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More on tree properties

by

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Abstract. Tree properties were introduced by Shelah, and it is well-known that a theory has TP (the tree property) if and only if it has TP_1 or TP_2. In any simple theory (i.e., a theory not having TP), forking supplies a good independence notion as it satisfies symmetry, full transitivity, extension, local character, and type-amalgamation, over sets. Shelah also introduced SOP_n (n-strong order property). Recently it has been proved that in any $NSOP_1$ theory (i.e. a theory not having SOP_1) having nonforking existence, Kim-forking also satisfies all the above mentioned independence properties except base monotonicity (one direction of full transitivity). These results are the sources of motivation for this paper.

Mainly, we produce type-counting criteria for SOP_2 (which is equivalent to TP_1) and SOP_1. In addition, we study relationships between TP_2 and Kim-forking, and show that a theory is supersimple iff there is no countably infinite Kim-forking chain.

1. Introduction. In this paper we study various notions of tree properties, and we mainly produce type-counting criteria for SOP_1 and SOP_2. TP (the tree property) was introduced by S. Shelah [17], who showed that in any simple theory (a theory not having TP), forking satisfies local character and finite character and extension; later in [10], [14], also symmetry, full transitivity, and type-amalgamation of Lascar types was shown over arbitrary sets.

In [16], it was stated that a theory has TP if and only if it has TP_1 or TP_2, and a complete proof was supplied in [13]. On the other hand, in [18], Shelah introduced the notions of n-strong order properties (SOP_n) for $n \geq 3$, which further classify theories having TP_1. More precisely, a theory has SOP_n if there is a formula $\varphi(x, y)$ (with $|x| = |y|$) defining a directed graph that has an infinite chain but no cycle of length $\leq n$. Hence SOP_{n+1} implies SOP_n, but the implication is reversible for no $n \geq 3$. As we are not dealing with SOP_n for $n \geq 3$ in this note, we omit the details.

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For $n = 1, 2$, Shelah defines SOP_n separately as follows.

DEFINITION 1.1.

(1) We say a formula $\varphi(x, y)$ has SOP_2 if there is a set $\{a_\alpha \mid \alpha \in 2^{<\omega}\}$ of tuples such that
   (a) for each $\beta \in 2^\omega$, $\{\varphi(x, a_{\beta\lceil n}) \mid n \in \omega\}$ is consistent, and
   (b) for each incomparable pair $\gamma, \gamma' \in 2^{<\omega}$, $\{\varphi(x, a_\gamma), \varphi(x, a_{\gamma'})\}$ is inconsistent.

   A theory T has SOP_2 if some formula in T has SOP_2.
(2) We say a formula $\varphi(x, y)$ has SOP_1 if there is a set $\{a_\alpha \mid \alpha \in 2^{<\omega}\}$ of tuples such that
   (a) for each $\beta \in 2^\omega$, $\{\varphi(x, a_{\beta\lceil n}) \mid n \in \omega\}$ is consistent, and
   (b) for each $\beta \in 2^{<\omega}$, $\{\varphi(x, a_\gamma), \varphi(x, a_{\beta^\frown 1})\}$ is inconsistent whenever $\beta^\frown 0 \trianglelefteq \gamma$.

   A theory T has SOP_1 if some formula in T has SOP_1. We say a theory T is NSOP_1 if T does not have SOP_1.

Hence it follows that SOP_2 implies SOP_1. It is known that for a theory, SOP_3 implies SOP_2, and SOP_2 is equivalent to TP_1. It is still an open question whether conversely, SOP_1 implies SOP_3 or SOP_2. The random parametrized equivalence relations (Example 4.4), an infinite-dimensional vector space over an algebraically closed field with a bilinear form, and ω-free PAC fields are typical examples having non-simple but NSOP_1 theories. Recently in [7], [8], it was shown that in any NSOP_1 theory, *over models*, 'Kim-forking' (here defined in Section 4) satisfies all the aforementioned axioms that forking satisfies in simple theories, except base monotonicity (one direction of full transitivity). Then it was proved in [6], [4] that the same axioms hold *over arbitrary sets* in any NSOP_1 theory having nonforking existence. The results summarized so far justify our study of various tree properties in this paper.

Throughout this note, we use standard notation. We work in a large saturated model $\mathcal{M}$ of a complete theory T in a language $\mathcal{L}$, and $a, b, \dots$ $(A, B, \dots)$ denote finite (small, resp.) tuples (sets, resp.) from $\mathcal{M}$, unless otherwise stated. We write $a \equiv_A b$ to mean $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$. As is customary, for cardinals κ, λ, we write λ^κ, $\lambda^{<\kappa}$ to denote $\{f \mid f : \kappa \to \lambda\}$, $\{f \mid f : \alpha \to \lambda,\ \alpha \in \kappa\}$ respectively, or their cardinalities, and it will be clear from context which we mean. As usual, we can look at $\lambda^{<\kappa} = \{f \mid f : \alpha \to \lambda,$ $\alpha < \kappa\}$ as a tree, and we equip it with the partial order $\alpha \trianglelefteq \beta$ iff $\alpha = \beta\lceil \mathrm{dom}(\alpha)$. We say α, β are *incomparable* if they are so in the ordering $\trianglelefteq$. Also $\alpha^\frown\beta$ denotes the concatenation of β after α. When $\beta = \langle i_0, \dots, i_n\rangle$ where $i_0, \dots, i_n \in \lambda$, we may simply write $\alpha i_0 \cdots i_n$ to mean $\alpha^\frown\beta$, so for example $\alpha^\frown 1$ or $\alpha 1$ indeed means $\alpha^\frown\langle 1\rangle$. In this note if we write a set as

$\{p_i \mid i \in I\}$ then $p_i \neq p_j$ for $i \neq j \in I$. Given a sequence $\langle c_i \mid i < \kappa \rangle$ and $j < \kappa$, we write $c_{<j}$, $c_{>j}$ for $\langle c_i \mid i < j \rangle$, $\langle c_i \mid j < i < \kappa \rangle$, respectively.

We now state some definitions and facts including those already mentioned that will be freely used throughout the paper.

Definition 1.2.

(1) Let $k \geq 2$. We say an $\mathcal{L}$-formula $\varphi(x, y)$ has the *k-tree property* (k-TP) if there is a set $\{c_\beta \mid \beta \in \omega^{<\omega}\}$ of tuples (from $\mathcal{M}$) such that for each $\alpha \in \omega^\omega$, $\{\varphi(x, c_{\alpha\lceil n}) \mid n \in \omega\}$ is consistent, while for any $\beta \in \omega^{<\omega}$, $\{\varphi(x, c_{\beta^\frown i}) \mid i \in \omega\}$ is k-inconsistent (i.e. any k-subset is inconsistent). A formula has the *tree property* (TP) if it has k-TP for some $k \geq 2$. We say T has TP if a formula in T has this property. We say T is *simple* if T does not have TP.
(2) A formula $\psi(x, y)$ has the *tree property of the first kind* (TP_1) if there are tuples a_α ($\alpha \in \omega^{<\omega}$) such that $\{\psi(x, a_{\beta\lceil n}) \mid n \in \omega\}$ is consistent for each $\beta \in \omega^\omega$, while $\psi(x, a_\alpha) \wedge \psi(x, a_\gamma)$ is inconsistent whenever $\alpha, \gamma \in \omega^{<\omega}$ are incomparable. A theory has TP_1 if a formula in it has TP_1.
(3) We say a formula $\psi(x, y) \in \mathcal{L}$ has the *tree property of the second kind* (TP_2) if there are tuples a^i_j ($i, j < \omega$) such that for each i, $\{\psi(x, a^i_j) \mid j < \omega\}$ is 2-inconsistent, whereas for any $f \in \omega^\omega$, $\{\psi(x, a^i_{f(i)}) \mid i < \omega\}$ is consistent. We say T has TP_2 if a formula in T has TP_2.

Fact 1.3.

(1) *The following are equivalent:*
  (a) *A theory T has TP.*
  (b) *T has 2-TP.*
  (c) *T has either* TP_1 *or* TP_2.
(2) *A formula has* TP_1 *iff it has* SOP_2.
(3) *If a formula has* SOP_1 *then it has 2-TP.*

In Fact 1.3(1), the equivalence of (a) and (b) is shown in [16], and that of (a) and (c) is claimed in [16], but a correct proof is given in [13]. The assertions of Fact 1.3(2, 3) easily come from the definitions.

Fact 1.4 ([5]). *The following are equivalent:*
(1) *A formula $\varphi(x, y) \in \mathcal{L}$ has* SOP_1.
(2) *There is a sequence $\langle a_i b_i \mid i < \omega \rangle$ such that*
  (a) *$a_i \equiv_{(ab)_{<i}} b_i$ for all $i < \omega$,*
  (b) *$\{\varphi(x, a_i) \mid i < \omega\}$ is consistent, and*
  (c) *$\{\varphi(x, b_i) \mid i < \omega\}$ is 2-inconsistent.*

In Section 2, we supply type-counting criteria for SOP_2. These are generalizations of those in [12], and we use similar techniques to [1] where analogous criteria for TP are stated.

In Section 3, in parallel, we produce type-counting criteria for SOP_1.

In Section 4, we recall the definition of Kim-forking, and study TP_2 in relation to Kim-forking and local weights. In particular we show that T is supersimple iff there is no Kim-forking chain of length ω.

2. Type-counting criteria for SOP_2**.** When Shelah introduced the class of simple theories in [17], he stated and proved type-counting criteria for TP. Then in [1], the first author improved those and suggested more elaborate criteria for TP. Later in [12], type-counting criteria for TP_1 (equivalently for SOP_2) analogous to the type-counting results of [17] were suggested. In [15], another type-counting criterion for SOP_2 was suggested. Now we supply more refined criteria for SOP_2, which are analogous to those for TP in [1].

DEFINITION 2.1. Let $\varphi(x,y)$ be an $\mathcal{L}$-formula. Assume infinite cardinals κ, λ are given. We define $\mathrm{NT}^2_\varphi(\kappa,\lambda)$ as the supremum of the cardinalities $|\mathcal{F}|$ of sets $\mathcal{F}$ of positive φ-types $p(x)$ over some fixed set A of cardinality λ satisfying

(1) $|p(x)| = \kappa$ for every $p(x) \in \mathcal{F}$, and
(2) for every subfamily $\{p_i \mid i < \lambda^+\} \subseteq \mathcal{F}$, there are disjoint subsets $\tau_j \subset \lambda^+$ with $|\tau_j| = \lambda^+$, and families $\{p'_i \mid p'_i \subseteq p_i,\ i \in \tau_j\}$ $(j = 0, 1)$ such that $|p_i \smallsetminus p'_i| < \kappa$ for each $i \in \tau_0 \cup \tau_1$, and every formula in $\bigcup_{i\in\tau_1} p'_i$ is inconsistent with every formula in $\bigcup_{i\in\tau_0} p'_i$.

Notice that if $|\mathcal{F}| \le \lambda$ then condition (2) is vacuous.

We define $\mathrm{NT}^2(\kappa,\lambda)$ in a similar way, with the only difference that each partial type $p(x) \in \mathcal{F}$ (with finite x) may contain any formula over A, not only instances of a fixed $\varphi(x,y)$, while still $|p(x)| = \kappa$.

Now given a formula φ, we give type-counting criteria for SOP_2, in terms of NT^2_φ.

THEOREM 2.2. *Let κ, λ be infinite cardinals. The following are equivalent for a formula $\varphi(x,y) \in \mathcal{L}$:*

(1) $\varphi(x,y)$ *has* SOP_2.
(2) $\mathrm{NT}^2_\varphi(\omega,\omega) \ge \omega_1$.
(3) $\mathrm{NT}^2_\varphi(\omega,\omega) \ge 2^\omega$.
(4) $\mathrm{NT}^2_\varphi(\kappa,\lambda) \ge \lambda^+$ *for some* κ, λ.
(5) $\mathrm{NT}^2_\varphi(\kappa,\lambda) \ge \lambda^+$ *for any* κ, λ *such that* $\lambda^{<\kappa} = \lambda$ *and* $\lambda^\kappa > \lambda$.
(6) $\mathrm{NT}^2_\varphi(\kappa,\lambda) \ge \lambda^\kappa$ *for any* κ, λ *such that* $\lambda^{<\kappa} = \lambda$ *and* $\lambda^\kappa > \lambda$.

Proof. (1)⇒(6). Assume $\varphi(x,y)$ has SOP_2. Suppose that for some infinite κ, λ, we have $\lambda^{<\kappa} = \lambda$ and $\lambda^\kappa > \lambda$. Hence $\kappa \le \lambda$. We will show that $\mathrm{NT}^2_\varphi(\kappa,\lambda) \ge \lambda^\kappa$.

Since φ has TP_1 as in Fact 1.3(2), by compactness, there is a tree of formulas $\{\varphi(x,a_\sigma) \mid \sigma \in \lambda^{<\kappa}\}$ witnessing TP_1 with respect to $\lambda^{<\kappa}$ (i.e. for each $\beta \in \lambda^\kappa$, $q_\beta(x) := \{\varphi(x,a_{\beta\lceil i}) \mid i < \kappa\}$ is consistent, while for any incomparable $\alpha,\gamma \in \lambda^{<\kappa}$, $\{\varphi(x,a_\alpha),\varphi(x,a_\gamma)\}$ is inconsistent). Let A be the set of parameters in the tree. We let $\mathcal{F} := \{q_\beta(x) \mid \beta \in \lambda^\kappa\}$. Note that $|\mathcal{F}| = \lambda^\kappa > \lambda = \lambda^{<\kappa} = |A|$.

We want to show that $\mathcal{F}$ satisfies condition (2) in Definition 2.1. Thus assume a set $\mathcal{G} = \{q_\beta \mid \beta \in \tau\}$ is given, where $\tau \subseteq \lambda^\kappa$ with $|\tau| = \lambda^+$. Now for each $\sigma \in \lambda^{<\kappa}$, we let $\mathcal{G}_\sigma := \{p \in \mathcal{G} \mid \varphi(x,a_\sigma) \in p\}$.

Claim. *There are $\mu \in \lambda^{<\kappa}$ and $s_0 < s_1 \in \lambda$ such that $|\mathcal{G}_{\mu^\frown\langle s_0\rangle}| = |\mathcal{G}_{\mu^\frown\langle s_1\rangle}| = \lambda^+$.*

Proof of Claim. Suppose not. Then for each $\sigma \in \lambda^{<\kappa}$ there is at most one $s < \lambda$ such that $|\mathcal{G}_{\sigma^\frown\langle s\rangle}| = \lambda^+$. Thus the only possibility is that there is $\delta \in \lambda^\kappa$ such that for each $i \in \kappa$, $|\mathcal{G}_{\delta\lceil i}| = \lambda^+$, while for each $j \in \lambda$ with $j \neq \delta(i)$, we have $|\mathcal{G}_{(\delta\lceil i)^\frown\langle j\rangle}| \le \lambda$. Since

$$\mathcal{G} = \{q_\delta\} \cup \bigcup\{\mathcal{G}_{(\delta\lceil i)^\frown\langle j_i\rangle} \mid i < \kappa,\, j_i < \lambda,\, j_i \neq \delta(i)\},$$

it follows that $|\mathcal{G}| \le 1+\lambda\cdot\lambda\cdot\kappa = \lambda$, a contradiction. Hence the claim follows.

Now let τ_0,τ_1 be the disjoint subsets of τ indexing the sets $\mathcal{G}_{\mu^\frown\langle s_0\rangle}$ and $\mathcal{G}_{\mu^\frown\langle s_1\rangle}$, respectively, so $\mathcal{G}_{\mu^\frown\langle s_j\rangle} = \{p_i \in \mathcal{G} \mid i \in \tau_j\}$ $(j = 0,1)$. For each $i \in \tau_0 \cup \tau_1$, we put $p'_i := p_i \setminus q_\mu$ where $q_\mu = \{\varphi(x,a_\sigma) \mid \sigma \trianglelefteq \mu\}$. Hence $|p_i \setminus p'_i| < \kappa$. Moreover clearly each formula in $\bigcup_{i\in\tau_1} p'_i$ is inconsistent with every formula in $\bigcup_{i\in\tau_0} p'_i$. Therefore (2) of Definition 2.1 holds.

(6)⇒(5)⇒(2)⇒(4) and (6)⇒(3)⇒(2) are clear.

(4)⇒(1). Assume $\mathrm{NT}^2_\varphi(\kappa,\lambda) \ge \lambda^+$ for some infinite κ and λ. Hence there is a family $\mathcal{F} = \{q_i \mid i < \lambda^+\}$ over a set A with $|A| \le \lambda$ satisfying (2) of Definition 2.1. We will produce an SOP_2 tree for φ from $\mathcal{F}$.

Claim. *There exist a function $f : 2^{<\omega} \to A$, a family $\{\mathcal{G}_\sigma \mid \sigma \in 2^{<\omega}\}$ of types, and a family $\{\tau_\sigma \subseteq \lambda^+ \mid \sigma \in 2^{<\omega}\}$ such that for all $\sigma \in 2^{<\omega}$,*

(i) *$|\tau_\sigma| = \lambda^+$; $\tau_{\sigma 0}$ and $\tau_{\sigma 1}$ are disjoint subsets of τ_σ,*
(ii) *$\mathcal{G}_\sigma$ is of the form $\{p_i \mid p_i \subseteq q_i,\, i \in \tau_\sigma\}$ (so $|\mathcal{G}_\sigma| = \lambda^+$) with $|q_i \setminus p_i| < \kappa$, and for $j \in \{0,1\}$, $\mathcal{G}_{\sigma j}$ is of the form $\{p'_i \mid p'_i \subseteq p_i \in \mathcal{G}_\sigma,\, i \in \tau_{\sigma j}\}$ with $|p_i \setminus p'_i| < \kappa$,*
(iii) *for $a_\sigma := f(\sigma)$ we have $\varphi(x,a_\sigma) \in \bigcap \mathcal{G}_\sigma$, and*
(iv) *each formula in $\bigcup \mathcal{G}_{\sigma 0}$ is inconsistent with every formula in $\bigcup \mathcal{G}_{\sigma 1}$.*

Proof of Claim. We construct such a function and sets by induction on the length of σ. When $\sigma = \emptyset$, choose $\varphi(x,b_i)$ from each $q_i \in \mathcal{F}$. Then since $|A| < \lambda^+$ and λ^+ is regular (or just by counting), there must be a subset

$\tau_\emptyset \subseteq \lambda^+$ of size λ^+ such that the b_i for $i \in \tau_\emptyset$ are all equal (say, to $a_\emptyset$). Then set $f(\emptyset) = a_\emptyset$. Also, set $\mathcal{G}_\emptyset := \{q_i \mid i \in \tau_\emptyset\}$, so $\varphi(x, a_\emptyset) \in \bigcap \mathcal{G}_\emptyset$.

Assume now the induction hypothesis for σ. We will find sets and function values corresponding to $\sigma 0$ and $\sigma 1$. Write $\mathcal{G}_\sigma = \{p_i \mid i \in \tau_\sigma\}$. Since $\mathcal{F}$ satisfies (2) of Definition 2.1, there exist disjoint subsets $\tau'_{\sigma j} \subseteq \tau_\sigma$ of size λ^+ $(j = 0, 1)$ and a subset $p'_i \subseteq q_i$ with $|q_i \setminus p'_i| < \kappa$ for each $i \in \tau'_{\sigma 0} \cup \tau'_{\sigma 1}$, such that every formula in $\bigcup_{i \in \tau'_{\sigma 0}} p'_i$ is inconsistent with each formula in $\bigcup_{i \in \tau'_{\sigma 1}} p'_i$. We now let $p''_i := p_i \cap p'_i$ for $i \in \bigcup_{j=0,1} \tau'_{\sigma j}$, and let $\mathcal{G}'_{\sigma j} := \{p''_i \mid i \in \tau'_{\sigma j}\}$. Then clearly $p''_i \subseteq p_i$, $|q_i \setminus p''_i| < \kappa$, and $|p_i \setminus p''_i| < \kappa$.

Now since again $|A| \le \lambda$, for $j \in \{0, 1\}$ there must be a set $\tau_{\sigma j} \subseteq \tau'_{\sigma j}$ with $|\tau_{\sigma j}| = \lambda^+$ such that for some $d_j \in A$, we have $\varphi(x, d_j) \in \bigcap_{i \in \tau_{\sigma j}} p''_i$ (and we then put $a_{\sigma j} = f(\sigma j)$). Therefore if we let $\mathcal{G}_{\sigma j} := \{p''_i \mid i \in \tau_{\sigma j}\}$, then $\tau_{\sigma j}$, $f(\sigma j)$ and $\mathcal{G}_{\sigma j}$, for $j = 0, 1$, satisfy all the required conditions for the induction step, and the proof of the Claim is complete.

Now, using the properties described in the Claim, we see that the tree $\{\varphi(x, a_\sigma) \mid \sigma \in 2^{<\omega}\}$ witnesses SOP_2. Indeed, given any $\sigma, \beta, \gamma \in 2^{<\omega}$, the formula $\varphi(x, a_{\sigma^\frown 0^\frown \beta})$ is inconsistent with $\varphi(x, a_{\sigma^\frown 1^\frown \gamma})$. ■

We now give type-counting criteria for SOP_2, for a theory.

Theorem 2.3. *Let κ, λ be infinite cardinals. The following are equivalent:*

(1) *T has SOP_2.*
(2) *For every regular $\kappa > |T|$, there is $\lambda \ge 2^\kappa$ such that $\mathrm{NT}^2(\kappa, \lambda) > \lambda$.*
(3) *For some regular $\kappa > |T|$ and some $\lambda \ge 2^\kappa$, we have $\mathrm{NT}^2(\kappa, \lambda) > \lambda$.*
(4) *For every κ, λ with $\lambda^{<\kappa} = \lambda$ and $\lambda^\kappa > \lambda$, we have $\mathrm{NT}^2(\kappa, \lambda) \ge \lambda^\kappa$.*
(5) *For every κ, λ with $\lambda^{<\kappa} = \lambda$ and $\lambda^\kappa > \lambda$, we have $\mathrm{NT}^2(\kappa, \lambda) > \lambda$.*

Proof. (1)⇒(4). The proof is the same as for (1)⇒(6) in Theorem 2.2.

(2)⇒(3) and (4)⇒(5) are clear.

(3)⇒(1). Assume (3). Hence there is a family $\mathcal{F}$ of arbitrary types over A with $|A| = \lambda$, satisfying conditions (2) and (3) of Definition 2.1. There is no harm in assuming that $|\mathcal{F}| = \lambda^+$ and we write $\mathcal{F} = \{q_i \mid i < \lambda^+\}$. Since $|q_i| = \kappa$, we write $q_i = \{\varphi^i_\alpha(x, a^i_\alpha) \mid \alpha < \kappa\}$, where $a^i_\alpha \in A$. Now since $|T|^\kappa = 2^\kappa < \lambda^+$, there must be a subset τ of λ^+ with $|\tau| = \lambda^+$ such that the sequence $\langle \varphi^i_\alpha(x, y^i_\alpha) \mid \alpha < \kappa \rangle$ stays the same, say $\langle \varphi_\alpha(x, y_\alpha) \mid \alpha < \kappa \rangle$, for every $i \in \tau$. Moreover since κ $(> |T|)$ is regular, there must be a subset $\sigma \subseteq \kappa$ of size κ such that $\varphi_\alpha(x, y_\alpha)$ stays the same, say $\varphi(x, y)$, for all $\alpha \in \sigma$. Now we let $\mathcal{F}_1 := \{\{\varphi(x, a^i_\alpha) \mid \alpha \in \sigma\} \mid i \in \tau\}$. Then it easily follows that $\mathcal{F}_1$ also satisfies (1) and (2) of Definition 2.1. Moreover each type in $\mathcal{F}_1$ is a positive φ-type. Therefore (1) follows by Theorem 2.2, (4)⇒(1).

(5)⇒(2). Assume (5). Now given regular $\kappa > |T|$, let $\lambda := \beth_\kappa(\kappa)$. Then $\lambda^{<\kappa} = \lambda < \lambda^\kappa$. Hence by (5), we have $\mathrm{NT}^2(\kappa,\lambda) > \lambda$. ∎

3. Type-counting criteria for SOP_1**.** As mentioned at the beginning of Section 2, type-counting criteria for SOP_2 are given in [12] as well. But for the first time, here we state and prove type-counting criteria for a formula to have SOP_1.

Definition 3.1. We say a formula $\varphi(x,y)$ has $\omega^{<\omega}$-SOP_1 if there is a set $\{a_\alpha \mid \alpha \in \omega^{<\omega}\}$ of tuples such that

(1) for each $\beta \in \omega^\omega$, $\{\varphi(x, a_{\beta\lceil n}) \mid n \in \omega\}$ is consistent, and
(2) for each $\beta \in \omega^{<\omega}$ and each pair $m < n \in \omega$, $\{\varphi(x,a_\gamma), \varphi(x,a_{\beta n})\}$ is inconsistent whenever $\beta m \trianglelefteq \gamma$.

Fact 3.2. *A formula has* SOP_1 *iff it has* $\omega^{<\omega}$-SOP_1.

Proof. (⇐) is clear.

(⇒) Assume $\varphi(x,y)$ and $\{a_\alpha \mid \alpha \in 2^{<\omega}\}$ witness SOP_1. Now for each $n > 1$, define a 1-1 map $f_n : n^{<\omega} \to 2^{<\omega}$ such that $f_n(\emptyset) := \emptyset$, and for $\alpha \in n^{<\omega}$ and $m < n$,

$$f_n(\alpha m) := f_n(\alpha)\, \overbrace{0\cdots 0}^{n-m-1}\, 1.$$

It follows that $A_n := \{a_{f_n(\alpha)} \mid \alpha \in n^{<\omega}\}$ forms an $n^{<\omega}$-SOP_1 tree for φ, and then compactness yields an $\omega^{<\omega}$-SOP_1 tree for the formula. ∎

Definition 3.3. Let $\varphi(x,y) \in \mathcal{L}$. For any infinite cardinals κ, λ, we define $\mathrm{NT}^1_\varphi(\kappa,\lambda)$ as the supremum of the cardinalities $|\mathcal{F}|$ of sets $\mathcal{F}$ of positive φ-types over some fixed set A of cardinality λ such that

(1) $|q(x)| = \kappa$ for every $q(x) \in \mathcal{F}$, and
(2) given any subfamily $\mathcal{G} = \{q_i \mid i < \lambda^+\}$ of $\mathcal{F}$ and a family $\mathcal{G}' = \{p_i \mid p_i \subseteq q_i, i < \lambda^+\}$ where $|q_i \smallsetminus p_i| < \kappa$ for each $i < \lambda^+$, there are disjoint subsets τ_0, τ_1 of λ^+ with $|\tau_j| = \lambda^+$ $(j = 0,1)$, and $\mathcal{G}'_j = \{p'_i \mid p'_i \subseteq p_i, i \in \tau_j\}$ with $|p_i \smallsetminus p'_i| < \kappa$ for each $i \in \tau_0 \cup \tau_1$, such that for every $p'_i \in \mathcal{G}'_1$ there is a formula in p'_i which is inconsistent with each formula in $\bigcup \mathcal{G}'_0$.

Notice that if $|\mathcal{F}| \le \lambda$ then condition (2) is vacuous.

Theorem 3.4. *Assume $\varphi(x,y)$ is an $\mathcal{L}$-formula, and κ, λ are infinite cardinals. The following are equivalent:*

(1) *$\varphi(x,y)$ has* SOP_1.
(2) $\mathrm{NT}^1_\varphi(\omega,\omega) \ge \omega_1$.
(3) $\mathrm{NT}^1_\varphi(\omega,\omega) \ge 2^\omega$.

(4) $\mathrm{NT}^1_\varphi(\kappa,\lambda) \geq \lambda^+$ *for some* κ,λ.

(5) $\mathrm{NT}^1_\varphi(\kappa,\lambda) \geq \lambda^+$ *for any* λ *and any regular* κ *with* $\lambda^{<\kappa} = \lambda$ *and* $\lambda^\kappa > \lambda$.

(6) $\mathrm{NT}^1_\varphi(\kappa,\lambda) \geq \lambda^\kappa$ *for any* λ *and any regular* κ *with* $\lambda^{<\kappa} = \lambda$ *and* $\lambda^\kappa > \lambda$.

Proof. (1)⇒(6). Assume that $\varphi(x,y)$ has SOP_1. Suppose that for a regular κ and an infinite λ, we have $\lambda^{<\kappa} = \lambda$ and $\lambda^\kappa > \lambda$. We will show that $\mathrm{NT}^1_\varphi(\kappa,\lambda) \geq \lambda^\kappa$.

Since φ has $\omega^{<\omega}$-SOP_1 as in Fact 3.2, by compactness, there is a tree $\{\varphi(x,a_\sigma) \mid \sigma \in \lambda^{<\kappa}\}$ of formulas witnessing SOP_1 with respect to $\lambda^{<\kappa}$ (i.e. for each $\beta \in \lambda^\kappa$, $q_\beta(x) := \{\varphi(x,a_{\beta\lceil i}) \mid i < \kappa\}$ is consistent, while for any $\alpha \in \lambda^{<\kappa}$ and $u < v \in \lambda$, $\{\varphi(x,a_\gamma),\varphi(x,a_{\alpha^\frown\langle v\rangle})\}$ is inconsistent for any $\gamma \trianglerighteq \alpha^\frown\langle u\rangle$). Let A be the set of parameters in the tree. We let $\mathcal{F} := \{q_\beta(x) \mid \beta \in \lambda^\kappa\}$. Note that $|\mathcal{F}| = \lambda^\kappa > \lambda = \lambda^{<\kappa} = |A|$.

We want to show that $\mathcal{F}$ satisfies condition (2) in Definition 3.3. So assume a set $\mathcal{G} = \{p_\beta \subseteq q_\beta \mid \beta \in \tau\}$ is given where $|q_\beta \smallsetminus p_\beta| < \kappa$ and $\tau \subseteq \lambda^\kappa$ with $|\tau| = \lambda^+$. Since $|q_\beta \smallsetminus p_\beta| < \kappa$ and κ is regular, for each $\beta \in \tau$ there must exist an ordinal $i_\beta < \kappa$ such that $\{\varphi(x,a_{\beta\lceil i}) \mid i_\beta \leq i < \kappa\} \subseteq p_\beta$. Note that $\lambda^{<\kappa} = \lambda$ implies $\kappa < \lambda^+$. Thus there exists a subset $\tau'' \subseteq \tau$ of size λ^+ such that i_β stays the same, say i_0, for every $\beta \in \tau''$. Once more, since $\lambda^{<\kappa} = \lambda$, for some subset $\tau' \subseteq \tau''$ of size λ^+, $\beta\lceil i_0$ stays the same for every $\beta \in \tau'$: there is $\sigma_0 \in \lambda^{<\kappa}$ such that $\sigma_0 = \beta\lceil i_0$ (and hence $a_{\sigma_0} = a_{\beta\lceil i_0}$) for all $\beta \in \tau'$.

Now let $\mathcal{G}' := \{p_\beta \in \mathcal{G} \mid \beta \in \tau'\}$, and for σ $(\trianglerighteq\sigma_0) \in \lambda^{<\kappa}$, let $\mathcal{G}'_\sigma := \{p \in \mathcal{G}' \mid \varphi(x,a_\sigma) \in p\}$.

Claim. *There are* μ $(\trianglerighteq\sigma_0) \in \lambda^{<\kappa}$ *and* $s_0 < s_1 \in \lambda$ *such that* $|\mathcal{G}'_{\mu^\frown\langle s_0\rangle}| = |\mathcal{G}'_{\mu^\frown\langle s_1\rangle}| = \lambda^+$.

Proof of Claim. Suppose not. Thus for each $\sigma \trianglerighteq\sigma_0 \in \lambda^{<\kappa}$ there is at most one $s < \lambda$ such that $|\mathcal{G}'_{\sigma^\frown\langle s\rangle}| = \lambda^+$. This leads to a contradiction by a similar cardinality computation to the proof of the Claim in the proof of Theorem 2.2, (1)⇒(6). Hence the claim follows.

Now let τ_0,τ_1 be the disjoint subsets of τ' indexing the sets $\mathcal{G}'_{\mu^\frown\langle s_0\rangle}$ and $\mathcal{G}'_{\mu^\frown\langle s_1\rangle}$, respectively, so $\mathcal{G}'_{\mu^\frown\langle s_j\rangle} = \{p_i \in \mathcal{G}' \mid i \in \tau_j\}$ $(j = 0,1)$. For each $i \in \tau_0 \cup \tau_1$, we now put $p'_i := p_i \smallsetminus q_\mu$ where $q_\mu = \{\psi(x,a_\sigma) \mid \sigma \trianglelefteq \mu\}$.

Notice that the formula $\varphi(x,a_{\mu s_1}) \in \bigcap_{i\in\tau_1} p'_i$ is inconsistent with any formula in $\bigcup_{i\in\tau_0} p'_i$. Hence (2) of Definition 3.3 holds.

(6)⇒(5)⇒(2)⇒(4) and (6)⇒(3)⇒(2) are clear.

(4)⇒(1). Assume $\mathrm{NT}^1_\varphi(\kappa,\lambda) \geq \lambda^+$ for some infinite λ and κ. Hence there is a family $\mathcal{F} = \{q_i \mid i < \lambda^+\}$ over a set A with $|A| \leq \lambda$ satisfying conditions (1) and (2) of Definition 3.3. We will produce an SOP_1 tree for φ from $\mathcal{F}$.

Claim. *There exist a function $f : 2^{<\omega} \to A$, a family $\{\mathcal{G}_\sigma \mid \sigma \in 2^{<\omega}\}$ of families of types, and a family $\{\tau_\sigma \subseteq \lambda^+ \mid \sigma \in 2^{<\omega}\}$ such that, for all $\sigma \in 2^{<\omega}$,*

(i) *$|\tau_\sigma| = \lambda^+$, and $\tau_{\sigma 0}$ and $\tau_{\sigma 1}$ are disjoint subsets of τ_σ,*
(ii) *$\mathcal{G}_\sigma$ is of the form $\{p_i \mid p_i \subseteq q_i,\, i \in \tau_\sigma\}$ (so $|\mathcal{G}_\sigma| = \lambda^+$) with $|q_i \smallsetminus p_i| < \kappa$, and for $j \in \{0,1\}$, $\mathcal{G}_{\sigma j}$ is of the form $\{p'_i \mid p'_i \subseteq p_i \in \mathcal{G}_\sigma,\, i \in \tau_{\sigma j}\}$ with $|p_i \smallsetminus p'_i| < \kappa$,*
(iii) *for $a_\sigma := f(\sigma)$ we have $\varphi(x, a_\sigma) \in \bigcap \mathcal{G}_\sigma$, and $\varphi(x, a_{\sigma 1}) \in \bigcap \mathcal{G}_{\sigma 1}$ is inconsistent with every formula in $\bigcup \mathcal{G}_{\sigma 0}$.*

Proof of Claim. We construct such a function and sets by induction on the length of σ. When $\sigma = \emptyset$, choose $\varphi(x, b_i)$ from each $q_i \in \mathcal{F}$. Then since $|A| \le \lambda$, there is a subset $\tau_\emptyset \subseteq \lambda^+$ of size λ^+ such that the b_i for $i \in \tau_\emptyset$ are all equal (say, to $a_\emptyset$). Then set $f(\emptyset) = a_\emptyset$. Also, set $\mathcal{G}_\emptyset := \{q_i \mid i \in \tau_\emptyset\}$, so $\varphi(x, a_\emptyset) \in \bigcap \mathcal{G}_\emptyset$.

Assume now the induction hypothesis for σ. We will find sets and function values corresponding to $\sigma 0$ and $\sigma 1$. Write $\mathcal{G}_\sigma = \{p_i \mid i \in \tau_\sigma\}$. Since $\mathcal{F}$ satisfies (2) of Definition 3.3, there exist disjoint subsets $\tau'_{\sigma j} \subseteq \tau_\sigma$ of size λ^+ ($j = 0, 1$), and a subset $p'_i \subseteq p_i$ with $|p_i \smallsetminus p'_i| < \kappa$ for each $i \in \tau'_{\sigma 0} \cup \tau'_{\sigma 1}$, such that for every $p'_i \in \mathcal{H}_1$, there is a formula $\varphi(x, a'_i) \in p'_i$ inconsistent with each formula in $\bigcup \mathcal{H}_0$, where $\mathcal{H}_j = \{p'_i \mid i \in \tau'_{\sigma j}\}$.

Now since again $|A| \le \lambda$, there must be a set $\tau_{\sigma 1} \subseteq \tau'_{\sigma 1}$ with $|\tau_{\sigma 1}| = \lambda^+$ such that the a'_i for $i \in \tau_{\sigma 1}$ are all equal; then we put $f(\sigma 1) = a_{\sigma 1}$. Thus if we let $\mathcal{G}_{\sigma 1} := \{p'_i \mid i \in \tau_{\sigma 1}\}$, then $\varphi(x, a_{\sigma 1}) \in \bigcap \mathcal{G}_{\sigma 1}$ is inconsistent with each formula in $\bigcup \mathcal{H}_0$. Similarly if we choose $\varphi(x, b'_i) \in q'_i \in \mathcal{H}_0$, there must be a subset $\tau_{\sigma 0} \subseteq \tau'_{\sigma 0}$ of size λ^+ such that b'_i stays the same for each $i \in \tau_{\sigma 0}$, and we let $f(\sigma 0) = a_{\sigma 0}$. Then let $\mathcal{G}_{\sigma 0} := \{p'_i \mid i \in \tau_{\sigma 0}\}$, so $\varphi(x, a_{\sigma 0}) \in \bigcap \mathcal{G}_{\sigma 0}$. Therefore, $\tau_{\sigma j}$, $f(\sigma j)$ and $\mathcal{G}_{\sigma j}$, for $j = 0, 1$, satisfy all the required conditions for the induction step, and the proof of the Claim is complete.

Now, using the properties described in the Claim, we see that the tree $\{\varphi(x, a_\sigma) \mid \sigma \in 2^{<\omega}\}$ witnesses SOP_1. Indeed given any $\sigma \in 2^{<\omega}$, the formula $\varphi(x, a_{\sigma 1})$ is inconsistent with any $\varphi(x, a_\gamma)$ where $\gamma \trianglerighteq \sigma 0$. ∎

We finish this section by asking the following: Given a theory, are there criteria for SOP_1 analogous to Theorem 2.3 for SOP_2?

4. Kim-forking and TP_2**.** We begin this section by recalling basic definitions.

Definition 4.1.

(1) We say a formula $\varphi(x, a_0)$ *divides over* a set A if there is an A-indiscernible sequence $\langle a_i \mid i < \omega\rangle$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent. A formula *forks* over A if the formula implies a finite disjunction of

formulas, each of which divides over A. A type *divides/forks* over A if the type implies a formula which divides/forks over A. We write $a \mathop{\smile\!\!\!\!\!\!|}_A B$ ($a \mathop{\smile\!\!\!\!\!\!|}^d_A B$) if $\mathrm{tp}(a/AB)$ does not fork (divide, resp.) over A.

(2) An A-indiscernible sequence $\langle a_i \mid i < \omega \rangle$ is said to be a *Morley sequence over* A if $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_{<i}$ for each $i < \omega$.

(3) We say a formula $\varphi(x, a_0)$ *Kim-divides over* A if $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent for some Morley sequence $\langle a_i \mid i < \omega \rangle$ over A. A formula *Kim-forks* over A if the formula implies a finite disjunction of formulas, each of which Kim-divides over A.

(4) A type *Kim-divides/forks* over A if the type implies a formula which Kim-divides/forks over A. We write $c \mathop{\smile\!\!\!\!\!\!|}^K_A B$ if $\mathrm{tp}(c/AB)$ *does not Kim-fork* over A. Hence $\mathop{\smile\!\!\!\!\!\!|} \Rightarrow \mathop{\smile\!\!\!\!\!\!|}^K$ and $\mathop{\smile\!\!\!\!\!\!|}^d$.

Originally in [7], the notion of *Kim-dividing* was introduced *over a model*, using the notion of a Morley sequence in a global invariant extension of a type over the model. There it was shown that in any NSOP_1 T, over a model, that notion is equivalent to the one stated in Definition 4.1(3). Since in general even in a simple theory, there need not exist a global invariant extension of a type over a set, instead in [6] the above definition in (3) is coherently given for NSOP_1 T as *Kim-dividing over an arbitrary set*.

As is well-known, in any simple T, $\mathop{\smile\!\!\!\!\!\!|}$ satisfies symmetry, full transitivity (that is, for any d and $A \subseteq B \subseteq C$, $d \mathop{\smile\!\!\!\!\!\!|}_A B$ and $d \mathop{\smile\!\!\!\!\!\!|}_B C$ iff $d \mathop{\smile\!\!\!\!\!\!|}_A C$; the left-to-right direction is called *transitivity*, and the converse is *base monotonicity*), extension, local character, finite character, and 3-amalgamation of Lascar types. Moreover in such T, $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^d = \mathop{\smile\!\!\!\!\!\!|}^K$ [10], and nonforking existence (that is, $d \mathop{\smile\!\!\!\!\!\!|}_A A$ for any d and A) holds. As we will not deal with these facts, we refer to [2] or [11] for details. Further advances were made in [7], [8], [6], [4]. Namely in [6], it was shown that in any NSOP_1 T having nonforking existence (as already mentioned, any simple T and all the known NSOP_1 T have this), the notions of Kim-forking and Kim-dividing coincide *over any set*, and $\mathop{\smile\!\!\!\!\!\!|}^K$ supplies a good independence notion *over any set* since in such T, $\mathop{\smile\!\!\!\!\!\!|}^K$ satisfies all the aforementioned properties (transitivity over sets is proved in [4]) that hold for $\mathop{\smile\!\!\!\!\!\!|}$ in simple theories, except base monotonicity (so there can exist d and $A \subseteq B \subseteq C$ such that $d \mathop{\smile\!\!\!\!\!\!|}^K_A C$ but $d \not\mathop{\smile\!\!\!\!\!\!|}^K_B C$). Indeed before [6], [4], the same properties were shown in [7] (and transitivity in [8]) *over models* in *any* NSOP_1 T.

In this section we study TP_2 in relation to Kim-forking. In particular we show that if T has TP_2 then there is a non-continuous Kim-forking chain of arbitrarily large length (Proposition 4.6), from which we deduce that T is supersimple iff there is no Kim-forking chain of length ω (Theorem 4.7). We also show that in any T with TP_2, there is a type having arbitrarily large local weight with respect to $\mathop{\smile\!\!\!\!\!\!|}^K$ (Proposition 4.8).

This section might be considered as an expository note, since all the results here are more or less straightforward consequences of known facts 4.3 and 4.5. In particular, the referee of this paper pointed out that Proposition 4.6 follows from a result in [3].

Recall that a sequence $\langle A_i \mid i < \kappa \rangle$ of sets is said to be *continuous* if for each limit $\delta < \kappa$, $A_\delta = \bigcup_{i<\delta} A_i$.

FACT 4.2 ([9]). *The following are equivalent:*

(1) T *is* NSOP_1.
(2) *There do not exist finite* d *and a continuous increasing sequence* $\langle M_i \mid i < |T|^+ \rangle$ *of* $|T|$*-sized models such that for each* $i < |T|^+$, $d \not\mathop{\smile}\limits^{K}_{M_i} M_{i+1}$.

Indeed, the following is implicitly shown in [9] using Fact 1.4. We supply a proof for completeness.

FACT 4.3. *If* T *has* SOP_1 *then for each infinite cardinal* κ*, there exist a finite tuple* d *and a continuous increasing sequence* $\langle A_\alpha \mid \alpha < \kappa \rangle$ *of sets such that for each* $\alpha < \kappa$, $|A_\alpha| \le |\alpha| \cdot \omega$ *and* $d \not\mathop{\smile}\limits^{K}_{A_\alpha} A_{\alpha+1}$.

Proof. Assume T has SOP_1. Given an infinite κ, by compactness, there are a formula $\varphi(x, y)$ and an indiscernible sequence $\langle a_i b_i \mid i \in \mathbb{Z}\cdot\kappa \rangle$ satisfying the conditions of Fact 1.4:

(∗) $a_i \equiv_{(ab)_{<i}} b_i$ for all $i \in \mathbb{Z} \cdot \kappa$, $\{\varphi(x, a_i) \mid i \in \mathbb{Z} \cdot \kappa\}$ is realized by say d, and $\{\varphi(x, b_i) \mid i \in \mathbb{Z} \cdot \kappa\}$ is 2-inconsistent.

Now for $n < \omega$, let $A_n = \{a_i b_i \mid i \in \mathbb{Z} \cdot (n + 1)\}$, and for $\omega \le \alpha < \kappa$, let $A_\alpha = \{a_i b_i \mid i \in \mathbb{Z} \cdot \alpha\}$. Then clearly $\langle A_\alpha \mid \alpha \in \kappa \rangle$ is a continuous increasing sequence with $|A_\alpha| \le \omega \cdot |\alpha|$. Moreover for each $b_i \in A_{\alpha+1} \setminus A_\alpha$, the countable sequence I_{b_i} ($\subset A_{\alpha+1} \setminus A_\alpha$) of successive b_j's starting from b_i is a finitely satisfiable indiscernible (so Morley) sequence in $\mathrm{tp}(b_i/A_\alpha)$. Thus by (∗), $\varphi(x, b_i)$ Kim-divides over A_α. Then since $a_i \equiv_{A_\alpha} b_i$, again by (∗) we have $d \not\mathop{\smile}\limits^{K}_{A_\alpha} a_i$. Note that $a_i \in A_{\alpha+1} \setminus A_\alpha$. Hence $d \not\mathop{\smile}\limits^{K}_{A_\alpha} A_{\alpha+1}$ as desired. ∎

In contrast to Fact 4.2(2), as seen in the following example, in NSOP_1 T, there can exist a *non-continuous* increasing Kim-forking sequence of length $|T|^+$ of $\le|T|$-sized sets, and *continuous* increasing Kim-forking sequences of arbitrary length.

EXAMPLE 4.4 ([4]). Let T be the theory of random parametrized equivalence relations, i.e., the Fraïssé limit of the class of finite models with two sorts (P, E) and a ternary relation $\sim$ on $P \times P \times E$ such that, for each $e \in E$, $x \sim_e y$ forms an equivalence relation on P.

So in a model of T, there are two sorts P and E as described above. Let $d \in P$. Given a cardinal κ, choose distinct $e_i \in E$, and $d_i \in P$ ($i < \kappa$) such that $d \sim_{e_i} d_i$, but $d_j \nsim_{e_k} d_i$ for all $j < i$ and $k \le i$. Let $D_i = ((ed)_{<i})e_i$. Note that the sequence $\langle D_i \mid i < \kappa \rangle$ is increasing but *not continuous* (for example, $D_{<\omega} \subsetneq D_\omega$). Notice further that $d \mathop{\not\perp}^K_{D_i} d_i$, so $d \mathop{\not\perp}^K_{D_i} D_{i+1}$ for each $i < \kappa$.

Moreover, there is a continuous increasing Kim-forking sequence of length κ of κ-sets. We work with the same chosen elements above. Let $C := \{e_i \mid i < \kappa\} \subset E$, and let $C_i := Cd_{<i}$. Clearly $\langle C_i \mid i < \kappa\rangle$ is a continuous increasing sequence of κ-sets. Now for each $i < \kappa$, it follows that $d \mathop{\not\perp}^K_{C_i} d_i$, and hence $d \mathop{\not\perp}^K_{C_i} C_{i+1}$.

Now we can ask whether such phenomena happen in any non-simple NSOP_1 T. We show that indeed in any theory with TP_2, such sequences can be found. The following fact is well-known and a proof can be found for example in [13]. Recall that an array $\langle a_{ij} \mid i < \kappa, j < \lambda\rangle$ is said to be *indiscernible* ([1]) over A if for $L_i := \langle a_{ij} \mid j < \lambda\rangle$, $\langle L_i \mid i < \kappa\rangle$ is A-indiscernible, and A-mutually indiscernible (i.e., each L_i is indiscernible over $\bigcup\{L_j \mid j\ (\neq i) < \kappa\}A$).

Fact 4.5. *The following are equivalent:*

(1) $\varphi(x, y)$ *has* TP_2.
(2) *There is an indiscernible array* $\langle a_{ij} \mid i, j < \omega\rangle$ *such that*
  (a) *for each* $i < \omega$, $\{\varphi(x, a_{ij}) \mid j < \omega\}$ *is 2-inconsistent, and*
  (b) *for any* $f : \omega \to \omega$, $\{\varphi(x, a_{if(i)}) \mid i < \omega\}$ *is consistent.*

Proposition 4.6. *Assume T has* TP_2. *Let κ be an infinite cardinal.*

(1) *There are a finite tuple d and an increasing* non-continuous *sequence of sets A_i $(i < \kappa)$ of size $|i|\cdot\omega$ $(< \kappa)$ such that $d \mathop{\not\perp}^K_{A_i} A_{i+1}$ for each $i < \kappa$. In particular there is an increasing countable sequence of countable sets B_i such that $d \mathop{\not\perp}^K_{B_i} B_{i+1}$ for each $i < \omega$.*
(2) *There are a finite tuple d and an increasing* continuous *sequence of sets E_i $(i < \kappa)$ of size κ such that $d \mathop{\not\perp}^K_{E_i} E_{i+1}$ for each $i < \kappa$.*

Proof. (1) Due to Fact 4.5 and compactness, there are a formula $\varphi(x, y)$ and an array $\langle a_{ij} \mid i < \kappa,\ j \in \omega + \omega^*\rangle$ where $\omega^* := \{i^* \mid i \in \omega\}$ with the reversed order of ω (so for $i^*, j^* \in \omega^*$, we have $n < i^*$ for all $n \in \omega$, and $j^* < i^*$ if $i < j$) such that

(a) for each $i < \kappa$, $\{\varphi(x, a_{ij}) \mid j \in \omega + \omega^*\}$ is 2-inconsistent,
(b) for any $f : \kappa \to \omega + \omega^*$, $\{\varphi(x, a_{if(i)}) \mid i < \kappa\}$ is consistent, and
(c) the array is mutually indiscernible, i.e., for any $i < \kappa$, $L_i := \langle a_{ij} \mid j \in \omega + \omega^*\rangle$ is indiscernible over $\bigcup\{L_j \mid j\ (\neq i) < \kappa\}$.

For each $i \in \kappa$, we let $I_i := \langle a_{ij} \mid j < \omega\rangle$, and let $J_i := \langle a^*_{ij} \mid j < \omega\rangle$ where $a^*_{ij} = a_{ij^*}$ with $j^* \in \omega^*$, so as a set $L_i = I_i \cup J_i$.

([1]) In some literature this notion is called *strongly indiscernible.*

Now due to (b), there is $d \models \{\varphi(x, a^*_{i0}) \mid i < \kappa\}$. Put $A_i = \{I_k \mid k \leq i\} \cup \{a^*_{k0} \mid k < i\}$. Then $|A_i| = |i| \cdot \omega$. Now by (c), J_i is finitely satisfiable, so Morley over A_i. Hence, by (a) we have

$$d \mathop{\not\perp\!\!\!\perp}^K_{A_i} a^*_{i0} \quad \text{and} \quad d \mathop{\not\perp\!\!\!\perp}^K_{A_i} A_{i+1}$$

for each $i < \kappa$. Notice that the sequence $\langle A_i \mid i < \kappa \rangle$ is not continuous, for example $A_{<\omega} = A_\omega \setminus I_\omega \subsetneq A_\omega$.

For the second statement of (1), simply put $B_i = A_i$ for $i < \omega$.

(2) We keep using the same d as in (1). Let $E := I_{<\kappa}$, and for $i < \kappa$ let $E_i := E \cup \{a^*_{k0} \mid k < i\}$. Now due to (c) again, for $i \in \kappa$, J_i is Morley over E_i. Therefore we have

$$d \mathop{\not\perp\!\!\!\perp}^K_{E_i} a^*_{i0} \quad \text{and hence} \quad d \mathop{\not\perp\!\!\!\perp}^K_{E_i} E_{i+1},$$

as desired. Note that clearly $\langle E_i \mid i < \kappa \rangle$ is a continuous increasing sequence with each $|E_i| = \kappa$. ∎

As mentioned before Fact 4.2, the referee pointed out that Proposition 4.6 directly follows from [3, proof of Lemma 4.7] as well. Thus the above proof might be considered as presenting the proposition as a straightforward consequence of Fact 4.5.

Now we recall that T is *supersimple* if for any finite a, and any set A, there is a finite subset A_0 of A such that $a \perp\!\!\!\perp_{A_0} A$. As is well-known, T is supersimple iff there does not exist a countably infinite forking chain (see for example [11]). The following theorem says that the same holds with a countably infinite Kim-forking chain.

THEOREM 4.7. *The following are equivalent:*

(1) *T is supersimple.*
(2) *There do not exist finite d and an increasing sequence of sets A_i ($i < \omega$) such that $d \not\perp\!\!\!\perp^K_{A_i} A_{i+1}$ for each $i < \omega$.*
(3) *There do not exist finite d and an increasing sequence of countable sets A_i ($i < \omega$) such that $d \not\perp\!\!\!\perp^K_{A_i} A_{i+1}$ for each $i < \omega$.*

Proof. (1)⇒(2) is well-known as mentioned before, and (2)⇒(3) is obvious.

(3)⇒(1). Suppose T is not supersimple. If T is simple, then since $\perp\!\!\!\perp = \perp\!\!\!\perp^K$, again it is well-known that there exist a tuple and a sequence as described in (3). If T is NSOP_1 but not simple, then T has TP_2 and Proposition 4.6 says there are such a tuple and sequence. If T has SOP_1 then the existence of such a tuple and sequence is guaranteed by Fact 4.3. ∎

As pointed out in [8], the following are equivalent:

(a) T has SOP_1.
(b) There exist tuples a_i $(i < \omega)$, a model M, and an $\mathcal{L}$-formula $\varphi(x, y)$ such that for each $i < \omega$, $a_i \equiv_M a_0$, $a_i \mathop{\smile\!\!\!\!\!\!|}^K_M a_{<i}$, $\varphi(x, a_i)$ Kim-divides over M, and $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent.

In particular if T is NSOP_1 and non-simple (so must have TP_2) then (b) does not hold. However, a condition only slightly weaker than (b) always holds in any T having TP_2.

Proposition 4.8. *Assume $\varphi(x, y)$ has TP_2. Then for each infinite κ, there are a set A with $|A| \leq \kappa$ and finite tuples d, c_i $(i < \kappa)$ such that*

(1) *$d \models \varphi(x, c_i)$,*
(2) *$\varphi(x, c_i)$ Kim-divides over A (so $d \not\mathop{\smile\!\!\!\!\!\!|}^K_A c_i$) witnessed by a Morley sequence $(c_i \in)$ J_i over A with $J_i \equiv J_0$ (so $c_i \equiv c_0$), and*
(3) *$c_i \mathop{\smile\!\!\!\!\!\!|}_A \{c_k \mid k < \kappa,\ k \neq i\}$ (so $c_i \mathop{\smile\!\!\!\!\!\!|}^K_A \{c_k \mid k < \kappa,\ k \neq i\}$).*

Proof. As in the proof of Proposition 4.6, there is an indiscernible array $\langle a_{ij} \mid i < \kappa,\ j \in \omega + \omega^* \rangle$ such that

(a) for each $i < \kappa$, $\{\varphi(x, a_{ij}) \mid j \in \omega + \omega^*\}$ is 2-inconsistent,
(b) for any $f : \kappa \to \omega + \omega^*$, $\{\varphi(x, a_{if(i)}) \mid i < \kappa\}$ is consistent, and
(c) for any $i < \kappa$, $L_i = \langle a_{ij} \mid j \in \omega + \omega^* \rangle$ is indiscernible over $\bigcup\{L_j \mid j < \kappa,\ j \neq i\}$.

Again for each $i \in \kappa$, let $I_i = \langle a_{ij} \mid j < \omega\}$ and $J_i = \langle a^*_{ij} \mid j < \omega \rangle$ where $a^*_{ij} = a_{ij^*}$ with $j^* \in \omega^*$. We further let $c_i := a^*_{i0}$. Now by (b), there is $d \models \{\varphi(x, c_i) \mid i < \kappa\}$.

We now put $A := I_{<\kappa}$, so $|A| = \kappa$. Now due to (c), each J_i is a Morley sequence over A, and $\mathrm{tp}(c_i/A\{c_k \mid k < \kappa,\ k \neq i\})$ is finitely satisfiable in A. Hence (3) follows, and (2) follows as well due to (a). ∎

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References

[1] E. Casanovas, *The number of types in simple theories*, Ann. Pure Appl. Logic 98 (1999), 69–86.
[2] E. Casanovas, *Simple Theories and Hyperimaginaries*, Cambridge Univ. Press, Cambridge, 2011.

[3] A. Chernikov, *Theories without the tree property of the second kind*, Ann. Pure Appl. Logic 165 (2014), 695–723.
[4] A. Chernikov, B. Kim, and N. Ramsey, *Transitivity, lowness, and ranks in* NSOP_1 *theories*, preprint.
[5] A. Chernikov and N. Ramsey, *On model-theoretic tree properties*, J. Math. Logic 16 (2016), no. 2, art.1650009, 41 pp.
[6] J. Dobrowolski, B. Kim, and N. Ramsey, *Independence over arbitrary sets in* NSOP_1 *theories*, arXiv:1909.08368 (2019).
[7] I. Kaplan and N. Ramsey, *On Kim-independence*, J. Eur. Math. Soc., to appear.
[8] I. Kaplan and N. Ramsey, *Transitivity of Kim-independence*, arXiv:1901.07026 (2019).
[9] I. Kaplan, N. Ramsey, and S. Shelah, *Local character of Kim-independence*, Proc. Amer. Math. Soc. 147 (2019), 1719–1732.
[10] B. Kim, *Forking in simple unstable theories*, J. London Math. Soc. 57 (1998), 257–267.
[11] B. Kim, *Simplicity Theory*, Oxford Univ. Press, Oxford, 2014.
[12] B. Kim and H.-J. Kim, *Notions around tree property 1*, Ann. Pure Appl. Logic 162 (2011), 698–709.
[13] B. Kim, H.-J. Kim, and L. Scow, *Tree properties, revisited*, Arch. Math. Logic 53 (2014), 211–232.
[14] B. Kim and A. Pillay, *Simple theories*, Ann. Pure Appl. Logic 88 (1997), 149–164.
[15] M. Malliaris and S. Shelah, *Model-theoretic applications of cofinality spectrum problems*, Israel J. Math. 220 (2017), 947–1014.
[16] S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*, North-Holland, Amsterdam, 1990.
[17] S. Shelah, *Simple unstable theories*, Ann. Math. Logic 19 (1980), 177–203.
[18] S. Shelah, *Toward classifying unstable theories*, Ann. Pure Appl. Logic 80 (1996), 229–255.

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