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Extendability and domains of holomorphy in infinite-dimensional spaces

Richard M. Aron (Kent, OH), Stéphane Charpentier (Marseille), Paul M. Gauthier (Montréal), Manuel Maestre (Valencia) and Vassili Nestoridis (Athens)

Dedicated to the memory of Professor Józef Siciak

Abstract. We study the notions of extendability and domain of holomorphy in the infinite-dimensional case. In this setting it is also true that the notions of domain of holomorphy and weak domain of holomorphy are equivalent. We also prove that the set of non-extendable functions belonging to some classes $X(B) \subset H(B)$, B being the open unit ball in a separable complex Banach space, is a lineable and dense G_δ. Moreover, when Ω is H_b-holomorphically convex (defined in the text), it is shown that the set of non-extendable holomorphic functions on Ω is a lineable and dense G_δ set.

1. Introduction. It is well-known (see, e.g., [20]) that the notion of domain of holomorphy and of weak domain of holomorphy are equivalent in $\mathbb{C}^d$; the first proof was constructive and by no means elementary. In [13], Baire's theorem was combined with a theorem of Banach to give a simpler proof of the above equivalence. In addition it was proved that the set of non-extendable functions contains a dense G_δ set. In [17, 18] Banach's theorem was replaced by Montel's theorem, and with a very simple proof it was also shown that the set of non-extendable functions is itself a dense G_δ set.

In the latter proof it was essential that the notion of extendable function be formulated in an equivalent way so that the holomorphic extension is bounded on balls. In the proof in [17, 18], the fact that closed balls in finite dimension are compact sets was used. But in infinite dimension closed balls are no longer compact. Therefore, in order to extend the above results to

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infinite-dimensional holomorphy, we need a different proof. In the present paper, we do this by using the fact that every holomorphic function on an open subset Ω of a complex Banach space is continuous; therefore, if $p \in \Omega$ then there exists a ball centered at p where the function is bounded.

This enables us to extend the result of [17, 18] to the case of infinite-dimensional holomorphy; in fact, we prove a more general version (Theorem 4.3) which allows us to reduce this case to that of one complex variable. As an application, we prove that, for every separable complex Banach space E, there are non-extendable holomorphic functions on its open ball $B \subset E$. Furthermore, the set of non-extendable functions belonging to the spaces $X(B) = H_b(B)$ or $X(B) = H^\infty(B) \cap A(B)$ (see Section 2 for the definitions) is a dense G_δ set in $X(B)$ endowed with its natural topology. Using ideas of Valdivia, we are also able to prove, under some assumptions on the open set $\Omega \subset E$, that the set of non-extendable functions in $H_b(\Omega)$ is lineable, i.e. it contains an infinite-dimensional vector subspace minus the zero function.

Section 2 contains preliminary results about infinite-dimensional holomorphy. Section 3 contains the extension to the infinite-dimensional case of an equivalent formulation of the notion of non-extendable function. Section 4 contains our main result mentioned above. Finally, in Section 5, we discuss lineability and "residuality" of some natural classes of non-extendable holomorphic functions on domains in separable Banach spaces.

2. Preliminaries. Let E be a complex Banach space (possibly of finite dimension). Let $\Omega \subset E$ be a domain (i.e. an open connected set). A function $f : \Omega \to \mathbb{C}$ is called *holomorphic* on Ω ($f \in H(\Omega)$) if it is complex Fréchet differentiable at every point of Ω [5], that is, for each $a \in \Omega$ there exists a continuous complex-linear form $\varphi_a : E \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \varphi_a(h)}{\|h\|} = 0.$$

Usually, φ_a is denoted by df_a and called the *differential* of f at a.

Equivalently, $f : \Omega \to \mathbb{C}$ is holomorphic on Ω if and only if it is continuous and *Gâteaux differentiable* [5, 16] at each point $a \in \Omega$, that is, for each $v \in E$ the limit

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

exists.

It follows that $f \in H(\Omega)$ if and only if f is continuous on Ω and $f \in H(\Omega \cap L)$ for every complex line $L \subset E$ intersecting Ω. Obviously, every $\varphi \in E^*$ (the topological dual of E) belongs to $H(E)$.

A function $P : E \to \mathbb{C}$ is called an *m-homogeneous* (continuous) *polynomial* if there exists an m-linear (continuous) form $A : E^m \to \mathbb{C}$ such that $P(x) = A(x, \ldots, x)$ for every $x \in E$.

It is also known that f belongs to $H(\Omega)$ if and only if it has a power series development around any point of Ω [5, 16], in the following sense: For every $a \in \Omega$ there exists a sequence $(P_m(a))_{m=1}^{\infty}$ of continuous m-homogeneous polynomials on E and $r = r(a) > 0$ such that the ball $B(a, r)$ centered at a and of radius r is included in Ω and, for every $x \in B(a, r)$,

$$f(x) - f(a) = \sum_{m=1}^{\infty} P_m(a)(x - a).$$

The infinite-dimensional version of Montel's theorem (see [6, Lemma 3.25], [7, Lemma 3.37] or [16, Proposition 9.16]) states that any given family $A \subset H(\Omega)$ that is uniformly bounded on each compact subset of Ω is relatively compact with respect to the compact-open topology in $H(\Omega)$. If moreover the Banach space E is separable, this result has the following formulation: Each sequence $(f_n) \subset H(\Omega)$ which is uniformly bounded on each compact subset of Ω has a subsequence $(f_{n_k})_{k=1}^{\infty}$ converging to some $f \in H(\Omega)$ uniformly on each compact subset of Ω ([5, Theorem 2.12] or [16, Exercise 9.F]). Montel's theorem has been used towards generic results of non-extendability in [3, 4, 9, 15].

It is also true that if $f_n \in H(\Omega)$, $n = 1, 2, \ldots$, converges to some function f uniformly on each compact subset of Ω, then $f \in H(\Omega)$. Indeed, if a sequence $z_j \in \Omega$, $j = 1, 2, \ldots$, converges to $a \in \Omega$, then f_n converges uniformly to f on the compact set $\{a\} \cup \{z_j : j = 1, 2, \ldots\}$, hence $\lim_j f(z_j) = f(a)$. This shows that f is continuous on Ω. Further, by the Weierstrass theorem in one complex variable, $f|_{L\cap\Omega} \in H(L\cap\Omega)$ for every complex line $L \subset E$. As already mentioned, this implies $f \in H(\Omega)$. Thus, the Weierstrass theorem extends to the infinite-dimensional case as well.

The space $H_b(\Omega)$ consists of all $f \in H(\Omega)$ which are bounded on each bounded set $M \subset \Omega$ such that $\operatorname{dist}(M, E \setminus \Omega) > 0$ [6, Section 6.3]. The space $H_b(\Omega)$ is a Fréchet space if we consider the following seminorms on it:

$$\|f\|_n = \sup_{z\in M_n} |f(z)|,$$

where $M_n = \{z \in \Omega : \|z\| \le n,\ \operatorname{dist}(z, E\setminus\Omega) \ge 1/n\}$ (see [6, Section 6.3] or [8]).

The set $H^{\infty}(\Omega)$ consists of all holomorphic functions $f \in H(\Omega)$ which are bounded on Ω. It becomes a Banach space if endowed with the norm $\|f\|_{\infty} = \sup_{z\in\Omega} |f(z)|$.

The set $A(\Omega)$ consists of all functions $f : \overline{\Omega} \to \mathbb{C}$ that are continuous on $\overline{\Omega}$ and holomorphic in its interior. The set $A(\Omega) \cap H^{\infty}(\Omega)$, with the norm $\|f\|_{\infty} = \sup_{z\in\Omega} |f(z)|$, is a Banach space.

3. Extendability, domain of holomorphy and weak domain of holomorphy. Let E be a complex Banach space, for instance let $E = \mathbb{C}^d$, $d \geq 1$, or let E be an infinite-dimensional Banach space. In order to extend the notions of domain of holomorphy and *weak* domain of holomorphy to the infinite-dimensional setting, we proceed as in [17, 18] by showing that the natural notion of holomorphic extendability can be defined in two equivalent ways.

PROPOSITION 3.1. *Let $\Omega \subset E$ be a domain and let f be a holomorphic function on Ω. The following assertions* (a) *and* (b) *are equivalent.*

(a) *There exists an open connected set $U \subset E$ with $U \cap \partial\Omega \neq \emptyset$ and a function $F : U \to \mathbb{C}$ holomorphic on U such that, for some connected component V of $U \cap \Omega$, we have $f(z) = F(z)$ for all $z \in V$.*
(b) *There exist open balls b, B in E with $b \subset \overline{b} \subset B \cap \Omega$ and $B \cap (E \setminus \Omega) \neq \emptyset$ and a bounded holomorphic function $F : B \to \mathbb{C}$ such that $f(z) = F(z)$ for all $z \in b$.*

The proof is based on the following simple lemma.

LEMMA 3.2. *Let $U \subset E$ be a domain such that $U \cap \partial\Omega \neq \emptyset$. For every connected component V of $U \cap \Omega$, we have $\overline{V} \cap \partial\Omega \cap U \neq \emptyset$.*

Proof. Let $z_1 \in V \subset U \cap \Omega$ and $z_2 \in U \cap (E \setminus \Omega)$. Since U is a domain, it is path connected, so there is a continuous path $\gamma : [0,1] \to U$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$. Since $z_1 \in \Omega$ and $z_2 \in E \setminus \Omega$, there exists $s \in (0,1]$ such that $P = \gamma(s) \in \partial\Omega \cap U$. Thus, upon defining t_0 as the infimum of such s in $[0,1)$, we clearly have $\gamma(t_0) \in \partial\Omega \cap U$ and $\gamma(t) \in V$ for all $t \in [0, t_0)$. It follows that $P := \lim_{t \to t_0^-} \gamma(t) \in \partial\Omega \cap \overline{V} \cap U$. ∎

Proof of Proposition 3.1. First assume that for a given $f \in H(\Omega)$, there exist U, V, F such that f and F coincide on V, as in (a). Let $P \in \overline{V} \cap \partial\Omega \cap U$ be given by the previous lemma. Since every holomorphic function is continuous, there exists $r \in (0, +\infty)$ such that the ball $B = B(P, r)$ with center P and radius r is contained in U and $|F(z) - F(P)| < 1$ for all $z \in B$. In particular, F is bounded on B. Moreover, since $P \in \overline{V}$ and V is open, there exist $w \in B \cap V$ and $\delta > 0$ such that the ball $b = B(w, \delta)$ is contained in V. Thus $f(z) = F(z)$ for all $z \in b$, hence (b) holds.

Conversely, if we start with the balls b and B and a bounded holomorphic function $F : B \to \mathbb{C}$ such that $f(z) = F(z)$ for all $z \in b$, then, by the principle of analytic continuation (see [5, Corollary 15.38] or [16, Proposition 5.7]), (a) is satisfied with $U = B$. ∎

Proposition 3.1 allows us to extend the notion of extendability in two natural equivalent ways.

Definition 3.3. Let $\Omega \subset E$ be a domain and let f be a holomorphic function on Ω. Then f is said to be *extendable* if it satisfies (a), or equivalently (b), in Proposition 3.1. Otherwise, f will be said to be *non-extendable.*

The interest of these two variants of the same definition is that each of them gives rise to a priori different notions of *domain of holomorphy* and *weak domain of holomorphy.*

Let $X(\Omega)$ be a subset of $H(\Omega)$.

Definition 3.4. Let $\Omega \subset E$ be a domain. We say that Ω is:

(i) an $X(\Omega)$-*domain of holomorphy* if there exists $f \in X(\Omega)$ which is non-extendable in the sense of Definition 3.3;
(ii) a *weak* $X(\Omega)$-*domain of holomorphy* if the following holds: for any open balls $b, B \subset E$ satisfying $b \subset \overline{b} \subset B \cap \Omega$ and $B \cap (E \setminus \Omega) \neq \emptyset$ there exists $f_{b,B} \in X(\Omega)$ such that the restriction $f_{b,B}|_b$ has no bounded holomorphic extension to B.

These definitions coincide with the finite-dimensional versions of domain of holomorphy and weak domain of holomorphy studied in [13, 20] for $X(\Omega) = H(\Omega)$, and more generally in [17, 18]. Obviously, any $X(\Omega)$-domain of holomorphy is a weak $X(\Omega)$-domain of holomorphy. As already mentioned, these notions in fact coincide in $\mathbb{C}^N$, $N \geq 1$, under some natural assumption on $X(\Omega)$. The purpose of the next section is to prove that they also do in the infinite-dimensional setting.

4. Main results. Let E be a complex Banach space and $\Omega \subset E$ a domain. The main result of the paper is the following.

Theorem 4.1. *Let V be a complete separable metric topological vector space. Let $T : V \to H(\Omega)$ be a linear mapping such that the convergence $a_n \to a$ of a sequence in V implies pointwise convergence $T(a_n)(z) \to T(a)(z)$ for all $z \in \Omega$. Assume that, for any open balls $b, B \subset E$ such that $b \subset \overline{b} \subset \Omega \cap B$ and $B \cap (E \setminus \Omega) \neq \emptyset$, there exists $a_{b,B} \in V$ such that $T(a_{b,B})|_b$ admits no bounded holomorphic extension on B. Then the set $S = \{a \in V : T(a)$ is non-extendable$\}$ is a dense G_δ subset of V. In particular, Ω is a $T(V)$-domain of holomorphy.*

Let now $X(\Omega)$ be a subset of $H(\Omega)$, and assume that $X(\Omega)$ is endowed with a distance which makes the corresponding topology complete and stronger than that of pointwise convergence on Ω. As a corollary of Theorem 4.1, we immediately get the following.

Corollary 4.2. *Under the above assumption, any weak $X(\Omega)$-domain of holomorphy is a domain of holomorphy.*

Moreover, the assumptions of Theorem 4.1 are trivially satisfied whenever there exists a single a in V such that $T(a)$ is non-extendable. Therefore, we have the following corollary.

COROLLARY 4.3. *Let V be a complete separable metric vector space. Let $T: V \to H(\Omega)$ be a linear mapping such that the convergence $a_n \to a$ of a sequence in V implies the pointwise convergence $T(a_n)(z) \to T(a)(z)$ for all $z \in \Omega$. If there exists $a \in V$ such that $T(a)$ is non-extendable, then the set of non-extendable functions is a dense G_δ subset of V.*

Proof of Theorem 4.1. Let P be the set of pairs (b, B) of balls as in the statement of Theorem 4.1. For a fixed $(b, B) \in P$, set

$$Y_{b,B} = \{a \in V : T(a)|_b \text{ admits a bounded and holomorphic extension on } B\}, \tag{1}$$

and for $M = 1, 2, \ldots$, define

$$Y_{b,B,M} = \{a \in V : T(a)|_b \text{ admits a holomorphic extension on } B \text{ bounded by } M\}. \tag{2}$$

Then, $Y_{b,B} = \bigcup_{M=1}^{\infty} Y_{b,B,M}$ and, by Proposition 3.1,

$$V \setminus S = \bigcup_{(b,B)\in P} Y_{b,B}.$$

We shall show that each $Y_{b,B,M}$ is closed in V with empty interior. Then, since E is separable, we can replace the pairs $(b, B) \in P$ by denumerably many (b_k, B_k), $k = 1, 2, \ldots$, so that $V \setminus S = \bigcup_{k=1}^{\infty} Y_{b_k,B_k}$. Hence $S = \bigcap_{k=1}^{\infty} V \setminus Y_{b_k,B_k}$ is a dense G_δ in V by the Baire theorem.

To prove that $Y_{b,B,M}$ is closed for each fixed b, B and M, let $(a_n) \subset Y_{b,B,M}$ be a sequence convergent to $a \in V$. There exists a sequence (F_n) of holomorphic functions bounded by M on B such that $T(a_n)(z) = F_n(z)$ for all $z \in b$. By Montel's theorem, there exists a subsequence n_k of natural numbers such that $F_{n_k}(z) \to F(z)$ for all $z \in B$, where F is holomorphic on B and bounded by M. By assumption, the convergence $a_n \to a$ in V implies $T(a_n)(z) \to T(a)(z)$ for all z in Ω, and in particular for all $z \in b$. Thus, for $z \in b$ we have

$$T(a)(z) = \lim_n T(a_n)(z) = \lim_k T(a_{n_k})(z) = \lim_k F(a_{n_k})(z) = F(z).$$

It follows that $a \in Y_{b,B,M}$; this proves that $Y_{b,B,M}$ is closed in V.

It remains to show that $\operatorname{int}(Y_{b,B,M}) = \emptyset$. Assume that $\operatorname{int}(Y_{b,B,M}) \neq \emptyset$. By hypothesis, there exists $a_{b,B} \in V$ such that $T(a_{b,B})|_b$ admits no bounded holomorphic extension on B. Let $a \in \operatorname{int}(Y_{b,B,M}) \subset Y_{b,B,M}$ and consider the sequence $a + \frac{1}{n} a_{b,B}$. This sequence converges to $a \in V$ as $n \to \infty$. Since $\operatorname{int}(Y_{b,B,M})$ is an open subset of V, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$a + \frac{1}{n_0} a_{b,B} \in \mathrm{int}(Y_{b,B,M}) \subset Y_{b,B,M}.$$

Thus, $T(a)|_b$ and $T\big(a+\frac{1}{n_0}a_{b,B}\big)\big|_b$ have holomorphic extensions on B bounded by M. But, by the linearity of T,

$$T\bigg(a + \frac{1}{n_0} a_{b,B}\bigg) = T(a) + \frac{1}{n_0} T(a_{b,B}).$$

It follows that $T(a_{b,B})|_b$ admits a holomorphic extension on B bounded by $2Mn_0$; this contradicts our assumption about $a_{b,B}$. Thus $\mathrm{int}(Y_{b,B,M}) = \emptyset$ and the proof is complete. ■

Remark 4.4. In fact, in the statement of Theorem 4.1, we do not need the completeness assumption on V if we only intend to prove that any set $S_{b,B} = \{a \in V : T(a)|_b$ has no bounded holomorphic extension on $B\}$, with b and B as in Theorem 4.1, is a dense G_δ in V. Indeed, without the use of the Baire theorem, one can always write $V \setminus S_{b,B}$ as an F_σ, and imitate the proof of Theorem 4.1 to check that this set is dense in V.

Let us give some applications of the above results.

Applications. Let E be a separable complex Banach space and $\widetilde{\Omega}$ be its open unit ball. Let $V = X(\widetilde{\Omega})$ where $X(\widetilde{\Omega}) = H_b(\widetilde{\Omega})$, $X(\widetilde{\Omega}) = A(\widetilde{\Omega}) \cap H^\infty(\widetilde{\Omega})$ or $X(\widetilde{\Omega}) = A_u(\widetilde{\Omega})$. Here, $H_b(\Omega)$ is the set of holomorphic functions $f : \Omega \to \mathbb{C}$ such that f is bounded on every Ω-bounded set; recall that a bounded subset $A \subset E$ is said to be *Ω-bounded* if $\inf_{x\in A} \mathrm{dist}(x, E \setminus \Omega) > 0$ (see [6, Section 6.3] or [8]). $A_u(\widetilde{\Omega})$ denotes the closed subalgebra of $A(\widetilde{\Omega}) \cap H^\infty(\widetilde{\Omega})$ of all f uniformly continuous on $\overline{\widetilde{\Omega}}$ and holomorphic in its interior $\widetilde{\Omega}$.

Let L be a complex line in E containing the origin. We set $V = X(\widetilde{\Omega})$ and consider the map $T : V \to H(\Omega)$ such that $T(f) = f|_{L\cap\widetilde{\Omega}}$, where $\Omega = L \cap \widetilde{\Omega}$. Let u be in $\overline{L \cap \widetilde{\Omega}}$ with $\|u\| = 1$, and let $\varphi \in E^*$ be such that $\varphi(u) = 1$ and $\|\varphi\| = 1$. By [2, 14, 19], there exists a non-extendable holomorphic function $g : \mathbb{D} \to \mathbb{C}$ in $A(\mathbb{D})$ $(= A_u(\mathbb{D}))$ (or even in $A^\infty(\mathbb{D})$). Then we set $f(z) = g(\varphi(z))$. One can easily verify that $f \in V$ and $f|_{L\cap\widetilde{\Omega}}$ is non-extendable. Thus Corollary 4.3 implies that the set $\{f \in V : T(f)$ is not extendable$\}$ is a dense G_δ subset of $V = X(\widetilde{\Omega})$.

Furthermore, since E is separable we can find a sequence $v_n \in E$, $n = 1, 2, \ldots$, $\|v_n\| = 1$, which is dense in the unit sphere $\partial\widetilde{\Omega}$ of E. We set $L_n = \{\lambda v_n : \lambda \in \mathbb{C}\}$. Since $V = X(\widetilde{\Omega})$ is complete, Baire's theorem and the preceding argument show that the set of functions $f \in X(\widetilde{\Omega})$ such that $f|_{L_n\cap\widetilde{\Omega}}$ is non-extendable for all n is a dense G_δ in $X(\widetilde{\Omega})$.

Clearly, these holomorphic functions on $\widetilde{\Omega}$ are non-extendable, and hence the set of non-extendable functions of $X(\widetilde{\Omega})$ is residual. In passing, we have also proved that $\widetilde{\Omega}$ is an $X(\widetilde{\Omega})$-domain of holomorphy. We should mention that a function $f \in H(\mathbb{B}_N)$ was constructed in [10, 11] such that $f|_{L\cap\mathbb{B}_N}$ is non-extendable for any complex line L (here, $\mathbb{B}_N$ is the unit ball in $\mathbb{C}^N$). We do not know whether the set of such functions is topologically generic.

5. Lineability of non-extendable holomorphic functions. In [1, §3], the authors proved the following result in the context of a domain of holomorphy $U \subsetneq \mathbb{C}^n$.

THEOREM 5.1. *There is a dense subspace $X \subset H(U)$ such that if f is in $X \setminus \{0\}$, then f is non-extendable.*

In this section, we prove a weaker form of this result in the infinite-dimensional context, about lineability for a certain natural set of non-extendable holomorphic functions on certain open subsets Ω of separable complex Banach spaces E. More precisely, we will prove under some natural assumption on Ω that there exists an infinite-dimensional vector subspace of $H_b(\Omega)$, every non-zero element of which is nowhere extendable.

Recall that the space $H_b(\Omega)$ of holomorphic functions was defined in the "Applications" subsection at the end of Section 4. We now describe properties of the open set $\Omega \subset E$. Given a subset $A \subset \Omega$ which is Ω-bounded, the set

$$\Big\{z \in \Omega : |f(z)| \le \sup_{x\in A} |f(x)| \text{ for all } f \in H_b(\Omega)\Big\}$$

is called the *H_b-hull of A*. The set Ω is said to be *H_b-holomorphically convex* provided the H_b-hull of every Ω-bounded set is also Ω-bounded. Examples of H_b-holomorphically convex sets include all open, convex sets, as well as all domains of holomorphy when E is finite-dimensional.

The following is our main result in connection with lineability of non-extendable holomorphic functions. We recall that a set S is said to be *lineable* if $S \cup \{0\}$ contains an infinite-dimensional vector subspace.

THEOREM 5.2. *Let E be a separable complex Banach space and $\Omega \neq E$ an H_b-holomorphically convex domain in E. Then the set of functions in $H_b(\Omega)$ that are non-extendable is lineable.*

Proof. The proof is based on Theorems 3 and 5 of [21]. Theorem 3 is an interpolation result: *Given an H_b-holomorphically convex subset Ω of a normed space E, if (x_n) is a sequence of distinct points of Ω such that* $\lim_{n\to\infty} \operatorname{dist}(x_n, \partial\Omega) = 0$ *and (u_n) is a sequence complex numbers, then there exists f in $H_b(\Omega)$ such that $f(x_n) = u_n$, $n = 1, 2, \ldots$.* On the other hand, Theorem 5 concerns the existence of non-extendable functions: *If Ω is an H_b-holomorphically convex subset of an arbitrary complex separable Banach* *space E, then there exists $f \in H_b(\Omega)$ such that f is non-extendable, and $f(\Omega)$ is dense in* $\mathbb{C}$. Let $A \subset \partial\Omega$ be a countable dense set. Given $n \in \mathbb{N}$, let $\mathcal{P}_n^*$ denote the (countable) set of connected components of $B(x, 1/n) \cap \Omega$, $x \in A$. Let $\mathcal{P} = \bigcup_n \mathcal{P}_n^*$. For each $D \in \mathcal{P}$, let $P(D)$ denote a countable dense subset of $\partial D \cap \partial\Omega$. Obviously the set $\mathcal{T}$ is also countable, where

$$\mathcal{T} = \{(z, D) : D \in \mathcal{P},\ z \in P(D)\}.$$

Let us enumerate $\mathcal{T}$ as $\mathcal{T} = \{(z_n, D_n) : n \in \mathbb{N}\}$.

For each $m \in \mathbb{N}$, we choose a sequence $(x_{m,n})_n \subset \Omega$ having the following properties:

(i) $x_{m,n} \in D_m$ for each n and m;
(ii) $\lim_{n\to\infty} x_{m,n}$ exists and equals z_m;
(iii) $x_{m,n} \neq x_{m',n'}$ whenever $(m, n) \neq (m', n')$;
(iv) for every m, $x_{m,n} \notin B_m$, where B_m is the H_b-hull of the set $\{z \in \Omega : \|z\| \leq m,\ \mathrm{dist}(z, E \setminus \Omega) \geq 1/m\}$.

Next, split $\mathbb{N}$ into a countable set of infinite, pairwise disjoint sets, $\mathbb{N} = \bigcup_{k=1}^{\infty} \{n_{k,\ell} : \ell = 1, 2, \ldots\}$, in such a way that for each k, the sequence $(n_{k,\ell})_\ell$ is strictly increasing. Thus, for each fixed k and m, we have $\lim_{\ell\to\infty} x_{m,n_{k,\ell}} = z_m$, where each $x_{m,n_{k,\ell}}$ belongs to $D_m \setminus B_m$. Now we apply [21, Theorem 3] with the dense sequence (u_n) chosen so that for each k, the set $\{u_{n_{k,\ell}} : \ell \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Thus, for each k, there is $f_k \in H_b(\Omega)$ such that

- $f_k(x_{1,n_{k,\ell}}) = u_{n_{k,\ell}}$ for $\ell \in \mathbb{N}$;
- $f_k(x_{m,n_{k,\ell}}) = n_{k,\ell} + m$ for $\ell \in \mathbb{N}$ and $m \geq 2$;
- $f_k(x_{m,n_{r,\ell}}) = 0$ for all ℓ and m whenever $r \neq k$.

We set $V = \mathrm{span}\{f_k : k \in \mathbb{N}\}$ and observe that by construction, V is infinite-dimensional. By [21, proof of Theorem 5], each f_k is non-extendable. Furthermore, each non-zero function in V is also non-extendable. To see this, let us pick $g = \alpha_1 f_1 + \cdots + \alpha_j f_j \in V \setminus \{0\}$ with $\alpha_j \neq 0$ for some j. Then

$$\frac{1}{\alpha_j} g = \frac{\alpha_1}{\alpha_j} f_1 + \cdots + f_j.$$

However, for all $\ell, m \in \mathbb{N}$, $\frac{1}{\alpha_j} g(x_{m,n_{j,\ell}}) = f_j(x_{m,n_{j,\ell}})$ for all $\ell, m \in \mathbb{N}$. By [21, proof of Theorem 5], we conclude that $\frac{1}{\alpha_j} g$ is not extendable. ∎

Combining Theorem 4.1 with Theorem 5.2, we deduce

COROLLARY 5.3. *Let E be a separable complex Banach space and $\Omega \neq E$ an H_b-holomorphically convex domain in E. Then the set of functions in $H_b(\Omega)$ that are non-extendable is a lineable dense G_δ set.*

REMARK 5.4. In [12, Proposition 2], Hirschowitz gives an example of a non-separable Banach space whose open unit ball is not an H_b-domain of holomorphy. Hence the above theorem on lineability cannot, in general, be extended to the non-separable case.

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References

[1] R. M. Aron, D. García, and M. Maestre, *Linearity in non-linear problems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 95 (2001), 7–12.

[2] L. Bernal-González, *Linear Kierst–Szpilrajn theorems*, Studia Math. 166 (2005), 55–69.

[3] E. Bolkas, V. Nestoridis and Ch. Panagiotis, *Non extendability from any side of the domain of definition as a generic property of smooth or simply continuous functions on an analytic curve*, arXiv:1511.08584 (2015).

[4] E. Bolkas, V. Nestoridis, Ch. Panagiotis and M. Papadimitrakis, *One side extendability and p-continuous analytic capacities*, J. Geom. Anal. 29 (2019), 1369–1406.

[5] A. Defant, D. García, M. Maestre and P. Sevilla-Peris, *Dirichlet Series and Holomorphic Functions in High Dimensions*, New Math. Monogr. Ser., Cambridge Univ. Press, to appear.

[6] S. Dineen, *Complex Analysis on Infinite-Dimensional Spaces*, Springer Monogr. Math., Springer London, London, 1999.

[7] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Stud. 57, North-Holland, Amsterdam, 1981.

[8] P. Galindo, D. García and M. Maestre, *Holomorphic mappings of bounded type*, J. Math. Anal. Appl. 166 (1992), 236–246.

[9] P. M. Gauthier, *Non-extendable zero sets of harmonic and holomorphic functions*, Canad. Math. Bull 59 (2016), 303–310.

[10] J. Globevnik and E. L. Stout, *Holomorphic functions with highly noncontinuable boundary behavior*, J. Anal. Math. 41 (1982), 211–216.

[11] J. Globevnik and E. L. Stout, *Highly noncontinuable functions on convex domains*, Bull. Sci. Math. (2) 104 (1980), 417–434.

[12] A. Hirschowitz, *Sur le non-plongement des variétés analytiques banachiques réelles*, C. R. Acad. Sci. Paris 266 (1969), 844–846.

[13] M. Jarnicki and P. Pflug, *Extension of Holomorphic Functions*, De Gruyter Expositions Math. 34, De Gruyter, Berlin, 2000.

[14] J.-P. Kahane, *Baire category theorem and trigonometric spaces*, J. Anal. Math. 80 (2000), 143–182.

[15] M. C. Matos, *Domains of τ-holomorphy in a separable Banach space*, Math. Ann. 195 (1972), 273–278.

[16] J. Mujica, *Complex Analysis in Banach Spaces*, Dover Publ., Mineola, NY, 2010.

[17] V. Nestoridis, *Domains of holomorphy*, Ann. Math. Québec 42 (2018), 101–105.

[18] V. Nestoridis, *Domains of holomorphy*, arXiv:1701.00734 (2017).

[19] V. Nestoridis, *Non extendable holomorphic functions*, Math. Proc. Cambridge Philos. Soc. 139 (2005), 351–360.

[20] R. M. Range, *Holomorphic Functions and Integral Representation in Several Variables*, Grad. Texts in Math. 108, Springer, New York, 1986.

[21] M. Valdivia, *Interpolation in certain function spaces*, Proc. Roy. Irish Acad. Sect. A 80 (1980), 173–189.

Richard M. Aron
Department of Mathematical Sciences
Kent State University,
Kent, OH 44242, U.S.A.
E-mail: aron@math.kent.edu

Stéphane Charpentier
Centre de Mathématiques
et Informatique (CMI)
Bureau 303
and
Aix-Marseille Université
Technopôle Château-Gombert
39, rue F. Joliot Curie
13453 Marseille Cedex 13, France
E-mail: stephane.charpentier.1@univ-amu.fr

Paul M. Gauthier
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, Que., H3C 3J7, Canada
E-mail: gauthier@dms.umontreal.ca

Manuel Maestre
Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjassot (Valencia), Spain
E-mail: manuel.maestre@uv.es

Vassili Nestoridis
Department of Mathematics
University of Athens
15784 Panepistemiopolis
Athens, Greece
E-mail: vnestor@math.uoa.gr