_{n\le x}\vartheta(n)^2 = o(x^{7/2}), \qquad x\to\infty,$$

it follows from Lemma 2.1(i) that

$$\sum_{n=0}^{\infty} n^2\vartheta(n)^2\rho^{2n} = o(N^{7/2}),$$

i.e.,

$$J_3 = o(m^{1/2}N^{7/4}).$$

Finally, we have

$$\vartheta(n) \le \Big(\sum_{j\le n}\vartheta(j)^2\Big)^{1/2} \le n^{3/4}$$

for any sufficiently large n. Thus

$$R_{A,B}(n) = cn + O(n^{3/4}),$$

i.e., $R_{A,B}(n) = \Theta(n)$. Under the assumptions of Theorem 1.1, we know that $J \gg mN^{3/2}$, $J_4 = o(N^2)$ and $J_1 = O(m^2N)$. It follows from $J \le J_1 + J_2 + J_3 + J_4$ that

$$mN^{3/2} \le Cm^2N + o(m^{3/2}N^{3/4}) + o(m^{1/2}N^{7/4}) + o(N^2)$$

for some constant $C > 1$. Setting $m = [C^{-2}N^{1/2}]$, we immediately get a contradiction when N is sufficiently large. ∎

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