

Continuous and other finitely generated canonical cofinal maps on ultrafilters

by

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Abstract. This paper investigates conditions under which canonical cofinal maps of the following three types exist: continuous, generated by finitary end-extension preserving maps, and generated by finitary maps. The main theorems prove that every monotone cofinal map on an ultrafilter from a certain class of ultrafilters is actually canonical when restricted to some cofinal subset. These theorems are then applied to find connections between Tukey, Rudin–Keisler, and Rudin–Blass reducibilities on large classes of ultrafilters.

The main theorems on canonical cofinal maps are the following. Under a mild assumption, basic Tukey reductions are inherited under Tukey reduction. In particular, every ultrafilter Tukey reducible to a p -point has continuous Tukey reductions. If \mathcal{U} is a Fubini iterate of p -points, then each monotone cofinal map from \mathcal{U} to some other ultrafilter is generated (on a cofinal subset of \mathcal{U}) by a finitary map on the base tree for \mathcal{U} which is monotone and end-extension preserving—the analogue of continuous in this context. Further, every ultrafilter which is Tukey reducible to some Fubini iterate of p -points has finitely generated cofinal maps. Similar theorems also hold for some other classes of ultrafilters.

1. Introduction. In this paper, ultrafilters on countable base sets are considered to be partially ordered by reverse inclusion. A map from an ultrafilter \mathcal{U} to another ultrafilter \mathcal{V} is *cofinal* if every image of a cofinal subset of \mathcal{U} is a cofinal subset of \mathcal{V} . We say that \mathcal{V} is *Tukey reducible* to \mathcal{U} and write $\mathcal{V} \leq_T \mathcal{U}$ if there is a cofinal map from \mathcal{U} to \mathcal{V} . When $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$, we say that \mathcal{U} is *Tukey equivalent* to \mathcal{V} and write $\mathcal{U} \equiv_T \mathcal{V}$. It is clear that \equiv_T is an equivalence relation, and \leq_T on the equivalence classes forms a partial ordering. The equivalence classes are called *Tukey types*. We point out that since \supseteq is a directed partial ordering on an ultrafilter, two ultrafilters are

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Tukey equivalent if and only if they are *cofinally similar*, that is, there is a partial ordering into which they both embed as cofinal subsets (see [18]). Thus, for ultrafilters, Tukey equivalence is the same as cofinal similarity. An equivalent formulation of Tukey reducibility, noticed by Schmidt in [14], shows that $\mathcal{V} \leq_T \mathcal{U}$ if and only if there is a *Tukey map* from \mathcal{V} to \mathcal{U} , that is, a map $g : \mathcal{V} \rightarrow \mathcal{U}$ such that every unbounded (with respect to the partial ordering \supseteq) subset of \mathcal{V} is unbounded in \mathcal{U} .

This paper focuses on the existence of canonical cofinal maps of three types: continuous, approximated by *basic* maps (end-extension and level preserving finitary maps—see Definitions 2.2 and 4.2), and approximated by monotone finitary maps (see Definition 6.2). In each of these cases, the original cofinal map is generated by the approximating finitary map.

The notion of Tukey reducibility between two directed partial orderings was first introduced by Tukey in [18] to study the Moore–Smith theory of net convergence in topology. This naturally led to investigations of Tukey types of more general partial orderings, directed and later non-directed. These investigations often reveal useful information for the comparison of different partial orderings. For example, Tukey reducibility preserves calibre-like properties, such as the countable chain condition, property K, precalibre \aleph_1 , σ -linked, and σ -centered (see [16]). For more on classification theories of Tukey types for certain classes of ordered sets, we refer the reader to [18], [3], [11], [15], and [16]. As the focus of this paper is canonical cofinal maps on ultrafilters, and as we have recently written a survey article giving an overview of the motivation and the state of the art of the Tukey theory of ultrafilters (see [5]), we present here only the background and motivations relevant for this work.

For ultrafilters, we may restrict our attention to monotone cofinal maps. A map $f : \mathcal{U} \rightarrow \mathcal{V}$ is *monotone* if for any $X, Y \in \mathcal{U}$, $X \supseteq Y$ implies $f(X) \supseteq f(Y)$. It is not hard to show that whenever $\mathcal{U} \geq_T \mathcal{V}$, then there is a *monotone* cofinal map witnessing this (see [8, Fact 6]).

As cofinal maps between ultrafilters have domain and range of size continuum, a priori, the Tukey type of an ultrafilter may have size 2^c . Indeed, this is the case for ultrafilters which have the maximum Tukey type ($[c]^{<\omega}, \subseteq$). However, if an ultrafilter has the property that every Tukey reduction from it to another ultrafilter may be witnessed by a continuous map, then it follows that its Tukey type, as well as the Tukey type of each ultrafilter Tukey reducible to it, has size at most continuum. This is the case for p -points.

DEFINITION 1.1. An ultrafilter \mathcal{U} on ω is a *p -point* if for each decreasing sequence $X_0 \supseteq X_1 \supseteq \dots$ of elements of \mathcal{U} , there is an $U \in \mathcal{U}$ such that $U \subseteq^* X_n$ for all $n < \omega$.

DEFINITION 1.2. An ultrafilter \mathcal{U} on ω has *continuous Tukey reductions* if whenever $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ such that $f \upharpoonright \mathcal{C}$ is continuous.

The following Theorem 20 in [8] has provided a fundamental tool for all subsequent research on the classification of Tukey types of p-points.

THEOREM 1.3 (Dobrinen/Todorcevic [8]). *Suppose \mathcal{U} is a p-point on ω . Given a monotone cofinal map f from \mathcal{U} into another ultrafilter, there is an $X \in \mathcal{U}$ such that f is continuous on $\mathcal{C} = \{Y \in \mathcal{U} : Y \subseteq X\}$. In particular, \mathcal{U} has continuous Tukey reductions. Moreover, these continuous Tukey reductions are generated by monotone basic functions.*

REMARK 1.4. The proof of [8, Theorem 20] shows that p-points have the stronger property of monotone basic Tukey reductions (see Definition 2.2).

It was later proved by Raghavan [13] ⁽¹⁾ that any ultrafilter Tukey reducible to some basically generated ultrafilter has Tukey type of cardinality at most \mathfrak{c} . The notion of a basically generated ultrafilter first appeared as [8, Definition 15]. The slightly modified version from [13] will be used in this paper, as it will simplify statements. In this paper, by a *filter base* for an ultrafilter \mathcal{U} we mean a cofinal subset of \mathcal{U} which is closed under finite intersections.

DEFINITION 1.5 ([13, Definition 13]). An ultrafilter \mathcal{U} on ω is *basically generated* if it has a filter base $\mathcal{B} \subseteq \mathcal{U}$ such that each sequence $\langle A_n : n < \omega \rangle$ of members of \mathcal{B} converging to another member of \mathcal{B} has a subsequence whose intersection is in \mathcal{U} .

It was shown in [8] that the class of basically generated ultrafilters contains all p-points and is closed under taking Fubini products. It is still unknown whether the class of all Fubini iterates of p-points is the same as or strictly contained in the class of all basically generated ultrafilters.

Continuous cofinal maps provide one of the main keys to the analysis of the structure of the Tukey types of p-points (see for instance [8], [13] and [12]). Moreover, continuous cofinal maps are crucial to providing a mechanism for applying Ramsey-classification theorems on barriers to classify the initial Tukey structures and Rudin–Keisler structures within these for a large class of p-points: selective ultrafilters in [13]; weakly Ramsey ultrafilters and a large hierarchy of rapid p-points satisfying partition relations in [9] and [10]; and k -arrow ultrafilters, hypercube ultrafilters, and a large class of p-points constructed using products of Fraïssé classes satisfying the Ramsey property

⁽¹⁾ In this joint paper of Raghavan and Todorcevic, it is clearly stated which results are due to each of the authors. In accordance with the wishes of the second author expressly stated to the author of this paper in 2011, whenever citing results from [13], we clearly state to whom the result is due.

in [7]. Similar methods were developed for a hierarchy of non-p-points above a Ramsey ultrafilter in [6].

Continuous cofinal maps are also used in the following theorem, which reveals the surprising fact that the Tukey and Rudin–Blass orders sometimes coincide. Recall that $\mathcal{V} \leq_{\text{RB}} \mathcal{U}$ if there is a finite-to-one map $f : \omega \rightarrow \omega$ such that $\mathcal{V} = f(\mathcal{U})$, where $f(\mathcal{U})$ is defined to be the set of all $X \subseteq \omega$ such that $f^{-1}[X]$ is a member of \mathcal{U} .

THEOREM 1.6 (Raghavan [13, Theorem 10]). *Let \mathcal{U} be any ultrafilter and let \mathcal{V} be a q -point. If $\mathcal{V} \leq_{\text{T}} \mathcal{U}$ and this is witnessed by a continuous, monotone cofinal map from \mathcal{U} to \mathcal{V} , then $\mathcal{V} \leq_{\text{RB}} \mathcal{U}$.*

In Section 2, we prove in Theorem 2.5 that, under a mild assumption, the property of having basic cofinal maps is inherited under Tukey reduction. The proof uses the Extension Lemma 2.4 showing that any basic monotone map on a cofinal subset of an ultrafilter may be extended to a basic monotone map on all of $\mathcal{P}(\omega)$. In particular, p-points satisfy the mild assumption; hence we obtain the following theorem.

THEOREM 2.6. *Every ultrafilter Tukey reducible to a p -point has basic, and hence continuous, Tukey reductions.*

Combined, Theorems 1.6 and 2.6 imply the following.

THEOREM 2.7. *If \mathcal{U} is Tukey reducible to some p -point, then any q -point Tukey below \mathcal{U} is actually Rudin–Blass below \mathcal{U} .*

The rest of the paper involves finding the analogues of Theorems 1.3 and 2.6 for countable iterations of Fubini products of p-points and applying them to connect Tukey reduction with Rudin–Keisler and Rudin–Blass reductions. We now delineate these results.

Section 3 is a primer, explicitly showing how any countable iteration of Fubini products of p-points, which we also simply call a *Fubini iterate of p -points*, can be viewed as an ultrafilter generated by trees on a front on ω . This precise way of viewing Fubini iterates of p-points sets the stage for finding the analogue of Theorem 1.3 for this more general class of ultrafilters. It is not possible to show that Fubini iterates of p-points have continuous Tukey reductions. In particular, Raghavan’s Corollary 11 in [13] shows that the Fubini product of two non-isomorphic selective ultrafilters does not have a continuous cofinal map into one of the selective ultrafilters. However, we will show that key properties of continuous maps still hold for this class of ultrafilters.

In Section 4 we define the notion of a *basic map* for Fubini iterates, which is in particular an end-extension preserving map from finite subsets of the tree \hat{B} of initial segments of members of a front B into finite subsets of ω

(see Definition 4.2). This is the analogue of continuity for Fubini iterates of p -points. One of the main results of this paper is the following.

THEOREM 4.4. *Fubini iterates of p -points have basic Tukey reductions.*

Thus, monotone cofinal maps on Fubini iterates of p -points are continuous, with respect to the product topology on the space $2^{\hat{B}}$. As basic maps on fronts have the key property (end-extension preserving) of continuous maps used to convert Tukey reduction to Rudin–Keisler reduction in [13], [9], [10], [7], and [6], it seems likely that they will play a crucial role in obtaining similar results for ultra-Ramsey spaces of Chapter 6 of Todorcevic’s book [17].

Sections 5 and 6 contain applications of Theorem 4.4 to a broad class of ultrafilters. In Section 5, we directly apply Theorem 4.4 to obtain an analogue of Raghavan’s Theorem 10 in [13]. In Theorem 5.1, we prove that if \mathcal{U} is a Fubini iterate of p -points and \mathcal{V} is a q -point Tukey reducible to \mathcal{U} , then there is a finite-to-one map on a large subset of \hat{B} , where B is the front base for \mathcal{U} , such that its image on \mathcal{U} generates a subfilter of \mathcal{V} . One of the consequences of this is the following.

THEOREM 5.3. *Suppose \mathcal{U} is a finite iteration of Fubini products of p -points. If \mathcal{V} is a q -point and $\mathcal{V} \leq_{\mathsf{T}} \mathcal{U}$, then $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}$.*

This improves one aspect of Raghavan’s Corollary 56 in [13] as \mathcal{V} is only required to be a q -point, not a selective ultrafilter. The improvement though comes at the expense of limiting \mathcal{U} to a finite Fubini iterate of p -points. It is unknown whether this can be extended to all Fubini iterates of p -points.

In Section 6 we prove the analogue of Theorem 2.5 for ultrafilters Tukey reducible to a Fubini iterate of p -points. (See Definition 6.2 for *finitely generated Tukey reductions*.)

THEOREM 6.3. *Let \mathcal{U} be any Fubini iterate of p -points. If $\mathcal{V} \leq_{\mathsf{T}} \mathcal{U}$, then \mathcal{V} has finitely generated Tukey reductions.*

These finitary maps are an improvement on the maps ψ_{φ} used in [13] (see [13, Definition 7]) in the sense that our finitary maps are shown to generate the original cofinal maps. Theorem 6.3 is used to extend Raghavan’s Theorem 17 in [13] to the class of all ultrafilters Tukey reducible to some Fubini iterate of p -points, in contrast to his result where \mathcal{U} is assumed to be basically generated. It is still open whether every ultrafilter Tukey reducible to a p -point is basically generated (see discussion around Problem 7.6), as the class of basically generated ultrafilters and the class of ultrafilters Tukey reducible to a Fubini iterate of p -points may be very different.

THEOREM 6.4. *If \mathcal{U} is Tukey reducible to a Fubini iterate of p -points, then for each $\mathcal{V} \leq_{\mathsf{T}} \mathcal{U}$, there is a filter $\mathcal{U}(P) \equiv_{\mathsf{T}} \mathcal{U}$ such that $\mathcal{V} \leq_{\mathsf{RK}} \mathcal{U}(P)$.*

The paper closes with a list of open problems in Section 7.

The results in Sections 2, 3 and 4 were completed in 2010, presented at the Logic Colloquium in Paris that year, and appeared in the unpublished preprint [4]. The present paper includes much revised presentations and proofs of those results, new extensions of them, and additional applications.

2. Basic Tukey reductions inherited under Tukey reducibility.

One of the crucial tools used to determine the structure of the Tukey types of p -points is the existence of continuous cofinal maps [8, Theorem 20]. Continuity contributes to the analysis of the structure of the Tukey types of p -points by essentially reducing the number of cofinal maps under consideration from $2^{\mathfrak{c}}$ to \mathfrak{c} , with the immediate consequence that there are at most \mathfrak{c} many ultrafilters Tukey reducible to any p -point. Continuity further contributes to finding exact Tukey and Rudin–Keisler structures below certain classes of p -points satisfying partition relations. The fact that each monotone cofinal map on a p -point is approximated by a finitary end-extension preserving function is what allows for application of Ramsey-classification theorems to find the exact Tukey and Rudin–Keisler structures below the p -points forced by certain topological Ramsey spaces (see [13], [9], [10], and [7]). Other applications of cofinal maps represented by finitary end-extension preserving maps appear in Dobrinen’s contributions in [2], and further in [6] where the precise Tukey and Rudin–Keisler structures below ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$ were found.

The notion of a basic map is a strengthening of continuity, and is the same as continuity when the domain is a compact subset of 2^ω (see Definition 2.2 below). The Extension Lemma 2.4 shows that all basic Tukey reductions on some cofinal subset of an ultrafilter extend to a basic map on $\mathcal{P}(\omega)$. This will be employed in the proof of the main theorem of this section, Theorem 2.5, which shows that, under mild assumptions, the property of having basic Tukey reductions is inherited under Tukey reducibility. Theorem 2.6 then follows: every ultrafilter Tukey reducible to a p -point has basic, and hence continuous, Tukey reductions. Combining Theorem 2.6 with Raghavan’s Theorem 10 in [13] shows that whenever \mathcal{W} is Tukey reducible to a p -point and \mathcal{V} is a q -point, then $\mathcal{W} \geq_T \mathcal{V}$ implies $\mathcal{W} \geq_{\text{RB}} \mathcal{V}$ (see Theorem 2.7).

We begin with some basic definitions. The following standard notation is used: $2^{<\omega}$ denotes the collection of finite sequences $s : n \rightarrow 2$ for $n < \omega$. We use s, t, u, \dots to denote members of $2^{<\omega}$. Note that given $s, t \in 2^{<\omega}$, the inclusion $s \subseteq t$ simply means that s is an initial segment of t , that is, $\text{dom}(s) \leq \text{dom}(t)$ and $t \upharpoonright \text{dom}(s) = s$. Write $s \subset t$ if and only if $s \subseteq t$ and $s \neq t$, that is, s is a proper initial segment of t . We use $a \sqsubseteq X$ for sets $a, X \subseteq \omega$ to denote

that, given their strictly increasing enumerations, a is an initial segment of X ; and $a \sqsubset X$ means that a is a proper initial segment of X .

We would like to identify subsets of ω with their characteristic functions, as this will greatly simplify notation in the proofs. For $X \subseteq \omega$, we let χ_X denote the characteristic function of X with domain ω . If X is infinite, we shall often abuse notation and use X to denote both the set X and its characteristic function χ_X . Given $m < \omega$, we let $\chi_X \upharpoonright m$ denote the characteristic function of $X \cap m$ with domain m . Whenever no ambiguity arises, we shall abuse notation and use $X \upharpoonright m$ to denote both the characteristic function $\chi_X \upharpoonright m$ and the set $X \cap m$. For precision throughout, for $s \in 2^{<\omega}$, we use $[[s]]$ to denote $s^{-1}(\{1\})$, the set of i in the domain of s for which $s(i) = 1$. For pointwise images of a function f on a set S , we shall use $f[S]$ to denote $\{f(x) : x \in S\}$. In Sections 3 and following, for a tree $T \subseteq [\omega]^{<\omega}$, we shall use $[T]$ to denote the set of all *branches* through T , meaning the collection of all maximal branches (if there are any finite maximal branches) and all cofinal branches (if there are any infinite branches) through T . This will not cause any ambiguity.

DEFINITION 2.1. Given a subset C of $2^{<\omega}$, we call a map $\hat{f} : C \rightarrow 2^{<\omega}$ *level preserving* if there is a strictly increasing sequence $(k_m)_{m < \omega}$ such that \hat{f} takes each member of $C \cap 2^{k_m}$ to a member of 2^m . A level preserving map \hat{f} is *end-extension preserving* if whenever $m < n$, $s \in C \cap 2^{k_m}$, and $t \in C \cap 2^{k_n}$, then $s \subseteq t$ implies $\hat{f}(s) \subseteq \hat{f}(t)$. A level and end-extension preserving map will be called *basic*. A level preserving map \hat{f} is *monotone* if for each $m < \omega$ and $s, t \in C \cap 2^{k_m}$, $[[s]] \subseteq [[t]]$ implies $[[\hat{f}(s)]] \subseteq [[\hat{f}(t)]]$.

DEFINITION 2.2. Let f be a map from a subset $\mathcal{C} \subseteq 2^\omega$ into 2^ω . We say that f is *represented by a basic map* if there is a strictly increasing sequence $(k_m)_{m < \omega}$ such that, with

$$(2.1) \quad C = \{X \upharpoonright k_m : X \in \mathcal{C}, m < \omega\},$$

there is a basic map $\hat{f} : C \rightarrow 2^{<\omega}$ such that for each $X \in \mathcal{C}$,

$$(2.2) \quad f(X) = \bigcup_{m < \omega} \hat{f}(X \upharpoonright k_m).$$

In this case, we say that \hat{f} *generates* f . If each monotone cofinal function from an ultrafilter \mathcal{U} to another ultrafilter is represented by a basic map on some cofinal subset of \mathcal{U} , then we say that \mathcal{U} *has basic Tukey reductions*.

Note that if $f \upharpoonright \mathcal{C}$ is generated by a basic map \hat{f} , then for each $X \in \mathcal{C}$ and $m < \omega$, $f(X) \upharpoonright m = \hat{f}(X \upharpoonright k_m)$. This fact will be used throughout.

For a set $C \subseteq 2^{<\omega}$, $[C]$ denotes the set of all branches through C . If $C = \{X \upharpoonright k_m : X \in \mathcal{C}, m < \omega\}$, where $\mathcal{C} \subseteq 2^\omega$ and $(k_m)_{m < \omega}$ is a strictly increasing sequence, then $[C] \subseteq 2^\omega$.

LEMMA 2.3. *Let \hat{f} be a monotone basic map with domain $C \subseteq 2^{<\omega}$. Then \hat{f} induces a monotone continuous map f^* on $[C]$ by $f^*(X) = \bigcup_{m < \omega} \hat{f}(X \upharpoonright k_m)$ for $X \in [C]$. Further, if \hat{f} generates f on \mathcal{C} and $C \supseteq \{X \upharpoonright k_m : X \in \mathcal{C}, m < \omega\}$, then $f^* \upharpoonright \mathcal{C} = f \upharpoonright \mathcal{C}$; hence $f \upharpoonright \mathcal{C}$ is continuous.*

Proof. That f^* is continuous on $[C]$ is trivial, since \hat{f} is basic. Since \hat{f} is monotone, it follows that f^* is monotone. If \hat{f} generates f on \mathcal{C} , then trivially $f^* \upharpoonright \mathcal{C}$ is simply $f \upharpoonright \mathcal{C}$. ■

In the next results, members of ultrafilters are identified with their characteristic functions. Then each function $f : \mathcal{U} \rightarrow \mathcal{V}$, where \mathcal{U} and \mathcal{V} are ultrafilters on ω , is identified with the function $F : \{\chi_U : U \in \mathcal{U}\} \rightarrow \{\chi_V : V \in \mathcal{V}\}$, where $F(\chi_U) = \chi_{f(U)}$. Rather than introducing more notation, we shall use f to denote both of these functions. Since all ultrafilters considered in this paper are non-principal, the notation $f(U) \subseteq f(U')$ will always refer to $f(U)$ and $f(U')$ as subsets of ω .

LEMMA 2.4 (Extension). *Suppose \mathcal{U} and \mathcal{V} are non-principal ultrafilters, $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $C \subseteq \mathcal{U}$ such that $f \upharpoonright C$ is represented by a monotone basic map \hat{f} . Let $(k_m)_{m < \omega}$ be the strictly increasing sequence such that the domain C of \hat{f} is contained in $\bigcup_{m < \omega} 2^{k_m}$. Then there is a continuous monotone map $\tilde{f} : 2^\omega \rightarrow 2^\omega$ such that*

- (1) \tilde{f} is represented by a monotone basic map \hat{g} on $\bigcup_{m < \omega} 2^{k_m}$;
- (2) $\tilde{f} \upharpoonright C = f \upharpoonright C$;
- (3) $\tilde{f} \upharpoonright \mathcal{U}$ is a cofinal map from \mathcal{U} to \mathcal{V} .

Proof. Let \hat{f} be a monotone basic map generating $f \upharpoonright C$, and let $(k_m)_{m < \omega}$ be the levels on which \hat{f} is defined. Thus, the domain of \hat{f} is $C = \{X \upharpoonright k_m : X \in \mathcal{C}, m < \omega\}$, and for each $s \in C \cap 2^{k_m}$, $\hat{f}(s) \in 2^{k_m}$.

CLAIM. *There is a monotone basic map \hat{g} , with domain $\bigcup_{m < \omega} 2^{k_m}$ which generates a function $\tilde{f} : 2^\omega \rightarrow 2^\omega$ such that $\tilde{f} \upharpoonright C = f \upharpoonright C$.*

Proof. Since C is cofinal in \mathcal{U} and \mathcal{U} is non-principal, the finite sequence of zeros of length k_m is in C for each $m < \omega$. Let $D = \bigcup_{m < \omega} 2^{k_m}$ and define \hat{g} on D as follows: for $t \in 2^{k_m}$, define $\hat{g}(t)$ to be the function from m into 2 such that for $i \in m$,

$$(2.3) \quad \hat{g}(t)(i) = \max\{\hat{f}(s)(i) : s \in C, |s| \leq k_m, \text{ and } [[s]] \subseteq [[t]]\}.$$

That is, $\hat{g}(t)(i) = 1$ if and only if there is some $s \in C$ such that $|s| \leq k_m$, $[[s]] \subseteq [[t]]$, and $\hat{f}(s)(i) = 1$. It follows from the definition that \hat{g} is monotone and level preserving. Since \hat{f} is monotone, $\hat{g} \upharpoonright C$ equals \hat{f} .

To see that \hat{g} is end-extension preserving, suppose $t \subset t'$, where $t \in 2^{k_m}$ and $t' \in 2^{k_{m'}}$ for some $m < m'$. Fix $i < m$. Suppose that $\hat{g}(t')(i) = 1$. Then there is some $s' \in C \cap 2^{k_n}$ such that $i < n \leq m'$, $[[s']] \subseteq [[t']]$, and

$\hat{f}(s')(i) = 1$. Letting $j = \min(m, n)$ and $s = s' \upharpoonright k_j$, we see that $s \in C$ and $[[s]] \subseteq [[t' \upharpoonright k_m]]$, where $t' \upharpoonright k_m = t$; moreover, $\hat{f}(s)(i) = 1$, since $\hat{f}(s) = \hat{f}(s') \upharpoonright j$. It follows that $\hat{g}(t)(i) = 1$. On the other hand, if $\hat{g}(t')(i) = 0$, then by the definition of \hat{g} , $\hat{f}(s)(i) = 0$ for all $s \in C \cap \bigcup_{n \leq m} 2^{k_n}$ such that $[[s]] \subseteq [[t]]$. Hence, $\hat{g}(t)(i) = 0$. Therefore, $\hat{g}(t') \upharpoonright m = \hat{g}(t)$.

Now define $\tilde{f} : 2^\omega \rightarrow 2^\omega$ by

$$(2.4) \quad \tilde{f}(Z) = \bigcup_{m < \omega} \hat{g}(Z \upharpoonright k_m).$$

Then \tilde{f} is generated by the basic map \hat{g} . It follows that \tilde{f} is monotone. Since $\hat{g} \upharpoonright C$ equals \hat{f} , it follows that $\tilde{f} \upharpoonright C = \hat{f} \upharpoonright C$. ■

Thus, \tilde{f} is continuous on 2^ω and (1) and (2) of the lemma hold. To show (3), it suffices to show that $\tilde{f} \upharpoonright \mathcal{U}$ has range inside of \mathcal{V} , since $\tilde{f} \upharpoonright C$ equals $\hat{f} \upharpoonright C$ which is monotone and cofinal in \mathcal{V} . Let $U \in \mathcal{U}$ be given. Fix $S \in C$ such that $U \supseteq S$. Then $\tilde{f}(U) \supseteq \tilde{f}(S) = \hat{f}(S) \in \mathcal{V}$, which concludes the proof of Lemma 2.4. ■

In the main theorem of this section, we show that the property of having basic Tukey reductions is inherited under Tukey reducibility below any ultrafilter having monotone basic Tukey reductions and satisfying the property (*) below.

THEOREM 2.5. *Suppose that the ultrafilter \mathcal{U} has monotone basic Tukey reductions. Suppose further that for each monotone cofinal map f from \mathcal{U} to another ultrafilter, there is some cofinal subset $C \subseteq \mathcal{U}$ such that $f \upharpoonright C$ is represented by a monotone basic function on some levels $(k_m)_{m < \omega}$ satisfying the following property:*

(*) *For each $X \in \bar{C}$ and each $m < \omega$, there is a $Z \in C$ such that $Z \supseteq X$ and $Z \upharpoonright k_m = X \upharpoonright k_m$.*

Then every ultrafilter \mathcal{V} Tukey reducible to \mathcal{U} also has basic Tukey reductions.

Proof. Suppose that \mathcal{U} satisfies the hypotheses and let $\mathcal{V} \leq_T \mathcal{U}$. We may assume that \mathcal{U} is non-principal, for the result holds immediately if \mathcal{U} is principal. By Lemma 2.4, there is a map $\tilde{f} : 2^\omega \rightarrow 2^\omega$ generated by a monotone basic map $\hat{f} : \bigcup_{m < \omega} 2^{k_m} \rightarrow 2^{<\omega}$, for some increasing sequence $(k_m)_{m < \omega}$, such that $\tilde{f} \upharpoonright \mathcal{U} : \mathcal{U} \rightarrow \mathcal{V}$ is a cofinal map. We shall let f denote the restricted map $\tilde{f} \upharpoonright \mathcal{U}$. Suppose $\mathcal{W} \leq_T \mathcal{V}$, and let $h : \mathcal{V} \rightarrow \mathcal{W}$ be a monotone cofinal map. Extend h to the map $\tilde{h} : 2^\omega \rightarrow 2^\omega$ defined as follows: for each $X \in 2^\omega$,

$$(2.5) \quad \tilde{h}(X) = \bigcap \{h(V) : V \in \mathcal{V} \text{ and } V \supseteq X\}.$$

Notice that \tilde{h} is monotone. Furthermore, it follows from h being monotone that $\tilde{h} \upharpoonright \mathcal{V} = h$.

Define $\tilde{g} = \tilde{h} \circ \tilde{f}$. Then $\tilde{g} : 2^\omega \rightarrow 2^\omega$ and is monotone. Letting g denote $\tilde{g} \upharpoonright \mathcal{U}$, we see that $g = h \circ f$; hence $g : \mathcal{U} \rightarrow \mathcal{W}$ is a monotone cofinal map. By the hypotheses, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ and a monotone basic map $\hat{g} : \mathcal{C} \rightarrow 2^{<\omega}$ generating $g \upharpoonright \mathcal{C}$ such that $(*)$ holds, where $(k_m)_{m < \omega}$ is the strictly increasing sequence associated with \hat{g} and $\mathcal{C} = \{X \upharpoonright k_m : X \in \mathcal{C} \text{ and } m < \omega\}$.

Without loss of generality, we may assume that \hat{f} and \hat{g} are defined on the same levels $(k_m)_{m < \omega}$ and that $k_0 = 0$: for if \hat{g} is defined on $\{X \upharpoonright j_m : X \in \mathcal{C} \text{ and } m < \omega\}$, we can take $l_m = \max(k_m, j_m)$ and define $\hat{f}'(s) = \hat{f}(s \upharpoonright k_m)$ for $s \in 2^{l_m}$ and $\hat{g}'(X \upharpoonright l_m) = \hat{g}(X \upharpoonright j_m)$ for $X \in \mathcal{C}$ and $m < \omega$. Notice that whenever $s \in C \cap 2^{k_m}$ and $s \sqsubset X \in \mathcal{C}$, then $\hat{f}(s) = f(X) \upharpoonright m$ and $\hat{g}(s) = g(X) \upharpoonright m$.

Define

$$(2.6) \quad D = \{\hat{f}(s) : s \in C\} \quad \text{and} \quad \mathcal{D} = f[C].$$

Notice that in fact $D = \{Y \upharpoonright m : Y \in \mathcal{D}, m < \omega\}$, and \mathcal{D} is cofinal in \mathcal{V} since $f : \mathcal{U} \rightarrow \mathcal{V}$ is monotone cofinal and \mathcal{C} is a cofinal subset of \mathcal{U} . Let $\overline{\mathcal{C}}$ denote the closure of \mathcal{C} in the topological space 2^ω . Since \tilde{f} is continuous on the compact space 2^ω and $f \upharpoonright \mathcal{C} = \tilde{f} \upharpoonright \mathcal{C}$, it follows that $\overline{\mathcal{D}} = f[\overline{\mathcal{C}}] = \tilde{f}[\overline{\mathcal{C}}]$.

CLAIM 1. *For each $Y \in \overline{\mathcal{D}}$ and each $m < \omega$, there is an $\tilde{m} \geq m$ satisfying the following: for each $Z \in \overline{\mathcal{C}}$ such that $\tilde{f}(Z) \upharpoonright \tilde{m} = Y \upharpoonright \tilde{m}$, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $X \upharpoonright k_m = Z \upharpoonright k_m$.*

Proof. Let $Y \in \overline{\mathcal{D}}$ and suppose the claim fails. Then there is an m such that for each $n \geq m$, there is a $Z_n \in \overline{\mathcal{C}}$ such that $\tilde{f}(Z_n) \upharpoonright n = Y \upharpoonright n$, but for each $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$, $Z_n \upharpoonright k_m \neq X \upharpoonright k_m$. As $\overline{\mathcal{C}}$ is compact, there is a subsequence $(Z_{n_i})_{i < \omega}$ which converges to some $X \in \overline{\mathcal{C}}$. Since \tilde{f} is continuous, $\tilde{f}(Z_{n_i})$ converges to $\tilde{f}(X)$. Since for each $i < \omega$, $\tilde{f}(Z_{n_i}) \upharpoonright n_i = Y \upharpoonright n_i$, it follows that $\tilde{f}(Z_{n_i})$ converges to Y . Therefore, $\tilde{f}(X) = Y$. Further, since $Z_{n_i} \rightarrow X$, there is a j such that for all $i \geq j$, $Z_{n_i} \upharpoonright k_m = X \upharpoonright k_m$. But this is a contradiction since $X \in \overline{\mathcal{C}}$ and $\tilde{f}(X) = Y$. ■

CLAIM 2. *There is a strictly increasing sequence $(j_m)_{m < \omega}$ such that for each $m < \omega$, for all $Y \in \overline{\mathcal{D}}$ and $Z \in \overline{\mathcal{C}}$ with $\tilde{f}(Z) \upharpoonright j_m = Y \upharpoonright j_m$, there is an $X \in \overline{\mathcal{C}}$ such that $\tilde{f}(X) = Y$ and $X \upharpoonright k_m = Z \upharpoonright k_m$.*

Proof. Let $j_0 = 0$ and note that j_0 vacuously satisfies the claim. Now suppose that $m \geq 1$ and suppose we have chosen $j_0 < \dots < j_{m-1}$ satisfying the claim. For each $Y \in \overline{\mathcal{D}}$, there is an $\tilde{m}(Y) \geq m$ satisfying Claim 1. The finite characteristic functions $Y \upharpoonright \tilde{m}(Y)$ determine basic open sets in 2^ω , and the union of these open sets (over all $Y \in \overline{\mathcal{D}}$) covers $\overline{\mathcal{D}}$. Since $\overline{\mathcal{D}}$ is compact, there is a finite subcover, determined by some $Y_0 \upharpoonright \tilde{m}(Y_0), \dots, Y_l \upharpoonright \tilde{m}(Y_l)$. Take $j_m > \max\{j_{m-1}, \tilde{m}(Y_0), \dots, \tilde{m}(Y_l)\}$. By this inductive construction, we obtain a sequence $(j_m)_{m < \omega}$ which satisfies the claim. ■

Recall that the notation $\tilde{g}(X)\upharpoonright m$ (and analogues) has the dual meaning of both a characteristic function with domain m and also the set $[[\tilde{g}(X)]] \cap m$. For clarity, in the next few claims, we shall use the precise notation of $[[\tilde{g}(X)]] \cap m$ to denote the set, and $\tilde{g}(X)\upharpoonright m$ to denote the characteristic function.

CLAIM 3. For each $X \in \bar{\mathcal{C}}$ and $m < \omega$,

$$(2.7) \quad [[\tilde{g}(X)]] \cap m \subseteq [[\hat{g}(X\upharpoonright k_m)]].$$

Proof. Let $X \in \bar{\mathcal{C}}$ and $m < \omega$ be given, and let s denote $X\upharpoonright k_m$. Then $s \in C$, and for any $Z \in \mathcal{C}$ with $Z\upharpoonright k_m = s$, we deduce that $\tilde{g}(Z)\upharpoonright m = \hat{g}(s)$. Since the property $(*)$ on \mathcal{C} implies there is a $Z \in \mathcal{C}$ such that $Z \supseteq X$ and $Z\upharpoonright k_m = s$, and since \tilde{g} is monotone, it follows that

$$[[\tilde{g}(X)]] \cap m \subseteq \bigcap \{ [[\tilde{g}(Z)]] \cap m : Z \in \mathcal{C} \text{ and } Z \supseteq X \} = [[\hat{g}(s)]] . \blacksquare$$

CLAIM 4. Let $Y \in \bar{\mathcal{D}}$ and m be given, and let $t = Y\upharpoonright j_m$. Then

$$[[\tilde{h}(Y)]] \cap m \subseteq [[\hat{g}(s)\upharpoonright m]]$$

for each $s \in C \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$.

Proof. Let $Y \in \bar{\mathcal{D}}$ and m be given, and let $t = Y\upharpoonright j_m$. Let s be any member of $C \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$. By Claim 2, there is an $X \in \bar{\mathcal{C}}$ such that $\hat{f}(X) = Y$ and $X\upharpoonright k_m = s\upharpoonright k_m$. We shall show that

$$(2.8) \quad [[\tilde{h}(Y)]] \cap m = [[\tilde{g}(X)]] \cap m \subseteq [[\hat{g}(X\upharpoonright k_m)]] = [[\hat{g}(s)\upharpoonright m]].$$

The first equality holds since $\tilde{f}(X) = Y$ and $\tilde{h} \circ \tilde{f}(X) = \tilde{g}(X)$, and the last since $\hat{g}(X\upharpoonright k_m) = \hat{g}(s\upharpoonright k_m) = \hat{g}(s)\upharpoonright m$. The inclusion holds by Claim 3. \blacksquare

Finally, we define the finitary function \hat{h} which will represent h on \mathcal{D} . Let $D' = \{t \in D : \exists m < \omega (|t| = j_m)\}$. For $t \in D' \cap 2^{j_m}$, define $\hat{h}(t)$ to be the function from m into 2 such that for $i \in m$,

$$(2.9) \quad \hat{h}(t)(i) = \min\{\hat{g}(s)(i) : s \in C \cap 2^{k_{j_m}} \text{ and } \hat{f}(s) = t\}.$$

In words, $\hat{h}(t)$ is the characteristic function with domain m of the intersection of the subsets a of m for which there is some $s \in C \cap 2^{k_{j_m}}$ with $\hat{f}(s) = t$ such that $\hat{g}(s)\upharpoonright m$ is the characteristic function of a with domain m . By definition, \hat{h} is level preserving.

CLAIM 5. The function \hat{h} is basic and generates $h\upharpoonright \mathcal{D}$.

Proof. Let $Y \in \mathcal{D}$, Z be a member of \mathcal{C} such that $\tilde{f}(Z) = Y$, and $m < \omega$ be given. Since $Z \in \mathcal{C}$, $\tilde{f}(Z) = f(Z)$. Let $t = Y\upharpoonright j_m$ and $u = Z\upharpoonright k_{j_m}$. Then $\hat{f}(u) = t$, so $[[\hat{g}(u)\upharpoonright m]] \supseteq [[\hat{h}(t)]]$. As $Z \in \mathcal{C}$, $[[\tilde{g}(Z)]] \cap m = [[\hat{g}(u)\upharpoonright m]]$. Thus,

$$(2.10) \quad [[h(Y)]] \cap m = [[h \circ f(Z)]] \cap m = [[\tilde{g}(Z)]] \cap m = [[\hat{g}(u)\upharpoonright m]] \supseteq [[\hat{h}(t)]] .$$

Now suppose $s \in C \cap 2^{k_{j_m}}$ and $\hat{f}(s) = t$. By Claim 4, since $\tilde{h} \upharpoonright \mathcal{V} = h$, we see that $[[h(Y)]] \cap m = [[\tilde{h}(Y)]] \cap m \subseteq [[\hat{g}(s) \upharpoonright m]]$. Since s was arbitrary, it follows that $[[h(Y)]] \cap m \subseteq [[\hat{h}(t)]]$. Therefore, $\hat{h}(Y \upharpoonright j_m) = h(Y) \upharpoonright m$.

Thus, \hat{h} generates h on \mathcal{D} . It follows that \hat{h} is end-extension preserving: If $t \subset t'$ are members of D' of lengths j_m and $j_{m'}$, respectively, then letting Y be any member of \mathcal{D} such that t' is an initial segment of Y , we see that $\hat{h}(t) = h(Y) \upharpoonright m = (h(Y) \upharpoonright m') \upharpoonright m = \hat{h}(t') \upharpoonright m$. Therefore, \hat{h} is basic. ■

Consequently, $h \upharpoonright \mathcal{D}$ is generated by the basic map \hat{h} on D' . Thus, \mathcal{V} has basic Tukey reductions. This finishes the proof of Theorem 2.5. ■

Every p-point has monotone basic Tukey reductions satisfying the additional property (*) of Theorem 2.5, as was shown in the proof of [8, Theorem 20], the cofinal set \mathcal{C} there being of the simple form $\mathcal{P}(X) \cap \mathcal{U}$ for some $X \in \mathcal{U}$. Hence, the following theorem holds.

THEOREM 2.6. *Every ultrafilter Tukey reducible to a p-point has basic, and hence continuous, Tukey reductions.*

Recall that an ultrafilter \mathcal{V} is *Rudin–Blass reducible* to an ultrafilter \mathcal{W} if there is a finite-to-one map $h : \omega \rightarrow \omega$ such that $\mathcal{V} = h(\mathcal{W})$. Thus, Rudin–Blass reducibility implies Rudin–Keisler reducibility. Our Theorem 2.6 combines with Raghavan’s Theorem 10 in [13] (see Theorem 1.6 for the statement) to yield the following.

THEOREM 2.7. *Suppose \mathcal{U} is Tukey reducible to a p-point. Then for each q-point \mathcal{V} , $\mathcal{V} \leq_{\text{T}} \mathcal{U}$ implies $\mathcal{V} \leq_{\text{RB}} \mathcal{U}$.*

REMARK 2.8. Stable ordered union ultrafilters are the analogues of p-points on the base set $\text{FIN} = [\omega]^{<\omega} \setminus \{\emptyset\}$ (see [1]). In [8, Theorems 71 and 72], it was shown that for each stable ordered union ultrafilter \mathcal{U} , both \mathcal{U} and its projection $\mathcal{U}_{\min, \max}$ have continuous Tukey reductions, with respect to the metric topology on the Milliken space of infinite block sequences. It is of interest that the ultrafilter $\mathcal{U}_{\min, \max}$ is rapid, but is neither a p-point nor a q-point, and condition (*) of Theorem 2.5 is satisfied. Further, all ultrafilters selective for some topological Ramsey space have monotone basic Tukey reductions, under a mild assumption which is satisfied in all known topological Ramsey spaces, as Dobrinen and Trujillo showed in [7, Theorem 56]. Many such ultrafilters are not p-points.

By arguments similar to those in Theorem 2.5, it should be possible to prove that every ultrafilter Tukey reducible to some stable ordered union ultrafilter, or more generally any ultrafilter selective for some topological Ramsey space, also has continuous Tukey reductions. We leave this as an open problem in Section 7.

3. Iterated Fubini products of ultrafilters represented as ultrafilters generated by \mathcal{U} -trees on fronts. Fubini products of ultrafilters on base set ω are commonly viewed as ultrafilters on base set $\omega \times \omega$. As was pointed out to us by Todorćević, Fubini products of non-principal ultrafilters on base set ω may also be viewed as ultrafilters on base set $[\omega]^2$. This view leads well to precise investigations of ultrafilters constructed by iterating the Fubini product construction. In this section, we review Fubini products of ultrafilters and countable iterations of this construction. After reviewing the notion of front, we show how every ultrafilter obtained by iterating the Fubini product construction can be viewed as an ultrafilter generated by certain subtrees of a base set which is a tree, and in particular a front. This section is a primer for the work in Section 4.

DEFINITION 3.1. Let \mathcal{U} and \mathcal{V}_n ($n < \omega$) be ultrafilters on ω . The *Fubini product* of \mathcal{V}_n over \mathcal{U} , denoted $\lim_{n \rightarrow \mathcal{U}} \mathcal{V}_n$, is defined as follows:

$$(3.1) \quad \lim_{n \rightarrow \mathcal{U}} \mathcal{V}_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{j \in \omega : (n, j) \in A\} \in \mathcal{V}_n\} \in \mathcal{U}\}.$$

When all \mathcal{V}_n are \mathcal{V} , we let $\mathcal{U} \cdot \mathcal{V}$ denote $\lim_{n \rightarrow \mathcal{U}} \mathcal{V}_n$.

Let \mathcal{U} and \mathcal{V} be ultrafilters on countable base sets I and J , respectively. We say that \mathcal{U} is *isomorphic* to \mathcal{V} if there exists a bijection $\pi : I \rightarrow J$ such that $\mathcal{V} = \{\pi[U] : U \in \mathcal{U}\}$. Up to isomorphism, Definition 3.1 also defines Fubini products of ultrafilters on arbitrary countable infinite sets.

The Fubini product construction can be iterated countably many times, each time producing an ultrafilter. For example, given an ultrafilter \mathcal{V} , let \mathcal{V}^1 denote \mathcal{V} , and let \mathcal{V}^{n+1} denote $\mathcal{V} \cdot \mathcal{V}^n$. Naturally, \mathcal{V}^ω denotes $\lim_{n \rightarrow \mathcal{V}} \mathcal{V}^n$. Continuing in this manner, we obtain \mathcal{V}^α for all $2 \leq \alpha < \omega \cdot 2$. At this point, it is ambiguous what is meant by $\mathcal{V}^{\omega \cdot 2}$. It is standard practice for a countable limit ordinal α to let \mathcal{V}^α denote any ultrafilter constructed by choosing (arbitrarily) an increasing sequence $(\alpha_n)_{n < \omega}$ converging to α and defining \mathcal{V}^α to be $\lim_{n \rightarrow \mathcal{V}} \mathcal{V}^{\alpha_n}$, but this is ambiguous, since the choice of the sequence $(\alpha_n)_{n < \omega}$ is completely arbitrary.

However, each countable iteration of Fubini products of ultrafilters (including the choice of sequence at limit stages) can be represented as an ultrafilter generated by $\vec{\mathcal{U}}$ -trees (see Definition 3.3) on a base set which is a front. This representation is unambiguous at limit stages. For this reason, Theorem 4.4 in the next section, showing that iterations of Fubini products of p-points have Tukey reductions which are as close to continuous as possible, will be formulated in the setting of $\vec{\mathcal{U}}$ -trees.

We now recall the definition of front. The reader desiring more background on fronts and $\vec{\mathcal{U}}$ -trees than presented here is referred to [17, pp. 12 and 90].

DEFINITION 3.2. A family B of finite subsets of some infinite subset I of ω is called a *front* on I if

- (1) $a \not\sqsubseteq b$ whenever a, b are in B , and
- (2) for every infinite $X \subseteq I$ there exists $b \in B$ such that $b \sqsubset X$.

Recall the following standard set-theoretic notation: $[\omega]^k$ denotes the collection of k -element subsets of ω , $[\omega]^{<k}$ denotes the collection of subsets of ω of size less than k , and $[\omega]^{\leq k} = [\omega]^{<k+1}$. It is easy to see that for each $k < \omega$, $[\omega]^k$ is a front.

Every front is lexicographically well-ordered, and hence has a unique lexicographic rank associated with it, namely the ordinal length of its lexicographical well-ordering. For example, $\text{rank}(\{\emptyset\}) = 1$, $\text{rank}([\omega]^1) = \omega$, and $\text{rank}([\omega]^2) = \omega \cdot \omega$. We shall usually drop the adjective ‘lexicographic’ when talking about ranks of fronts.

Given a front B , for each $n \in \omega$, we define $B_n = \{b \in B : n = \min b\}$ and $B_{\{n\}} = \{b \setminus \{n\} : b \in B_n\}$. Then $B = \bigcup_{n \in \omega} B_n$, and $B_n = \{\{n\} \cup a : a \in B_{\{n\}}\}$ for each n . Note that each $B_{\{n\}}$ is a front on $[n+1, \omega)$ with rank strictly less than the rank of B . Conversely, given any collection of fronts $B_{\{n\}}$ on $[n+1, \omega)$, the union $\bigcup_{n \in \omega} B_n$ is a front on ω , where B_n is defined as above.

Given any front B , we let \hat{B} denote the collection of all initial segments of members of B . Let \hat{B}^- denote the collection of all proper initial segments of members of B , that is, $\hat{B}^- = \hat{B} \setminus B$. Both \hat{B} and \hat{B}^- form trees under the partial ordering \sqsubseteq .

DEFINITION 3.3. Given a front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ of non-principal ultrafilters on ω , a $\vec{\mathcal{U}}$ -tree is a tree $T \subseteq \hat{B}$ such that $\emptyset \in T$ and for each $c \in T \cap \hat{B}^-$, $\{n \in \omega : c \cup \{n\} \in T\} \in \mathcal{U}_c$.

NOTATION. Given a front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ of non-principal ultrafilters on ω , let $\mathfrak{T} = \mathfrak{T}(\vec{\mathcal{U}})$ denote the collection of all $\vec{\mathcal{U}}$ -trees. For any $c \in \hat{B}^-$ and $T \in \mathfrak{T}$, let $T_c = \{t \in T : t \sqsubseteq c \text{ or } t \sqsupset c\}$, the tree with stem c consisting of all nodes in T comparable with c . If T is a \mathcal{U} -tree, then the set of maximal branches through T , denoted $[T]$, is exactly $T \cap B$.

Todorćević pointed out to us the following correspondence between Fubini iterates and ultrafilters on fronts. Start by fixing a collection \mathcal{P}_0 of non-principal ultrafilters on ω . Given $\alpha < \omega_1$, define $\mathcal{P}_{\alpha+1} = \{\lim_{n \rightarrow \mathcal{U}} \mathcal{V}_n : \mathcal{U} \in \mathcal{P}_0 \text{ and } \mathcal{V}_n \in \mathcal{P}_\alpha\}$. For each limit ordinal α , define $\mathcal{P}_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}_\beta$. Then $\mathcal{P}_{<\omega_1} := \bigcup\{\mathcal{P}_\alpha : \alpha < \omega_1\}$ is the collection of all iterated Fubini products of non-principal ultrafilters on ω . Each $\mathcal{W} \in \mathcal{P}_{<\omega_1}$ has a well-defined notion of rank, namely $\text{rank}(\mathcal{W})$ is the least $\alpha < \omega_1$ for which \mathcal{W} is a member of \mathcal{P}_α . The following lemma will be used in the next section, with \mathcal{P}_0 being the collection of all p-points.

LEMMA 3.4. *If $\mathcal{W} \in \mathcal{P}_{<\omega_1}$, then there is a front B and ultrafilters $\mathcal{U}_c \in \mathcal{P}_0$, $c \in \hat{B}^-$, such that \mathcal{W} is isomorphic to the ultrafilter on B generated by the $(\mathcal{U}_c : c \in \hat{B}^-)$ -trees.*

Proof. We prove by induction on $\alpha < \omega_1$ that the conclusion holds for every ultrafilter in \mathcal{P}_α . If $\mathcal{W} \in \mathcal{P}_0$, then \mathcal{W} is represented on the front $B = [\omega]^1$ via the obvious isomorphism $n \mapsto \{n\}$.

Let $1 \leq \alpha < \omega_1$ and assume the fact holds for each ultrafilter in $\bigcup_{\gamma < \alpha} \mathcal{P}_\gamma$. If α is a limit ordinal, then there is nothing to prove, so assume $\alpha = \beta + 1$ for some $\beta < \omega_1$. Suppose that $\mathcal{W} \in \mathcal{P}_\alpha$. Then $\mathcal{W} = \lim_{n \rightarrow \mathcal{U}} \mathcal{W}_n$, where $\mathcal{U} \in \mathcal{P}_0$ and for each n , $\mathcal{W}_n \in \mathcal{P}_\beta$. By the induction hypothesis, for each $n < \omega$ there is a front $B_{\{n\}}$ on ω and there are ultrafilters $\mathcal{U}_{n,c} \in \mathcal{P}_0$, $c \in \widehat{B_{\{n\}}}^-$, such that \mathcal{W}_n is isomorphic to the ultrafilter generated by $(\mathcal{U}_{n,c} : c \in \widehat{B_{\{n\}}}^-)$ -trees on $B_{\{n\}}$. Without loss of generality, we may assume that $B_{\{n\}}$ is a front on $[n + 1, \omega)$. In the standard way, we glue the fronts together to obtain a new front: the set $B = \bigcup_{n < \omega} \{\{n\} \cup b : b \in B_{\{n\}}\}$ is a front on ω .

Let $\mathcal{U}_\emptyset = \mathcal{U}$ and $\mathcal{U}_{\{n\} \cup c} = \mathcal{U}_{n,c}$ for each $c \in B_{\{n\}}$. It is straightforward to check that \mathcal{W} is isomorphic to the ultrafilter on B generated by the $(\mathcal{U}_b : b \in \hat{B}^-)$ -trees. ■

The case $\alpha = 2$ of the proof of Lemma 3.4 yields the following result, which seems interesting in its own right.

LEMMA 3.5. *The Fubini product $\lim_{n \rightarrow \mathcal{U}} \mathcal{V}_n$ of non-principal ultrafilters on ω is isomorphic to the ultrafilter on $B = [\omega]^2$ generated by $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in [\omega]^{\leq 1})$ -trees, where $\mathcal{U}_\emptyset = \mathcal{U}$ and for each $n \in \omega$, $\mathcal{U}_{\{n\}} = \mathcal{V}_n$.*

4. Basic cofinal maps on iterated Fubini products of p-points.

Fubini products of p-points do not in general have continuous Tukey reductions, as pointed out in the introduction (see below for more discussion). However, we will show that they do have canonical cofinal maps satisfying many of the properties of continuous maps, which we call *basic* (see Definition 4.2 below). Making use of the natural representation of a Fubini iterate of p-points as an ultrafilter generated by $\vec{\mathcal{U}}$ -trees on some front B (recall Lemma 3.4), we show in Theorem 4.4 that countable iterates of Fubini products of p-points have *basic Tukey reductions*. Such Tukey reductions are continuous on the space $2^{\hat{B}}$ with the Cantor topology, where \hat{B} is the tree consisting of all initial segments of members of the front B . This extends a key property of p-points (recall Theorem 1.3) to a large class of ultrafilters. Theorem 4.4 will be applied in Sections 5 and 6.

Raghavan's Corollary 11 in [13] shows that it is impossible for a Fubini product of two non-isomorphic selective ultrafilters to have continuous cofinal Tukey reductions. The next proposition shows that Fubini products of non-

principal ultrafilters do not have monotone basic Tukey reductions given by an approximating map on the finite subsets of the base for the ultrafilter. Thus, it is impossible to attain Theorem 1.3 for Fubini products of non-principal ultrafilters.

Recall that the *Cantor topology* on $2^{\omega \times \omega}$ is the topology generated by basic open sets of the form $\{h \in 2^{\omega \times \omega} : s \subset h\}$, where s is a function from some finite subset of $\omega \times \omega$ into 2. Equivalently, if we let $\langle w_i : i < \omega \rangle$ be any linear order of the members of $\omega \times \omega$ in order type ω , the Cantor topology on $2^{\omega \times \omega}$ is generated by basic open sets of the form $\{h \in 2^{\omega \times \omega} : s \subset h\}$, where s is a function from $\langle w_i : i < k \rangle$ into 2 for some $k < \omega$. For $k < \omega$, let W_k denote $\{w_i : i < k\}$, so that 2^{W_k} is the set of all functions with domain $\{w_i : i < k\}$ into 2. Given $k < \omega$ and $X \subseteq \omega \times \omega$, let $X \upharpoonright W_k$ denote the characteristic function of $\{w_i \in X : i < k\}$ on the domain W_k .

PROPOSITION 4.1. *Let \mathcal{U} and \mathcal{V} be any non-principal ultrafilters and let $\pi : \omega \times \omega \rightarrow \omega$ be defined by $\pi(m, n) = m$. Let $f : 2^{\omega \times \omega} \rightarrow 2^\omega$ be defined by $f(A) = \pi[A]$. Then for any cofinal subset \mathcal{C} of $\mathcal{U} \cdot \mathcal{V}$, $f \upharpoonright \mathcal{C}$ is not generated by a monotone basic map in the sense of Definition 2.2: there is no strictly increasing sequence $(k_m)_{m < \omega}$ and monotone basic map $\hat{f} : \bigcup_{m < \omega} 2^{W_{k_m}} \rightarrow 2^{< \omega}$ generating $f \upharpoonright \mathcal{C}$.*

Proof. Let g denote the restriction of f to the ultrafilter $\mathcal{U} \cdot \mathcal{V}$. Notice that g is a monotone cofinal map onto \mathcal{U} . Suppose toward a contradiction that there is a cofinal subset \mathcal{C} of $\mathcal{U} \cdot \mathcal{V}$ for which $g \upharpoonright \mathcal{C}$ is represented by a monotone basic function. In this context, with the Cantor topology on $2^{\omega \times \omega}$ in place of 2^ω , Definitions 2.1 and 2.2 are interpreted as follows: there is a strictly increasing sequence $(k_m)_{m < \omega}$ such that if we let $C = \{X \upharpoonright W_{k_m} : X \in \mathcal{C}, m < \omega\}$, then there is a monotone basic map $\hat{g} : C \rightarrow 2^{< \omega}$ such that for each $X \in \mathcal{C}$,

$$(4.1) \quad g(X) = \bigcup_{m < \omega} \hat{g}(X \upharpoonright W_{k_m}).$$

To be clear in this context, \hat{g} is *level preserving* means that for each $s \in C \cap 2^{W_{k_m}}$, $\hat{g}(s)$ is a member of 2^m . For two members s and t of C , t *end-extends* s , written $s \sqsubseteq t$, if the domain of s is W_j and the domain of t is W_l for some $j \leq l$, and for each $i < j$, $s(w_i) = t(w_i)$.

By the Extension Lemma 2.4, there is a monotone map $\tilde{f} : 2^{\omega \times \omega} \rightarrow 2^\omega$ which is represented by a monotone basic map \hat{f} on $\bigcup_{m < \omega} 2^{W_{k_m}}$ such that $\tilde{f} \upharpoonright \mathcal{C} = g \upharpoonright \mathcal{C}$. As seen in the proof of Lemma 2.4, modified to the current context, the map \hat{f} is defined by

$$(4.2) \quad \hat{f}(t)(w_i) = \max\{\hat{g}(s)(w_i) : s \in C, |s| \leq k_m \text{ and } [[s]] \subseteq [[t]]\}$$

for $t \in 2^{W_{k_m}}$ and $i < m$.

CLAIM. There is a $j < \omega$ and an infinite collection $\{s_l : l < \omega\} \subseteq C$, satisfying the following: for each $l < \omega$, if d_l denotes $[[s_l]]$, $j = \min(\pi[d_l])$, and i_l denotes the least i such that both $w_i \in d_l$ and $j = \pi(w_i)$, then $(i_l)_{l < \omega}$ forms a strictly increasing sequence.

Proof. For $s \in C$, let d_s denote $[[s]]$. Suppose toward a contradiction that for each $j < \omega$, there is an \tilde{i}_j such that for each $s \in C$ with $j = \min(\pi[d_s])$, there is an $i < \tilde{i}_j$ such that $w_i \in d_s$ and $j = \pi(w_i)$. Fix $X \in \mathcal{C}$. Since \mathcal{C} is a cofinal subset of the Fubini product of two non-principal ultrafilters, the set

$$(4.3) \quad J = \{\min(\pi[d_s]) : s \in C \text{ and } d_s \subseteq X\}$$

is infinite. For $j \in J$, let

$$(4.4) \quad S_j = \{s \in C : d_s \subseteq X \text{ and } \min(\pi[d_s]) = j\}.$$

Then for each $j \in J$ and $s \in S_j$, there is some $w_i \in d_s$ with $i < \tilde{i}_j$. Thus, every member of \mathcal{C} has infinite intersection with $\{w_i : \exists j (j = \min(\pi[d_s]) \text{ and } i < \tilde{i}_j)\}$. This contradicts the fact that \mathcal{V} is non-principal.

Thus, the negation of the supposition holds: there is a $j < \omega$ such that for each $\tilde{i} < \omega$, there is an $s \in C$ with $\min(\pi[d_s]) = j$ such that, whenever $w_i \in d_s$ and $j = \pi(w_i)$, then $i \geq \tilde{i}$. Fix such a j . Take $s_0 \in C$ such that $\min(\pi[d_{s_0}]) = j$. Take i_0 least such that $w_{i_0} \in d_{s_0}$ and $\pi(w_{i_0}) = j$. Using i_0 as the next \tilde{i} , there is an $s_1 \in C$ with $\min(\pi[d_{s_1}]) = j$ and $i_1 > i_0$, where i_1 is least such that $w_{i_1} \in d_{s_1}$ and $\pi(w_{i_1}) = j$. In this way, one constructs a collection $\{s_l : l < \omega\}$ satisfying the Claim. ■

Take $j < \omega$ and $\{s_l : l < \omega\}$ as in the Claim. Define $Y_l \in \mathcal{U} \cdot \mathcal{V}$ by $Y_l = s_l \cup (j, \omega) \times \omega$. Then $Y_l \rightarrow Y$, where $Y = (j, \omega) \times \omega$, which is a member of $\mathcal{U} \cdot \mathcal{V}$. Since \tilde{f} is generated by a basic map, \tilde{f} is continuous on $2^{\omega \times \omega}$. Hence, $\tilde{f}(Y_l)$ converges to $\tilde{f}(Y)$.

On the other hand, we shall show that for each $l < \omega$, j is in $\tilde{f}(Y_l)$ while j is not in $\tilde{f}(Y)$, contradicting continuity of \tilde{f} . Note that for each $l < \omega$, $s_l \in C$ implies there is an $X \in \mathcal{C}$ whose characteristic function extends s_l . Since g is generated by the basic map \hat{g} , and $j \in \pi[X]$ implies $j \in g(X)$, it follows that $j \in [[\hat{g}(s_l)]]$. By the definition of \tilde{f} in (4.2), it follows that $j \in [[\tilde{f}(s_l)]] \subseteq \tilde{f}(Y_l)$. However, j is not in $\tilde{f}(Y)$, since \tilde{f} is generated by \hat{f} , and it follows from the definition of \hat{f} in (4.2) that $j \notin [[\hat{f}(Y \upharpoonright k_m)]]$ for any $m < \omega$.

Thus, there is no cofinal $\mathcal{C} \subseteq \mathcal{U} \cdot \mathcal{V}$ for which $g \upharpoonright \mathcal{C}$ is generated by a monotone basic map on the topological space $2^{\omega \times \omega}$. ■

However, we will soon show that each ultrafilter \mathcal{W} which is an iterated Fubini product of p-points has finitely generated Tukey reductions which, moreover, are basic, and hence continuous, with respect to the topology on

the appropriate tree space. Toward this end, we proceed to give the definition of basic for this context, and then prove the main results of this section.

NOTATION. For any subset $A \subseteq [\omega]^{<\omega}$, recall that \hat{A} denotes the set of all initial segments of members of A . For any front B , we let \hat{B}^- denote $\hat{B} \setminus B$. For any subset $A \subseteq [\omega]^{<\omega}$ and $k < \omega$, let $A \upharpoonright k$ denote $\{a \in A : \max a < k\}$. For $A \subseteq \hat{B}$ and $k < \omega$, we shall abuse notation and also use $A \upharpoonright k$ to denote the characteristic function of the set $A \upharpoonright k$ on the domain $\hat{B} \upharpoonright k$. For each $k < \omega$, let $2^{\hat{B} \upharpoonright k}$ denote the set of all functions from $\hat{B} \upharpoonright k$ into $\{0, 1\}$. Notice that this is exactly the collection of characteristic functions of subsets of $\hat{B} \upharpoonright k$ on the domain $\hat{B} \upharpoonright k$.

DEFINITION 4.2. Let B be a front on ω , $\tilde{T} \subseteq \hat{B}$ be a tree, and $(n_k)_{k < \omega}$ be an increasing sequence. We say that a function $\hat{f} : \bigcup_{k < \omega} 2^{\tilde{T} \upharpoonright n_k} \rightarrow 2^{<\omega}$ is *level preserving* if $\hat{f} : 2^{\tilde{T} \upharpoonright n_k} \rightarrow 2^k$ for each $k < \omega$; \hat{f} is *end-extension preserving* if whenever $k < m$ and $A \subseteq \tilde{T}$ then $\hat{f}(A \upharpoonright n_k) = \hat{f}(A \upharpoonright n_m) \upharpoonright k$; \hat{f} is *basic* if it is level and end-extension preserving; and \hat{f} is *monotone* if whenever $A \subseteq C \subseteq \tilde{T}$ and $k < \omega$ then $[[\hat{f}(A \upharpoonright n_k)]] \subseteq [[\hat{f}(C \upharpoonright n_k)]]$.

Let \mathcal{U} be an ultrafilter on B generated by $(\mathcal{U}_c : c \in \hat{B}^-)$ -trees, let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a monotone cofinal map, where \mathcal{V} is an ultrafilter on base ω , and let $\tilde{T} \in \mathfrak{T}(\mathcal{U})$. Let $\mathfrak{T} \upharpoonright \tilde{T}$ denote the set of all \vec{U} -trees contained in \tilde{T} . We say that $\hat{f} : \bigcup_{k < \omega} 2^{\tilde{T} \upharpoonright n_k} \rightarrow 2^{<\omega}$ *generates* f on $\mathfrak{T} \upharpoonright \tilde{T}$ if for each $T \in \mathfrak{T} \upharpoonright \tilde{T}$,

$$(4.5) \quad f([T]) = \bigcup_{k < \omega} \hat{f}(T \upharpoonright n_k).$$

We say that \mathcal{U} has *basic Tukey reductions* if whenever $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map then there is a $\tilde{T} \in \mathfrak{T}(\mathcal{U})$ and a basic map \hat{f} which generates f on $\mathfrak{T} \upharpoonright \tilde{T}$.

REMARK 4.3. Note that if \hat{f} witnesses that f is basic on $\mathfrak{T} \upharpoonright \tilde{T}$, then \hat{f} generates a continuous map on the collection of *trees* in $\mathfrak{T} \upharpoonright \tilde{T}$, continuity being with respect to the Cantor topology on $2^{\hat{B}}$. Moreover, we may define a map \hat{g} on B as follows: For each finite subset $A \subseteq B$, define $\hat{g}(A) = \hat{f}(\hat{A})$, where \hat{A} is the collection of all initial segments of members of A . Then \hat{g} is finitary, but not necessarily continuous on 2^B , and \hat{g} generates f on $\{[T] : T \in \mathfrak{T} \upharpoonright \tilde{T}\}$, which is a base for the ultrafilter. Thus, for ultrafilters generated by \mathcal{U} -trees, basic Tukey reductions imply finitely represented Tukey reductions on the original base set B .

Now we prove the main theorem of this section. Fix a total order of $[\omega]^{<\omega}$ in order type ω such that $\max a < \max b$ implies $a \prec b$ for all $a, b \in [\omega]^{<\omega}$. Note that for each $k < \omega$, the set $\{c \in [\omega]^{<\omega} : \max c = k\}$ forms a finite interval in $([\omega]^{<\omega}, \prec)$.

THEOREM 4.4. *Let B be any front and $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in \hat{B}^-)$ be a sequence of p -points. Then the ultrafilter \mathcal{U} on base B generated by the $\vec{\mathcal{U}}$ -trees has basic Tukey reductions. Therefore, every countable iteration of Fubini products of p -points has monotone basic Tukey reductions.*

Proof. Let \mathcal{V} be some ultrafilter Tukey reducible to \mathcal{U} , and let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a monotone cofinal map. If \mathcal{V} is a principal ultrafilter, say generated by a singleton $\{r\}$, then we claim that the theorem trivially holds. Since the set consisting of $\{r\}$ is cofinal in \mathcal{V} , there is some $X \in \mathcal{U}$ such that $f(X) = \{r\}$. Set $n_k = k$ and define $\hat{f} : \bigcup_{k < \omega} 2^{\hat{B}|k} \rightarrow 2^{<\omega}$ as follows: for $k < \omega$ and $s \in 2^{\hat{B}|k}$, $\hat{f}(s)$ is the sequence in 2^k such that for each $i < k$, $\hat{f}(s)(i) = 0$ if and only if $i \neq r$. Then \hat{f} is monotone basic and generates f on $\mathcal{U}|X$. Thus, we shall assume from now on that \mathcal{V} is non-principal.

We let \mathfrak{T} denote $\mathfrak{T}(\mathcal{U}_c : c \in \hat{B}^-)$, the set of all $\vec{\mathcal{U}}$ -trees. Recall that \mathfrak{T} is a base for the ultrafilter \mathcal{U} . We make the convention that $\max \emptyset = -1$. For each $k < \omega$, let $\hat{B}|k$ denote the collection of all $b \in \hat{B}$ with $\max b < k$. Thus, $\hat{B}|0 = \{\emptyset\}$, $\hat{B}|1 = \{\emptyset, \{0\}\}$, and so forth. Fix an enumeration of the finite, non-empty \sqsubseteq -closed subsets of \hat{B} as $\langle A_i : i < \omega \rangle$ such that for each $i < j$, $\max \bigcup A_i \leq \max \bigcup A_j$. Let $(p_k)_{k < \omega}$ denote the strictly increasing sequence such that for each k , the sequence $\langle A_i : i < p_k \rangle$ lists all \sqsubseteq -closed subsets of $\hat{B}|k$. (For example, if $B = [\omega]^2$, then $\hat{B} = [\omega]^{\leq 2}$ and $\hat{B}^- = [\omega]^{\leq 1}$, and we may let $A_0 = \{\emptyset\}$, $A_1 = \{\emptyset, \{0\}\}$, $A_2 = \{\emptyset, \{0\}, \{1\}\}$, $A_3 = \{\emptyset, \{0\}, \{0, 1\}\}$, $A_4 = \{\emptyset, \{0\}, \{0, 1\}, \{1\}\}$, $A_5 = \{\emptyset, \{1\}\}$. Note that $p_0 = 1$, $p_1 = 2$, and $p_2 = 6$.)

For $k < \omega$ and $i < p_k$, define

$$(4.6) \quad \hat{B}_i^k = A_i \cup \{b \in \hat{B} : \exists a \in A_i (b \sqsupset a \text{ and } \min(b \setminus a) \geq k)\}.$$

Thus, \hat{B}_i^k is the maximal tree in \mathfrak{T} for which $T|k = A_i$. For a tree $T \subseteq \hat{B}$ and $c \in T \cap \hat{B}^-$, define the notation

$$(4.7) \quad U_c(T) = \{l > \max c : c \cup \{l\} \in T\}.$$

We refer to $U_c(T)$ as the *set of immediate extensions of c in T* . Note that if $T \in \mathfrak{T}$, then for each $c \in T \cap \hat{B}^-$, $U_c(T)$ is a member of \mathcal{U}_c . For $c \in \hat{B}^-$, recall that \hat{B}_c denotes the tree of all $a \in \hat{B}$ such that either $a \sqsubseteq c$ or $a \sqsupset c$.

Our goal is to construct a tree $\tilde{T} \in \mathfrak{T}$ and find a sequence $(n_k)_{k < \omega}$ of good cut-off points such that the following holds:

(*) For each $T \subseteq \tilde{T}$ in \mathfrak{T} , $k < \omega$, and $i < p_{n_k}$ such that $A_i = T|n_k$, for every $j \leq k$,

$$j \in f([T]) \iff j \in f([\tilde{T} \cap \hat{B}_i^{n_k}]).$$

CLAIM 1. *The property (*) implies that f has monotone basic Tukey reductions.*

Proof. For $k < \omega$ and $T \in \mathfrak{T}|\tilde{T}$, define

$$(4.8) \quad \hat{f}(T \upharpoonright n_k) = f([\tilde{T} \cap \hat{B}_i^{n_k}]) \upharpoonright k,$$

where i is *the* integer below p_{n_k} such that $T \upharpoonright n_k = A_i$. By definition, \hat{f} is level preserving. Let $l > k$ and let m be such that $T \upharpoonright n_l = A_m$. Then $A_m \upharpoonright n_k = A_i$. For $j < k$, (\otimes) implies that $j \in f([\tilde{T} \cap \hat{B}_i^{n_k}])$ if and only if $j \in f([T])$ if and only if $j \in f([\tilde{T} \cap \hat{B}_m^{n_l}])$. Thus, $j \in [[\hat{f}(T \upharpoonright n_k)]]$ if and only if $j \in [[\hat{f}(T \upharpoonright n_l)]]$. Therefore, \hat{f} is end-extension preserving, that is, $\hat{f}(T \upharpoonright n_l) \upharpoonright k = \hat{f}(T \upharpoonright n_k)$. Furthermore, $f([T]) = \bigcup_{k < \omega} \hat{f}(T \upharpoonright n_k)$; thus, \hat{f} generates f on $\mathfrak{T}|\tilde{T}$. To see that \hat{f} is monotone, suppose that $S \upharpoonright n_k \subseteq T \upharpoonright n_k$ for some $S, T \in \mathfrak{T}|\tilde{T}$. Let $i, j < p_k$ be such that $A_i = S \upharpoonright n_k$ and $A_j = T \upharpoonright n_k$. Then

$$(4.9) \quad \hat{f}(S \upharpoonright n_k) = f([\tilde{T} \cap \hat{B}_i^{n_k}]) \upharpoonright k \subseteq f([\tilde{T} \cap \hat{B}_j^{n_k}]) \upharpoonright k = \hat{f}(T \upharpoonright n_k),$$

where \subseteq holds because of (\otimes) and the fact that f is monotone and $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq \tilde{T} \cap \hat{B}_j^{n_k}$. Therefore, \hat{f} is a monotone basic map generating f on $\mathfrak{T}|\tilde{T}$. ■

The construction of \tilde{T} and $(n_k)_{k < \omega}$ is in three stages.

STAGE 1. We will choose some $R_i^k \in \mathfrak{T}$ with $A_i \subseteq R_i^k$ such that for all $k < \omega$, the following holds:

$(*)_k$ For all $i < p_k$ and $T \subseteq R_i^k$ in \mathfrak{T} with $T \upharpoonright k = A_i$, for each $j \leq k$, $j \in f([T]) \Leftrightarrow j \in f([R_i^k])$.

We point out that for any front, A_0 is always $\{\emptyset\}$ and $p_0 = 1$. Since we are assuming \mathcal{V} is non-principal, choose an R_0^0 in \mathfrak{T} so that $0 \notin f([R_0^0])$. Now let $k > 0$, and suppose we have chosen R_j^l for all $l < k$ and $j < p_l$. For $i < p_{k-1}$, if there is an $R \subseteq R_i^{k-1}$ in \mathfrak{T} such that $R \upharpoonright k = A_i$ and $k \notin f([R])$, then let R_i^k be such an R ; if not, let $R_i^k = R_i^{k-1} \cap \hat{B}_i^k$. Now suppose that $p_{k-1} \leq i < p_k$. If there is an $R \in \mathfrak{T}$ such that $R \upharpoonright k = A_i$ and $0 \notin f([R])$, let $R_{i,0}^k$ be such an R ; if not, let $R_{i,0}^k = \hat{B}_i^k$. Given $R_{i,j}^k$ for $j < k$, if there is an $R \in \mathfrak{T}$ such that $R \subseteq R_{i,j}^k$, $R \upharpoonright k = A_i$, and $j+1 \notin f([R])$, then let $R_{i,j+1}^k$ be such an R ; if not, then let $R_{i,j+1}^k = R_{i,j}^k$. Finally, let $R_i^k = R_{i,k}^k$.

It follows from the construction that for all $1 \leq l \leq k$ and $p_{l-1} \leq i < p_l$,

$$(4.10) \quad R_{i,0}^l \supseteq R_{i,1}^l \supseteq \cdots \supseteq R_{i,l}^l = R_i^l \supseteq R_i^k,$$

and moreover, for any $j \leq l$,

$$(4.11) \quad R_{i,j}^l \upharpoonright l = R_i^l \upharpoonright l = R_i^k \upharpoonright k = A_i.$$

Fix $k < \omega$; we check that $(*)_k$ holds. Let $i < p_k$, $T \subseteq R_i^k$ in \mathfrak{T} with $T \upharpoonright k = A_i$, and $j \leq k$ be given. If $j \in f([T])$, then j must be in $f([R_i^k])$, since $T \subseteq R_i^k$ and f is monotone. Now suppose that $j \notin f([T])$; we will show that $j \notin f([R_i^k])$. Let $p_{-1} = 0$, and let $l \leq k$ be the integer satisfying

$p_{l-1} \leq i < p_l$. Note that $\max \bigcup A_i = l - 1$. Thus, $T \upharpoonright k = A_i = T \upharpoonright l$. We now have two cases to check.

CASE 1: $j \leq l$. Notice that $T \subseteq R_i^k \subseteq R_i^l \subseteq R_{i,j}^l$. If $j = 0$, then $R_{i,j}^l \subseteq \hat{B}_i^l$ and we let R' denote \hat{B}_i^l ; if $j > 0$, then $R_{i,j}^l \subseteq R_{i,j-1}^l$ and we let R' denote $R_{i,j-1}^l$. Since j is not in $f([T])$ and $T \upharpoonright l = A_i$, T is a witness that there is an $R \subseteq R'$ with $R \upharpoonright l = A_i$ such that $j \notin f([R])$. Thus, $R_{i,j}^l$ was chosen so that $j \notin f([R_{i,j}^l])$. It follows that $j \notin f([R_i^k])$, since $R_i^k \subseteq R_{i,j}^l$ and f is monotone.

CASE 2: $l < j \leq k$. In this case, $T \subseteq R_i^k \subseteq R_i^j \subseteq R_i^{j-1}$. Since T is a witness that there is an $R \subseteq R_i^{j-1}$ with $R \upharpoonright l = A_i$ and $j \notin f([R])$, R_i^j was chosen so that $j \notin f([R_i^j])$. Thus, $j \notin f([R_i^k])$, since $R_i^k \subseteq R_i^j$ and f is monotone.

Therefore, $j \in f([T])$ if and only if $j \in f([R_i^k])$; hence $(*)_k$ holds. This concludes Stage 1 of our construction.

Given $k < \omega$ and $c \in \hat{B}^- \upharpoonright k$, define

$$(4.12) \quad S_c^k = \bigcap \{R_i^l : l \leq k, i < p_l, \text{ and } c \in R_i^l\}.$$

We claim that S_c^k is a member of \mathfrak{T} and that $c \in S_c^k$. To see this, notice that for $c \in \hat{B}^- \upharpoonright k$, if we let $l \leq k$ be least such that $l > \max c$, then c is in A_i for at least one $i < p_l$. Since $A_i \subseteq R_i^l$, the set $\{(l, i) : l \leq k, i < p_l, \text{ and } c \in R_i^l\}$ is non-empty; hence, $c \in S_c^k$. Moreover, S_c^k is a member of \mathfrak{T} , since it is a finite intersection of members of \mathfrak{T} . It follows that for each $a \in S_c^k \cap \hat{B}^-$, the set $\{l > \max a : a \cup \{l\} \in S_c^k\}$ is a member of the ultrafilter \mathcal{U}_c . Define

$$(4.13) \quad U_c^k := U_c(S_c^k) = \{l > \max c : c \cup \{l\} \in S_c^k\}.$$

Thus, for each $c \in \hat{B}^-$ and $j = \max c + 1$, we have $S_c^j \supseteq S_c^{j+1} \supseteq \dots$, each of which is a member of \mathfrak{T} ; and $U_c^j \supseteq U_c^{j+1} \supseteq \dots$, each of which is a member of the p -point \mathcal{U}_c .

STAGE 2. In this stage we construct a tree T^* in \mathfrak{T} which will be thinned down one more time in Stage 3 to obtain a subtree $\tilde{T} \subseteq T^*$ in \mathfrak{T} such that $f \upharpoonright (\mathfrak{T} \upharpoonright \tilde{T})$ is basic. The tree T^* which we construct in the present stage will have sets of immediate successors

$$(4.14) \quad U_c := U_c(T^*) = \{l \geq \max c + 1 : c \cup \{l\} \in T^*\}$$

for $c \in T^* \cap \hat{B}^-$. The sets U_c will have interval gaps which have right endpoints which line up often and in a useful way (meshing). This will aid in finding good cut-off points n_k needed in Stage 3 to thin T^* down to \tilde{T} . Toward obtaining these interval gaps, we will construct a family of functions which we call *meshing functions* $m(c, \cdot) : \omega \rightarrow \omega$ satisfying the following ‘meshing property’:

(†) For each $c \in \hat{B}^-$ and $j < \omega$, if $a \in \hat{B}^-$ is such that $a \prec c$, then there exists $i < \omega$ such that $m(a, 2i) = m(c, 2j)$.

The meshing functions of (†) will aid in obtaining a tree $T^* \in \mathfrak{T}$ with the following properties:

(‡) For all $c \in T^* \cap \hat{B}^-$,

(a) $U_c \subseteq U_c^{\max c+1}$;

(b) for all $i < \omega$, $U_c \setminus m(c, 2i) = U_c \setminus m(c, 2i+1) \subseteq U_c^{m(c, 2i)}$.

We now begin the construction of the meshing functions $m(c, \cdot)$ and the sets U_c , proceeding by recursion on the well-ordering (\hat{B}^-, \prec) . Since \emptyset is \prec -minimal in \hat{B}^- , we start by choosing g_\emptyset , $m(\emptyset, \cdot)$, and Y_\emptyset as follows. Since \mathcal{U}_\emptyset is a p-point, we may choose a $U_\emptyset^* \in \mathcal{U}_\emptyset$ such that $U_\emptyset^* \subseteq^* U_\emptyset^k$ for each k . (Recall the definition of U_c^k in (4.13).) Let $g_\emptyset : \omega \rightarrow \omega$ be a strictly increasing function such that for each k , $U_\emptyset^* \setminus g_\emptyset(k+1) \subseteq U_\emptyset^{g_\emptyset(k)}$, and $g_\emptyset(0) > 0$. If $\bigcup_{i \in \omega} [g_\emptyset(2i), g_\emptyset(2i+1)] \in \mathcal{U}_\emptyset$, then define $m(\emptyset, k) = g_\emptyset(k+1)$; otherwise, $\bigcup_{i \in \omega} [g_\emptyset(2i+1), g_\emptyset(2i+2)] \in \mathcal{U}_\emptyset$, and define $m(\emptyset, k) = g_\emptyset(k)$. Let $Y_\emptyset = \bigcup_{i \in \omega} [m(\emptyset, 2i+1), m(\emptyset, 2i+2)]$ and define

$$(4.15) \quad U_\emptyset = U_\emptyset^0 \cap U_\emptyset^* \cap Y_\emptyset.$$

Note that for each k we have $U_\emptyset \setminus m(\emptyset, k+1) \subseteq U_\emptyset^{m(\emptyset, k)}$.

Now suppose $c \in \hat{B}^-$ and for all $b \prec c$ in \hat{B}^- , g_b and $m(b, \cdot)$ have been defined. Since \mathcal{U}_c is a p-point, there is a $U_c^* \in \mathcal{U}_c$ such that $U_c^* \subseteq^* U_c^k$ for all $k > \max c$. Let a denote the immediate \prec -predecessor of c in \hat{B}^- . Let $g_c : \omega \rightarrow \omega$ be a strictly increasing function such that $g_c(0) > \max\{\max c, g_a(2)\}$, and

(1 $_{g_c}$) for each $i < \omega$, $U_c^* \setminus g_c(i+1) \subseteq U_c^{g_c(i)}$;

(2 $_{g_c}$) for each $j < \omega$, there is an $i > 0$ such that $g_c(j) = m(a, 2i)$.

Let Y_c be the one of the two sets $\bigcup_{i \in \omega} [g_c(2i), g_c(2i+1)]$ or $\bigcup_{i \in \omega} [g_c(2i+1), g_c(2i+2)]$ which is in \mathcal{U}_c . In the first case define $m(c, i) = g_c(i+1)$; in the second case set $m(c, i) = g_c(i)$. Then

$$(4.16) \quad Y_c = \bigcup_{i < \omega} [m(c, 2i+1), m(c, 2i+2)],$$

and this set is in \mathcal{U}_c . Let

$$(4.17) \quad U_c = U_c^{\max c+1} \cap U_c^* \cap Y_c.$$

This concludes the recursive definition.

We check that (†) holds. Let $c \in \hat{B}^-$ and let $a_0 \prec \dots \prec a_l \prec c$ be the enumeration of all \prec -predecessors of c in \hat{B}^- . Let $j < \omega$. Then either $m(c, 2j) = g_c(2j)$ or $m(c, 2j) = g_c(2j+1)$. By (2 $_{g_c}$), $g_c(2j) = m(a_l, 2i)$ for some i , and $g_c(2j+1) = m(a_l, 2i)$ for some i . Thus, there is an i such

that $m(a_l, 2i) = m(c, 2j)$. Let i_l denote this i . Likewise, either $m(a_l, 2i_l) = g_{a_l}(2i_l)$ or $m(a_l, 2i_l) = g_{a_l}(2i_l + 1)$. By $(2_{g_{a_l}})$, $g_{a_l}(2i_l) = m(a_{l-1}, 2i)$ for some i , and $g_{a_l}(2i_l + 1) = m(a_{l-1}, 2i)$ for some i . Let i_{l-1} denote the i such that $m(a_{l-1}, 2i_{l-1}) = m(a_l, 2i_l)$. Continuing in this manner, we obtain numbers i_k , $k \leq l$, such that

$$(4.18) \quad m(c, 2j) = m(a_l, 2i_l) = m(a_{l-1}, 2i_{l-1}) = \cdots = m(a_0, 2i_0).$$

Hence, (\dagger) holds.

CLAIM 2. *There exists a strictly increasing sequence $(m_k)_{k < \omega}$ such that, for all k ,*

$$(4.19) \quad \forall c \in \hat{B}^- \upharpoonright m_k \exists r (m(c, 2r) = m_{k+1}).$$

Proof. Let m_0 be arbitrary, and let c_0 be the \prec -maximum of $\hat{B}^- \upharpoonright m_0$. Since $m(c_0, \cdot)$ is strictly increasing, we can fix $j_0 \in \omega$ such that $m(c_0, 2j_0) > m_0$. Let $m_1 = m(c_0, 2j_0)$. In general, given m_k , let c_k be the \prec -maximum of $\hat{B}^- \upharpoonright m_k$. Since $m(c_k, \cdot)$ is strictly increasing, we can fix some $j_k < \omega$ such that $m(c_k, 2j_k) > m_k$, and let $m_{k+1} = m(c_k, 2j_k)$. In this manner, we inductively construct the sequence $(m_k)_{k < \omega}$. To check that this sequence has the desired property, let $k < \omega$ be given and fix $c \in \hat{B}^- \upharpoonright m_k$. Since $c \preceq c_k$, it follows from (\dagger) that there exists an r such that $m(c, 2r) = m(c_k, 2j_k) = m_{k+1}$. ■

Let T^* be the tree in \mathfrak{T} defined by declaring $U_c(T^*) = U_c$ for each $c \in \hat{B}^- \cap T^*$. If the reader is not satisfied with this top-down construction (which is precise as \emptyset is in every member of \mathfrak{T} and this completely determines the rest of T^*), we point out that T^* can also be seen as being constructed level by level as follows. Let $\emptyset \in T^*$, and for each $l \in U_\emptyset$, put $\{l\}$ in T^* , so that the first level of T^* is exactly $\{\{l\} : l \in U_\emptyset\}$. Suppose we have constructed the tree T^* up to level k , meaning that we know exactly what $T^* \cap \hat{B} \cap [\omega]^{\leq k}$ is. For each $c \in \hat{B}^- \cap T^* \cap [\omega]^k$, let the immediate successors of c in T^* be exactly the set U_c ; in other words, for each $l > \max c$, put $c \cup \{l\} \in T^*$ if and only if $l \in U_c$. Recalling that $\max c < g_c(0) \leq m(c, 0)$ and $U_c \subseteq Y_c = Y_c \setminus m(c, 0)$, we see that each element of U_c is strictly greater than $\max c$. Hence, by constructing T^* in this manner, we obtain a member of \mathfrak{T} such that for each $c \in T^* \cap \hat{B}^-$, $U_c(T^*)$ is exactly U_c .

We now check that (\ddagger) holds. Let $c \in T^* \cap \hat{B}^-$ be given. By (4.17), $U_c \subseteq U_c^{\max c + 1}$, so $(\ddagger)(a)$ holds. By (4.16), we see that $Y_c \cap [m(c, 2i), m(c, 2i+1)) = \emptyset$ for each i . Thus,

$$(4.20) \quad U_c \cap [m(c, 2i), m(c, 2i+1)) = \emptyset,$$

since $U_c \subseteq Y_c$ by (4.17). Recall that U_c^* diagonalizes the collection of the sets U_c^k for all $k > \max c$, and the function g_c was chosen to witness this diagonalization so that (1_{g_c}) holds. Either $m(c, i) = g_c(i)$ and $m(c, i+1) = g_c(i+1)$, or else $m(c, i) = g_c(i+1)$ and $m(c, i+1) = g_c(i+2)$. In either

case, (1_{g_c}) implies that $U_c^* \setminus m(c, i+1) \subseteq U_c^{m(c,i)}$. Thus

$$(4.21) \quad U_c \setminus m(c, i+1) \subseteq U_c^{m(c,i)},$$

since $U_c \subseteq U_c^*$ by (4.17). Now $(\ddagger)(b)$ follows from (4.20) and (4.21). This finishes Stage 2 of our construction.

STAGE 3. We will show that there is a strictly increasing sequence $(n_k)_{k < \omega}$ and a subtree $\tilde{T} \subseteq T^*$ in \mathfrak{T} such that for all k ,

$$(4.22) \quad \forall c \in \tilde{T} \cap (\hat{B}^- \upharpoonright n_k) \exists r (m(c, 2r) = n_k).$$

LEMMA 4.5. *Let B be a front on ω , and let T^* be a \vec{U} -tree, where $\vec{U} = \langle \mathcal{U}_c : c \in \hat{B}^- \rangle$ is a sequence of non-principal ultrafilters on ω . Suppose that strictly increasing functions $m(c, \cdot) : \omega \rightarrow \omega$ for every $c \in \hat{B}^-$ and a strictly increasing sequence $(m_k)_{k < \omega}$ are given such that, for all k ,*

$$(4.23) \quad \forall c \in \hat{B}^- \upharpoonright m_k \exists r (m(c, 2r) = m_{k+1}).$$

Then there exist $\tilde{T} \subseteq T^$ such that \tilde{T} is a \vec{U} -tree and a subsequence $(n_k)_{k < \omega}$ of $(m_k)_{k < \omega}$ such that, for all k ,*

$$(4.24) \quad \forall c \in \tilde{T} \cap (\hat{B}^- \upharpoonright n_k) \exists r (m(c, 2r) = n_k).$$

Proof. The proof is by induction on the rank of the front B . First assume $\text{rank}(B) = 1$. In this case, $B = \hat{B} = \{\emptyset\}$ and $\hat{B}^- = \{\}$. Also notice that the only $\langle \rangle$ -tree is $\{\emptyset\}$. Therefore, setting $\tilde{T} = \{\emptyset\}$ and $(n_k)_{k < \omega} = (m_k)_{k < \omega}$ will work.

Now assume that $\text{rank}(B) > 1$. Then, for every $l < \omega$, $B_{\{l\}}$ is a front on $[l+1, \omega)$ of rank strictly less than $\text{rank}(B)$. Given $l < \omega$, set $m_l(c, \cdot) = m(\{l\} \cup c, \cdot)$ for $c \in \hat{B}_{\{l\}}^-$. For $l \in U_\emptyset(T^*)$, define

$$(4.25) \quad T_l^* = \{c \in \hat{B}_{\{l\}}^- : \{l\} \cup c \in T^*\},$$

and observe that T_l^* is a \vec{U}_l -tree, where $\vec{U}_l = \langle \mathcal{U}_{\{l\} \cup c} : c \in \hat{B}_{\{l\}}^- \rangle$.

Next, we will inductively define subsequences $(n_k^j)_{k < \omega}$ of $(m_k)_{k < \omega}$ for every $j < \omega$. We will also make sure that $(n_k^h)_{k < \omega}$ is a subsequence of $(n_k^j)_{k < \omega}$ whenever $h \geq j$. Furthermore, we will obtain a \vec{U}_l -tree $\tilde{T}_l \subseteq T_l^*$ for every $l \in U_\emptyset(T^*)$, such that, for all k ,

$$(4.26) \quad \forall c \in \tilde{T}_l \cap (\hat{B}_{\{l\}}^- \upharpoonright n_k^l) \exists r (m_l(c, 2r) = n_k^l).$$

Let $(m_k^0)_{k < \omega} = (m_k)_{k < \omega}$. If $0 \notin U_\emptyset(T^*)$, simply let $(n_k^0)_{k < \omega} = (m_k^0)_{k < \omega}$. If $0 \in U_\emptyset(T^*)$, apply the induction hypothesis to $B_{\{0\}}$ and T_0^* , with respect to the functions $m_0(c, \cdot)$ and the sequence $(m_k^0)_{k < \omega}$. This will yield a \vec{U}_0 -tree $\tilde{T}_0 \subseteq T_0^*$ and a subsequence $(n_k^0)_{k < \omega}$ of $(m_k^0)_{k < \omega}$.

Let $(m_k^1)_{k < \omega} = (n_k^0)_{k < \omega}$. If $1 \notin U_\emptyset(T^*)$, simply let $(n_k^1)_{k < \omega} = (m_k^1)_{k < \omega}$. If $1 \in U_\emptyset(T^*)$, apply the induction hypothesis to $B_{\{1\}}$ and T_1^* , with respect to

the functions $m_1(c, \cdot)$ and the sequence $(m_k^1)_{k < \omega}$. Continue as in these first two steps to complete the inductive construction.

Set $h(0) = n_0^0$. Given $h(k)$, fix $q(k)$ such that $n_{q(k)}^{h(k)} > h(k)$, then set $h(k+1) = n_{q(k)}^{h(k)}$. Notice that exactly one of $\bigcup_{k < \omega} [h(2k), h(2k+1))$ and $\bigcup_{k < \omega} [h(2k+1), h(2k+2))$ will belong to \mathcal{U}_\emptyset , and denote it by Z . Set $U_\emptyset(\tilde{T}) = U_\emptyset(T^*) \cap Z$, then define

$$(4.27) \quad \tilde{T} = \{\emptyset\} \cup \bigcup_{l \in U_\emptyset(\tilde{T})} \{\{l\} \cup c : c \in \tilde{T}_l\}.$$

It is straightforward to check that \tilde{T} is a \vec{U} -tree contained in T^* . If $Z = \bigcup_{k < \omega} [h(2k), h(2k+1))$, then define $n_k = h(2k+2)$, otherwise define $n_k = h(2k+1)$. We will only complete the proof in the first case, as the other case is similar.

Fix $k < \omega$ and $c \in \tilde{T} \cap (\hat{B}^- \upharpoonright n_k)$. If $c = \emptyset$, then $c \in \hat{B}^- \upharpoonright m_{j-1}$, where $j > 0$ is such that $m_j = n_k$. Therefore, there exists r such that $m(c, 2r) = m_j = n_k$. Now assume that $c \neq \emptyset$, and let $l = \min c$. Since $l \in Z$ and $l < h(2k+2)$, we must have $l < h(2k+1)$. In particular, $(n^{h(2k+1)})_{k < \omega}$ is a subsequence of $(n_k^l)_{k < \omega}$. Therefore, there exists q such that $n_{q(2k+1)}^{h(2k+1)} = n_q^l$. Finally, since $c \setminus \{l\} \in \tilde{T}_l \cap (\hat{B}_{\{l\}}^- \upharpoonright n_q^l)$, we see that there exists r such that

$$(4.28) \quad m(c, 2r) = m_l(c \setminus \{l\}, 2r) = n_q^l = n_{q(2k+1)}^{h(2k+1)} = h(2k+2) = n_k,$$

which is what we needed to show. ■

Taking \tilde{T} as in Lemma 4.5 concludes Stage 3 of the construction.

Finally, we check that (\otimes) holds. Toward this, we first show that for all $k < \omega$ and $i < p_{n_k}$, $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$. It will follow that for each $T \in \mathfrak{T} \upharpoonright \tilde{T}$ with $T \upharpoonright n_k = A_i$, we in fact have $T \subseteq R_i^{n_k}$. This along with $(*)_{n_k}$ for all $k < \omega$ will yield (\otimes) .

CLAIM 3. *Let $k < \omega$ and $i < p_{n_k}$ be given, and suppose that $A_i \subseteq \tilde{T}$. Then $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$.*

Proof. Let Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$. Since $A_i \subseteq \tilde{T}$, we see that $Q \upharpoonright n_k = A_i$, which equals $R_i^{n_k} \upharpoonright n_k$. Thus, to prove the claim it is enough to show that for each $c \in Q \cap \hat{B}^-$,

$$(4.29) \quad U_c(Q) \setminus n_k \subseteq U_c(R_i^{n_k}).$$

Since $Q \subseteq \tilde{T} \subseteq T^*$, we see that for all $c \in Q \cap \hat{B}^-$,

$$(4.30) \quad U_c(Q) \setminus n_k \subseteq U_c(\tilde{T}) \setminus n_k \subseteq U_c(T^*) \setminus n_k.$$

We have two cases for c .

CASE 1: $c \in Q \cap (\hat{B}^- \upharpoonright n_k)$. Then by Lemma 4.5, there is an r such that $m(c, 2r) = n_k$. By (\ddagger) (b), we have

$$(4.31) \quad U_c(T^*) \setminus n_k = U_c(T^*) \setminus m(c, 2r + 1) \subseteq U_c^{m(c, 2r)} = U_c^{n_k}.$$

Since $Q \upharpoonright n_k = A_i \subseteq R_i^{n_k}$, c is in $R_i^{n_k}$, and hence $S_c^{n_k} \subseteq R_i^{n_k}$, by (4.12). Therefore, recalling (4.13), we have

$$(4.32) \quad U_c^{n_k} = U_c(S_c^{n_k}) \subseteq U_c(R_i^{n_k}).$$

By (4.30)–(4.32), we see that

$$(4.33) \quad U_c(Q) \setminus n_k \subseteq U_c(T^*) \setminus n_k \subseteq U_c^{n_k} \subseteq U_c(R_i^{n_k}).$$

CASE 2: $c \in Q \cap \hat{B}^-$ with $\max c \geq n_k$. Then it follows from (\ddagger) (a) that $U_c(T^*) \subseteq U_c(S_c^{\max c + 1})$. The proof will proceed by induction on the cardinality of $c \setminus n_k$.

Suppose $|c \setminus n_k| = 1$. Let $l = \max c$, and let a denote $c \setminus \{l\}$. Then $l \in U_a(Q) \setminus n_k$. Further, a is a member of $R_i^{n_k}$, since $a \in Q \upharpoonright n_k = A_i \subseteq R_i^{n_k}$. Since by Case 1, $U_a(Q) \setminus n_k$ is contained in $U_a(R_i^{n_k})$, we see that $c \in R_i^{n_k}$. Further,

$$(4.34) \quad U_c(Q) \subseteq U_c(T^*) \subseteq U_c(S_c^{l+1}) \subseteq U_c(R_i^{n_k}),$$

since $l + 1 > n_k$, $i < p_{n_k}$ and $c \in R_i^{n_k}$ imply that $S_c^{l+1} \subseteq R_i^{n_k}$, by the definition (4.12) of S_c^{l+1} . Thus, Case 2 holds for the basis of our induction scheme.

Now assume that Case 2 holds for all $c \in Q \cap \hat{B}^-$ with $1 \leq |c \setminus n_k| \leq m$. Suppose $c \in Q \cap \hat{B}^-$ with $|c \setminus n_k| = m + 1$. Letting $l = \max c$ and $a = c \setminus \{l\}$, the induction hypothesis applied to a yields $a \in R_i^{n_k}$ and $U_a(Q) \subseteq U_a(R_i^{n_k})$. Thus, $c \in R_i^{n_k}$. Again, as in (4.34), we find that $U_c(Q) \subseteq U_c(R_i^{n_k})$, which finishes the proof of Case 2. ■

To finish the proof of the theorem, we prove that (\otimes) holds. Let $T \in \mathfrak{T} \upharpoonright \tilde{T}$, $k < \omega$, and suppose $i < p_{n_k}$ is the integer such that $T \upharpoonright n_k = A_i$. Letting Q denote $\tilde{T} \cap \hat{B}_i^{n_k}$, we have $T \subseteq Q$. By Claim 3, $Q \subseteq R_i^{n_k}$; so for all $j \leq k$,

$$(4.35) \quad j \in f([T]) \iff j \in f([R_i^{n_k}])$$

by $(*)_{n_k}$. Since $Q \upharpoonright n_k = A_i$, it follows from $(*)_{n_k}$ that for all $j \leq k$,

$$(4.36) \quad j \in f([Q]) \iff j \in f([R_i^{n_k}]).$$

Equations (4.35) and (4.36) complete the proof of (\otimes) . By Claim 1, f is monotone basic on $\mathfrak{T} \upharpoonright T$. This concludes the proof of Theorem 4.4. ■

We conclude this section by pointing out how f may be basic on $2^{\hat{B}}$ while being only finitely generated on 2^B . Given a front B and a set $\mathcal{C} \subseteq 2^B$, let $\hat{\mathcal{C}}$ denote $\{\hat{X} : X \in \mathcal{C}\}$, a subset of $2^{\hat{B}}$. Letting C denote $\{\hat{X} \upharpoonright k_m : X \in \mathcal{C} \text{ and } m < \omega\}$, we observe that any finitary function $\hat{f} : C \rightarrow 2^{<\omega}$ determines functions $f' : \mathcal{C} \rightarrow 2^\omega$ and $f^* : \hat{\mathcal{C}} \rightarrow 2^\omega$ by setting $f'(X) = f^*(\hat{X}) =$

$\bigcup_{m < \omega} \hat{f}(\hat{X} \upharpoonright k_m)$. In particular, $f'(X) = f^*(\hat{X})$ for each $X \in \mathcal{C}$. The following is straightforward to prove.

PROPOSITION 4.6. *Suppose that B is a front, \mathcal{C} is a subset of 2^B , and $C = \{\hat{X} \upharpoonright k_m : X \in \mathcal{C}, m < \omega\}$. If $\hat{f} : C \rightarrow 2^{<\omega}$ is a basic map, then f^* is continuous on \hat{C} as a subspace of $2^{\hat{B}}$.*

In particular, given a map $f : \mathcal{U} \rightarrow \mathcal{V}$ in the setting of Theorem 4.4, the map $f^* : \mathfrak{T} \upharpoonright \tilde{T} \rightarrow \mathcal{V}$ defined by $f^*(T) = \bigcup_{m < \omega} \hat{f}(T \upharpoonright k_m)$ is a continuous map on its domain $\mathfrak{T} \upharpoonright \tilde{T}$. This map f^* is equivalent to f in the following sense: for each $T \in \mathfrak{T} \upharpoonright \tilde{T}$, $f^*(T) = f([T])$. In contrast, if we let $\mathcal{C} = \{[T] : T \in \mathfrak{T} \upharpoonright \tilde{T}\}$, then the map $f : \mathcal{C} \rightarrow \mathcal{V}$ is not necessarily continuous. However, $f \upharpoonright \mathcal{C}$ is still represented by the monotone finitary map \hat{f} ; given $X \in \mathcal{C}$, $f(X) = \bigcup_{m < \omega} [[\hat{f}(\hat{X} \upharpoonright k_m)]]$.

5. Further connections between Tukey, Rudin–Blass, and Rudin–Keisler reductions. In [13, Lemma 9], Raghavan distilled properties of cofinal maps which, when satisfied, imply that Tukey reducibility above a q -point implies Rudin–Blass reducibility. He then showed that continuous cofinal maps satisfy these properties, thus yielding [13, Theorem 10] (see Theorem 1.6). The proof of the next theorem follows the general structure of Raghavan’s proofs. The key differences are that we start with a weaker assumption, basic maps on Fubini iterates of p -points, and obtain a finite-to-one map on the base tree \hat{B} for the ultrafilter rather than the base B itself. While the following theorem may be of interest in itself, we will apply it to prove Theorems 5.3 and 5.4 stating that for finite Fubini iterates of p -points, and for generic ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$, Tukey reducibility above a q -point is equivalent to Rudin–Keisler reducibility.

THEOREM 5.1. *Suppose \mathcal{U} is a Fubini iterate of p -points and \mathcal{V} is a q -point. If $\mathcal{V} \leq_{\text{T}} \mathcal{U}$, then there is a finite-to-one function $\tau : \tilde{T} \rightarrow \omega$, for some $\tilde{T} \in \mathfrak{T}$, such that $\{\tau[T] : T \in \mathfrak{T} \upharpoonright \tilde{T}\} \subseteq \mathcal{V}$.*

Proof. Let B be the front which is a base for \mathcal{U} , and as usual let \mathfrak{T} denote the set of all \vec{U} -trees on \hat{B} . Given $m < n$ and $T \subseteq [\omega]^{<\omega}$, let $T \upharpoonright [m, n)$ denote the set of all $a \in T$ such that $m \leq \max a < n$.

Let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a monotone cofinal map. Let $\tilde{T} \in \mathfrak{T}$ be given by Theorem 4.4, so that f is generated on $\mathfrak{T} \upharpoonright \tilde{T}$ by some monotone basic map $\hat{f} : \bigcup_{m < \omega} 2^{\tilde{T} \upharpoonright k_m} \rightarrow 2^{<\omega}$. Let $\psi : \mathcal{P}(\tilde{T}) \rightarrow \mathcal{P}(\omega)$ be defined by

$$(5.1) \quad \psi(A) = \{k \in \omega : \forall T \in \mathfrak{T} \upharpoonright \tilde{T} (A \subseteq T \rightarrow k \in f([T]))\}$$

for each $A \subseteq \tilde{T}$. Note that ψ is monotone and further that for each $T \in \mathfrak{T} \upharpoonright \tilde{T}$ and each $m < \omega$, $[[\hat{f}(T \upharpoonright k_m)]] = \psi(T \upharpoonright k_m) \cap m$. By the same argument as in [13, Lemma 8], we may assume that for each finite $A \subseteq \tilde{T}$, $\psi(A)$ is finite.

Notice that for any pair $m, j < \omega$, if $j \notin \psi(\tilde{T} \upharpoonright m)$, then there is a $T \in \mathfrak{T} \upharpoonright \tilde{T}$ such that $T \sqsupset \tilde{T} \upharpoonright m$ and $j \notin f([T])$; hence, $j \notin [[\hat{f}(T \upharpoonright k_j)]]$. It follows that for all $S \in \mathfrak{T} \upharpoonright \tilde{T}$ satisfying $S \sqsupseteq T \upharpoonright k_j$, $j \notin f([S])$ and hence also $j \notin \psi(S)$. Without loss of generality, assume that $T \upharpoonright [m, k_j] = \emptyset$; that is, each $a \in T$ has $\max a \notin [m, k_j]$. Now if A is a finite \sqsubset -closed subset of \tilde{T} and $A \upharpoonright [m, k_j] = \emptyset$, then there is an $S \in \mathfrak{T} \upharpoonright \tilde{T}$ such that $\tilde{T} \upharpoonright m \cup A \subseteq S$, $S \upharpoonright [m, k_j] = \emptyset$, and $S \sqsupset T \upharpoonright k_j$. Then $j \notin f([S])$, so $j \notin \psi(\tilde{T} \upharpoonright m \cup A)$.

Define $g : \omega \rightarrow \omega$ by $g(0) = 0$; and given $g(n)$, choose $g(n+1) > g(n)$ so that

$$(5.2) \quad g(n+1) > \max\{k_{g(n)}, \max(\psi(\tilde{T} \upharpoonright k_{g(n)}))\}.$$

Since \mathcal{V} is a q-point, there is a $V_0 \in \mathcal{V}$ such that $|V_0 \cap [g(n), g(n+1))| = 1$ for each $n < \omega$. We may, without loss of generality, assume that $V_1 = \bigcup_{n \in \omega} [g(2n), g(2n+1))$ is in \mathcal{V} , and let $V = V_0 \cap V_1$. Enumerate V as $\{v_i : i < \omega\}$. Notice that for each $i < \omega$,

$$(5.3) \quad g(2i) \leq v_i < g(2i+1).$$

Without loss of generality, assume that $v_0 > 0$. Then $v_0 \notin \psi(\emptyset)$, since \mathcal{V} being non-principal, $\psi(\emptyset)$ must be empty.

Our construction ensures that for all $i < \omega$,

- (1) $g(i+1) > \max(k_{g(i)}, \max(\psi(\tilde{T} \upharpoonright k_{g(i)})))$;
- (2) $g(2i) \leq v_i < g(2i+1)$;
- (3) $k_{v_i} < g(2i+2)$;
- (4) $\psi(\tilde{T} \upharpoonright k_{v_i}) \subseteq g(2i+2)$.

We will now define a strictly increasing function $h : \omega \rightarrow \omega$ such that for each $i < \omega$,

- (a) $h(i) < k_{v_i}$;
- (b) $v_i \notin \psi(\tilde{T} \upharpoonright h(i))$;
- (c) for each finite, \sqsubset -closed set $A \subseteq \tilde{T}$, if $v_i \in \psi(A)$ then $A \upharpoonright [h(i), h(i+1)) \neq \emptyset$.

Define $h(0) = 0$. Then (a)–(c) trivially hold. Suppose $h(i)$ has been defined so that (a)–(c) hold. Define $h(i+1) = k_{v_i}$. Then $h(i+1) > h(i)$, since $k_{v_i} > h(i)$ by (a) of the induction hypothesis. (a) holds since $k_{v_i} < k_{v_{i+1}}$. To check (b), note that

$$(5.4) \quad \psi(\tilde{T} \upharpoonright h(i+1)) = \psi(\tilde{T} \upharpoonright k_{v_i}) \subseteq \psi(\tilde{T} \upharpoonright k_{g(2i+1)}) \subseteq g(2i+2) \leq v_{i+1},$$

where the inclusions hold by (3) and (1), and the inequality holds by (2). Thus, $v_{i+1} \notin \psi(\tilde{T} \upharpoonright h(i+1))$.

To check (c), fix a finite \sqsubset -closed set $A \subseteq \tilde{T}$ with $A \upharpoonright [h(i), h(i+1)) = \emptyset$; that is, for all $a \in A$, $\max a \notin [h(i), h(i+1))$. Let $A' = A \setminus (T \upharpoonright h(i+1))$. We claim that $v_i \notin \psi(\tilde{T} \upharpoonright h(i) \cup A')$. By (b), $v_i \notin \psi(\tilde{T} \upharpoonright h(i))$. Therefore,

$v_i \notin [[\hat{f}(\tilde{T} \upharpoonright h(i))]]$. Now $(\tilde{T} \upharpoonright h(i) \cup A') \upharpoonright h(i+1) = \tilde{T} \upharpoonright h(i)$, since $A' \upharpoonright [h(i), h(i+1)) = \emptyset$. So $v_i \notin [[\hat{f}((\tilde{T} \upharpoonright h(i) \cup A') \upharpoonright h(i+1))]]$. Therefore, for each $S \in \mathfrak{T} \upharpoonright \tilde{T}$ such that $S \upharpoonright h(i+1) = (\tilde{T} \upharpoonright h(i) \cup A') \upharpoonright h(i+1)$, which we point out is the same as $\tilde{T} \upharpoonright h(i)$, we have $v_i \notin f([S])$. Thus, if $S \sqsupseteq \tilde{T} \upharpoonright h(i) \cup A'$ and $S \upharpoonright [h(i), h(i+1)) = \emptyset$, then $v_i \notin f([S])$, since this gets decided by height k_{v_i} which equals $h(i+1)$. Therefore, $v_i \notin \psi(\tilde{T} \upharpoonright h(i) \cup A')$, which proves (c).

Now we define a function $\tau : \tilde{T} \rightarrow \omega$ as follows: for $i < \omega$ and $a \in \tilde{T}$, if $\max a \in [h(i), h(i+1))$, then $\tau(a) = v_i$.

CLAIM. For each $T \in \mathfrak{T} \upharpoonright \tilde{T}$, $\tau[T] \in \mathcal{V}$.

Proof. Suppose not. Then there is a $T \in \mathfrak{T} \upharpoonright \tilde{T}$ such that $\tau[T] \notin \mathcal{V}$; so $\tau[T] \in \mathcal{V}^*$. Then there is an $S \in \mathfrak{T} \upharpoonright T$ with $f([S]) \subseteq (\omega \setminus \tau[T]) \cap V \in \mathcal{V}$. Let j be least such that $\hat{f}(\chi_S \upharpoonright k_j) \neq \emptyset$ and let $s = S \upharpoonright k_j$. Then $\emptyset \neq [[\hat{f}(\chi_S \upharpoonright k_j)]] \subseteq \psi(s)$. Fix some $v_i \in \psi(s)$. Then $v_i \notin \tau[T]$ since $v_i \in \psi(s) \subseteq f([S]) \subseteq \omega \setminus \tau[T]$. However, $v_i \in \psi(s)$ implies $s \upharpoonright [h(i), h(i+1)) \neq \emptyset$, by (c). For each $a \in s \upharpoonright [h(i), h(i+1))$, $\tau(a) = v_i$. Since $s \subseteq T$, we have $v_i \in \tau[T]$, a contradiction. Thus, $\tau[T] \in \mathcal{V}$. ■

Therefore, τ is a finite-to-one map from \tilde{T} into ω , and the set of τ -images of members of $\mathfrak{T} \upharpoonright \tilde{T}$ generates a filter contained inside \mathcal{V} . ■

The previous theorem does not necessarily imply that the τ -image of $\mathfrak{T} \upharpoonright \tilde{T}$ generates \mathcal{V} . However, under certain conditions, it does. If $\mathfrak{T} \upharpoonright \tilde{T}$ generates an ultrafilter on base set the tree \tilde{T} , as is the case when all the p-points are the same selective ultrafilter, then the τ -image of \mathcal{U} is \mathcal{V} . The following is a Rudin–Blass analogue, but on \hat{B} instead of B .

COROLLARY 5.2. If \mathcal{U} is a Fubini power of some selective ultrafilter \mathcal{V} , where the base set for \mathcal{U} is the front B , then there is a finite-to-one map $\tau : \hat{B} \rightarrow \omega$ such that $\{\tau[T] : T \in \mathfrak{T}\}$ generates \mathcal{V} .

It is useful to point out the connection and contrast between this corollary and the following previously known results. Every Fubini power of some selective ultrafilter is Tukey equivalent to that selective ultrafilter [8, Corollary 37]. Thus, if \mathcal{U} is a Fubini iterate of one selective ultrafilter, then the q-point \mathcal{V} in Theorem 5.1 must be that selective ultrafilter. On the other hand, the only ultrafilters Tukey reducible to a selective ultrafilter are those ultrafilters isomorphic to some Fubini power of the selective ultrafilter (Todorćević’s Theorem 24 in [13]). Thus, the collection of ultrafilters Tukey equivalent to a given selective ultrafilter is exactly the collection of Fubini powers of that selective ultrafilter; and any Fubini power of a selective ultrafilter is trivially Rudin–Keisler above that same selective ultrafilter.

In [13, Corollary 56], Raghavan showed that if \mathcal{U} is some Fubini iterate of p-points and \mathcal{V} is selective, then $\mathcal{V} \leq_T \mathcal{U}$ implies $\mathcal{V} \leq_{RK} \mathcal{U}$. We now

generalize this to q-points, though at the cost of assuming \mathcal{U} is only a finite Fubini iterate of p-points.

THEOREM 5.3. *Suppose \mathcal{U} is a finite iterate of Fubini products of p-points. If \mathcal{V} is a q-point and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V} \leq_{\text{RK}} \mathcal{U}$.*

Proof. Let k denote the length of the Fubini iteration, so $[\omega]^k$ is the front which is a base for \mathcal{U} . Let τ be the finite-to-one map from Theorem 5.1, and without loss of generality, assume τ is defined on all of $[\omega]^{\leq k}$. For each $T \in \mathfrak{T}$, we notice that $\bigcup_{1 \leq l \leq k} \tau[T \cap [\omega]^l] = \tau[T] \in \mathcal{V}$. For each $T \in \mathfrak{T}$, let $L(T) = \{1 \leq l \leq k : \tau[T \cap [\omega]^l] \in \mathcal{V}\}$. Then there is a $T \in \mathfrak{T}$ such that for all $S \in \mathfrak{T} \upharpoonright T$, $L(S) = L(T)$. Let $l = \max(L(T))$.

Now $\{S \cap [\omega]^l : S \in \mathfrak{T} \upharpoonright T\}$ generates an ultrafilter on base set $[\omega]^l \cap T$; further, for each $S \in \mathfrak{T} \upharpoonright T$, $\tau[S \cap [\omega]^l]$ is a member of \mathcal{V} (since $l \in L(S)$). Thus, $\{\tau[S \cap [\omega]^l] : S \in \mathfrak{T} \upharpoonright T\}$ generates an ultrafilter, and each of these τ -images is in \mathcal{V} . It follows that $\{\tau[S \cap [\omega]^l] : S \in \mathfrak{T} \upharpoonright T\}$ generates \mathcal{V} . If $l = k$, we are done, and in fact we have a Rudin–Blass map from \mathcal{U} to \mathcal{V} . If $l < k$, then define $\sigma : [\omega]^k \rightarrow \omega$ by $\sigma(a) = \tau(\pi_l(a))$. Then σ is a Rudin–Keisler map from \mathcal{U} into \mathcal{V} . ■

We point out that the *basic* maps on the generic ultrafilters \mathcal{G}_k forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$, $2 \leq k < \omega$, in [6] have exactly the same properties as basic maps on $[\omega]^k$ in the present paper (see [6, Definition 37 and Theorem 38]). Hence, Theorem 5.1 also applies to these ultrafilters. Since for each $1 \leq l \leq k$, the projection of \mathcal{G}_k to $[\omega]^l$ yields the generic ultrafilter \mathcal{G}_l , the same proof as in Theorem 5.3 yields the following theorem.

THEOREM 5.4. *Suppose \mathcal{G}_k is a generic ultrafilter forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$ for any $2 \leq k < \omega$. If \mathcal{V} is a q-point and $\mathcal{V} \leq_T \mathcal{G}_k$, then $\mathcal{V} \leq_{\text{RK}} \mathcal{G}_k$.*

REMARK 5.5. We cannot in general weaken the requirement of q-point to rapid in Theorem 5.1. In [10], it is shown that there are Tukey equivalent rapid p-points, and hence a Fubini iterate of such p-points Tukey equivalent to a rapid p-point, which are Rudin–Keisler incomparable.

6. Ultrafilters Tukey reducible to Fubini iterates of p-points have finitely generated Tukey reductions. In this section, we prove the analogue of Theorem 2.6 for the class of all ultrafilters which are Tukey reducible to some Fubini iterate of p-points. Namely, in Theorem 6.3, we prove that every such ultrafilter has finitely generated Tukey reductions (Definition 6.2). This sharpens a result of Raghavan [13, Lemma 16] by obtaining finitary maps which generate the original cofinal map on some filter base rather than some possibly different cofinal map. Also, the class on which we obtain finitely generated Tukey reductions is closed under Tukey

reduction, whereas the class where his result applies (basically generated ultrafilters) is not known to be closed under Tukey reduction. Theorem 6.3 allows us to extend Raghavan's Theorem 17 in [13] relating Tukey reduction to Rudin–Keisler reduction for basically generated ultrafilters to the class of all ultrafilters Tukey reducible to some Fubini iterate of p-points (see Theorem 6.4 and the discussion preceding it).

The next lemma is the analogue of Lemma 2.4 for the space $2^{\hat{B}}$ in place of 2^ω . As the proof is an almost verbatim repetition with obvious changes, we omit it.

LEMMA 6.1 (Extension lemma for fronts). *Suppose \mathcal{U} is a non-principal ultrafilter with base set a front B . Suppose $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{C} \subseteq \mathcal{U}$ such that $f \upharpoonright \mathcal{C}$ is represented by a monotone basic function, in the sense of Definition 4.2. Then there is a continuous, monotone $\tilde{f} : 2^{\hat{B}} \rightarrow 2^\omega$ such that*

- (1) \tilde{f} is represented by a monotone basic map $\hat{f} : \bigcup_{m < \omega} 2^{\hat{B} \upharpoonright k_m} \rightarrow 2^{< \omega}$, in the sense of Definition 4.2.

Moreover, if we define $f' : \mathcal{U} \rightarrow 2^\omega$ by $f'(U) = \tilde{f}(\hat{U})$ for $U \in \mathcal{U}$ then

- (2) $f' \upharpoonright \mathcal{C} = f \upharpoonright \mathcal{C}$;
- (3) $f' \upharpoonright \mathcal{U}$ is a cofinal map from \mathcal{U} to \mathcal{V} .

DEFINITION 6.2 (Finitely generated Tukey reduction). We say that an ultrafilter \mathcal{V} on base set ω has *finitely generated Tukey reductions* if for each monotone cofinal map $f : \mathcal{V} \rightarrow \mathcal{W}$, there is a cofinal subset $\mathcal{C} \subseteq \mathcal{V}$, a strictly increasing sequence $(k_m)_{m < \omega}$, and a function $\hat{f} : \mathcal{C} \rightarrow 2^{< \omega}$, where $\mathcal{C} = \{X \upharpoonright k_m : X \in \mathcal{C}, m < \omega\}$, such that

- (1) \hat{f} is level preserving: for each $m < \omega$ and $s \in \mathcal{C}$, $|s| = k_m$ implies $|\hat{f}(s)| = m$;
- (2) \hat{f} generates f on \mathcal{C} : for each $X \in \mathcal{C}$,

$$(6.1) \quad f(X) = \bigcup_{m < \omega} [[\hat{f}(X \upharpoonright k_m)]].$$

The difference between a basic Tukey reduction and a finitely generated Tukey reduction is that the map \hat{f} in Definition 6.2 is not required to be end-extension preserving.

Now we prove the main theorem of this section. This is the analogue of Theorem 2.5 (which holds for ultrafilters Tukey reducible to some p-point) to the setting of all ultrafilters Tukey reducible to some Fubini iterate of p-points.

THEOREM 6.3. *Let \mathcal{U} be any Fubini iterate of p-points. If $\mathcal{V} \leq_T \mathcal{U}$, then \mathcal{V} has finitely generated Tukey reductions.*

Proof. Suppose that \mathcal{U} is an iterate of Fubini products of p-points and that $\mathcal{V} \leq_T \mathcal{U}$. Let B be a front which is a base for \mathcal{U} , and without loss of generality, assume that ω is the base set for the ultrafilter \mathcal{V} . By Theorem 4.4, \mathcal{U} has basic Tukey reductions. Applying Lemma 6.1, we obtain a continuous monotone map $\tilde{f} : 2^{\hat{B}} \rightarrow 2^\omega$ which is generated by a monotone basic map $\hat{f} : \bigcup_{m < \omega} 2^{\hat{B} \upharpoonright k_m} \rightarrow 2^{<\omega}$ for some increasing sequence $(k_m)_{m < \omega}$. Hence, for each $A \subseteq \hat{B}$, $\tilde{f}(A) = \bigcup_{m < \omega} [[\hat{f}(A \upharpoonright k_m)]]$. Furthermore, defining $f(U) = \tilde{f}(\hat{U})$ for $U \in \mathcal{U}$, we see that $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map.

Suppose $\mathcal{W} \leq_T \mathcal{V}$, and let $h : \mathcal{V} \rightarrow \mathcal{W}$ be a monotone cofinal map. Extend h to the map $\tilde{h} : 2^\omega \rightarrow 2^\omega$ defined for each $X \in 2^\omega$ by

$$(6.2) \quad \tilde{h}(X) = \bigcap \{h(V) : V \in \mathcal{V} \text{ and } V \supseteq X\}.$$

It follows from h being monotone that \tilde{h} is monotone and $\tilde{h} \upharpoonright \mathcal{V} = h$.

Letting \tilde{g} denote $\tilde{h} \circ \tilde{f}$, we see that the map $\tilde{g} : 2^{\hat{B}} \rightarrow 2^\omega$ is monotone. For $U \in \mathcal{U}$, $\tilde{g}(\hat{U}) = \tilde{h}(\tilde{f}(\hat{U})) = \tilde{h}(f(U)) = h \circ f(U)$. Thus, letting g denote $h \circ f$, we see that $g : \mathcal{U} \rightarrow \mathcal{W}$ is a monotone cofinal map with $g(U) = \tilde{g}(\hat{U})$ for each $U \in \mathcal{U}$. By Theorem 4.4, there is a \mathcal{U} -tree \tilde{T} and an increasing sequence $(k_m)_{m < \omega}$ such that the restriction of g to $\mathcal{C} = \{[T] : T \in \mathfrak{T} \upharpoonright \tilde{T}\}$ is generated by some monotone basic map $\hat{g} : \mathcal{C} \rightarrow 2^{<\omega}$, where

$$\mathcal{C} = \{T \upharpoonright k_m : T \in \mathfrak{T} \upharpoonright \tilde{T} \text{ and } m < \omega\}.$$

Notice that $\mathcal{C} = \bigcup_{m < \omega} 2^{\tilde{T} \upharpoonright k_m}$. Without loss of generality, we may assume that the levels k_m are the same for \hat{f} and \hat{g} , by taking the minimum of the two m th levels.

Let $g^* : 2^{\tilde{T}} \rightarrow 2^\omega$ be determined by \hat{g} as follows: for $A \in 2^{\tilde{T}}$,

$$(6.3) \quad g^*(A) = \bigcup_{m < \omega} \hat{g}(A \upharpoonright k_m).$$

Since \hat{g} is end-extension preserving, it follows that for each $A \in 2^{\tilde{T}}$ and $m < \omega$, $g^*(A) \upharpoonright m = \hat{g}(A \upharpoonright k_m)$. We point out that the restriction of g^* to $\mathfrak{T} \upharpoonright \tilde{T}$ yields exactly \tilde{g} , since \hat{g} generates \tilde{g} on $\mathfrak{T} \upharpoonright \tilde{T}$.

CLAIM 1. For each $A \in 2^{\tilde{T}}$,

$$(6.4) \quad g^*(A) = \bigcap \{g(X) : X \in \mathcal{C} \text{ and } \hat{X} \supseteq A\} \supseteq \tilde{g}(A).$$

Proof. Let A be any member of $2^{\tilde{T}}$. Let m be given and note that $A \upharpoonright k_m \in \mathcal{C}$. For any $X \in \mathcal{C}$ such that $\hat{X} \upharpoonright k_m = A \upharpoonright k_m$, we have

$$(6.5) \quad \hat{g}(A \upharpoonright k_m) = \hat{g}(\hat{X} \upharpoonright k_m) = \tilde{g}(\hat{X}) \upharpoonright m = g(X) \upharpoonright m.$$

Since $\mathcal{C} = \{[T] : T \in \mathfrak{T} \upharpoonright \tilde{T}\}$, there is an $X \in \mathcal{C}$ such that $\hat{X} \supseteq A$ and $\hat{X} \upharpoonright k_m = A \upharpoonright k_m$. (This is the key property of \mathcal{C} needed for this proof.) Thus,

$$(6.6) \quad [[\hat{g}(A \upharpoonright k_m)]] = Y \upharpoonright m \supseteq \tilde{g}(A) \upharpoonright m,$$

where $Y = \bigcap \{g(X) : X \in \mathcal{C} \text{ and } \hat{X} \supseteq A\}$. Taking the union over all $m < \omega$ in (6.6) yields the claim. ■

Define

$$(6.7) \quad \mathcal{D} = \{\hat{f}(s) : s \in C\} \quad \text{and} \quad \mathcal{D} = f[\mathcal{C}].$$

Then \mathcal{D} is cofinal in \mathcal{V} , and every member of \mathcal{D} is a limit of members of D . We point out that $\mathfrak{T} \upharpoonright \tilde{T}$ is a subspace of $2^{\hat{B}}$, and the closure of $\mathfrak{T} \upharpoonright \tilde{T}$ in $2^{\hat{B}}$ is the compact space $2^{\tilde{T}}$. Define a function $\hat{h} : D \rightarrow 2^{<\omega}$ as follows: for $t \in D \cap 2^m$ and $i \in m$,

$$(6.8) \quad \hat{h}(t)(i) = \min\{\hat{g}(s)(i) : s \in C \cap 2^{\hat{B} \upharpoonright k_m} \text{ and } \hat{f}(s) = t\}.$$

That is, $\hat{h}(t)$ is the function from m into 2 such that for $i \in m$, $\hat{h}(t)(i) = 1$ if and only if $\hat{g}(s)(i) = 1$ for all $s \in C \cap 2^{\hat{B} \upharpoonright k_m}$ satisfying $\hat{f}(s) = t$. By definition, \hat{h} is level preserving.

We proceed to prove that \hat{h} represents h on \mathcal{D} . Fix $Y \in \mathcal{D}$. Then there is an $X \in \mathcal{C}$ such that $f(X) = Y$. For each $m < \omega$, $\hat{f}(\hat{X} \upharpoonright k_m) = Y \upharpoonright m$, so

$$(6.9) \quad \begin{aligned} [[\hat{h}(Y \upharpoonright m)]] &\subseteq [[\hat{g}(\hat{X} \upharpoonright k_m)]] = \tilde{g}(\hat{X}) \upharpoonright m = g(X) \upharpoonright m \\ &= h \circ f(X) \upharpoonright m = h(Y) \upharpoonright m. \end{aligned}$$

Thus, $\bigcup_{m < \omega} [[\hat{h}(Y \upharpoonright m)]] \subseteq h(Y)$.

Next we show that for each $l \in h(Y)$, there is some n such that $l \in [[\hat{h}(Y \upharpoonright n)]]$. Let $m = l + 1$ and $t = Y \upharpoonright m$. Let

$$(6.10) \quad S_l = \{s \in C \cap 2^{\hat{B} \upharpoonright k_m} : \forall A \in 2^{\tilde{T}} \text{ with } s \sqsubset A \text{ } (\hat{f}(A) \neq Y)\}.$$

CLAIM 2. *For each $s \in S_l$, there is an n_s such that each $s' \in 2^{\hat{B} \upharpoonright n_s}$ with $s' \sqsupset s$ satisfies $\hat{f}(s') \not\sqsubset Y$.*

Proof. If not, then for some $s \in S_l$, for each n there is an $s' \in C$ of length n with $s' \supseteq s$ and some $A \in 2^{\tilde{T}}$ with $A \sqsupset s'$ such that $\hat{f}(A) = Y$. By König's Lemma, there is a sequence $s \sqsubset s_0 \sqsubset s_1 \sqsubset \dots$ of strictly increasing lengths such that for each $i < \omega$, there is some $A_i \in 2^{\tilde{T}}$ such that $A_i \sqsupset s_i$ and $\hat{f}(A_i) = Y$. Letting $A' = \bigcup_{i < \omega} s_i$, we see that $A' \in 2^{\tilde{T}}$ and $\hat{f}(A') = Y$, by continuity of f and the fact that \hat{f} generates \tilde{f} on $2^{\tilde{T}}$. But this contradicts $s \in S_l$. ■

Since S_l is finite, we may take $n = \max\{n_s : s \in S_l\}$. Then for all s' of length k_n , if $\hat{f}(s') \sqsubset Y$ then s' does not end-extend any s in S_l .

CLAIM 3. *For all $s' \in C$ of length k_n such that $\hat{f}(s') \sqsubset Y$, $l \in [[\hat{g}(s')]]$.*

Proof. For each $s' \in C \cap 2^{\hat{B} \upharpoonright k_n}$ satisfying $\hat{f}(s') = Y \upharpoonright n$, we see that $s' \upharpoonright k_m$ is not in S_l . So there is some $A \in 2^{\tilde{T}}$ such that $A \sqsupset s' \upharpoonright k_m$ and $\hat{f}(A) = Y$. It

follows that

$$(6.11) \quad [[\hat{g}(s') \upharpoonright m]] = [[\hat{g}(s' \upharpoonright k_m)]] \supseteq \tilde{g}(A) \upharpoonright m = \tilde{h} \circ \tilde{f}(A) \upharpoonright m = h(Y) \upharpoonright m,$$

where the \supseteq follows from Claim 1, since $[[\hat{g}(s' \upharpoonright k_m)]] = g^*(A) \upharpoonright m \supseteq \tilde{g}(A) \upharpoonright m$. Thus, $l \in [[\hat{g}(s')]]$ for each $s' \in 2^{\hat{T} \upharpoonright k_n}$ satisfying $\hat{f}(s') = Y \upharpoonright n$. ■

Therefore, $l \in [[\hat{h}(Y \upharpoonright n)]]$. Thus, for each $l \in h(Y)$, there is an n_l such that $l \in [[\hat{h}(Y \upharpoonright n_l)]]$. It follows that, for any $j < \omega$, there is an n such that $\hat{h}(\chi_Y \upharpoonright n) \upharpoonright j = h(Y) \upharpoonright j$. This n may be obtained by taking the maximum of the n_s over all $s \in \bigcup \{S_l : l < j\}$. Hence, $\bigcup_{m < \omega} [[\hat{h}(Y \upharpoonright m)]] = h(Y)$.

Thus, $h \upharpoonright \mathcal{D}$ is finitely represented by \hat{h} on D . ■

Theorem 6.3 is now applied to extend Raghavan's Theorem 17 in [13] to all ultrafilters Tukey reducible to some Fubini iterate of p -points. Raghavan showed that for any basically generated ultrafilter \mathcal{U} , if $\mathcal{V} \leq_T \mathcal{U}$ then there is a filter $\mathcal{U}(P)$ which is Tukey equivalent to \mathcal{U} such that $\mathcal{V} \leq_{\text{RK}} \mathcal{U}(P)$. It is routine to check that the maps in Theorem 6.3 satisfy the conditions of the maps in [13, Theorem 17]. Thus, we obtain the following.

THEOREM 6.4. *If \mathcal{U} is Tukey reducible to a Fubini iterate of p -points, then for each $\mathcal{V} \leq_T \mathcal{U}$, there is a filter $\mathcal{U}(P) \equiv_T \mathcal{U}$ such that $\mathcal{V} \leq_{\text{RK}} \mathcal{U}(P)$.*

Here, assuming without loss of generality that the base set for \mathcal{U} is ω , we see that P is the collection of \sqsubset -minimal finite subsets s of ω for which $\hat{h}(s) \neq \emptyset$, where \hat{h} witnesses that a given monotone cofinal $h : \mathcal{U} \rightarrow \mathcal{V}$ is finitely generated. And $\mathcal{U}(P)$ is the collection of all sets of the form $\{s \in P : s \subseteq U\}$ for $U \in \mathcal{U}$.

REMARK 6.5. The same proof of Theorem 6.3 works for the basic cofinal maps for the generic ultrafilters \mathcal{G}_k forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$, $2 \leq k < \omega$, in [6]. Thus, Theorem 6.4 also holds when \mathcal{U} is an ultrafilter forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}$.

7. Open problems. We conclude this paper by highlighting some of the more important open problems in this area. Theorem 2.6 showed that every ultrafilter Tukey below a p -point has continuous Tukey reductions.

PROBLEM 7.1. Determine the class of all ultrafilters that have continuous Tukey reductions.

In particular, are there ultrafilters not Tukey reducible to a p -point which satisfy the conditions of Theorem 2.5?

By Theorem 56 of Dobrinen and Trujillo [7] ⁽²⁾, under very mild conditions, any ultrafilter selective for some topological Ramsey space has basic, and hence continuous (with respect to its metric topology), Tukey reductions. This is especially of interest when the ultrafilter associated with the

⁽²⁾ Attributions of the results of the various authors are made clear in [7].

topological Ramsey space is not a p-point. By arguments similar to those in the present paper it should be possible to prove the following.

PROBLEM 7.2. Prove the analogues of Theorems 2.5, 4.4, and 6.3 for stable ordered union ultrafilters and their iterated Fubini products, and more generally for ultrafilters selective for some topological Ramsey space, with respect to the correct topologies.

More generally, we would like to know the following.

PROBLEM 7.3. Determine the class of all ultrafilters which have finitely generated Tukey reductions. Is this the same as the class of all ultrafilters with Tukey type strictly below the maximum Tukey type?

In Section 5, we applied Theorem 5.1 to obtain more examples when Tukey reducibility implies Rudin–Keisler reducibility. Theorem 5.3 improves on one aspect of Raghavan’s Corollary 56 in [13] provided that there are q-points which are not selective and which are Tukey below some Fubini iterate of p-points. Do such ultrafilters ever exist?

PROBLEM 7.4. Is there a q-point which is not selective and which is Tukey reducible to some finite Fubini iterate of p-points? Or does $\mathcal{V} \leq_T \mathcal{U}$ with \mathcal{U} a Fubini iterate of p-points and \mathcal{V} a q-point imply that \mathcal{V} is actually selective?

PROBLEM 7.5. Can Theorem 5.3 be extended to all countable iterates of Fubini products of p-points? Are similar results true for all ultrafilters Tukey reducible to some Fubini iterate of p-points?

Question 25 in [8] asks whether every ultrafilter Tukey reducible to a p-point is basically generated. Question 26 in [8] asks whether the classes of basically generated and Fubini iterates of p-points are the same, or whether the former is strictly larger than the latter. Though these questions in general are still open, we ask even more general questions:

PROBLEM 7.6. Is the property of being basically generated inherited under Tukey reduction? That is, if \mathcal{V} is Tukey reducible to a basically generated ultrafilter, is \mathcal{V} necessarily basically generated?

Or the possibly weaker problem: If \mathcal{V} is Tukey reducible to some Fubini iterate of p-points, is \mathcal{V} necessarily basically generated?

There are certain collections of p-points, in particular those associated with the topological Ramsey spaces in [9], [10], and [7], for which every ultrafilter Tukey below some Fubini iterate of these p-points is again a Fubini iterate of these p-points and hence basically generated. However, the above questions are in general still open.

Work in this paper and work in [13] found conditions when Tukey reducibility implies Rudin–Keisler or even Rudin–Blass reducibility.

PROBLEM 7.7. When in general does $\mathcal{U} \geq_T \mathcal{V}$ imply $\mathcal{U} \geq_{\text{RK}} \mathcal{V}$ or $\mathcal{U} \geq_{\text{RB}} \mathcal{V}$?

Finally, how closely related are the properties of having finitely generated Tukey reductions and having Tukey type below the maximum?

PROBLEM 7.8. Does $(\mathcal{U}, \supseteq) <_T ([\omega]^{<\omega}, \subseteq)$ imply that \mathcal{U} has finitely generated Tukey reductions?

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References

- [1] A. Blass, *Ultrafilters related to Hindman's finite-unions theorem and its extensions*, in: *Contemp. Math.* 65, Amer. Math. Soc., 1987, 89–124.
- [2] A. Blass, N. Dobrinen, and D. Raghavan, *The next best thing to a P-point*, *J. Symbolic Logic* 80 (2015), 866–900.
- [3] M. M. Day, *Oriented systems*, *Duke Math. J.* 11 (1944), 201–229.
- [4] N. Dobrinen, *Continuous cofinal maps on ultrafilters*, arXiv:1110.4154v1 (2011).
- [5] N. Dobrinen, *Survey on the Tukey theory of ultrafilters*, *Zb. Rad.* 17 (25) (2015), 53–80.
- [6] N. Dobrinen, *High dimensional Ellentuck spaces and initial chains in the Tukey structure of non-p-points*, *J. Symbolic Logic* 81 (2016), 237–263.
- [7] N. Dobrinen, J. G. Mijares, and T. Trujillo, *Topological Ramsey spaces from Fraïssé classes, Ramsey-classification theorems, and initial structures in the Tukey types of p-points*, *Arch. Math. Logic* 56 (2017), 733–782.
- [8] N. Dobrinen and S. Todorcevic, *Tukey types of ultrafilters*, *Illinois J. Math.* 55 (2011), 907–951.
- [9] N. Dobrinen and S. Todorcevic, *A new class of Ramsey-classification theorems and their application in the Tukey theory of ultrafilters, Part 1*, *Trans. Amer. Math. Soc.* 366 (2014), 1659–1684.
- [10] N. Dobrinen and S. Todorcevic, *A new class of Ramsey-classification theorems and their applications in the Tukey theory of ultrafilters, Part 2*, *Trans. Amer. Math. Soc.* 367 (2015), 4627–4659.
- [11] J. R. Isbell, *The category of cofinal types. II*, *Trans. Amer. Math. Soc.* 116 (1965), 394–416.
- [12] D. Raghavan and S. Shelah, *On embedding certain partial orders into the P-points under Rudin–Keisler and Tukey reducibility*, *Trans. Amer. Math. Soc.* 369 (2017), 4433–4455.
- [13] D. Raghavan and S. Todorcevic, *Cofinal types of ultrafilters*, *Ann. Pure Appl. Logic* 163 (2012), 185–199.
- [14] J. Schmidt, *Konfinalität*, *Z. Math. Logik Grundlag. Math.* 1 (1955), 271–303.
- [15] S. Todorcevic, *Directed sets and cofinal types*, *Trans. Amer. Math. Soc.* 290 (1985), 711–723.

- [16] S. Todorcevic, *A classification of transitive relations on ω_1* , Proc. London Math. Soc. (3) 73 (1996), 501–533.
- [17] S. Todorcevic, *Introduction to Ramsey Spaces*, Princeton Univ. Press, 2010.
- [18] J. W. Tukey, *Convergence and Uniformity in Topology*, Princeton Univ. Press, 1940.

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