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A converse theorem for Jacobi cusp forms of degree two

by

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1. Introduction. Explicit relations between families of Dirichlet series and different types of automorphic forms have been investigated for a long time. Starting with Hamburger's work [5], which connects the Riemann zeta function with the classical theta series, and after the most influential result of this class, Hecke's converse theorem [6], many researchers have obtained generalizations to various kinds of automorphic forms.

The meaning of a converse theorem in this context is best illustrated by presenting the beautiful result of Hecke.

Theorem (Hecke). *Let $\{c(n) \mid n \geq 1\}$ be a sequence of complex numbers satisfying the estimate $c(n) = O(n^{\mu})$ for some $\mu > 0$, and k a positive integer. Then the following two statements are equivalent:*

(A) *The Fourier series $f(\tau) = \sum_{n\geq 1} c(n)\exp(2\pi i n\tau)$ ($\tau \in \mathcal{H}$ = the complex upper half-plane) defines a weight k cusp form over $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.*

(B) *The completed Dirichlet series $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)\sum_{n\geq 1} c(n)n^{-s}$ has an analytic continuation to the whole s-plane, it is a bounded function on any vertical strip, and satisfies the functional equation*

$$\Lambda(f, k-s) = i^k \Lambda(f,s).$$

Other results of this kind are Weil's converse theorem [25] for cusp forms over congruence subgroups $\Gamma_0(N)$ (here one has to consider infinitely many Dirichlet series twisted with Dirichlet characters), Maass' converse theorem [15] for Maass waveforms, and a string of papers [3], [10], [11] proving the converse theorem for automorphic forms on $\mathrm{GL}(n)$ by Jacquet, Langlands, Piatetski-Shapiro, Shalika and Cogdell.

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The theory of modular forms over Γ has been extended to the realm of several complex variables in many ways. Among the most classical ones we find the theory of Siegel modular forms and the theory of Jacobi forms. Standard references for these matters are [12], [17], [4], [27]. In [8] K. Imai proved a converse theorem for Siegel cusp forms of degree two. This result was extended to Siegel cusp forms of any degree by Weissauer [26], and revisited for not necessarily cuspidal Siegel forms of degree two by Arakawa, Makino and Sato [2]. As for Jacobi forms, the second named author proved a converse theorem for Jacobi cusp forms of degree 1 in [18]. This result has been extended to Jacobi cusp forms of degree 1 over congruence subgroups in [20]. There is also a study on Dirichlet series for Jacobi forms of higher degree in [19].

Despite the time elapsed from the publication of [8] and [18], to the best of our knowledge no much work has been done on a possible converse theorem for Jacobi forms of degree 2. The purpose of this article is to fill this gap.

1.1. Main result. In order to describe our results, we need some notation: If R is a commutative ring with 1 and i, j are positive integers, we let $R^{i,j}$ be the set of all $i \times j$ matrices with entries in R. For any $X \in R^{i,j}$ we write X^t for the transpose of X, and if $i = j$, the trace (resp. determinant) of X is denoted by $\operatorname{trace} X$ (resp. $\det X$). Moreover, we let $e(X) = \exp(2\pi i \cdot \operatorname{trace} X)$.

Let $\mathcal{I} = \mathbb{Z}^{2,1}/2m\mathbb{Z}^{2,1}$ for a fixed positive integer m. This is an abelian group of order $n = 4m^2$. Let J be the set of all 2×2, half-integral, symmetric, positive-definite matrices. This is a Γ-set under the action $T \mapsto g^t T g$ for any $T \in J$ and $g \in \Gamma$. (Note that all cosets in $\mathcal{I}$ are usually identified with their representatives in $\mathbb{Z}^{2,1}$ throughout the article.)

Let $\mathcal{H}_2$ be the Siegel space of degree two. The Jacobi cusp forms in this article are certain holomorphic functions $F : \mathcal{H}_2 \times \mathbb{C}^2 \to \mathbb{C}$ which are invariant under an action of the Jacobi group $\Gamma_2^J = \mathrm{Sp}_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \times \mathbb{Z}^2)$ indexed by two positive integers: the weight k and the index m (see (2.1) and Definition 2.1). The set of Jacobi cusp forms is a finite-dimensional $\mathbb{C}$-vector space which we denote as $J_{2,k,m}^{\mathrm{cusp}}$.

As we explain below, every $F(Z, z_1, z_2) \in J_{2,k,m}^{\mathrm{cusp}}$ has a Fourier series representation of type

$$F(Z, z_1, z_2) = \sum_{R\in\mathcal{I}} F_R(Z)\Theta_{m,R}(Z, z_1, z_2), \tag{1.1}$$

where $Z \in \mathcal{H}_2$, $z_1, z_2 \in \mathbb{C}$, $\Theta_{m,R}$ is the theta series (2.4), and

$$F_R(Z) = \sum_{T\in J,\, T+RR^t\in 4mJ} c_R(T) e\left(\frac{1}{4m} TZ\right) \tag{1.2}$$

for every $R \in \mathcal{I}$. These series representations show that $F(Z, z_1, z_2)$ is completely determined by n sequences of complex numbers indexed by the cosets in $\mathcal{I}$. Namely, the Fourier coefficients of the series $F_R(Z)$ in (1.2). Such sets of coefficients have polynomial growth of the determinant of the index matrices, and satisfy functional equations derived from the action of a certain subgroup of $\mathrm{Sp}_2(\mathbb{Z}) \subseteq \Gamma_2^J$.

In view of these facts, it is natural to study collections $\{F_R \mid R \in \mathcal{I}\}$ of sequences $F_R = \{c_R(T) \mid T \in J,\ T + RR^t \in 4mJ\}$ in $\mathbb{C}$ such that

$$c_R(T) = O((\det T)^{k/2}), \tag{1.3a}$$

$$c_{g^tR}(g^tTg) = (\det g)^k c_R(T) \quad \text{for all } g \in \mathrm{GL}_2(\mathbb{Z}). \tag{1.3b}$$

To each of these sequences we associate a formal Fourier series. Namely, to F_R we associate the series $F_R(Z)$ in (1.2), and with all of them we construct the series $F(Z, z_1, z_2)$ in (1.1). The estimate in (1.3a) implies that $F_R(Z)$ (resp. $F(Z, z_1, z_2)$) defines a holomorphic function on $\mathcal{H}_2$ (resp. on $\mathcal{H}_2 \times \mathbb{C}^2$).

We also attach certain twisted Dirichlet series to the set $\{F_R \mid R \in \mathcal{I}\}$ of sequences. To this end we consider any vector-valued Grössencharacter $\mathcal{U}$ associated to the permutation representation ρ defined by the action of Γ on $\mathcal{I}$ and given in (2.10) (see Definition 3.1), and let $L(F, \mathcal{U}, s)$ be the twisted Dirichlet series introduced in Definition 3.2. We write $\Lambda(F, \mathcal{U}, s)$ for the completion of $L(F, \mathcal{U}, s)$ with certain gamma factors (Definition 3.4).

The basic problem is to relate the Fourier and Dirichlet series mentioned above in the context of Jacobi forms of degree two. Our answer requires the matrix $A_m = (A_m(R', R))_{R', R \in \mathcal{I}} \in \mathbb{C}^{n,n}$ whose entries are

$$A_m(R', R) = \frac{1}{2m} e\left(\frac{1}{2m}(R')^t S R\right) \quad \text{with} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Note that $\mathcal{U}A_m$ is a Grössencharacter whenever $\mathcal{U}$ is a Grössencharacter.

Main Theorem 1.1. *Let $\{F_R \mid R \in \mathcal{I}\}$ be a collection of n sequences $F_R = \{c_R(T) \mid T \in J,\ T + RR^t \in 4mJ\}$ of complex numbers which satisfy* (1.3a) *and* (1.3b). *Let $\mathcal{G}$ be the set of vector-valued Grössencharacters associated to the representation ρ of* $\mathrm{PSL}_2(\mathbb{Z})$ *such that the parameter α in Definition* 3.1 (*which controls the growth of a Grössencharacter at infinity*) *satisfies $0 \le \alpha < 1$. Then the following two statements are equivalent:*

(A) *The series $F(Z, z_1, z_2)$ in* (1.1) *defines a Jacobi cusp form in $J^{\mathrm{cusp}}_{2,k,m}$.*
(B) *For each $\mathcal{U}$ in $\mathcal{G}$ the completed Dirichlet series $\Lambda(F, \mathcal{U}, s)$ has an analytic continuation to the whole s-plane; the continuation is a bounded function of s on any vertical strip, and satisfies the functional equation*

$$\Lambda(F, \mathcal{U}, k - s - 1/2) = (-1)^k \Lambda(F, \mathcal{U}A_m, s).$$

Our proof of the theorem rests on two key ideas. We first consider the theta decomposition of Jacobi forms to obtain certain vector-valued Siegel modular forms. This process is well-known in a much more general setting by the work of Shimura [22]. Secondly, we use the spectral theory of the hyperbolic surface $\Gamma\backslash\mathcal{H}$ in the vectorial setting in order to deduce statement (A) from statement (B). Imai [8] and Arakawa, Makino and Sato [2] take a similar approach in the scalar case for their converse theorems.

In this paper we give a detailed account of all technical issues involved in the proof of Theorem 1.1. We also note that the proof of the converse direction in the theorem given in [8] for Siegel cusp forms requires an extra condition on some Mellin transforms, mentioned explicitly in [8, p. 929 statement (2) about $\widetilde{f}_s(W)$]. In our article we dispense with this extra condition via the arguments presented in 6.3.

It is worth noticing that the estimate (1.3a) is not very stringent for Siegel cups forms, but it is more precise than the one used in other converse theorems (see Hecke's theorem above [2], [8]). It is easy to check that there is no loss of generality in having the exponent $k/2$ in (1.3a) as opposed to an arbitrary positive exponent.

The proof of Theorem 1.1 presented here can be extended without major changes to the case of cuspidal Jacobi forms of degree 2 whose index is a matrix. We only consider the scalar index case for simplicity.

There is a straightforward application of Theorem 1.1 to some half-integral weight Siegel cusp forms of degree 2 by way of an isomorphism proved by Ibukiyama [7], as we now explain. Let $S^+_{k-1/2}(\Gamma_{2,0}(4))$ be Kohnen's plus subspace of weight $k-1/2$ Siegel cusp forms over the congruence subgroup $\Gamma_{2,0}(4)$ of $\mathrm{Sp}_2(\mathbb{Z})$ (see (8.1)). Then any $h(Z) \in S^+_{k-1/2}(\Gamma_{2,0}(4))$ has a Fourier series representation

$$h(Z) = \sum_{T\in J} d(T)e(TZ) \tag{1.5}$$

with some additional condition on the index set.

As in the case of Siegel Jacobi forms, we here consider sequences

$$\{d(T) \mid T \in J,\, T + \mu\mu^t \in 4J \text{ for some } \mu \in \mathbb{Z}^{2,1}\} \subseteq \mathbb{C}$$

with

$$d(T) = O((\det T)^{k/2}), \tag{1.6a}$$

$$d(g^tTg) = (\det g)^k d(T) \quad \text{for all } g \in \mathrm{GL}_2(\mathbb{Z}). \tag{1.6b}$$

To each such sequence we associate a formal Fourier series, namely (1.5) with Z in $\mathcal{H}_2$.

We also associate twisted Dirichlet series to both the sequence $\{d(T)\}_T$ given above and its subsequence $\{d(T) \mid T \in 4J\}$. Namely, if $\mathcal{V}$ is a scalar-valued Grössencharacter for the 1-dimensional trivial representation of Γ, we attach to those sequences the completed, twisted Dirichlet series $\Lambda(h, \mathcal{V}, s)$ and $\Lambda_0(h, \mathcal{V}, s)$ respectively (see Definition 8.1). Then we get this result:

COROLLARY 1.2. *Let* $\{d(T) \mid T \in J,\ T + \mu\mu^t \in 4J$ *for some* $\mu \in \mathbb{Z}^{2,1}\}$ *be a sequence of complex numbers satisfying* (1.6a), (1.6b) *with* k *a positive even integer. Let* $\widetilde{\mathcal{G}}$ *be the set of scalar Grössencharacters associated to the trivial representation of* $\mathrm{PSL}_2(\mathbb{Z})$ *such that the parameter* α (*which controls the Grössencharacter's growth at infinity*) *satisfies* $0 \leq \alpha < 1$. *Then the following two statements are equivalent:*

(A) *The series* $h(Z)$ *in* (1.5) *defines a cusp form in Kohnen's plus space* $S^+_{k-1/2}(\Gamma_{2,0}(4))$.
(B) *For each* $\mathcal{V}$ *in* $\widetilde{\mathcal{G}}$ *the completed Dirichlet series* $\Lambda(h, \mathcal{V}, s)$ *and* $\Lambda_0(h, \mathcal{V}, s)$ *have an analytic continuation to the whole* s*-plane; the continuations are bounded functions of* s *on any vertical strip, and satisfy the functional equation*

$$\Lambda(h, \mathcal{V}, k - 1/2 - s) = \Lambda_0(h, 2\mathcal{V}, s).$$

This article is organized as follows: In the next section we recall basic facts about Jacobi forms of degree 2. In Section 3 we introduce the notions of Grössencharacters and twisted Dirichlet series associated to collections of n sequences of complex numbers satisfying (1.3a), (1.3b). Section 4 is devoted to Mellin transforms and an integral representation of the twisted Dirichlet series introduced earlier. In Section 5 we give a proof of (A)$\Rightarrow$(B) in Theorem 1.1. The converse is shown in Section 6. Finally, Section 7 contains the proofs of the technical results used in preceding arguments, and Section 8 is devoted to half-integral weight Siegel cusp forms of degree 2, and the proof of the corollary.

NOTATION. We denote by I_2 the 2×2 identity matrix. For any ring R, $R^2 = R^{1,2}$ is the space of row vectors, and for matrices A, B, C of suitable sizes we put $A[B] = B^t AB$. Any M in $\mathrm{Sp}_2(\mathbb{R})$ is written as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with A, B, C, D in $\mathbb{R}^{2,2}$. Throughout, k, m are positive integers and $\mathcal{P}$, $\mathcal{SP}$ are the sets of matrices

$$\mathcal{P} = \{Y \in \mathbb{R}^{2,2} \mid Y^t = Y,\ Y > 0\}, \quad \mathcal{SP} = \{W \in \mathcal{P} \mid \det W = 1\}.$$

Always the real coordinates for the matrices $T \in J$, $Y \in \mathcal{P}$ and $W \in \mathcal{SP}$, and those of $\tau \in \mathcal{H}$, are given as

$$(1.7) \qquad T = \begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix},\ Y = \begin{pmatrix} y_1 & y \\ y & y_2 \end{pmatrix},\ W = \begin{pmatrix} w_1 & w \\ w & w_2 \end{pmatrix},\ \tau = x + iy.$$

2. Jacobi cusp forms of degree two. The symplectic modular group of degree two is

$$\Gamma_2 = \left\{ M \in \mathbb{Z}^{4,4} \,\middle|\, \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}[M] = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}.$$

It acts on the abelian group $\{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{Z}^2\}$ via matrix multiplication, and the corresponding semidirect product $\Gamma_2^J = \Gamma_2 \ltimes (\mathbb{Z}^2 \times \mathbb{Z}^2)$ is the Jacobi group. A natural generalization of Poincaré's upper half-plane $\mathcal{H}$ is the Siegel space of degree two, $\mathcal{H}_2 = \{Z \in \mathbb{C}^{2,2} \mid Z^t = Z,\ \Im(Z) > 0\}$. We use it to build the complex analytic manifold $\mathcal{H}_2 \times \mathbb{C}^2$, and observe that Γ_2^J acts on it via

$$\begin{aligned}(M, \lambda, \mu) &\cdot (Z, z_1, z_2) \\ &= \big((AZ + B)(CZ + D)^{-1}, ((z_1, z_2) + \lambda Z + \mu)(CZ + D)^{-1}\big).\end{aligned}$$

Here $Z \in \mathcal{H}_2$, $(z_1, z_2) \in \mathbb{C}^2$, $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$ and $(\lambda, \mu) \in \mathbb{Z}^2 \times \mathbb{Z}^2$. This action and the integers k and m define a right action of Γ_2^J on the set of holomorphic functions $F : \mathcal{H}_2 \times \mathbb{C}^2 \to \mathbb{C}$, namely,

$$\begin{aligned}&F|_{k,m}[M, \lambda, \mu](Z, z_1, z_2) \\ &\quad = \det{(CZ + D)}^{-k} \mathcal{E}(M, \lambda, \mu, Z, z_1, z_2) F((M, \lambda, \mu) \cdot (Z, z_1, z_2)),\end{aligned} \tag{2.1}$$

where the factor $\mathcal{E}(M, \lambda, \mu, Z, z_1, z_2)$ is the exponential map

$$e\big(-m((CZ + D)^{-1}C)[(z_1, z_2)^t + Z\lambda^t + \mu^t] + m(Z[\lambda^t] + 2\lambda(z_1, z_2)^t)\big).$$

Definition 2.1. A *Jacobi form* of degree 2, weight k and index m over Γ_2^J is a holomorphic function $F : \mathcal{H}_2 \times \mathbb{C}^2 \to \mathbb{C}$ such that $F|_{k,m}[M, \lambda, \mu] = F$ for all $(M, \lambda, \mu) \in\ \Gamma^J$.

Such an F is called a *cusp* form if it has a Fourier series representation

$$F(Z, z_1, z_2) = \sum_N \sum_R c(N, R) e(NZ) e(R(z_1, z_2)), \tag{2.2}$$

where N runs over J, R runs over the set of column vectors $\mathbb{Z}^{2,1}$, and $c(N, R) \in \mathbb{C}$ is zero whenever the matrix $T = 4mN - RR^t$ is not positive-definite.

It is easy to see that the invariance of any $F \in J_{2,k,m}^{\text{cusp}}$ with respect to the subgroup $\{I_2\} \times (\mathbb{Z}^2 \times \mathbb{Z}^2)$ of Γ_2^J is equivalent to the set of equations

$$c(N, R) = c\big(N + 2^{-1}(R\lambda + \lambda^t R^t) + m\lambda^t\lambda, R + 2m\lambda^t\big) \tag{2.3}$$

for all $\lambda \in \mathbb{Z}^2$. If we introduce the complex numbers $c_R(T) = c(N, R)$ for $T = 4mN - RR^t$, and put $c_R(T) = 0$ otherwise, equation (2.3) can be written as $c_R(T) = c_{R'}(T)$ whenever $R - R' \in 2m\mathbb{Z}^{2,1}$. More importantly, the numbers $c_R(T)$ allow us to express F as a vector of Fourier series. In order to do so, put them as coefficients in a series $F_R(Z)$ as in (1.2) for every $R \in \mathcal{I} = (\mathbb{Z}^{2,1}/2m\mathbb{Z}^{2,1})$, and consider the theta series

$$\Theta_{m,R}(Z, z_1, z_2) = \sum_{\lambda \in \mathbb{Z}^{2,1}} e\big(m(\lambda + R)^t Z(\lambda + R) + (z_1, z_2)(\lambda + R)\big). \tag{2.4}$$

A formal manipulation of (2.2) using (2.3) yields the identity

$$F(Z, z_1, z_2) = \sum_{R \in \mathcal{I}} F_R(Z)\Theta_{m,R}(Z, z_1, z_2), \tag{2.5}$$

which shows that every Jacobi cusp form $F \in J^{\mathrm{cusp}}_{2,k,m}$ is completely determined by a vector-valued, holomorphic function, which by convenience we also denote by F; say $F : \mathcal{H}_2 \to \mathbb{C}^n$, where

$$F(Z) = (\ldots, F_R(Z), \ldots)_{R \in \mathcal{I}}. \tag{2.6}$$

In order to be more precise about this vector-valued map, we observe that the group $\mathrm{GL}_2(\mathbb{Z})$ is mapped into Γ_2 via $g \mapsto M_g = \left(\begin{smallmatrix} g^t & 0 \\ 0 & g^{-1} \end{smallmatrix}\right)$. Consequently, for any $F \in J^{\mathrm{cusp}}_{2,k,m}$ one has $F|_{k,m}[M_g] = F$, and so

$$(\det g)^k F(Z[g], (z_1, z_2)g) = F(Z, z_1, z_2).$$

This relation is equivalent to the set of identities $c_{gR}(T[g^t]) = (\det g)^k c_R(T)$ for all T and R. In turn, all of them can be expressed in a single equality of two vectors of Fourier series,

$$(\ldots, F_R(Z[g]), \ldots)_R = (\det g)^k (\ldots, F_{gR}(Z), \ldots)_R. \tag{2.7}$$

In particular, these identities yield $F_R(Z) = F_{-R}(Z)$ for all R (take $g = -I_2$). Hence the image of $\mathcal{H}_2$ under the map $Z \mapsto F(Z)$ is contained in the $\mathbb{C}$-vector space

$$V = \{(\ldots, v_R, \ldots)_{R \in \mathcal{I}} \in \mathbb{C}^n \mid v_R = v_{-R}\} \subseteq \mathbb{C}^n. \tag{2.8}$$

For any $R \in \mathcal{I}$ let $\widetilde{u}_R \in V$ be the vector with 1 in the $\pm R$th positions and 0 elsewhere. Then $\{\widetilde{u}_R \mid R \in \mathcal{I}\}$ is a basis of V, the vector $F(Z)$ in (2.6) can be written as $F(Z) = \sum_{R \in \mathcal{I}/\pm 1} F_R(Z)\widetilde{u}_R$, and the identity (2.7) for $g \in \Gamma$ is

$$F(Z[g]) = \sum_{R \in \mathcal{I}/\pm 1} F_R(Z[g])\widetilde{u}_R = \sum_{R \in \mathcal{I}/\pm 1} F_R(Z)\widetilde{u}_{g^{-1}R}. \tag{2.9}$$

Let $\rho : \Gamma \to \mathrm{Aut}_{\mathbb{C}}(V) = \mathrm{GL}(V)$ be the linear representation given by

$$\rho(g)\widetilde{u}_R = \widetilde{u}_{gR} \quad \text{for all } g \in \Gamma. \tag{2.10}$$

Clearly ρ is a representation of $\mathrm{PSL}_2(\mathbb{Z}) = \Gamma/\{\pm I_2\}$ and, as any permutation representation, is unitary with respect to the Hermitian form

$$\langle (\ldots, v_R, \ldots)_R, (\ldots, w_R, \ldots)_R \rangle = \sum_{R \in \mathcal{I}} v_R \overline{w_R}. \tag{2.11}$$

The adjoint of $\rho(g)$ is $\rho^*(g) = \rho(g^{-1}) = \rho(g)^t$, which obviously satisfies the relation $\rho^*(gh) = \rho^*(h)\rho^*(g)$. Another feature of ρ is that its kernel contains the principal congruence subgroup $\Gamma(2m)$, as $gR = R$ for all $R \in \mathcal{I}$ whenever $g \in \Gamma(2m)$. Furthermore, the vector-valued function $F(Z)$ in (2.6) satisfies $F(Z[g]) = \rho^*(g)F(Z)$ for all $g \in \Gamma$, as we see from (2.9). For convenience we notice that

$$\rho(g)(\ldots, v_R, \ldots)_R = (\ldots, v_{g^{-1}R}, \ldots)_R. \tag{2.12}$$

In 1978 G. Shimura [22] proved that $F(Z, z_1, z_2) \mapsto (\ldots, F_R(Z), \ldots)_{R\in\mathcal{I}}$ is an isomorphism from the vector space $J_{2,k,m}^{\mathrm{cusp}}$ onto the space of V-valued Siegel cusp forms on $\mathcal{H}_2$ of weight $k - 1/2$ such that

$$F_R(Z + B) = e\biggl(\frac{-1}{4m}B[R]\biggr)F_R(Z) \tag{2.13}$$

for every symmetric matrix $B \in \mathbb{Z}^{2,2}$, and

$$F_R(-Z^{-1}) = (\det(Z/i))^{-1/2}\frac{(\det Z)^k}{2m}\sum_{R'\in\mathcal{I}} e\biggl(\frac{1}{2m}R^tR'\biggr)F_{R'}(Z) \tag{2.14}$$

for all $R \in \mathcal{I}$. (See also [27, p. 210].)

We close this section with a technical remark. The Fourier coefficients of any F in $J_{2,k,m}^{\mathrm{cusp}}$ have polynomial growth. Indeed, the usual Hecke argument yields $c_R(T) \ll_F (\det T)^{k/2}$ for all $T \in J$ and $R \in \mathbb{Z}^{2,1}$.

3. Grössencharacters and Dirichlet series. In the introduction of this article we associate certain Fourier series to a collection $\{F_R \mid R \in \mathcal{I}\}$ of sequences F_R in $\mathbb{C}$. In this section we attach twisted Dirichlet series to such a collection. To this end, we first recall the notion of a vector-valued Grössencharacter, and then introduce Dirichlet series twisted by them.

Definition 3.1. Let V be a finite-dimensional vector space over $\mathbb{C}$ and $\rho : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ a unitary representation. A *Grössencharacter* (with respect to ρ) is any C^∞-function $\mathcal{U} : \mathcal{H} \to V$ such that

(1) $\mathcal{U}(g\tau) = \rho(g)\mathcal{U}(\tau)$ for any $g \in \Gamma$,
(2) $\mathcal{U}$ is an eigenfunction of the hyperbolic Laplacian

$$\Delta = -y^2\biggl(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\biggr), \qquad \text{where } \tau = x + iy \in \mathcal{H},$$

(3) $\mathcal{U} = (\ldots, \mathcal{U}_R, \ldots)_{R\in I}$ has *moderate growth at infinity*, that is, there exists $\alpha > 0$ such that $\mathcal{U}_R(\tau) = O(y^\alpha)$ for any $R \in \mathcal{I}$ as $y \to \infty$.

From this point on, the symbol ρ denotes the unitary representation (2.10), and every Grössencharacter is associated to such a representation.

Just as $\mathcal{H}_2$ is a generalization of the complex upper half-plane $\mathcal{H}$, the cone $\mathcal{P}$ of real symmetric matrices is a generalization of the imaginary axis in $\mathcal{H}$.

Notice that the index set J in (1.2) is a subset of $\mathcal{P}$, and $\mathcal{P} = \mathbb{R}_{>0} \times \mathcal{SP}$ where $\mathcal{SP} = \{W \in \mathcal{P} \mid \det W = 1\}$. On both $\mathcal{P}$ and $\mathcal{SP}$, the group $\mathrm{GL}_2(\mathbb{Z})$ acts via $Y \mapsto Y[g]$, and this action leaves J invariant. Furthermore, the maps

$$\tau = x + iy \mapsto W_\tau = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right], \tag{3.1}$$

$$W = \begin{pmatrix} w_1 & w \\ w & w_2 \end{pmatrix} \mapsto \tau_W = \frac{-w}{w_1} + i \frac{1}{w_1}$$

identify $\mathcal{H}$ with $\mathcal{SP}$ as Γ-sets, and they are inverse of each other with $W_{g\tau} = W_\tau[g^{-1}]$ for all $\tau \in \mathcal{H}$ and $g \in \Gamma$. The identification in (3.1) allows us to consider a map $\widetilde{\mathcal{U}}$ on $\mathcal{SP}$ as a function $\mathcal{U}$ on $\mathcal{H}$ via $\mathcal{U}(\tau) = \widetilde{\mathcal{U}}(W_\tau)$. Similarly, a map $\mathcal{U}$ on $\mathcal{H}$ defines a function $\widetilde{\mathcal{U}}$ on $\mathcal{P}$ via $\widetilde{\mathcal{U}}(Y) = \mathcal{U}(\tau_{(\det Y)^{-1/2}Y})$. In many cases we use the same symbol for both functions. Notice also that

$$\widetilde{\mathcal{U}}(W[g]) = \rho^*(g)\widetilde{\mathcal{U}}(W) \quad \text{and} \quad \mathcal{U}(g\tau) = \rho(g)\mathcal{U}(\tau) \quad \text{for } g \in \Gamma, \tag{3.2}$$

are equivalent conditions whenever $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ are related as above.

Definition 3.2. Let $\{F_R \mid R \in \mathcal{I}\}$ be a collection of n sequences $F_R = \{c_R(T) \mid T \in J,\ T + RR^t \in 4mJ\}$ of complex numbers which satisfy (1.3a) and (1.3b). For any Grössencharacter $\mathcal{U}$ and all $s \in \mathbb{C}$ with $\Re(s) \gg 0$, set

$$L(F, \mathcal{U}, s) = \sum_{T \in J/\Gamma} \frac{1}{\xi_1(T)} \langle (\ldots, c_R(T), \ldots)_R, \mathcal{U}(T^{-1}) \rangle (\det T)^{-s}, \tag{3.3}$$

where $\xi_1(T) = \#\{g \in \Gamma \mid T[g] = T\}$ and $\langle\ ,\ \rangle$ denotes the Hermitian product (2.11).

Notice that the index T in (3.3) runs through the Γ-orbits in J. This is meaningful as one sees from (1.3b), (2.12), (3.2) that $(\ldots, c_R(T[g]), \ldots) = \rho(g^t)(\ldots, c_R(T), \ldots)$ and

$$(\ldots, \mathcal{U}_R((T[g])^{-1}), \ldots) = \rho^*((g^t)^{-1})(\ldots, \mathcal{U}_R(T^{-1}), \ldots) \quad \text{for all } g \in \Gamma.$$

Lemma 3.3. *The Dirichlet series $L(F, \mathcal{U}, s)$ is absolutely convergent in the half-plane $\Re(s) > (k + \alpha + 3)/2$, where $\alpha > 0$ is the real parameter in Definition* 3.1.

Proof. Write $T \in J$ as in (1.7). Then $\sqrt{\det T}\, T^{-1} \in \mathcal{SP}$, and we can identify this matrix with the number $(t + i\sqrt{4 \det T})/(2t_2)$ in $\mathcal{H}$ by (3.1). Therefore,

$$\mathcal{U}_R(\sqrt{\det T}\, T^{-1}) = O((\sqrt{\det T}/t_2)^\alpha) = O((\det T)^{\alpha/2}) \quad \text{for any } R \in \mathcal{I}.$$

This fact and (1.3a) yield

$$\langle (\ldots, c_R(T), \ldots)_R, \mathcal{U}(\sqrt{\det T}\; T^{-1})\rangle = O((\det T)^{(k+\alpha)/2}).$$

On the other hand, we have $\mathrm{GL}_2(\mathbb{Z}) = \Gamma \cup \Gamma\epsilon$ where $\epsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

If we denote by O_T (resp. U_T) the orbit of $T \in J$ under Γ (resp. $\mathrm{GL}_2(\mathbb{Z})$), we clearly have $U_T = O_T \cup O_{T[\epsilon]}$ and the Γ-orbits on the right hand side are either equal or disjoint.

Let $\xi(T) = \#\{g \in \mathrm{GL}_2(\mathbb{Z}) \mid T[g] = T\}$. It is not difficult to see that $O_T = O_{T[\epsilon]}$ if and only if $\xi(T) = 2\xi_1(T)$, and $O_T \cap O_{T[\epsilon]} = \emptyset$ if and only if $\xi(T) = \xi_1(T)$. Consequently,

$$\sum_{T \in J/\Gamma} \frac{1}{\xi_1(T)} (\det T)^{-s} = 2 \sum_{T \in J/\mathrm{GL}_2(\mathbb{Z})} \frac{1}{\xi(T)} (\det T)^{-s}. \tag{3.4}$$

Since the series on the right hand side of (3.4) is absolutely convergent for $\Re(s) > 3/2$ (see for example the argument in [13, p. 149]), the lemma follows. ∎

Definition 3.4. Let $\{F_R \mid R \in \mathcal{I}\}$ and $\mathcal{U}$ be as in Definition 3.2. The *completed Dirichlet series* associated to them is

$$\Lambda(F, \mathcal{U}, s) = \sqrt{\pi} \left(\frac{\pi}{2m}\right)^{-2s} \Gamma(s-a)\Gamma(s-b) L(F, \mathcal{U}, s), \tag{3.5}$$

where $\{a, b\} = \{(1 \pm \sqrt{1+4\lambda})/4\}$, with $\Delta\overline{\mathcal{U}} = \lambda\overline{\mathcal{U}}$.

By Lemma 3.3 we know that $\Lambda(F, \mathcal{U}, s)$ is a meromorphic function of s on $\Re(s) > (k+\alpha+3)/2$.

Remark. Let $\{F_R \mid R \in \mathcal{I}\}$ and $\mathcal{U}$ be as above.

If $\beta > \max\{(k+\alpha+3)/2, (k+5)/2\}$, then $\Lambda(F, \mathcal{U}, s) = O(|\Im(s)|^{-\mu})$ for all $\mu > 0$ as $|\Im(s)| \to \infty$ on the vertical line $\Re(s) = \beta$.

Indeed, from Stirling's estimate one has

$$|\Gamma(s-a)\Gamma(s-b)| \ll_\beta |s-a|^{\Re(s-a)-1/2} |s-b|^{\Re(s-b)-1/2} e^{-|\Im(s-a)|\pi/2} e^{-|\Im(s-b)|\pi/2}$$

as $|\Im(s)| \to \infty$ on any vertical line $\Re(s) = \beta > 0$. On the other hand, $L(F, \mathcal{U}, s)$ is also absolutely convergent on the line $\Re(s) = \beta > (k+\alpha+3)/2$, with

$$\sum_{T \in J/\Gamma} \frac{1}{\xi_1(T)} |\langle (\ldots, c_R(T), \ldots)_R, \mathcal{U}(T^{-1})\rangle (\det T)^{-s}| \le C \sum_{T \in J/\Gamma} (\det T)^{-\beta+(k+\alpha)/2} \tag{3.6}$$

for some $C > 0$, as shown in the proof of Lemma 3.3. Since the bound is independent of $\Im(s)$, we have

$$|\Lambda(F,\mathcal{U},s)| \ll_\beta \Im(s-a)^{\Re(s-a)-1/2}\Im(s-b)^{\Re(s-b)-1/2}e^{-\pi(|\Im(s-a)|+|\Im(s-b)|)/2}$$

on the line $\Re(s) = \beta$ as $|\Im(s)| \to \infty$, with a and b depending only on $\mathcal{U}$.

4. A family of Mellin transforms attached to the Fourier series $F_R(Z)$. Our proof of Theorem 1.1 rests on the decomposition $\mathcal{P} = \mathbb{R}_{>0} \times \mathcal{SP}$, the identification of $\mathcal{SP}$ with $\mathcal{H}$, and a certain collection of Mellin transforms of the Fourier series $\{F_R(iY) \mid R \in \mathcal{I}\}$ which constitutes a bridge between the Fourier series (1.2) and the Dirichlet series (3.5). In this section we introduce those Mellin transforms, study some of their properties, and give an integral representation of (3.5) in terms of them.

We point out here that our method is very classical, and amounts to setting the problem in a way that the well-known Fourier analysis on $\Gamma\backslash\mathcal{H}$ can be applied. This approach was used by Maass [16] in his work on Siegel modular forms and Dirichlet series, by Terras [23] in her work on integral formulas for series of positive-definite matrices, and by many others. (See for example [2], [8], and the references indicated in the introduction of [23]. See Moeglin–Waldspurger [21] for a very general discussion of this.)

Definition 4.1. Let $\{F_R \mid R \in \mathcal{I}\}$ be a collection of sequences as in Definition 3.2 and $F_R(Z)$ the corresponding Fourier series (1.2). For every $W \in \mathcal{SP}$ define the function $F_R(W;\cdot) : \mathbb{R}_{>0} \to \mathbb{C}$ by

$$F_R(W;t) = F_R(it^{1/2}W), \tag{4.1}$$

and let

$$G_R(W;s) = \int_0^\infty F_R(W;t)t^{s-1}dt \qquad (\text{for } \Re(s) \gg 0) \tag{4.2}$$

be its Mellin transform. For convenience, also put

$$\begin{aligned} F(W;t) &= (\ldots, F_R(W;t), \ldots)_{R\in\mathcal{I}}, \\ G(W;s) &= (\ldots, G_R(W;s), \ldots)_{R\in\mathcal{I}}. \end{aligned} \tag{4.3}$$

In Lemma 4.3 we establish properties of $G_R(W;s)$ (in particular we find an explicit region where the integral converges and the function is holomorphic), but first we present a bound for the absolute value of its integrand.

Lemma 4.2. *Let $\{F_R \mid R \in \mathcal{I}\}$ and $F_R(Z)$ be as in Definition 4.1. Then there exist positive real numbers B_1, B_2 and B such that*

$$|F_R(iY)| \le \big(B_1(\det Y)^{-(k+5)/2} + B_2(\det Y)^{-(k+3)/2}\big)e^{-B\sqrt{\det Y}}$$

for all $Y \in \mathcal{P}$ and $R \in \mathcal{I}$.

Proof. The equalities in (1.3b) imply $F_R(iY[g]) = (\det g)^k F_{gR}(iY)$ for all $g \in \mathrm{GL}_2(\mathbb{Z})$. Hence, it suffices to prove the lemma for Y in a $\mathrm{GL}_2(\mathbb{Z})$-fundamental domain of $\mathcal{P}$. Here and for the rest of the article we use Minkowski's fundamental domain $\mathcal{R}$ (see for example [12, pp. 12, 13, 20]). There exists $\delta > 0$ such that the matrix $Y - \delta Y^D$ is positive-definite for all $Y = \left(\begin{smallmatrix} y_1 & y \\ y & y_2 \end{smallmatrix}\right) \in \mathcal{R}$, where $Y^D = \left(\begin{smallmatrix} y_1 & 0 \\ 0 & y_2 \end{smallmatrix}\right)$. For all real, symmetric $T > 0$ we conclude that

$$\text{(4.4)} \qquad \begin{aligned} \operatorname{trace}(TY) > \delta \operatorname{trace}(TY^D) &= \delta(t_1 y_1 + t_2 y_2) \\ &\geq 2\delta\sqrt{t_1 t_2}\sqrt{y_1 y_2} \geq 2\delta\sqrt{t_1 t_2}\sqrt{\det Y}. \end{aligned}$$

Also, the estimate $c_R(T) = O(\det T^{k/2})$ implies $c_R(T) = O((t_1 t_2)^{k/2})$. Thus

$$\begin{aligned} |F_R(iY)| &\leq \sum_{T \in J} |c_R(T)| \exp\left(\frac{-\pi}{m}\delta\sqrt{t_1 t_2}\sqrt{\det Y}\right) \leq C \sum_{T \in J} (t_1 t_2)^{k/2} \\ &\quad \times \exp\left(\frac{-\pi}{m}\delta\sqrt{t_1 t_2}\sqrt{\det Y}\right) \leq C' \sum_{l=1}^{\infty} l^{(k+3)/2} \exp\left(\frac{-\pi}{m}\delta\sqrt{l}\sqrt{\det Y}\right) \end{aligned}$$

for some real constants C, C', as $\#\{T \in J \mid t_1 t_2 = l\}$ is bounded by $5l^{3/2}$. Then

$$\text{(4.5)} \qquad \begin{aligned} &|F_R(iY)| \\ &\quad \leq C' \exp\left(\frac{-\pi\delta}{2m}\sqrt{\det Y}\right) \sum_{l=1}^{\infty} l^{(k+3)/2} \exp\left(\frac{-\pi\delta\sqrt{l}}{2m}\sqrt{\det Y}\right). \end{aligned}$$

Consider the function $h(x) = x^{m_0} e^{-\beta\sqrt{x}}$ of the variable $x > 0$ for some $m_0, \beta > 0$. Then h is positive, has a maximum at $x = (2m_0/\beta)^2$, and goes to zero as $x \to \infty$. Therefore

$$\begin{aligned} \sum_{l=1}^{\infty} h(l) &\leq \int_0^{\infty} h(x)\,dx + 2h((2m_0/\beta)^2) \\ &= 2\beta^{-2m_0}\left(\beta^{-2}\Gamma(2m_0 + 2) + (2m_0)^{2m_0} e^{-2m_0}\right). \end{aligned}$$

Using this expression in (4.5) with $m_0 = (k+3)/2$ and $\beta = \pi\delta\sqrt{\det Y}/(2m)$, one obtains

$$\begin{aligned} |F_R(iY)| &\leq 2C' \exp\left(\frac{-\pi}{2m}\delta\sqrt{\det Y}\right) \\ &\quad \times \left(\frac{2m}{\pi\delta\sqrt{\det Y}}\right)^{k+3} \left\{\left(\frac{2m}{\pi\delta\sqrt{\det Y}}\right)^2 \Gamma(k+5) + (k+3)^{k+3} e^{-k-3}\right\}. \end{aligned}$$

This is exactly the inequality in the lemma, with closed formulas for B, B_1 and B_2. ∎

LEMMA 4.3. *The map* $s \mapsto G_R(W;s)$ *is holomorphic on the half-plane* $\Re(s) > (k+5)/2$. *Moreover*

$$G_R(W;s) = O(|\Im(s)|^{-\mu})$$

for all $\mu > 0$ *as* $|\Im(s)| \to \infty$ *on the vertical line* $\Re(s) = \beta$ *whenever* $\beta > (k+5)/2$. *This estimate holds uniformly in* W. *One also has*

$$F_R(W;t) = \frac{1}{2\pi i} \int\limits_{\Re(s)=\beta} G_R(W;s) t^{-s}\, ds$$

for any $t > 0$, $W \in \mathcal{SP}$, *and* β *as above.*

Proof. By Lemma 4.2 and a comparison with the standard integral representation of the gamma function, the integral (4.2) is absolutely convergent for $\Re(s) > (k+5)/2$, uniformly in W.

The holomorphicity of (4.2) follows from standard arguments.

In order to get the estimate for $G_R(W;s)$, we first prove

$$\sum_{T\in J} \int\limits_0^\infty \left| c_R(T) e\left(\frac{i}{4m} T t^{1/2} W\right) t^{s-1}\right| dt < \infty \tag{4.6}$$

for any $R \in \mathcal{I}$. By the identities in (1.3b), for any $g \in \mathrm{GL}_2(\mathbb{Z})$ we have

$$\begin{aligned}
&\sum_{T\in J} \int\limits_0^\infty \left| c_R(T) e\left(\frac{i}{4m} T t^{1/2} W[g]\right) t^{s-1}\right| dt \\
&\qquad = \sum_{T\in J} \int\limits_0^\infty \left| (\det g)^k c_{gR}(T[g^t]) e\left(\frac{i}{4m} T[g^t] t^{1/2} W\right) t^{s-1}\right| dt \\
&\qquad = \sum_{T\in J} |c_{gR}(T)| \int\limits_0^\infty \exp\left(\frac{-\pi}{2m} T t^{1/2} W\right) t^{\Re(s)-1}\, dt.
\end{aligned}$$

Note that for the second equality we have used the change of variable $T[g^t] \to T$. Hence, it suffices to prove (4.6) with W in Minkowski's fundamental domain $\mathcal{R}$. In particular, we can use the existence of a $\delta > 0$ such that

$$\operatorname{trace} TW \geq 2\delta\sqrt{t_1 t_2}\sqrt{\det W} = 2\delta\sqrt{t_1 t_2} > 0$$

for all $T \in J$ written as in (1.7) (see (4.4)). This fact, the estimate in (1.3a), and the computation of the last integral yield

$$\sum_{T\in J}\int_0^\infty \left|c_R(T)e\left(\frac{i}{4m}Tt^{1/2}W\right)t^{s-1}\right|dt$$
$$\le C\left(\frac{\pi}{2m}\right)^{-2\Re(s)}\Gamma(2\Re(s))\sum_{T\in J}(\det T)^{k/2}(\operatorname{trace} TW)^{-2\Re(s)}$$
$$\le C\left(\frac{\delta\pi}{m}\right)^{-2\Re(s)}\Gamma(2\Re(s))\sum_{\substack{0<t_1,t_2\\ t^2<t_1t_2}}(t_1t_2)^{-\Re(s)+k/2}$$

whenever $\Re(s) > (k+3)/2$. If we take the (rough) upper bound $3\sqrt{t_1t_2}$ for the number of integers in the interval $[-\sqrt{t_1t_2}, \sqrt{t_1t_2}]$ we obtain

$$\sum_{T\in J}\int_0^\infty \left|c_R(T)e\left(\frac{i}{4m}Tt^{1/2}W\right)t^{s-1}\right|dt \le 3C\left(\frac{\delta\pi}{m}\right)^{-2\Re(s)}\Gamma(2\Re(s))\zeta\left(\Re(s)-\frac{k+1}{2}\right), \tag{4.7}$$

which shows that the series (4.6) is finite if $\Re(s) = \beta > (k+5)/2$. One particular consequence is that we can use the Fubini–Tonelli theorem to get

$$\begin{aligned} G_R(W;s) &= \int_0^\infty \sum_{T\in J} c_R(T)e\left(\frac{i}{4m}Tt^{1/2}W\right)t^{s-1}\,dt \\ &= 2\sum_{T\in J} c_R(T)\int_0^\infty e\left(\frac{i}{4m}TWu\right)u^{2s-1}\,du \\ &= 2\left(\frac{\pi}{2m}\right)^{-2s}\Gamma(2s)\sum_{T\in J} c_R(T)(\operatorname{trace} TW)^{-2s} \end{aligned} \tag{4.8}$$

for $\Re(s) = \beta$. This expression and the argument above for the finiteness of (4.6) yield

$$|G_R(W;s)| \le 3C\left(\frac{\delta\pi}{m}\right)^{-2\Re(s)}|\Gamma(2s)|\zeta\left(\Re(s)-\frac{k+1}{2}\right).$$

This inequality and Stirling's estimate imply that $G_R(W;s) = O(|\Im(s)|^{-\mu})$ for all $\mu > 0$ as $|\Im(s)| \to \infty$ on the vertical line $\Re(s) = \beta$, uniformly in W.

The last claim of the lemma is that $F_R(W;t)$ is the inverse Mellin transform of $G_R(W;s)$, which can be proved as before using the Fubini–Tonelli theorem. ∎

Remark. Notice that for any set $\{F_R \mid R \in \mathcal{I}\}$ of sequences as in Definition 4.1 the integral

$$\widetilde{G_R}(W;s) = \int_1^\infty F_R(W;t)t^{s-1}\,dt \tag{4.9}$$

defines an entire function of s for all $R \in \mathcal{I}$ and $W \in \mathcal{SP}$. This is bounded on any vertical strip of $\mathbb{C}$, uniformly in W. The proof of these facts is completely analogous to the given for Lemma 4.3.

Remark. From (1.3b) we easily deduce that

$$\begin{aligned} F_R(W[g];t) &= \sum_{T\in J} c_R(T)e\left(\frac{1}{4m}Tit^{1/2}W[g]\right) \\ &= \sum_{T\in J} c_R(T)e\left(\frac{1}{4m}T[g^t]it^{1/2}W\right) \\ &= \sum_{T\in J} c_R(T[(g^t)^{-1}])e\left(\frac{1}{4m}Tit^{1/2}W\right) \\ &= \sum_{T\in J} c_{gR}(T)e\left(\frac{1}{4m}Tit^{1/2}W\right) = F_{gR}(W;t) \end{aligned} \tag{4.10}$$

for all $g \in \mathit{\Gamma}$. This is equivalent to $F(W[g];t) = \rho^*(g)F(W;t)$ for all $g \in \mathit{\Gamma}$, and is just another way of writing (2.9).

Remark. The identification (3.1) between the $\mathit{\Gamma}$-sets $\mathcal{SP}$ and $\mathcal{H}$ allows us to index the functions (4.1)–(4.3) in terms of the variable $\tau \in \mathcal{H}$. That is why we also denote them as $F_R(\tau;t)$, $G_R(\tau;s)$, $F(\tau;t)$ and $G(\tau;s)$ resp. In particular, with this notation (4.10) reads $F(g\tau;t) = \rho(g)F(\tau;t)$ for all $g \in \mathit{\Gamma}$, as established in (3.2).

We emphasize that the purpose of introducing $G(\tau;s)$ is to get a manageable link between the vector of Fourier series $F(iY)$ in (2.6) and the Dirichlet series $\mathit{\Lambda}(F,\mathcal{U},s)$. The connection is made explicit in the next proposition, but first some preliminaries are necessary: The sets $\mathcal{P}$, $\mathcal{SP}$ and $\mathcal{H}$ have the respective $\mathit{\Gamma}$-invariant volume elements

$$d\mu(Y) = (\det Y)^{-3/2}dydy_1dy_2, \quad d\mu(W) = w_1^{-1}dwdw_1, \quad d\mu(\tau) = y^{-2}dxdy,$$

where the corresponding real coordinates are the ones described in (1.7), and dy_1, dy_2, dy, dw_1, dw, dx are Lebesgue measure on $\mathbb{R}$. These measures are related as follows:

$$d\mu(Y) = \frac{dt}{t}d\mu(W), \qquad d\mu(W) = d\mu(\tau), \tag{4.11}$$

where $Y = t^{1/2}W,\ t = \det Y,\ W = W_\tau$.

By (4.10) and (3.2), we have $G_R(W[g];s) = G_{gR}(W;s)$ for all $R \in \mathcal{I}$, and $G(W[g];s) = \rho(g^{-1})G(W;s) = \rho^*(g)G(W;s)$ for $W \in \mathcal{SP}$, $g \in \Gamma$, and $s \in \mathbb{C}$ with $\Re(s) \gg 0$. These relations are equivalent to $G(g\tau;s) = \rho(g)G(\tau;s)$ for all $g \in \Gamma$, as observed in (3.2). This comment, and the fact that ρ is unitary, imply that the complex value $\langle G(\tau;s), \mathcal{U}(\tau)\rangle$ is constant on any Γ-orbit of $\mathcal{H}$ and

$$\int_{\Gamma\backslash\mathcal{H}} \langle G(\tau;s), \mathcal{U}(\tau)\rangle \, d\mu(\tau)$$

is well-defined. In order to prove the convergence of this integral we write $c(T) = (\ldots, c_R(T), \ldots)_{R\in\mathcal{I}} \in \mathbb{C}^n$ for every $T \in J$, and use the Cauchy–Schwarz inequality to get

$$\begin{aligned}
|\langle c(T), \mathcal{U}((\det Y)^{-1/2}Y)\rangle| &\le \|c(T)\| \, \|\mathcal{U}((\det Y)^{-1/2}Y)\| \\
&= \Big(\sum_{R\in\mathcal{I}} |c_R(T)|^2 \sum_{R'\in\mathcal{I}} |\mathcal{U}_{R'}((\det Y)^{-1/2}Y)|^2\Big)^{1/2} \\
&\le C(\det T)^{k/2}\left(\frac{\sqrt{\det Y}}{y_1}\right)^{\alpha}
\end{aligned}$$

for some real constant C, where we have used the estimate (1.3a) and the estimate for the Grössencharacters. Since the diagonal entries of a matrix $Y \in \mathcal{P}$ (written as in (1.7)) are positive, we have $y_1^{-1} < y_1^{-1} + y_2^{-1} = (y_1+y_2)/(y_1y_2)$. These inequalities yield

$$|\langle c(T), \mathcal{U}((\det Y)^{-1/2}Y)\rangle| \le C(\det T)^{k/2}\left(\frac{\operatorname{trace} Y}{\det Y}\right)^{\alpha}(\det Y)^{\alpha/2}.$$

From this expression, the $\mathrm{GL}_2(\mathbb{Z})$-invariance of J, and $[\mathrm{GL}_2(\mathbb{Z}) : \Gamma] = 2$, one gets the upper bound

$$\begin{aligned}
&\int_{\mathcal{P}/\Gamma} \sum_{T\in J} |\langle c(T), \mathcal{U}((\det Y)^{-1/2}Y)\rangle| \exp\left(\frac{-\pi}{2m}TY\right)(\det Y)^{\Re(s)}\, d\mu(Y) \\
&\qquad \le 2C \int_{\mathcal{P}/\mathrm{GL}_2(\mathbb{Z})} \sum_{T\in J} (\det T)^{k/2} \exp\left(\frac{-\pi}{2m}TY\right) \\
&\qquad\qquad \times (\operatorname{trace} Y)^{\alpha}(\det Y)^{\Re(s)-\alpha/2}\, d\mu(Y).
\end{aligned}$$

The last integral is finite, as can be shown using the computations in [17, pp. 210–213]. For the reader's convenience we sketch the proof.

We pick Minkowski's fundamental domain $\mathcal{R}$ as the domain of integration, and recall the existence of $\delta > 0$ such that $Y - \delta Y^D$ is positive-definite for all $Y \in \mathcal{R}$ (see the proof of Lemma 4.2). Then (4.4) holds, so $\delta \cdot \operatorname{trace} Y < \operatorname{trace} TY$, $2\delta\sqrt{\det TY} < \operatorname{trace} TY$, and therefore

$$\int_{\mathcal{P}/\mathrm{GL}_2(\mathbb{Z})} \sum_{T\in J} (\det T)^{k/2} \exp\left(\frac{-\pi}{2m} TY\right) (\operatorname{trace} Y)^{\alpha} (\det Y)^{\Re(s)-\alpha/2}\, d\mu(Y)$$
$$\leq 2^{-k}\delta^{-k-\alpha} \int_{\mathcal{R}} \sum_{T\in J} (\operatorname{trace} TY)^{k+\alpha} \exp\left(\frac{-\pi}{2m} TY\right) \times (\det Y)^{\Re(s)-(k+\alpha)/2} d\mu(Y).$$

For $T \in J$ and $Y \in \mathcal{R}$ one has $\operatorname{trace} TY > 0$. Hence, there is C' in $\mathbb{R}$ such that

$$(\operatorname{trace} TY)^{k+\alpha} \exp\left(\frac{-\pi}{4m} TY\right) = (t_1 y_1 + ty + t_2 y_2)^{k+\alpha} \exp\left(\frac{-\pi(t_1 y_1 + ty + t_2 y_2)}{4m}\right) \leq C'$$

for all T and Y. Consequently,

$$\int_{\mathcal{R}} \sum_{T\in J} (\operatorname{trace} TY)^{k+\alpha} \exp\left(\frac{-\pi}{2m} TY\right) (\det Y)^{\Re(s)-(k+\alpha)/2}\, d\mu(Y)$$
$$\leq C' \int_{\mathcal{R}} \sum_{T\in J} \exp\left(\frac{-\pi}{4m} TY\right) (\det Y)^{\Re(s)-(k+\alpha)/2}\, d\mu(Y).$$

On the other hand, there is a real constant C'' such that

$$\sum_{T\in J} \exp\left(\frac{-\pi}{4m} TY\right) \leq C'' (\det Y)^{-3/2} \exp\left(\frac{-\pi}{2^4 m} \delta \sqrt{\det Y}\right),$$

as shown for example in [17, p. 213]. Thus

$$\int_{\mathcal{R}} \sum_{T\in J} \exp\left(\frac{-\pi}{4m} TY\right) (\det Y)^{\Re(s)-(k+\alpha)/2}\, d\mu(Y)$$
$$\leq C'' \int_{\mathcal{P}/\Gamma} \exp\left(\frac{-\pi}{2^4 m} \delta \sqrt{\det Y}\right) (\det Y)^{\Re(s)-(k+\alpha+3)/2}\, d\mu(Y).$$

Here we have also used the facts that $\mathcal{R}$ is a $\mathrm{GL}_2(\mathbb{Z})$-fundamental domain in $\mathcal{P}$, and that the integrand is a positive function. Finally, we put $t = \det Y$ in the last integral and get

$$\int_{\mathcal{P}/\Gamma} \exp\left(\frac{-\pi}{2^4 m} \delta \sqrt{\det Y}\right) (\det Y)^{\Re(s)-(k+\alpha+3)/2}\, d\mu(Y)$$

$$\leq \int_{\mathcal{SP}/\Gamma} \int_0^\infty \exp\left(\frac{-\pi}{2^4 m}\delta\sqrt{t}\right) t^{\Re(s)-(k+\alpha+3)/2}\,\frac{dt}{t}\,d\mu(W)$$

$$= \int_{\Gamma\backslash\mathcal{H}} d\mu(\tau)\cdot\int_0^\infty \exp\left(\frac{-\pi}{2^4 m}\delta\sqrt{t}\right) t^{\Re(s)-(k+\alpha+3)/2}\,\frac{dt}{t}.$$

The first factor in this product is the finite volume of the surface $\Gamma\backslash\mathcal{H}$. The second one is a finite integral whenever $\Re(s) > (k+\alpha+3)/2$. This shows that

$$\int_{\mathcal{P}/\Gamma}\sum_{T\in J} |\langle c(T), \mathcal{U}((\det Y)^{-1/2}Y)\rangle| \exp\left(\frac{-\pi}{2m}TY\right)(\det Y)^{\Re(s)}\,d\mu(Y) \tag{4.12}$$

is a finite integral for all $s\in\mathbb{C}$ with $\Re(s) > (k+\alpha+3)/2$, as claimed above.

Recall that every component function $\mathcal{U}_R : \mathcal{H} \to \mathbb{C}$ of a Grössencharacter $\mathcal{U}$ can be seen as a function of $W \in \mathcal{SP}$ by the identification (3.1). As such, it satisfies the conditions listed in [17, pp. 84–85]. Namely $\mathcal{U}_R$ is a C^∞-map, an eigenfunction of the hyperbolic Laplacian, and

$$\left(w_1\frac{\partial}{\partial w_1} + w\frac{\partial}{\partial w} + w_2\frac{\partial}{\partial w_2}\right)\mathcal{U}_R(W) = 0 \quad \text{and}$$
$$\left|\frac{\partial^l \mathcal{U}_R}{\partial w_1^\alpha \partial w^\beta \partial w_2^\gamma}\right| \leq C_l(\operatorname{trace} W)^{\kappa_l}$$

for all positive integers α,β,γ, where the constants C_l and κ_l depend on $l=\alpha+\beta+\gamma$. These properties allow us to use the computations in [17, p. 85] to conclude that

$$\int_{\mathcal{P}} e\left(\frac{i}{2\pi}YX^{-1}\right)\overline{\mathcal{U}_R((\det Y)^{-1/2}Y)}(\det Y)^s d\mu(Y) = \pi^{1/2}\Gamma(s-a)\Gamma(s-b)(\det X)^s\,\overline{\mathcal{U}_R((\det X)^{-1/2}X)} \tag{4.13}$$

for all $R\in\mathcal{I}$, where a and b are the complex numbers described in Definition 3.4. This relation plays a crucial role in the next proof.

Proposition 4.4. *Let $\{F_R \mid R\in\mathcal{I}\}$ be a collection of sequences as in Definition* 3.2 *and $\mathcal{U}$ a Grössencharacter associated to the representation ρ. Then*

$$\Lambda(F,\mathcal{U},s) = \int_{\Gamma\backslash\mathcal{H}} \langle G(\tau;s), \mathcal{U}(\tau)\rangle\,d\mu(\tau)$$

for all $s\in\mathbb{C}$ with $\Re(s) > (k+\alpha+5)/2$.

Proof. The identification (3.1) between $\mathcal{H}$ and $\mathcal{SP}$ (see also (4.11)) yields

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}} &\langle G(\tau;s),\mathcal{U}(\tau)\rangle\, d\mu(\tau)\\ &= \int_{\mathcal{SP}/\Gamma}\int_0^\infty \langle(\ldots,F_R(it^{1/2}W),\ldots),\mathcal{U}(W)\rangle t^s\,\frac{dt}{t}\,d\mu(W)\\ &= \int_{\mathcal{P}/\Gamma}\langle F(iY),\mathcal{U}((\det Y)^{-1/2}Y)\rangle(\det Y)^s\,d\mu(Y).\end{aligned}$$

In the last expression we now insert the Fourier series (1.2) of the component functions in (2.6) and the notation $c(T)=(\ldots,c_R(T),\ldots)_{R\in\mathcal{I}}$ to obtain

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}}\langle G(\tau;s),\mathcal{U}(\tau)\rangle\,d\mu(\tau) &= \int_{\mathcal{P}/\Gamma}\sum_{T\in J/\Gamma}\frac{1}{\xi_1(T)}\\ \times\Bigg(\sum_{g\in\Gamma}\langle c(T[g]),\mathcal{U}((\det Y)^{-1/2}Y)\rangle e&\bigg(\frac{i}{4m}T[g]Y\bigg)\Bigg)(\det Y)^s\,d\mu(Y).\end{aligned}\tag{4.14}$$

Since the integral (4.12) is finite, we can use the Fubini–Tonelli theorem and interchange in (4.14) the leftmost sum and integral symbols. Then

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}}\langle G(\tau;s),\mathcal{U}(\tau)\rangle\,d\mu(\tau) &= \sum_{T\in J/\Gamma}\frac{1}{\xi_1(T)}\\ \times\int_{\mathcal{P}/\Gamma}\sum_{g\in\Gamma}\langle c(T[g]),\mathcal{U}((\det Y)^{-1/2}Y)\rangle e&\bigg(\frac{i}{4m}T[g]Y\bigg)(\det Y)^s\,d\mu(Y).\end{aligned}$$

By (1.3b), (2.12) we have $c(T[g])=(\ldots,c_{(g^t)^{-1}R}(T),\ldots)=\rho(g^t)c(T)$. Hence $\langle c(T[g]),\mathcal{U}((\det Y)^{-1/2}Y)\rangle = \langle c(T),\rho^*(g^t)\mathcal{U}((\det Y)^{-1/2}Y)\rangle$, and by (3.2),

$$\begin{aligned}\int_{\Gamma\backslash\mathcal{H}}&\langle G(\tau;s),\mathcal{U}(\tau)\rangle d\mu(\tau)\\ &= \sum_{T\in J/\Gamma}\frac{1}{\xi_1(T)}\int_{\mathcal{P}/\Gamma}\sum_{g\in\Gamma}\langle c(T),\mathcal{U}((\det Y)^{-1/2}Y[g^t])\rangle e\bigg(\frac{i}{4m}TY[g^t]\bigg)\\ &\qquad\times(\det Y)^s\,d\mu(Y)\\ &= \sum_{T\in J/\Gamma}\frac{1}{\xi_1(T)}\int_{\mathcal{P}}\langle c(T),\mathcal{U}((\det Y)^{-1/2}Y)\rangle e\bigg(\frac{i}{4m}TY\bigg)(\det Y)^s\,d\mu(Y).\end{aligned}$$

Unfolding the Euclidean inner product in the last expression we get

$$\int_{\Gamma\backslash\mathcal{H}} \langle G(\tau;s),\mathcal{U}(\tau)\rangle d\mu(\tau)$$

$$= \sum_{T\in J/\Gamma} \frac{1}{\xi_1(T)} \sum_{R\in\mathcal{I}} c_R(T) \int_{\mathcal{P}} \overline{\mathcal{U}((\det Y)^{-1/2}Y)}\, e\Big(\frac{i}{4m}TY\Big)(\det Y)^s\, d\mu(Y).$$

Now we use (4.13) with $X = (2m/\pi)T^{-1}$ to obtain

$$\int_{\Gamma\backslash\mathcal{H}} \langle G(\tau;s),\mathcal{U}(\tau)\rangle\, d\mu(\tau) = \pi^{1/2}\Big(\frac{2m}{\pi}\Big)^{2s} \Gamma(s-a)\Gamma(s-b)$$

$$\times \sum_{T\in J/\Gamma} \frac{1}{\xi_1(T)} \sum_{R\in\mathcal{I}} c_R(T)(\det T)^{-s}\, \overline{\mathcal{U}_R((\det T)^{1/2}T^{-1})}.$$

The proposition follows. ∎

To close this section we note that each side of the equality in Proposition 4.4 is absolutely convergent for s as indicated. (For the left hand side use Lemma 3.3, and for the right hand side, use the statement about the integral (4.12).)

5. Proof of the main result: (A)⇒(B). In this section we show that the twisted Dirichlet series $\Lambda(F,\mathcal{U},s)$ satisfy the properties stated in Theorem 1.1(B). In order to do so we first prove that similar properties hold for the Mellin transforms $G_R(W;s)$, and then use Proposition 4.4.

Indeed, for fixed $R\in\mathcal{I}$ and $W\in\mathcal{SP}$ we consider the map $s\mapsto G_R(W;s)$ given in Definition 4.1. It is holomorphic on the half-plane $\Re(s) > (k+5)/2$ (Lemma 4.3). Our first goal here is to show that $G_R(W;s)$ has a holomorphic continuation to the whole s-plane, it is bounded on vertical strips, and satisfies a particular functional equation.

As in the classical case, we write the Mellin transform $G_R(W;s)$ as a sum of two integrals, and make the change of variable $t\mapsto t^{-1}$ to get

$$G_R(W;s) = \int_1^\infty F_R(W;t^{-1})t^{-s-1}\,dt + \int_1^\infty F_R(W;t)t^{s-1}\,dt.$$

Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as above. Then $W[S^t] = W^{-1}$, and by (4.10) we have $F_R(W;t^{-1}) = F_{SR}(W[S^t];t^{-1}) = F_{SR}(-(it^{1/2}W)^{-1})$. Since the series $F(Z,z_1,z_2)$ is a Jacobi cusp form, its theta components $\{F_R(Z)\mid R\in\mathcal{I}\}$ satisfy (2.14). Thus

$$F_R(W;t^{-1}) = \frac{(-1)^k}{2m} t^{k-1/2} \sum_{R'\in\mathcal{I}} e\Big(\frac{(R')^tSR}{2m}\Big) F_{R'}(W;t).$$

If we use this identity in the expression for $G_R(W;s)$, we have

$$G_R(W;s) = \frac{(-1)^k}{2m} \sum_{R'\in\mathcal{I}} e\left(\frac{(R')^t SR}{2m}\right) \widetilde{G_{R'}}(W;k-s-1/2) + \widetilde{G_R}(W;s), \tag{5.1}$$

where $\widetilde{G_R}(W;s)$ is the integral introduced in (4.9). From this equality and the first remark following the proof of Lemma 4.3 we deduce that $G_R(W;s)$ admits a continuation to an entire function of s, which is bounded on any vertical strip. Moreover, such a bound is uniform with respect to W.

Next we make the change of variable $t \mapsto t^{-1}$ in the integral $\widetilde{G_R}(W;s)$ of (5.1), and get from (2.14) the functional equation

$$G_R(W;s) = \frac{(-1)^k}{2m} \sum_{R'\in\mathcal{I}} e\left(\frac{(R')^t SR}{2m}\right) G_{R'}(W;k-s-1/2). \tag{5.2}$$

In terms of the vector-valued function $G(W;s)$ in (4.3), this identity is

$$G(W;s) = (-1)^k G(W;k-s-1/2)A_m, \tag{5.3}$$

with A_m the matrix in (1.4).

Now we look at the Dirichlet series $\Lambda(F,\mathcal{U},s)$ for any Grössencharacter $\mathcal{U}$ as in Theorem 1.1. By Proposition 4.4,

$$\begin{aligned}\Lambda(F,\mathcal{U},s) &= \int_{\Gamma\backslash\mathcal{H}} \langle G(\tau;s),\mathcal{U}(\tau)\rangle\, d\mu(\tau) \\ &= \sum_{R\in\mathcal{I}} \int_{\Gamma\backslash\mathcal{H}} G_R(\tau;s)\overline{\mathcal{U}_R(\tau)}\, d\mu(\tau).\end{aligned} \tag{5.4}$$

We have just shown that $G_R(W;s)$ is bounded on any vertical strip $\Omega_{a,A} = \{s\in\mathbb{C} \mid a \le \Re(s) \le A\}$ of $\mathbb{C}$, uniformly with respect to W (or τ), and so

$$\begin{aligned}|\Lambda(F,\mathcal{U},s)| &\le \sum_{R\in\mathcal{I}} \int_{\Gamma\backslash\mathcal{H}} |G_R(\tau;s)\overline{\mathcal{U}_R(\tau)}|\, d\mu(\tau) \\ &\le C_{a,A,F} \sum_{R\in\mathcal{I}} \int_{\Gamma\backslash\mathcal{H}} |\mathcal{U}_R(\tau)|\, d\mu(\tau)\end{aligned}$$

for some real constant $C_{a,A,F}$. By hypothesis there are $0 \le \alpha < 1$ and C such that $|\mathcal{U}_R(\tau)| < Cy^\alpha$ for all $R \in \mathcal{I}$. Thus

$$|\Lambda(F,\mathcal{U},s)| \le nCC_{a,A,F} \int_{\Gamma\backslash\mathcal{H}} y^{\alpha-2}\, dx\, dy < \infty \quad \text{for all } s \in \Omega_{a,A}.$$

From this fact and (5.4) it is not difficult to get the analytic continuation of $\Lambda(F,\mathcal{U},s)$ to the whole s-plane and that such a function is bounded on any vertical strip.

Finally, it remains to prove the functional equation in (B).

We have

$$\Lambda(F,\mathcal{U},k-s-1/2)=\int\limits_{\Gamma\backslash\mathcal{H}}\langle G(\tau;k-s-1/2),\mathcal{U}(\tau)\rangle\,d\mu(\tau).$$

Inserting the relation (5.3) in this expression and using the fact that A_m is a Hermitian matrix, one gets

$$\begin{aligned}\Lambda(F,\mathcal{U},k-s-1/2)&=(-1)^k\int\limits_{\Gamma\backslash\mathcal{H}}\langle G(\tau;s)A_m,\mathcal{U}(\tau)\rangle\,d\mu(\tau)\\&=(-1)^k\Lambda(F,\mathcal{U}A_m,s).\end{aligned}$$

6. Proof of the main result: (B)⇒(A)

6.1. Spectral theory of the hyperbolic surface. We prove that (B) implies (A) using the vectorial spectral theory of the hyperbolic surface $\Gamma\backslash\mathcal{H}$. Similar arguments in the scalar case are used in [2] and [8] for Siegel modular forms of degree 2. For convenience we recall some basic facts, and refer the reader to [24] for details.

Let $\rho:\Gamma\to\mathrm{GL}(V)$ be the linear representation (2.10). A function $f:\mathcal{H}\to V\subseteq\mathbb{C}^n$ is said to be *automorphic* relative to ρ if $f(g\tau)=\rho(g)f(\tau)$ for all $g\in\Gamma$ and $\tau\in\mathcal{H}$. Let $\mathcal{F}$ be the standard (open) fundamental domain of Γ on $\mathcal{H}$. We denote by $\mathcal{C}^\infty(\mathcal{F},\rho)$ (resp. $\mathcal{C}_0^\infty(\mathcal{F},\rho)$) the $\mathbb{C}$-vector space of automorphic functions which are smooth on $\mathcal{H}$ and bounded (resp. compactly supported) on $\mathcal{F}$. In $\mathcal{C}_0^\infty(\mathcal{F},\rho)$ we consider the scalar product

$$\langle f,g\rangle_{\mathcal{F},\rho}=\int\limits_{\mathcal{F}}\langle f(\tau),g(\tau)\rangle\,d\mu(\tau)$$

and the associated norm $\|f\|^2_{\mathcal{F},\rho}=\langle f,f\rangle_{\mathcal{F},\rho}$.

Let $L^2(\mathcal{F},\rho,d\mu(\tau))$ be the completion of $\mathcal{C}_0^\infty(\mathcal{F},\rho)$ in the norm $\|\ \|_{\mathcal{F},\rho}$. This is a Hilbert space with the inner product $\langle\ ,\ \rangle_{\mathcal{F},\rho}$. The set

$$\mathcal{D}(\mathcal{F},\rho)=\{f\in\mathcal{C}^\infty(\mathcal{F},\rho)\mid f(\tau),\Delta f(\tau)\text{ are bounded in the norm of }V\}$$

is invariant under the Laplacian Δ (see Definition 3.1), and $\mathcal{C}_0^\infty(\mathcal{F},\rho)\subseteq\mathcal{D}(\mathcal{F},\rho)\subseteq L^2(\mathcal{F},\rho,d\mu(\tau))$. Moreover Δ has an extension to $L^2(\mathcal{F},\rho,d\mu(\tau))$.

Definition 6.1. Let V be the vector space in (2.8) and

$$\widetilde{V}=\{v\in V\mid\rho(g_T)v=v\}\quad\text{where}\quad g_T=\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

Let $\widetilde{B}=\{\widetilde{e}_1,\dots,\widetilde{e}_d\}$ be an orthonormal basis of $\widetilde{V}$, fixed from now on. To a compactly supported C^∞-map $\psi:[0,\infty)\to\mathbb{C}$, and $1\le j\le d$, we associate the vector-valued, incomplete Eisenstein series $\theta(\tau;\psi;j):\mathcal{H}\to V$ given by

$$\theta(\tau;\psi;j)=\sum_{g\in\Gamma_\infty\backslash\Gamma}\psi(\Im(g\tau))\rho^*(g)\widetilde{e}_j. \tag{6.1}$$

As usual, Γ_∞ denotes the Γ-stabilizer of infinity and ρ^* the adjoint representation of ρ.

For every $\tau\in\mathcal{H}$ the series (6.1) has only finitely many non-zero terms. Furthermore, $\theta(h\tau;\psi;j)=\rho(h)\theta(\tau;\psi;j)$ if $h\in\Gamma$ (as $\rho^*(h^{-1})=\rho(h)$), and $\theta(\tau;\psi;j)$ belongs to $\mathcal{C}^\infty(\mathcal{F},\rho)\cap L^2(\mathcal{F},\rho,d\mu(\tau))$.

Let now $\langle\theta(\tau;\psi;j)\mid\psi,j\rangle$ be the linear $\mathbb{C}$-span of the incomplete Eisenstein series (6.1) with ψ and j varying as in Definition 6.1, and

$$\mathcal{E}(\Gamma,\rho)=\overline{\langle\theta(\tau;\psi;j)\mid\psi,j\rangle}.$$

This is a subspace of $L^2(\mathcal{F},\rho,d\mu(\tau))$ whose orthogonal complement is called the space of *cusp vector-functions*, or *parabolic forms*, and is denoted as $L_0^2(\mathcal{F},\rho,d\mu(\tau))$. Then

$$L^2(\mathcal{F},\rho,d\mu(\tau))=L_0^2(\mathcal{F},\rho,d\mu(\tau))\oplus\mathcal{E}(\Gamma,\rho)\quad\text{(orthogonal sum)}. \tag{6.2}$$

The rest of this section is devoted to the properties of the subspaces in (6.2) that are necessary in our arguments below. Some of these facts are well-known and we just recall them. Others are more technical, and we give a proof.

Regarding the subspace of cusp forms we observe that f belongs to $L^2(\mathcal{F},\rho,d\mu(\tau))$ if and only if

$$\int_0^1\langle f(\tau),v\rangle\,dx=0\qquad\text{identically in } y\in(0,\infty),\text{ for any } v\in\widetilde{V}.$$

Every $f(\tau)$ in $L_0^2(\mathcal{F},\rho,d\mu(\tau))$ is an n-tuple $f(\tau)=(\ldots,f_R(\tau),\ldots)_{R\in\mathcal{I}}$ of smooth functions $f_R:\mathcal{H}\to\mathbb{C}$ which are invariant under the subgroup $\Gamma(2m)$ (since $\Gamma(2m)\subseteq\operatorname{Ker}\rho$). In fact, each f_R belongs to $L^2(\mathcal{F}_{\Gamma(2m)},d\mu(\tau))$ where $\mathcal{F}_{\Gamma(2m)}$ denotes a $\Gamma(2m)$-fundamental domain in $\mathcal{H}$.

Furthermore, if f is an eigenfunction of Δ, then f_R is also an eigenfunction, say $\Delta f_R=s(1-s)f_R$, and we can write

$$f_R(\tau)=\sum_{0\neq n\in\mathbb{Z}}a(f_R;n)\cdot 2\sqrt{|n|y}\,K_{s-1/2}(2\pi|n|y)e(nx)\quad\text{with }\tau=x+iy,$$

for some complex numbers $a(f_R;n)$, where $K_s(z)$ is the standard Bessel function with exponential decay at infinity. This Fourier expansion implies that $f_R(\tau)=O(1)$ as $y\to\infty$. This argument shows that every $f\in L_0^2(\mathcal{F},\rho,d\mu(\tau))$ which is an eigenfunction of Δ satisfies the conditions in Definition 3.1, and so it is a vector-valued Grössencharacter with $\alpha=0$.

It is known that $L_0^2(\mathcal{F},\rho,d\mu(\tau))$ has a countable orthonormal basis of eigenfunctions of the hyperbolic Laplacian, say $\{\mathcal{U}_j(\tau)\}_{j=1}^{\infty}$.

As for the subspace $\mathcal{E}(\Gamma,\rho)$ in (6.2), we proceed differently. All incomplete Eisenstein series admit an integral representation in terms of a particular class of eigenfunctions of the Laplacian, as we now recall: The Mellin transform $L_\psi(s)$ of a compactly supported smooth function $\psi:[0,\infty)\to\mathbb{C}$ is

$$L_\psi(s)=\int_0^\infty \psi(y)y^{-s-1}\,dy \quad \Big(\text{hence } \psi(y)=\int_{\Re(s)=s_0} L_\psi(s)y^s\,ds\Big). \tag{6.3}$$

This function is entire, and there exists $C_\psi>0$ such that

$$|L_\psi(s)|\ll_{\psi,N}\frac{C_\psi^{|\Re(s)|}}{(1+|\Im(s)|)^N}\quad \text{on } \Omega_{a,A}=\{s\in\mathbb{C}\mid a\le\Re(s)\le A\} \tag{6.4}$$

for any positive integer N and any real $a<A$. (The implied constant in $\ll$ can be taken to be C_ψ^N: see [14, p. 75]). If we consider the vector-valued Eisenstein series

$$E_{s,j}(\tau)=\sum_{g\in\Gamma_\infty\backslash\Gamma}\Im(g\tau)^s\rho^*(g)\widetilde{e}_j \tag{6.5}$$

for any $1\le j\le d$ and $s\in\mathbb{C}$ with $\Re(s)>1$, where $\widetilde{e}_j$ belongs to the basis $\widetilde{B}$ fixed in Definition 6.1, we have

$$\theta(\tau;\psi;j)=\frac{1}{2\pi i}\int_{\Re(u)=u_0}L_\psi(u)E_{u,j}(\tau)\,du \tag{6.6}$$

for arbitrary $u_0>1$. (This integral is independent of u_0.) The key point about (6.6) is that $E_{s,j}(\tau)$ is a well-known eigenfunction of the Laplacian. For future reference we here collect some technical facts about $E_{s,j}(\tau)$ whose proofs are given in the last section of this article.

Lemma 6.2. *Let $\{g_l\}_{l=1}^{\kappa}$ be a set of representatives for the cosets in the quotient $\{\pm I_2\}\Gamma_1(2m)\backslash\Gamma$. For every $\epsilon>0$ there is a continuous function $(u,\tau)\mapsto C_{\Re(u),\Im(\tau),\epsilon}$ from the cartesian product $\{u\in\mathbb{C}\mid 3/4<\Re(u)<3/2\}\times\mathcal{H}$ into $\mathbb{R}_{>0}$ (independent of $\Im(u)$ and $\Re(\tau)$) such that $\lim_{y\to\infty}C_{\Re(u),y,\epsilon}=0$ and*

$$\|E_{u,j}(\tau)\|\le\sum_{l=1}^{\kappa}\Im(g_l\tau)^{\Re(u)}+A'\zeta(2\Re(u))\sqrt{|\Im(u)|}\sum_{l=1}^{\kappa}\Im(g_l\tau)^{1-\Re(u)} + \kappa C_{\Re(u),y,\epsilon}/|\Gamma(u)|$$

for some absolute constant $A'>0$, whenever $3/4<\Re(u)<3/2$ and $|\Im(u)|\ge 1$. Moreover, if $E_{u,j,R}$ is the Rth component function of $E_{u,j}$, i.e.

$E_{u,j} = (\ldots, E_{u,j,R}, \ldots)$, *then*

$$|E_{u,j,R}(\tau)| \le \kappa y^{\Re(u)} + A'\kappa\zeta(2\Re(u))\sqrt{|\Im(u)|}\, y^{1-\Re(u)} + \kappa C_{\Re(u),y,\epsilon}/|\Gamma(u)|$$

for all $R \in \mathcal{I}$, *provided that* $3/4 < \Re(u) < 1$, $|\Im(u)| > 1$, *and* $\tau = x+iy \in \mathcal{F}$.

REMARK. The vector-valued map $E_{u,j}$ is a Grössencharacter associated to the representation ρ for every $u \in \mathbb{C}$ with $3/4 < \Re(u) < 1$.

LEMMA 6.3. *Let* $\beta > (k+6)/2$ *and* $3/4 < u_1 < 1$. *Then the double integral*

$$\int_{\Re(s)=\beta} \int_{\Re(u)=u_1} |\Lambda(F, L_\psi(u)E_{u,j}, s)t^s|\, du\, ds$$

is finite.

LEMMA 6.4. *Let* $E_{u,j,R}$ *be as above. Let* L_ψ *be the Mellin transform of* ψ *in* (6.3), $u_0 > 1$ *and* $\tau \in \mathcal{H}$. *Then*

$$\int_{\Re(u)=u_0} L_\psi(u)E_{u,j,R}(\tau)\, du = 2\pi i \lim_{u\to 1}\, (u-1)L_\psi(u)E_{u,j,R}(\tau) + \int_{\Re(u)=u_1} L_\psi(u)E_{u,j,R}(\tau)\, du$$

for some $u_1 \in \mathbb{R}$ *with* $3/4 < u_1 < 1$.

6.2. The proof: Grössencharacters. Now that some necessary technical tools have been established, we can spell out why condition (A) follows from (B) in Theorem 1.1.

First we notice that each sequence in $\{F_R \mid R \in \mathcal{I}\}$ defines a holomorphic function on $\mathcal{H}_2$ via the Fourier series (1.2). If we put them together as in (1.1), we get a holomorphic function $F(Z, z_1, z_2) : \mathcal{H}_2 \times \mathbb{C}^2 \to \mathbb{C}$.

Clearly each $F_R(Z)$ satisfies (2.13). Hence, for the proof of (A) it suffices to prove the set of equations (2.14). Moreover, as all functions involved in the latter are analytic, one just needs to verify (2.14) for $Z = iY$ with $Y \in \mathcal{P}$. If we rephrase these equations with the parametrization $\mathbb{R}_{>0} \times \mathcal{SP}$ of $\mathcal{P}$ and the relation $F_R(-(it^{1/2}W)^{-1}) = F_R(W^{-1}; t^{-1}) = F_R(W[S]; t^{-1}) = F_{SR}(W; t^{-1})$, where S is the matrix in (1.4) (see (4.10)), then the identities to prove are

$$F_R(W; t^{-1}) = (-1)^k t^{k-1/2} \sum_{R'\in\mathcal{I}} \frac{1}{2m} e\biggl(\frac{(R')^t SR}{2m}\biggr) F_{R'}(W; t)$$

for all $t \in \mathbb{R}_{>0}$, $W \in \mathcal{SP}$ and $R \in \mathcal{I}$. They are equivalent to the vector-valued identity

$$F(W; t^{-1}) = (-1)^k t^{k-1/2} F(W; t) A_m \quad \text{for all } (t, W) \in \mathbb{R}_{>0} \times \mathcal{SP}. \tag{6.7}$$

In order to deduce (6.7) from (B) we fix $t \in \mathbb{R}_{>0}$ and consider $F(W;t)$ as a map on $\mathcal{H}$ via (3.1). Then $F(\tau;t)$ is a smooth function of τ which satisfies

$$F(g\tau;t) = (\ldots, F_R(W_\tau[g^{-1}];t), \ldots)_R = \rho(g)F(\tau;t)$$

for all $g \in \Gamma$ (recall (2.12) and (4.10)). Moreover, by Lemma 4.2 one has

$$\int_{\mathcal{F}} \|F(\tau;t)\|^2 \, d\mu(\tau) < ne^{-2B\sqrt{t}}(B_1 t^{-(k+5)/2} + B_2 t^{-(k+3)/2})^2 \int_{\Gamma\backslash\mathcal{H}} d\mu(\tau)$$

for some real constants B, B_1, B_2. As the right hand side of this expression is finite, we conclude $F(\tau;t) \in L^2(\mathcal{F}, \rho, d\mu)$ for all $t > 0$. In particular, both sides of (6.7) are continuous functions of τ in the Hilbert space $L^2(\mathcal{F}, \rho, d\mu)$.

Consequently, (6.7) holds for every t and W (pointwise in W) if it holds in the L^2-sense for every fixed $t > 0$. By (6.2) and our remarks about the subspaces $L_0^2(\mathcal{F}, \rho, d\mu(\tau))$ and $\mathcal{E}(\Gamma, \rho)$, the latter follows from the equalities

$$(6.8) \qquad \langle F(\tau;t^{-1}), f(\tau)\rangle_{\mathcal{F},\rho} = \langle (-1)^k t^{k-1/2} F(\tau;t) A_m, f(\tau)\rangle_{\mathcal{F},\rho}$$

whenever $f(\tau) = \mathcal{U}_j(\tau)$ for $j = 1, 2, \ldots$, and $f(\tau) = \theta(\tau;\psi;l)$ for any compactly supported C^∞-map $\psi : [0,\infty) \to \mathbb{C}$, $1 \le l \le d$.

The rest of this subsection is devoted to proving (6.8) in the case that the function $f(\tau) = \mathcal{U}(\tau) = (\ldots, \mathcal{U}_R(\tau), \ldots)_{R\in\mathcal{I}}$ is a Grössencharacter associated to ρ with parameter $\alpha < 1$.

By Lemma 4.3 and the identification (3.1) between $\mathcal{SP}$ and $\mathcal{H}$, one has

$$(6.9) \qquad 2\pi i \, \langle F(\tau;t^{-1}), \mathcal{U}(\tau)\rangle_{\mathcal{F},\rho} = 2\pi i \sum_{R\in\mathcal{I}} \int_{\mathcal{F}} F_R(\tau;t^{-1})\overline{\mathcal{U}_R(\tau)} \, d\mu(\tau) = \int_{\mathcal{F}} \int_{\Re(s)=\beta} \langle G(\tau;s), \mathcal{U}(\tau)\rangle t^s \, ds \, d\mu(\tau).$$

Here β is any real number such that $\beta > (k+5)/2$.

Since we want to change the order of integration in the expression above, we check first that the Fubini–Tonelli theorem is applicable. By the Cauchy–Schwarz inequality,

$$\int_{\mathcal{F}} \int_{\Re(s)=\beta} |\langle G(\tau;s), \mathcal{U}(\tau)\rangle t^s| \, ds \, d\mu(\tau) \le t^\beta \int_{\mathcal{F}} \Big(\int_{\beta-i\infty}^{\beta-iL} + \int_{\beta-iL}^{\beta+iL} + \int_{\beta+iL}^{\beta+i\infty} \|G(\tau;s)\| \, \|\mathcal{U}(\tau)\| \, ds \Big) \, d\mu(\tau)$$

for any positive real number L.

Pick $\mu > 1$. By Lemma 4.3 there exists $L > 0$ large enough that $|\Im(s)| > L$ implies $|G_R(\tau;s)| \le C_\mu |\Im(s)|^{-\mu}$ on the vertical line $\Re(s) = \beta$ for some $C_\mu > 0$. This bound is uniform in W and valid for all $R \in \mathcal{I}$. These choices, the inequality $\|G(\tau;s)\|^2 = \sum_{R\in\mathcal{I}} |G_R(\tau;s)|^2 \le nC_\mu^2 |\Im(s)|^{-2\mu}$ for all such s, and the existence of a bound $K_\beta > 0$ for $\|G(\tau;s)\|$ along the whole line $\Re(s) = \beta$ yield

$$\int_{\mathcal{F}} \int_{\Re(s)=\beta} |\langle G(\tau;s), \mathcal{U}(\tau)\rangle t^s| \, ds \, d\mu(\tau) \tag{6.10}$$
$$\le t^\beta \int_{\mathcal{F}} \Big(\int_{\beta-iL}^{\beta+iL} \|G(\tau;s)\| \, ds + 2\sqrt{n}\, C_\mu \int_{\beta+iL}^{\beta+i\infty} |\Im(s)|^{-\mu} \, ds\Big) \|\mathcal{U}(\tau)\| \, d\mu(\tau)$$
$$\le \Big(2LK_\beta + 2\sqrt{n}\, C_\mu \int_{\beta+iL}^{\beta+i\infty} |\Im(s)|^{-\mu} \, ds\Big) t^\beta \int_{\mathcal{F}} \|\mathcal{U}(\tau)\| \, d\mu(\tau).$$

The growth condition of any Grössencharacter $\mathcal{U}(\tau)$ states that its component functions $\mathcal{U}_R(\tau)$ are bounded by Cy^α as $y \to \infty$ for a real constant $C > 0$. Thus

$$\int_{\mathcal{F}} \|\mathcal{U}(\tau)\| \, d\mu(\tau) = \int_{\mathcal{F}} \sqrt{\sum_{R\in\mathcal{I}} |\mathcal{U}_R(\tau)|^2 \, y^{-2} \, dx \, dy} < nC \int_{1/2}^{\infty} y^{\alpha-2} \, dy < \infty.$$

As $\mu > 1$, the integral along the half-line $\Re(s) = \beta$, $L < \Im(s)$ in (6.10) is finite, and so the double integral on the left hand side of (6.10) is finite. These computations show that the Fubini–Tonelli theorem can be used on the right hand side of (6.9). This fact and Proposition 4.4 yield

$$\langle F(\tau;t^{-1}), \mathcal{U}(\tau)\rangle_{\mathcal{F},\rho} = \frac{1}{2\pi i} \int_{\Re(s)=\beta} \Lambda(F,\mathcal{U},s) t^s \, ds. \tag{6.11}$$

At this point we make use of the hypothesis on $\Lambda(F,\mathcal{U},s)$ in (B), and conclude

$$\langle F(\tau;t^{-1}), \mathcal{U}(\tau)\rangle_{\mathcal{F},\rho} = \frac{(-1)^k}{2\pi i} \int_{\Re(s)=\beta} \Lambda(F,\mathcal{U}A_m, k-s-1/2) t^s \, ds \tag{6.12}$$
$$= \frac{(-1)^k}{2\pi i} \int_{\Re(s)=k-\beta-1/2} \Lambda(F,\mathcal{U}A_m, s) t^{k-s-1/2} \, ds.$$

The rest of the argument is quite common in the ambit of converse theorems and we just give it for completeness' sake.

Since $\alpha < 1$ and $\beta > (k+5)/2$, the Remark following Definition 3.4 applies and the estimate $\Lambda(F,\mathcal{U},s) = O(|\Im(s)|^{-\mu})$ holds for all $\mu > 0$ as $|\Im(s)| \to \infty$ on the vertical line $\Re(s) = \beta$. Of course, the same is true for $\Lambda(F,\mathcal{U}A_m,s)$. This fact and the functional equation in (B) imply $\Lambda(F,\mathcal{U}A_m,s) = (-1)^k\Lambda(F,\mathcal{U},k-s-1/2) = O(|\Im(s)|^{-\mu})$ for all $\mu > 0$ as $|\Im(s)| \to \infty$ on the vertical line $\Re(s) = k-\beta-1/2$. Now we recall that the analytic continuation of $\Lambda(F,\mathcal{U}A_m,s)$ is bounded on any vertical strip by (B), in particular on $\varOmega_{k-\beta-1/2,\beta} = \{s \in \mathbb{C} \mid k-\beta-1/2 \le \Re(s) \le \beta\}$, and conclude from the Phragmén–Lindelöf theorem that

$$\Lambda(F,\mathcal{U}A_m,s) = O(|\Im(s)|^{-\mu}) \quad \text{for all } \mu > 0 \text{ as } |\Im(s)| \to \infty, \tag{6.13}$$

uniformly on $\varOmega_{k-\beta-1/2,\beta}$. As $\Lambda(F,\mathcal{U}A_m,s)$ is entire by hypothesis, shifting contours of integration and (6.13) yield

$$\int\limits_{\Re(s)=k-\beta-1/2} \Lambda(F,\mathcal{U}A_m,s)t^{-s}\,ds = \int\limits_{\Re(s)=\beta} \Lambda(F,\mathcal{U}A_m,s)t^{-s}\,ds.$$

From this equality and (6.12) we get

$$\langle F(\tau;t^{-1}),\mathcal{U}(\tau)\rangle_{\mathcal{F},\rho} = \frac{(-1)^k}{2\pi i}t^{k-1/2} \int\limits_{\Re(s)=\beta} \Lambda(F,\mathcal{U}A_m,s)t^{-s}\,ds.$$

In turn this identity, (6.11) with t (resp. $\mathcal{U}(\tau)A_m$) instead of t^{-1} (resp. $\mathcal{U}(\tau)$), the definition of $\langle f,g\rangle_{\mathcal{F},\rho}$, and the fact that A_m is Hermitian imply

$$\begin{aligned}\langle F(\tau;t^{-1}),\mathcal{U}(\tau)\rangle_{\mathcal{F},\rho} &= (-1)^k t^{k-1/2}\langle F(\tau;t),\mathcal{U}(\tau)A_m\rangle_{\mathcal{F},\rho} \\ &= (-1)^k t^{k-1/2}\int\limits_{\mathcal{F}}\langle F(\tau;t),\mathcal{U}(\tau)A_m\rangle\,d\mu(\tau) \\ &= (-1)^k t^{k-1/2}\langle F(\tau;t)A_m,\mathcal{U}(\tau)\rangle_{\mathcal{F},\rho}.\end{aligned}$$

This finishes the proof of (6.8) for all Grössencharacters $f(\tau) = \mathcal{U}(\tau)$ with parameter $\alpha < 1$. In particular, those Grössencharacters include the elements $\mathcal{U}_j(\tau)$ in the basis of the parabolic subspace $L_0^2(\mathcal{F},\rho,d\mu(\tau))$ mentioned above.

6.3. The proof: The incomplete Eisenstein series. As mentioned above, the series $\theta(\tau;\psi;j)$ is not a Grössencharacter in general. However, the integral representation (6.6) is available, and here we show how to use it in order to get (6.8) for $f(\tau) = \theta(\tau;\psi;j)$.

Exactly the same argument that yields (6.9) also gives

$$\begin{aligned}\langle F(\tau;t^{-1}),\theta(\tau;\psi;j)\rangle_{\mathcal{F},\rho} &= \sum_{R\in\mathcal{I}}\int\limits_{\mathcal{F}} F_R(\tau;t^{-1})\overline{\theta_R(\tau;\psi,j)}\,d\mu(\tau) \\ &= \frac{1}{2\pi i}\int\limits_{\mathcal{F}}\int\limits_{\Re(s)=\beta}\langle G(\tau;s),\theta(\tau;\psi;j)\rangle t^s\,ds\,d\mu(\tau)\end{aligned} \tag{6.14}$$

for $\beta > (k+6)/2$. As before, we want to change the order of integration in the last double integral and so we consider

$$\begin{aligned}&\int\limits_{\mathcal{F}}\int\limits_{\Re(s)=\beta}|\langle G(\tau;s),\theta(\tau;\psi;j)\rangle t^s|\,ds\,d\mu(\tau) \\ &\qquad\le t^\beta\int\limits_{\mathcal{F}}\|\theta(\tau;\psi;j)\|\Big(\int\limits_{\Re(s)=\beta}\|G(\tau;s)\|\,ds\Big)\,d\mu(\tau)\end{aligned}$$

(use the Cauchy–Schwarz inequality). Just as prior to (6.10), for any $\mu > 1$ there is $L > 0$ large enough that $\|G(\tau;s)\|^2 \le nC_\mu^2|\Im(s)|^{-2\mu}$ for some $C_\mu > 0$, on the vertical line $\Re(s) = \beta$ with $|\Im(s)| > L$. Thus

$$\int_{\Re(s)=\beta} \|G(\tau;s)\|\,ds \le \int_{\beta-iL}^{\beta+iL} \|G(\tau;s)\|\,ds + 2\sqrt{n}\,C_\mu \int_{\beta+iL}^{\beta+i\infty} |\Im(s)|^{-\mu}\,ds$$
$$\le 2LK_\beta + 2\sqrt{n}\,C_\mu \int_{\beta+iL}^{\beta+i\infty} |\Im(s)|^{-\mu}\,ds,$$

where $K_\beta > 0$ is any bound for $\|G(\tau;s)\|$ on the segment $\Re(s) = \beta$, $-L \le \Im(s) \le L$. Putting these expressions together one obtains

$$\int_{\mathcal{F}} \int_{\Re(s)=\beta} |\langle G(\tau;s), \theta(\tau;\psi;j)\rangle t^s|\,ds\,d\mu(\tau)$$
$$\le \Big(2LK_\beta + 2\sqrt{n}\,C_\mu \int_{\beta+iL}^{\beta+i\infty} |\Im(s)|^{-\mu}\,ds\Big) t^\beta \int_{\mathcal{F}} \|\theta(\tau;\psi;j)\|\,d\mu(\tau). \tag{6.15}$$

Since $\mu > 1$, the integral along the half-line $\Re(s) = \beta$ with $L < \Im(s)$ in (6.15) is finite.

As for the other integral, we recall that $\theta(\tau;\psi;j)$ is in $L^2(\mathcal{F},\rho,d\mu)$, observe that $(\mathcal{F},d\mu)$ is a measure space with finite measure, and conclude that $\theta(\tau;\psi;j)$ is in $L^1(\mathcal{F},\rho,d\mu)$. Thus, the left hand side of (6.15) is finite. Consequently, the Fubini–Tonelli theorem can be used in (6.14) to get

$$\langle F(\tau;t^{-1}), \theta(\tau;\psi;j)\rangle_{\mathcal{F},\rho}$$
$$= \frac{1}{2\pi i} \sum_{R\in\mathcal{I}} \int_{\Re(s)=\beta} \Big(\int_{\mathcal{F}} G_R(\tau;s)\overline{\theta_R(\tau;\psi;j)}\,d\mu(\tau)\Big) t^s\,ds. \tag{6.16}$$

Until now, our arguments have been very similar to the ones prior to (6.11), but the analysis diverges from this point on. By Lemma 6.4, we get from (6.6) the identity

$$\int_{\mathcal{F}} G_R(\tau;s)\overline{\theta_R(\tau;\psi;j)}\,d\mu(\tau)$$
$$= \overline{L_\psi(1)} \lim_{u\to 1} \overline{(u-1)E_{u,j,R}(\tau)} \int_{\mathcal{F}} G_R(\tau;s)\,d\mu(\tau)$$
$$+ \frac{1}{2\pi i} \int_{\mathcal{F}} G_R(\tau;s) \int_{\Re(u)=u_1} \overline{L_\psi(u)E_{u,j,R}(\tau)}\,du\,d\mu(\tau) \tag{6.17}$$

for some $u_1 \in \mathbb{R}$ with $3/4 < u_1 < 1$. The order of integration in the last expression can be switched. Indeed, one has

$$\int_{\Re(u)=u_1}\int_{\mathcal{F}} |G_R(\tau;s)|\,|L_\psi(u)E_{u,j,R}(\tau)|\,d\mu(\tau)\,du \le C_{F,\beta}\int_{\Re(u)=u_1} |L_\psi(u)|\int_{\mathcal{F}} |E_{u,j,R}(\tau)|\,d\mu(\tau)\,du \tag{6.18}$$

for some constant $C_{F,\beta} > 0$, using the estimate for $G_R(\tau,s)$ proved in Lemma 4.3. From the second part of Lemma 6.2 we get the following inequality for the last inner integral:

$$\int_{\mathcal{F}} |E_{u,j,R}(\tau)|\,d\mu(\tau) \le \kappa\int_{\mathcal{F}} y^{u_1}\,d\mu(\tau) + A'\kappa\zeta(2u_1)\sqrt{\Im(u)}\int_{\mathcal{F}} y^{1-u_1}\,d\mu(\tau) + \frac{\kappa}{|\Gamma(u)|}\int_{\mathcal{F}} C_{u_1,y,\epsilon}\,d\mu(\tau).$$

Now we observe that the last three integrals are finite and independent of $\Im(u)$ (use $3/4 < u_1 < 1$ for the first two integrals, and $\lim_{y\to\infty} C_{u_1,y,\epsilon} = 0$ for the third one). Thus

$$\begin{aligned}(\kappa C_{F,\beta})^{-1}&\int_{\Re(u)=u_1}\int_{\mathcal{F}} |G_R(\tau;s)|\,|L_\psi(u)E_{u,j,R}(\tau)|\,d\mu(\tau)\,du\\ &\le \int_{\mathcal{F}} y^{u_1}\,d\mu(\tau)\int_{\Re(u)=u_1} |L_\psi(u)|\,du + A'\zeta(2u_1)\int_{\mathcal{F}} y^{1-u_1}\,d\mu(\tau)\\ &\quad\times\int_{\Re(u)=u_1} |L_\psi(u)|\Im(u)^{1/2}\,du + \int_{\mathcal{F}} C_{u_1,y,\epsilon}\,d\mu(\tau)\int_{\Re(u)=u_1}\left|\frac{L_\psi(u)}{\Gamma(u)}\right| du.\end{aligned}$$

All integrals on the vertical line $\Re(u) = u_1$ on the right hand side of this relation are finite, as we can see from Stirling's estimate of the gamma function and from the asymptotic

$$|L_\psi(u)e^{\pi|\Im(u)|/2}| \to 0 \quad \text{as } |\Im(u)| \to \infty,$$

which holds uniformly for $\Re(u)$ in a closed interval containing u_1 and u_0 (see remark in the proof of Lemma 6.4). Thus, the double integral on the left hand side of (6.18) is finite, and we can use the Fubini–Tonelli theorem to interchange the integrations on the right hand side of (6.17). Consequently,

$$\begin{aligned}\sum_{R\in\mathcal{I}}\int_{\mathcal{F}} G_R(\tau;s)\overline{\theta_R(\tau;\psi;j)}\,d\mu(\tau) &= \overline{L_\psi(1)}\sum_{R\in\mathcal{I}}\lim_{u\to1}\overline{(u-1)E_{u,j,R}(\tau)}\\ \times\int_{\mathcal{F}} G_R(\tau;s)\,d\mu(\tau) &+ \frac{1}{2\pi i}\int_{\Re(u)=u_1}\overline{L_\psi(u)}\sum_{R\in\mathcal{I}}\int_{\mathcal{F}} G_R(\tau;s)\overline{E_{u,j,R}(\tau)}\,d\mu(\tau)\,du.\end{aligned} \tag{6.19}$$

Since $E_{u,j}$ is a Grössencharacter for ρ with parameter $\alpha = \Re(u)$ (see Remark following Lemma 6.2), the same is true for the function $L_\psi(u)E_{u,j}$. On the other hand, if we set $C_{j,R} = \lim_{u\to1}(u-1)E_{u,j,R}(\tau)$ and $C_{\psi,j} =$

$(\ldots, L_\psi(1)C_{j,R}, \ldots) \in V$ for every j, ψ, we deduce that $C_{\psi,j}$ is a (constant) Grössencharacter for ρ with parameter $\alpha = 0$.

Thus Proposition 4.4 and the choice $\Re(s) = \beta > (k+6)/2$ made above show that (6.19) can be written as

$$\int_{\mathcal{F}} \langle G(\tau; s), \theta(\tau; \psi; j)\rangle \, d\mu(\tau) = \Lambda(F, C_{\psi,j}, s) + \frac{1}{2\pi i} \int_{\Re(u)=u_1} \Lambda(F, L_\psi(u)E_{u,j}, s)\, du.$$

In turn, this identity allows us to express (6.16) as

$$\langle F(\tau; t^{-1}), \theta(\tau; \psi; j)\rangle_{\mathcal{F},\rho} = \frac{1}{2\pi i} \int_{\Re(s)=\beta} \Lambda(F, C_{\psi,j}, s) t^s \, ds - \frac{1}{4\pi^2} \int_{\Re(s)=\beta} \int_{\Re(u)=u_1} \Lambda(F, L_\psi(u)E_{u,j}, s) t^s \, du\, ds. \tag{6.20}$$

The next step in our argument requires switching the order of integration in the last double integral, and we can do so by Lemma 6.3. Indeed, this result allows us to use the Fubini–Tonelli theorem on the right hand side of (6.20) to get

$$\begin{aligned} \langle F(\tau; t^{-1}), \theta(\tau; \psi; j)\rangle_{\mathcal{F},\rho} &= \frac{1}{2\pi i} \int_{\Re(s)=\beta} \Lambda(F, C_{\psi,j}, s) t^s \, ds \\ &\quad - \frac{1}{4\pi^2} \int_{\Re(u)=u_1} L_\psi(u) \int_{\Re(s)=\beta} \Lambda(F, E_{u,j}, s) t^s \, ds\, du \\ &= \langle F(\tau; t^{-1}), C_{\psi,j}\rangle_{\mathcal{F},\rho} + \frac{1}{2\pi i} \int_{\Re(u)=u_1} L_\psi(u) \langle F(\tau; t^{-1}), E_{u,j}(\tau)\rangle_{\mathcal{F},\rho}\, du. \end{aligned} \tag{6.21}$$

For the last step we have used (6.11) with $\mathcal{U} = C_{\psi,j}$ and $\mathcal{U} = E_{u,j}$.

Now we recall that (6.8) holds for every Grössencharacter associated to ρ with parameter $\alpha < 1$ (see Subsection 6.2), and use it to obtain from (6.21) the equality

$$\begin{aligned} &(-1)^k t^{-k+1/2} \langle F(\tau; t^{-1}), \theta(\tau; \psi; j)\rangle_{\mathcal{F},\rho} \\ &= \langle F(\tau; t)A_m, C_{\psi,j}\rangle_{\mathcal{F},\rho} + \frac{1}{2\pi i} \int_{\Re(u)=u_1} L_\psi(u) \langle F(\tau; t)A_m, E_{u,j}(\tau)\rangle_{\mathcal{F},\rho}\, du \\ &= \langle F(\tau; t)A_m, \theta(\tau; \psi; j)\rangle_{\mathcal{F},\rho}. \end{aligned} \tag{6.22}$$

The last identity is just (6.21) with t instead of t^{-1}, and $F(\tau; t)A_m$ instead of $F(\tau; t)$. Notice that we have obtained (6.8) for all $f(\tau) = \theta(\tau; \psi; j)$, as desired.

This result and the arguments in the previous subsection show that (B) implies (A) in Theorem 1.1, and so the proof of the latter is complete.

7. Proof of auxiliary results. Here we prove Lemmas 6.2–6.4, and the Remark following Lemma 6.2.

Proof of Lemma 6.2. As $\Gamma(2m)$ is a normal subgroup of Γ and satisfies $\{\pm I_2\}\Gamma_1(2m) = \Gamma_\infty\Gamma(2m) \subseteq \operatorname{Ker}\rho$, we can write the function $E_{u,j}(\tau)$ in terms of the scalar-valued, real analytic Eisenstein series

$$\mathcal{E}_{\Gamma_1(2m)}(\tau,u) = \sum_{g\in\Gamma_\infty\backslash\{\pm I_2\}\Gamma_1(2m)} \Im(g\tau)^u. \tag{7.1}$$

Indeed, as $\{g_l\}_{l=1}^{\kappa}$ are representatives for the cosets in $\{\pm I_2\}\Gamma_1(2m)\backslash\Gamma$, we have

$$\begin{aligned} E_{u,j}(\tau) &= \sum_{l=1}^{\kappa} \sum_{g\in\Gamma_\infty\backslash\{\pm 1\}\Gamma_1(2m)} \Im(gg_l\tau)^u\rho^*(g_l)\widetilde{e}_j \\ &= \sum_{l=1}^{\kappa} \mathcal{E}_{\Gamma_1(2m)}(g_l\tau,u)\rho^*(g_l)\widetilde{e}_j. \end{aligned} \tag{7.2}$$

The series (7.1) is an absolutely, locally uniformly convergent series on $\Re(u) > 1$. From its well-known properties one finds that the Rth component $E_{u,j,R}(\tau)$ of $E_{u,j}(\tau)$ has a meromorphic continuation to the whole complex u-plane. Its singularities on the half-plane $\Re(u) \ge 1/2$ are all simple, finitely many, and located in the real interval $(1/2,1]$. The meromorphic continuation of $E_{u,j,R}(\tau)$ is either holomorphic or has a simple pole at $u=1$. In the latter case, its residue at 1 is a constant function.

The Fourier expansion of $\mathcal{E}_{\Gamma_1(2m)}(\tau,u)$ (at infinity) is found in many places in the literature. In particular, from [9, pp. 22, 52, 66] we get

$$\begin{aligned} \mathcal{E}_{\Gamma_1(2m)}(\tau,u) &= y^u + \pi^{1/2}\frac{\Gamma(u-1/2)\zeta(2u-1)}{\Gamma(u)\zeta(2u)}y^{1-u} \\ &+ \frac{2\pi^u}{\Gamma(u)} \sum_{n\in\mathbb{Z},\, n\neq 0} |n|^{u-1/2}y^{1/2}\Big(\sum_{c=1}^{\infty} c^{-2u}\mathcal{S}(0,n;c)\Big)K_{u-1/2}(2\pi|n|y)e(nx), \end{aligned} \tag{7.3}$$

where

$$\mathcal{S}(m,n;c) = \sum_{ad\equiv 1\,(\mathrm{mod}\,c)} e\Big(\frac{dm+an}{c}\Big), \qquad K_\nu(y) = \int_0^\infty e^{-y\cosh t}\cosh(\nu t)\,dt$$

are the classical Kloosterman sum and the K-Bessel function. For the desired asymptotic we recall $|\mathcal{S}(m,n;c)| \le \sqrt{c}\sqrt{\gcd(m,n,c)}\,\sigma_0(c) \le \sigma_0(c)\sqrt{c|n|}$, where $\sigma_0(c)$ is the number of positive divisors of c, which satisfies $\sigma_0(c) \ll c^{\epsilon}$ (see for example [9, p. 52]), and observe that for $y > 0$ and $\nu \in \mathbb{C}$

with $\Re(\nu) > 0$,

$$|K_\nu(y)| \le \int_0^\infty e^{-y\cosh t}|\cosh(\nu t)|\,dt \le \sqrt{2}\int_0^\infty e^{-y\cosh t}\cosh(\Re(\nu)t)\,dt.$$

Hence, the infinite series on the right hand side of (7.3) is bounded above by

$$\frac{2^{3/2}\pi^{\Re(u)}C_\epsilon}{|\Gamma(u)|}\sum_{n\in\mathbb{Z},\,n\neq 0}|n|^{\Re(u)}y^{1/2}\sum_{c=1}^\infty c^{-2\Re(u)+1/2+\epsilon}K_{\Re(u)-1/2}(2\pi|n|y)$$

where the constant C_ϵ comes from the estimate $\sigma_0(c) \ll c^\epsilon$. For every $u \in \mathbb{C}$ with $\Re(u) > 3/4+\epsilon$, write $C_{\Re(u),y,\epsilon}$ for

$$2^{3/2}\pi^{\Re(u)}C_{2\epsilon}\zeta(2\Re(u)-1/2-2\epsilon)\sum_{n\in\mathbb{Z}-\{0\}}|n|^{\Re(u)}y^{1/2}K_{\Re(u)-1/2}(2\pi|n|y).$$

This is a positive real number for every fixed $y, \epsilon > 0$, as the K-Bessel function has exponential decay. Moreover, $\lim_{y\to\infty} C_{\Re(u),y,\epsilon} = 0$ and

$$\left|\frac{2\pi^u}{\Gamma(u)}\right|\sum_{n\in\mathbb{Z},\,n\neq 0}\left||n|^{u-1/2}y^{1/2}\Big(\sum_{c=1}^\infty c^{-2u}\mathcal{S}(0,n;c)\Big)K_{u-1/2}(2\pi|n|y)e(nx)\right| \le \frac{C_{\Re(u),y,\epsilon}}{|\Gamma(u)|}.$$

This inequality and (7.3) show that the map

$$u \mapsto \mathcal{E}_{\Gamma_1(2m)}(\tau,u) - y^u - \pi^{1/2}\frac{\Gamma(u-1/2)\zeta(2u-1)}{\Gamma(u)\zeta(2u)}y^{1-u}$$

is holomorphic on the half-plane $\Re(u) > 3/4$ for any $\tau \in \mathcal{H}$.

This argument yields a meromorphic continuation of $\mathcal{E}_{\Gamma_1(2m)}(\tau,u)$, as a function of u, from $\Re(u) > 1$ to $\Re(u) > 3/4$ with a unique singularity in the larger half-plane. The singularity is a simple pole at $u = 1$ with residue $3/\pi$ which exactly comes from the second term on the right hand side of (7.3).

Now, the meromorphic continuation of $\Gamma(u-1/2)\zeta(2u-1)$ to the whole u-plane has exactly two singularities, one simple pole at $u = 1/2$ and another simple pole at $u = 1$. From [1, p. 270, eq. 30] we know that there is a real constant A (independent of u) such that $|\zeta(2u-1)| \le 1 + A\sqrt{|\Im(u)|}$ if $3/4 \le \Re(u) \le 3/2$ and $|\Im(u)| \ge 1$. Without loss of generality we can take $A > 1$, and change the notation to conclude $|\zeta(2u-1)| \le A\sqrt{|\Im(u)|}$ if $3/4 \le \Re(u) \le 3/2$ and $|\Im(u)| \ge 1$. Then

$$\left|\frac{\Gamma(u-1/2)}{\Gamma(u)}\zeta(2u-1)\right| \le 2A\sqrt{|\Im(u)|}$$

for all $u \in \mathbb{C}$ such that $3/4 \le \Re(u) \le 3/2$ and $|\Im(u)| \ge 1$. Here we have also used $|\Gamma(s-1/2)|/|\Gamma(s)| \le |\Gamma(\Re(s)-1/2)|/|\Gamma(\Re(s)|$, which one can see from the relations

$$\left|\frac{\Gamma(x)}{\Gamma(x+iy)}\right|^2 = \prod_{j=0}^{\infty}\left(1+\frac{y^2}{(x+j)^2}\right)$$

for $x, y \in \mathbb{R}$, and $1/2 < \Gamma(x_1) \le \Gamma(x_2)$ for $0 < x_1 \le x_2$. On the other hand,

$$\left|\frac{1}{\zeta(2u)}\right| \le \sum_{n=1}^{\infty}\left|\frac{\mu(n)}{n^{2u}}\right| \le \sum_{n=1}^{\infty}\frac{1}{n^{2\Re(u)}} = \zeta(2\Re(u))$$

whenever $\Re(u) > 1/2$. Thus

$$|\Gamma(u-1/2)\zeta(2u-1)/\Gamma(u)\zeta(2u)| \le 2A\zeta(2\Re(u))\sqrt{|\Im(u)|}$$

for all $u \in \mathbb{C}$ such that $3/4 \le \Re(u) \le 3/2$ and $|\Im(u)| \ge 1$.

Putting these inequalities and (7.3) together, we get

$$|\mathcal{E}_{\Gamma_1(2m)}(\tau,u)| \le y^{\Re(u)} + 2\pi^{1/2}A\zeta(2\Re(u))\sqrt{|\Im(u)|}\,y^{1-\Re(u)} + \frac{C_{\Re(u),y,\epsilon}}{|\Gamma(u)|}$$

for all $u \in \mathbb{C}$ as above. Finally, we combine this relation with (7.2), the fact that $\{\widetilde{e}_1, \ldots, \widetilde{e}_d\}$ is an orthonormal set, and that ρ^* is unitary, to get the first inequality in Lemma 6.2,

$$\|E_{u,j}(\tau)\| \le \sum_{l=1}^{\kappa}\Im(g_l\tau)^{\Re(u)} + 2\pi^{1/2}A\zeta(2\Re(u))\sqrt{|\Im(u)|}\sum_{l=1}^{\kappa}\Im(g_l\tau)^{1-\Re(u)} + \frac{\kappa C_{\Re(u),y,\epsilon}}{|\Gamma(u)|}.$$

The second inequality follows from the previous one, using the fact that $\Im(g\tau) \le \Im(\tau)$ for all $g \in \Gamma$ whenever $\tau \in \mathcal{F}$. ■

Proof of the Remark following Lemma 6.2. From (6.5) we get $E_{u,j}(g\tau) = \rho(g)E_{u,j}(\tau)$ for all $g \in \Gamma$, provided that $\Re(u) > 1$. Clearly this symmetry also holds for the meromorphic continuation of $E_{u,j}(\tau)$. On the other hand, equation (7.2), the Fourier representation (7.3), and the fact that y^u, y^{1-u}, and $2y^{1/2}K_{u-1/2}(2\pi y)e(x)$ are eigenfunctions of the Laplacian with eigenvalue $u(1-u)$ imply that $\Delta E_{u,j} = u(1-u)E_{u,j}$. Finally, the second statement in Lemma 6.2 yields the growth condition $E_{u,j,R}(\tau) = O(y^\alpha)$ as $y \to \infty$, with $\alpha = \Re(u)$, for the component functions of $E_{u,j}(\tau)$. These properties show that $E_{u,j}(\tau)$ is a Grössencharacter associated to ρ. ■

Proof of Lemma 6.3. From (3.5) one has

$$|\Lambda(F, L_\psi(u)E_{u,j}, s)| \le \sqrt{\pi}\left(\frac{\pi}{2m}\right)^{-2\Re(s)}|\Gamma(s-a)\Gamma(s-b)L_\psi(u)L(F,E_{u,j},s)|$$

where $a+b = 1/2$ and $(4a-1)^2 = 1+4\overline{u(1-u)}$, with $\Delta\overline{E_{u,j}} = \overline{u(1-u)}\;\overline{E_{u,j}}$.

If $\Re(u) = u_1 < 1$, the relations among a, b and u given above yield $\Re(a) \to \infty$ and $\Im(a) \to 0$ as $|\Im(u)| \to \infty$. These facts and Stirling's estimate for the gamma function imply the following:

For any $0 < \epsilon_1 < 1$ there exists a constant $C' > 0$ such that

$$(7.4)\quad |\Gamma(s-a)\Gamma(s-b)| \le C'(1-\epsilon_1)^{\beta-|\Im(u)|-1/2}|\Im(s+a)|^{2\beta-3/2}e^{-\pi|\Im(s)|}$$

for $s, u \in \mathbb{C}$ with $\Re(s) = \beta$ and $\Re(u) = u_1$, provided that $\Im(s) \gg 0$ and $\Im(u) \gg 0$. Now we consider (7.4) with $\epsilon_1 = 1/2$ and incorporate $\sqrt{2}$ in the constant. Then

$$|\Gamma(s-a)\Gamma(s-b)| \le C' 2^{|\Im(u)|-\beta}|\Im(s+a)|^{2\beta-3/2}e^{-\pi|\Im(s)|}.$$

If we use this relation in (3.5) and recall $\Im(a) \to 0$ as $|\Im(u)| \to \infty$, we conclude that there exists a constant $C_{\beta,u_1} > 0$ such that

$$(7.5)\quad |\Lambda(F, L_\psi(u)E_{u,j}, s)| \le C_{\beta,u_1} 2^{|\Im(u)|}|\Im(s)|^{2\beta-3/2}e^{-\pi|\Im(s)|}|L_\psi(u)L(F, E_{u,j}, s)|$$

whenever $|\Im(u)| \gg 0$. Furthermore, (3.3) and the Cauchy–Schwarz inequality yield

$$(7.6)\quad |L(F, E_{u,j}, s)| \le \sum_{T\in J/\Gamma} \|c(T)\|\,\|(\ldots, E_{u,j,R}(T^{-1}), \ldots)\|(\det T)^{-\beta}.$$

We have already seen the matrix identity $T^{-1} = (\det T)^{-1}T[S]$ for S as in (1.4), and we deduce from it $E_{u,j}(T^{-1}) = E_{u,j}((\det T)^{-1/2}T[S]) = \rho^*(S)E_{u,j}((\det T)^{-1/2}T)$. Therefore $\|E_{u,j}(T^{-1})\| = \|E_{u,j}((\det T)^{-1/2}T)\| = \|E_{u,j}(\tau)\|$ for $\tau = (-t + i\sqrt{4\det T})/(2t_1)$, where the entries of T are as in (1.7). Notice that $0 < \Im(\tau) = \sqrt{\det T}/t_1 \le \sqrt{\det T}$. Hence $\Im(\tau)^{u_1} \le (\det T)^{u_1/2}$ and $\Im(\tau)^{1-u_1} \le (\det T)^{(1-u_1)/2}$ for $3/4 < u_1 < 1$. Since τ can be taken in $\mathcal{F}$, we obtain from Lemma 6.2, the estimate in (1.3a), and (7.6), the existence of a constant $C_F > 0$ such that

$$(7.7)\quad |L(F, E_{u,j}, s)| \le C_F \sum_{T\in J/\Gamma}\left\{(\det T)^{u_1/2} + \frac{\zeta(2u_1)|\Im(u)|^{1/2}}{(\det T)^{(u_1-1)/2}} + \frac{C_{u_1,T}}{|\Gamma(u)|}\right\}(\det T)^{k/2-\beta}.$$

Here we have denoted by $C_{u_1,T}$ the map $C_{u_1,\sqrt{\det T}/t_1,\epsilon}$ in Lemma 6.2 which satisfies $\lim_{\det T\to\infty} C_{u_1,T} = 0$. (Observe that the condition $|\Im(u)| > 1$ has been dropped since $E_{u,j,R}(\tau)$ has no pole on the vertical line $\Re(u) = u_1$.)

Putting together (7.5) and (7.7) we get an explicit bound for the absolute value $|\Lambda(F, L_\psi(u)E_{u,j}, s)|$. This bound shows that the double integral

$$\int_{\Re(s)=\beta}\int_{\Re(u)=u_1} |\Lambda(F, L_\psi(u)E_{u,j}, s)t^s|\,du\,ds$$

is finite provided that the following conditions hold: $S(r)=\sum_{T\in J/\Gamma}(\det T)^{-r}$ is finite in the cases $r=\beta-(k+u_1)/2$ and $r=\beta-(k+1-u_1)/2$, the series $S'(r)=\sum_{T\in J/\Gamma}C_{u_1,T}(\det T)^{-r}$ is finite if $r=\beta-k/2$, and the double integrals

$$\int\limits_{\Re(s)=\beta}\int\limits_{\Re(u)=u_1} 2^{|\Im(u)|}|\Im(s)|^{2\beta-3/2}e^{-\pi|\Im(s)|}|L_\psi(u)|h(u)\,du\,ds$$

are finite whenever $h(u)$ is one of the functions in $\{1,|\Im(u)|^{1/2},|\Gamma(u)|^{-1}\}$.

The convergence of the series $S(r)$ and $S'(r)$ at the given arguments follows from the estimate $\#\{T\in J\mid \det(2T)=j\}/\mathrm{GL}_2(\mathbb{Z})\ll_\epsilon j^{1/2+\epsilon}$, a comparison with the Riemann zeta function, and the hypothesis $\beta>(k+6)/2$. For the convergence of the last three integrals it suffices to check the one with $h(u)=|\Gamma(u)|^{-1}$. By Stirling's estimate the integral is bounded above by

$$C'\int\limits_{\Re(s)=\beta}\int\limits_{\Re(u)=u_1}|\Im(s)|^{2\beta-3/2}e^{-\pi|\Im(s)|}|\Im(u)|^{u_1-1/2}|L_\psi(u)| \times e^{|\Im(u)|(\log 2+\pi/2)}\,du\,ds$$

for some constant $C'>0$. By the hypothesis on u_1 and β we know that $(u_1-1/2)\log|\Im(u)|<(2\beta-3/2)\log|\Im(u)|$ for $\log|\Im(u)|>0$, and we are able to pick a suitable constant C'' such that the last double integral is bounded above by

$$C''\int\limits_{\Re(s)=\beta}|\Im(s)|^{2\beta-3/2}e^{-\pi|\Im(s)|}\,ds\int\limits_{\Re(u)=u_1}|L_\psi(u)|\,|\Im(u)|^{2\beta-3/2}e^{3|\Im(u)|}\,du.$$

Both line integrals are finite (see (7.8) and the argument for it), and so Lemma 6.3 follows. ∎

Proof of Lemma 6.4 (sketch). The comments below (7.2) about the component functions of $E_{u,j}$ yield the existence of $3/4<u_1<1$ such that each $E_{u,j,R}(\tau)$ has at most one pole in the region $\Re(u)\ge u_1$, namely at $u=1$. Pick any such u_1.

By (6.4) there exists a constant $C=C_\psi>0$ such that

$$|L_\psi(u)e^{\pi|\Im(u)|/2}|\le C^N e^{1+2|\Im(u)|}(1+|\Im(u)|)^{-N} \tag{7.8}$$

for any positive integer N and all $u\in\mathbb{C}$ with $3/4\le\Re(u)\le 3/2$. From (7.8) a straightforward argument yields $|L_\psi(u)e^{\pi|\Im(u)|/2}|\to 0$ as $|\Im(u)|\to\infty$, uniformly for $\Re(u)$ in $[3/4,3/2]$. By this asymptotic relation, the last inequality in Lemma 6.2 and Stirling's estimate we have

$$|L_\psi(u)E_{u,j,R}(\tau)|\to 0 \quad \text{as } |\Im(u)|\to\infty, \tag{7.9}$$

uniformly for $\Re(u)$ in a closed interval containing u_1 and u_0, for any $\tau\in\mathcal{H}$.

Now we recall that $L_\psi(u)E_{u,j,R}(\tau)$ is a holomorphic function of u in a region containing the vertical strip $u_1\le\Re(u)\le u_0$, except perhaps at $u=1$.

This fact and (7.9) allow us to get the formula in Lemma 6.4 by shifting the contour of integration. ∎

8. An application to half-integral weight Siegel cusp forms. To prove Corollary 1.2 we recall some notation and a few facts. For further details we refer to [7]. The congruence subgroup $\Gamma_{2,0}(4)$ and the theta series $\theta(Z) : \mathcal{H}_2 \to \mathbb{C}$ are

$$\Gamma_{2,0}(4) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \;\middle|\; C \equiv 0 \pmod 4 \right\} \tag{8.1}$$

and $\theta(Z) = \sum_{v \in \mathbb{Z}^{2,1}} e(Z[v])$.

Then $\theta(M \cdot Z)^2/\theta(Z)^2 = \operatorname{sgn}(\det D) \det(CZ + D)$ for all $M \in \Gamma_{2,0}(4)$, and for any even positive integer k and any function $h : \mathcal{H}_2 \to \mathbb{C}$ the expression

$$h|_{k-1/2}[M](Z) = \frac{\theta(M \cdot Z)}{\theta(Z)} \det(CZ + D)^{-k} h(M \cdot Z)$$

defines a right action of $\Gamma_{2,0}(4)$ on the set of functions on $\mathcal{H}_2$.

Recall that a weight $k-1/2$ Siegel cusp form over $\Gamma_{2,0}(4)$ is any holomorphic function $h : \mathcal{H}_2 \to \mathbb{C}$ such that $h|_{k-1/2}[M] = h$ for all $M \in \Gamma_{2,0}(4)$ and h vanishes at every cusp. It has a Fourier series representation of type (1.5). The set of such forms is a finite-dimensional $\mathbb{C}$-vector space which we denote as $S_{k-1/2}(\Gamma_{2,0}(4))$. Kohnen's plus space $S^+_{k-1/2}(\Gamma_{2,0}(4))$ is the subspace given as

$$\{h \in S_{k-1/2}(\Gamma_{2,0}(4)) \mid d(T) = 0 \text{ if } T + \mu\mu^t \notin 4J \text{ for any } \mu \in \mathbb{Z}^{2,1}\}.$$

(In [7] this set is defined by the condition that $d(T) = 0$ if $(T + \mu\mu^t)/4$ is not half-integral and symmetric. In fact, both conditions are equivalent since $T > 0$ and $\mu\mu^t \geq 0$.)

In [7] T. Ibukiyama exhibits an explicit Hecke-invariant linear isomorphism between $J^{\mathrm{cusp}}_{2,k,1}$, the space of Jacobi cusp forms of weight k and index 1 over Γ_2, and $S^+_{k-1/2}(\Gamma_{2,0}(4))$ (Ibukiyama's proof is for arbitrary degree). Indeed, every $F(Z, z_1, z_2)$ in $J^{\mathrm{cusp}}_{2,k,1}$ has a theta decomposition

$$F(Z, z_1, z_2) = \sum_{R \in \mathcal{I}} F_R(Z) \Theta_{1,R}(Z, z_1, z_2), \quad \text{where} \quad \mathcal{I} = \{R_0, R_1, R_2, R_3\}$$

with $R_0^t = (0, 0)$, $R_1^t = (0, 1)$, $R_2^t = (1, 0)$ and $R_3^t = (1, 1)$. The isomorphism takes such an F to

$$h_F(Z) = \sum_{R \in \mathcal{I}} F_R(4Z) \in S^+_{k-1/2}(\Gamma_{2,0}(4)).$$

To define the inverse, take any $h(Z)$ in $S^+_{k-1/2}(\Gamma_{2,0}(4))$ with series representation (1.5) and define

$$h_R(Z) = \sum_N d(4N - RR^t)e((N - RR^t/4)Z) \quad \text{for every } R \in \mathcal{I}, \tag{8.2}$$

where N runs over the set J such that $4N - RR^t \in J$; the inverse then maps h to

$$F_h(Z, z_1, z_2) = \sum_{R\in\mathcal{I}} h_R(Z)\Theta_{1,R}(Z, z_1, z_2) \in J^{\text{cusp}}_{2,k,1}. \tag{8.3}$$

Now we recall the notion of a Grössencharacter for the 1-dimensional trivial representation of $\mathrm{PSL}_2(\mathbb{Z})$. This is any C^∞-function $\mathcal{V} : \mathcal{H} \to \mathbb{C}$ such that: (a) $\mathcal{V}(g\tau) = \mathcal{V}(\tau)$ for any $g \in \Gamma$, (b) $\mathcal{V}$ is an eigenfunction of the hyperbolic Laplacian Δ, and (c) $\mathcal{V}$ has moderate growth at infinity, i.e. there exists $\alpha > 0$ such that $\mathcal{V}(\tau) = O(y^\alpha)$ as $y = \Im(\tau) \to \infty$.

Definition 8.1. Let $\{d(T) \mid T \in J,\ T + \mu\mu^t \in 4J$ for some $\mu \in \mathbb{Z}^{2,1}\}$ be a sequence of complex numbers satisfying (1.6a), (1.6b).

For any Grössencharacter $\mathcal{V}$ associated to the trivial representation set

$$L(h, \mathcal{V}, s) = \sum_{T\in J/\Gamma} \frac{1}{\xi_1(T)} d(T)\overline{\mathcal{V}(T^{-1})}(\det T)^{-s} \quad (s \in \mathbb{C},\ \Re(s) \gg 0) \tag{8.4}$$

where $\xi_1(T)$ is the positive integer introduced after (3.3), and $\mathcal{V}$ is evaluated at T^{-1} via the identification (3.1). The completed Dirichlet series for h and $\mathcal{V}$ is

$$\Lambda(h, \mathcal{V}, s) = (\pi/2)^{-2s}\Gamma(s - a)\Gamma(s - b)L(h, \mathcal{V}, s), \tag{8.5}$$

where a, b are defined by the eigenvalue λ in $\Delta\overline{\mathcal{V}} = \lambda\overline{\mathcal{V}}$, as in Definition 3.4. Similarly, for the subsequence $\{d(T) \mid 4T \in J\}$ and any Grössencharacter $\mathcal{V}$ as above one puts

$$L_0(h, \mathcal{V}, s) = \sum_{4T\in J/\Gamma} \frac{1}{\xi_1(T)} d(T)\overline{\mathcal{V}(T^{-1})}(\det T)^{-s},$$

and $\Lambda_0(h, \mathcal{V}, s) = (\pi/2)^{-2s}\Gamma(s - a)\Gamma(s - b)L_0(h, \mathcal{V}, s)$.

Proof of Corollary 1.2 (sketch). First, we note that any Grössencharacter $\mathcal{V} : \mathcal{H} \to \mathbb{C}$ associated to the trivial representation defines a Grössencharacter $\widetilde{\mathcal{V}} : \mathcal{H} \to V$ for the representation ρ in (2.10) with $m = 1$ by way of $\widetilde{\mathcal{V}}(\tau) = (\mathcal{V}(\tau), \mathcal{V}(\tau), \mathcal{V}(\tau), \mathcal{V}(\tau))$.

On the other hand, if $\mathcal{U} : \mathcal{H} \to V$ is any Grössencharacter associated to ρ as in Definition 3.1 with $\mathcal{U} = (\mathcal{U}_{R_0}, \mathcal{U}_{R_1}, \mathcal{U}_{R_2}, \mathcal{U}_{R_3})$, then $\mathcal{U}'(\tau) = \sum_{R\in\mathcal{I}} \mathcal{U}_R(\tau)$ defines a scalar Grössencharacter for the trivial representation.

Let $\{d(T) \mid T \in J\}$ be any sequence as in Definition 8.1. If we recall the Fourier series $h(Z)$ in (1.5), $h_R(Z)$ in (8.2), and $F_h(Z)$ in (8.3) plus the corresponding Dirichlet series, it is easy to check that $L(h, \mathcal{V}, s) = L(F_h, \widetilde{\mathcal{V}}, s)$ for any scalar Grössencharacter $\mathcal{V}$ as in the corollary, and also $L(F_h, \mathcal{U}, s) = L(h, \mathcal{U}', s)$ for any Grössencharacter $\mathcal{U}$ associated to ρ. (Here one uses the fact that for every $T \in J$ there is at most one $0 \leq j \leq 3$ such that $T + R_j R_j^t \in 4J$.) Consequently, we have $\Lambda(h, \mathcal{V}, s) = \pi^{-1/2} \Lambda(F_h, \widetilde{\mathcal{V}}, s)$ and $\Lambda_0(h, \mathcal{V}, s) = \pi^{-1/2} \Lambda(F_h, (\mathcal{V}, 0, 0, 0), s)$. Finally, we observe that $\widetilde{\mathcal{V}} A_1 = (2\mathcal{V}, 0, 0, 0)$ where A_1 is the matrix defined in (1.4), and so the functional equation in Theorem 1.1 can be written as

$$\Lambda(F_h, \widetilde{\mathcal{V}}, k - s - 1/2) = \Lambda(F_h, (2\mathcal{V}, 0, 0, 0), s).$$

This is equivalent to the identity $\Lambda(h, \mathcal{V}, k - 1/2 - s) = \Lambda_0(h, 2\mathcal{V}, s)$. ∎

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