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MATHEMATICAE
Online First version

The descriptive complexity of approximation properties in an admissible topology

by

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Abstract. We show that the set of all separable Banach spaces that have the π-property and the set of all separable Banach spaces that have the BAP are Borel sets of class 6 whenever the set of non-empty closed subsets of $C(\Delta)$ (where Δ is the Cantor space) is equipped with an admissible topology.

1. Introduction. A topological space is *Polish* if it is homeomorphic to a separable metric complete space. The separable Banach space $C(\Delta)$ of real-valued continuous functions on the Cantor space Δ is isometrically universal for separable Banach spaces. We denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$. To provide the set $\mathcal{SE}$ with a topology, G. Godefroy and J. Saint-Raymond [10] equip the set $\mathcal{F}(C(\Delta))$ of non-empty closed subsets of $C(\Delta)$ with an admissible Polish topology. Although this topology is not canonical, the Borel σ-field is the same for all admissible topologies (it is called the *Effros–Borel structure*) and the Borel class of a subset of $\mathcal{F}(C(\Delta))$ is essentially independent of the choice of the admissible topology. The subset $\mathcal{SE}$ of $\mathcal{F}(C(\Delta))$ is shown to be a Polish subset of $\mathcal{F}(C(\Delta))$ for any admissible topology [10]. Admissible topologies on $\mathcal{SE}$ lead to distinguishing between different levels of complexity of Borel sets. When $\mathcal{F}(C(\Delta))$ has Effros–Borel structure, there are many results about descriptive complexity of some classes of Banach spaces, for which we refer the reader to [7], [8], [1], [13], [12], [6].

Concerning Borel sets, we follow the notation [11, pp. 68–69]. Hence for instance, F_σ-sets are Σ^0_2-sets, G_δ-sets are Π^0_2-sets, the notation Σ^0_3 means $G_{\delta\sigma}$, Π^0_3 means $F_{\sigma\delta}$ and so on. Suppose X is a Polish space and $E^+(U) =$

2010 *Mathematics Subject Classification*: 46B20, 54H05.
Key words and phrases: π_λ-property, Σ^0_6-set, Borel set, admissible topology, bounded approximation property.
Received 2 July 2019.
Published online 20 December 2019.

DOI: 10.4064/fm758-8-2019

$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$, $E^-(U) = \{F \in \mathcal{F}(X) : F \subset U\}$, where U is an arbitrary open subset of X. A Polish topology τ on the set $\mathcal{F}(X)$ is *admissible* if it satisfies the following conditions:

(i) For every open subset U of X, the set $E^+(U)$ is τ-open.
(ii) There exists a subbasis of open sets of $\mathcal{F}(X)$ such that every set in that subbasis is a countable union of sets of the form $E^+(U) \setminus E^+(V)$, where U and V are open in X.
(iii) The set $\{(x, F) \in X \times \mathcal{F}(X) : x \in F\}$ is closed in $X \times \mathcal{F}(X)$.

Suppose an admissible topology is chosen on the set $\mathcal{F}(C(\Delta))$, and the subset $\mathcal{SE}$ is equipped with the (Polish) restriction of this topology. It is shown in [10] that the isomorphism class of the Hilbert space is a Borel set of class 2, the isomorphism classes of certain near-Hilbert spaces are Borel sets of class 4, and the isomorphism classes of the spaces ℓ_p $(1 < p < \infty, p \neq 2)$ are Borel sets of class $\omega + 1$, where ω is the first infinite ordinal. Moreover, we need the following result from [10]:

THEOREM 1.1. *There exists a sequence $(f_i)_{i=1}^{\infty}$ of continuous functions on $\mathcal{SE}$ with values in $C(\Delta)$ such that for all $F \in \mathcal{SE}$, the set F is the closure in $C(\Delta)$ of the set $\{f_i(F) : i \in \mathbb{N}\}$.*

We recall that a Banach space X has the *π_λ-property* if there is a net (S_α) of finite rank projections on X converging strongly to the identity on X with $\limsup_\alpha \|S_\alpha\| \leq \lambda$ (see [2]). We say that X has the *π-property* if it has the π_λ-property for some $\lambda \geq 1$. Moreover, X has the *λ-Bounded Approximation Property* (λ-BAP), $\lambda \geq 1$, if for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator $T : X \to X$ with $\|T\| \leq \lambda$ and $\|T(x) - x\| < \epsilon$ for every $x \in K$.

Let us recall results from [4] and [5]: if $\mathcal{SE}$ is equipped with the standard Effros–Borel structure, then the set of all separable Banach spaces that have the bounded approximation property (BAP) is a Borel subset of $\mathcal{SE}$, and the set of all separable Banach spaces that have the metric approximation property (MAP) is also Borel. Also, the set of all separable Banach spaces that have the π-property is Borel.

This note provides quantitative refinements of these results. Indeed, we show that the set of all separable Banach spaces that have the π-property and the set of all separable Banach spaces that have the BAP are Borel sets of class 6 whenever $\mathcal{F}(C(\Delta))$ is equipped with an admissible topology.

2. Main results. We first need the following lemma of [5] and we recall its proof.

LEMMA 2.1. *Suppose $(x_n)_{n=1}^{\infty}$ is a dense sequence in a Banach space X and $\lambda > 1$. Then X has the π-property if and only if*

- $\forall c \in (0,1/4)\cap\mathbb{Q}\ \forall K\ \forall\epsilon>0\ \forall\lambda'>\lambda\ \exists R\ \forall N\geq R\ \exists\sigma_1,\dots,\sigma_N\in\mathbb{Q}^R$ $\forall\alpha_1,\dots,\alpha_N\in\mathbb{Q}$*:*

$$\Big\|\sum_{i=1}^{N}\alpha_i\Big[\sum_{i=1}^{R}\sigma_i(j)x_j\Big]\Big\|\leq\lambda'\Big\|\sum_{i=1}^{N}\alpha_i x_i\Big\|, \tag{2.1}$$

$$\forall i\leq K,\qquad \Big\|x_i-\sum_{j=1}^{R}\sigma_i(j)x_j\Big\|\leq\epsilon, \tag{2.2}$$

$$\Big\|\sum_{i=1}^{N}\alpha_i\Big[\sum_{j=1}^{R}\sigma_i(j)x_j\Big]-\sum_{i=1}^{N}\alpha_i\Big[\sum_{t=1}^{R}\Big[\sum_{j=1}^{R}\sigma_i(j)\sigma_j(t)\Big]x_t\Big]\Big\|\leq c\Big\|\sum_{i=1}^{N}\alpha_i x_i\Big\|, \tag{2.3}$$

where K, R, N vary over $\mathbb{N}$ and ϵ, λ' vary over $\mathbb{Q}$.

Proof. Suppose X has the π_λ -property. Then there is a sequence (P_n) of finite rank projections such that $\|P_n\|<\lambda$ for all n and P_n converges strongly to the identity.

By perturbing P_n, we may suppose that P_n maps into the finite-dimensional subspace $[x_1,\dots,x_{R_n}]$ for some R_n in $\mathbb{N}$ but we still have (2.2) and $\|P_n\|<\lambda$. Pick $\lambda'\in\mathbb{Q}$ with $\|P_n\|<\lambda'<\lambda$. Then, for every N, we may perturb P_n slightly so that $\|P_n\|<\lambda'$ and $P_n(x_i)$ belongs to the $\mathbb{Q}$-linear span of the x_i for all $i\leq N$, such that (2.1)–(2.3) still hold, and $P_n^2(x_i)=P_n(x_i)$. Define now $(\sigma_i^{(n)})\in(\mathbb{Q}^{R_n})^N$ such that $P_n(x_i)=\sum_{j=1}^{R_n}\sigma_i^{(n)}(j)x_j$. Since

$$P_n^2(x_i)=\sum_{t=1}^{R_n}\Big[\sum_{j=1}^{R_n}\sigma_i(j)\sigma_j(t)\Big]x_t,$$

the three inequalities hold for all $\alpha_1,\dots,\alpha_N\in\mathbb{Q}$ and $i\leq K$.

Conversely, suppose that the above criterion holds and that $\epsilon'>0$. Pick a rational $\frac{\epsilon'}{3\lambda}>\epsilon>0$ and a K. Let λ' and R be as above. Then for every N and $i\leq N$, define

$$y_i^N=\sum_{j=1}^{R}\sigma_i(j)x_j\quad\text{and}\quad z_i^N=\sum_{t=1}^{R}\Big[\sum_{j=1}^{R}\sigma_i(j)\sigma_j(t)\Big]x_t$$

in $[x_1,\dots,x_R]$, where the σ_i depend on N. We have

$$\Big\|\sum_{i=1}^{N}\alpha_i y_i^N\Big\|\leq\lambda'\Big\|\sum_{i=1}^{N}\alpha_i x_i\Big\|\qquad\text{for all }\alpha_i\in\mathbb{Q}, \tag{2.4}$$

$$\|x_i-y_i^N\|\leq\epsilon\qquad\text{for all }i\leq K, \tag{2.5}$$

$$\Big\|\sum_{i=1}^{N}\alpha_i y_i^N-\sum_{i=1}^{N}\alpha_i z_i^N\|\leq c\Big\|\sum_{i=1}^{N}\alpha_i x_i\Big\|\qquad\text{for all }c\in(0,1/4)\cap\mathbb{Q}. \tag{2.6}$$

In particular, for every i, the sequences $(y_i^N)_{N=i}^\infty$, and $(z_i^N)_{N=i}^\infty$ are contained in a bounded set in a finite-dimensional space. So by a diagonal procedure, we may find some subsequence (N_l) such that $y_i = \lim_{l\to\infty} y_i^{N_l}$ and $z_i = \lim_{l\to\infty} z_i^{N_l}$ exist for all i. In consequence

$$\Big\|\sum_{i=1}^{\infty}\alpha_i y_i\Big\| \le \lambda'\Big\|\sum_{i=1}^{\infty}\alpha_i x_i\Big\| \quad \text{for all } \alpha_i \in \mathbb{Q}, \tag{2.7}$$

$$\|x_i - y_i\| \le \epsilon \quad \text{for all } i \le K, \tag{2.8}$$

$$\Big\|\sum_{i=1}^{\infty}\alpha_i y_i - \sum_{i=1}^{\infty}\alpha_i z_i\Big\| \le c\Big\|\sum_{i=1}^{\infty}\alpha_i x_i\Big\| \quad \text{for all } c \in (0,1/4)\cap\mathbb{Q}. \tag{2.9}$$

Now, since the x_i are dense in X, there are uniquely defined bounded linear operators $T_{K,\epsilon} : X \to [x_1,\dots,x_R]$ satisfying $T_{K,\epsilon}(x_i) = y_i$ and then $T^2_{K,\epsilon} : X \to [x_1,\dots,x_R]$ satisfies $T^2_{K,\epsilon}(x_i) = z_i$ with $\|T_{K,\epsilon}\| \le \lambda' < \lambda$ and $\|x_i - T_{K,\epsilon}(x_i)\| \le \epsilon$ for all $i \le K$. Let $K \to \infty$ and $\epsilon \to 0$. Then $T_{K,\epsilon}(x_i) \to x_i$ strongly for all i. Since (x_i) is a dense sequence in X and the operators $T_{K,\epsilon}$ are uniformly bounded, $T_{K,\epsilon}(x) \to x$ strongly for all $x \in X$. Also, $\limsup_{n\to\infty}\|T_{K,\epsilon} - T^2_{K,\epsilon}\| < 1/4$. Therefore, by [3, Theorem (3.7)], X has the $\pi_{\lambda+1}$-property. ■

THEOREM 2.2. *The set of all separable Banach spaces that have the π-property is a Σ^0_6-set in $\mathcal{SE}$.*

Proof. Let K, R, N vary over $\mathbb{N}$ and ϵ, λ' vary over $\mathbb{Q}$. Let also $c \in (0,1/4)\cap\mathbb{Q}$, $\sigma \in (\mathbb{Q}^R)^N$, and $\alpha \in \mathbb{Q}^N$. Then the set

$$\{(x_n)_{n=1}^{\infty} \in C(\Delta)^{\mathbb{N}} : (2.1)\text{–}(2.3) \text{ hold}\}$$

is closed in $C(\Delta)^{\mathbb{N}}$. The set

$$E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha} = \{X \in \mathcal{SE} : (f_n(X))_{n=1}^{\infty} \text{ satisfies } (2.1)\text{–}(2.3)\}$$

is closed in $\mathcal{SE}$, because f_n is continuous for every n. Therefore, for $\lambda \in \mathbb{R}$,

$$E_\lambda = \bigcap_c\bigcap_K\bigcap_\epsilon\bigcap_{\lambda'>\lambda}\bigcup_R\bigcup_N\bigcap_{\sigma\in(\mathbb{Q}^R)^N}\bigcup_{\alpha\in\mathbb{Q}^N} E_{c,K,\epsilon,\lambda',R,N,\sigma,\alpha}$$

is a Π^0_5-set in $\mathcal{SE}$. Hence $E = \bigcup_{\lambda\in\mathbb{Q}} E_\lambda$ is a Σ^0_6-set in $\mathcal{SE}$. But $E = \{X \in \mathcal{SE} : X \text{ has the } \pi\text{-property}\}$. ■

THEOREM 2.3. *The set of all separable Banach spaces which have BAP is a Σ^0_6-set in $\mathcal{SE}$.*

Proof. Let $(x_k)_{k\ge0} \in C(\Delta)^{\mathbb{N}}$ and $K,l,t,M,N \in \mathbb{N}$ and $\sigma = (\lambda_{j,k})_{j\le N,\,k\le M} \in \mathbb{Q}^{N\times M}$. Now we define the set $G_{K,l,t,N,M,\sigma} \subseteq C(\Delta)^{\mathbb{N}}$ such that $(x_k)_{k\ge0} \in G_{K,l,t,N,M,\sigma}$ if and only if:

(a-1) $x_0 = 0$.
(a-2) $\|\sum_{j=1}^{N}\lambda_{j,k}x_j - x_k\| \le 1/l,\ \forall k \le t$.
(a-3) $\|\sum_{j=1}^{N}\lambda_{j,k}x_j - \sum_{j=1}^{N}\lambda_{j,m}x_j\| \le K\|x_k - x_m\|,\ \forall k,m \le M$.

The set $G_{K,l,t,N,M,\sigma}$ is closed in $C(\Delta)^{\mathbb{N}}$.

Consider

$$G = \bigcup_K \bigcap_{t,l} \bigcup_N \bigcap_M \bigcup_{\sigma \in \mathbb{Q}^{N\times M}} G_{K,l,t,N,M,\sigma}.$$

Claim 1. *$(x_k)_{k\geq 0} \in G \Leftrightarrow \exists \Phi_n : \overline{(x_k)_{k\geq 0}} \to C(\Delta)$ such that:*

(b-1) $\Phi_n(0) = 0,\ \forall n.$
(b-2) $\|\Phi_n(x) - x\| \xrightarrow[n\to\infty]{} 0,\ \forall x \in \overline{(x_k)_{k\geq 0}}.$
(b-3) $\Phi_n(\overline{(x_k)_{k\geq 0}}) \subseteq \mathrm{span}\{x_j : j \leq L\}$ *for some* $L \in \mathbb{N}$.
(b-4) $(\Phi_n)_{n\geq 1}$ *is uniformly Lipschitz (that is, there exists $K \in \mathbb{N}$ such that Φ_n is K-Lipschitz for every n).*

Indeed, if $(x_k)_{k\geq 0} \in G$, then for all (l,t,N,M) we can define a K-Lipschitz map $\phi = \phi_{l,t,N,M}$ by

$$\phi : \{x_k : k \leq M\} \to \mathrm{span}\{x_j : j \leq N\},$$
$$x_k \mapsto \sum_{j=1}^{N} \lambda_{j,k} x_j,\ (\lambda_{j,k})_{j\leq N,\, k\leq M} \in \mathbb{Q}^{N\times M}.$$

The map ϕ has the following properties:

(c-1) $\phi(0) = 0$.
(c-2) $\|\phi(x_k) - x_k\| \leq 1/l,\ \forall k \leq t.$
(c-3) $\phi(\{x_k : k \leq M\}) \subseteq \mathrm{span}\{x_j : j \leq N\}.$
(c-4) ϕ is K-Lipschitz.

Since M is arbitrary, a diagonal argument shows that there exist K-Lipschitz maps $\Phi = \Phi_{l,t,N}$ such that:

(d-1) $\Phi(0) = 0$.
(d-2) $\|\Phi(x_k) - x_k\| \leq 1/l,\ \forall k \leq t.$
(d-3) $\Phi(\overline{(x_k)_{k\geq 0}}) \subseteq \mathrm{span}\{x_j : j \leq N\}.$
(d-4) Φ is K-Lipschitz.

Since l and t are arbitrary the existence of a sequence $(\Phi_n)_{n\geq 1}$ satisfying (b-1)–(b-4) follows.

Conversely, assume that there exist uniformly Lipschitz maps $(\Phi_n)_{n\geq 1}$ which satisfy (b-1)–(b-4). Now, pick $\{x_k : k \leq M\} \subset (x_k)_{k\geq 0}$; without loss of generality we can assume that $x_0 = 0$. We have $\Phi_n(x_k) = \sum_{j=1}^{N} \mu_{j,k} x_j$ for some $N \in \mathbb{N}$ and $(\mu_{j,k})_{j\leq N,\, k\leq M} \in \mathbb{R}^{N\times M}$ by using (b-3). The condition (b-2) says that

$$\forall l\ \exists n_0\ \forall n \geq n_0\ \forall k \leq t : \|\Phi_n(x_k) - x_k\| \leq 1/l,$$

i.e. we can pick N such that $\|\sum_{j=1}^{N} \mu_{j,k} x_j - x_k\| \leq 1/l$ for all $k \leq t$. In addition, from (b-4), there exists $K \in \mathbb{N}$ such that

$$\Big\|\sum_{j=1}^{N} \mu_{j,k} x_j - \sum_{j=1}^{N} \mu_{j,m} x_j\Big\| \le K\|x_k - x_m\|, \quad \forall k, m \le M.$$

Since we can approximate $(\mu_{j,k})_{j \le N,\, k \le M} \in \mathbb{R}^{N \times M}$ by a rational sequence $\sigma = (\lambda_{j,k})_{j \le N,\, k \le M} \in \mathbb{Q}^{N \times M}$, there exists K such that

$$\forall l, t\ \exists N\ \forall M\ \exists \sigma \in \mathbb{Q}^{N \times M} : (x_k)_{k \ge 0} \in G_{K,l,t,N,M,\sigma}.$$

Hence, $(x_k)_{k \ge 0} \in G$.

Claim 2. *X has BAP $\Leftrightarrow$ there exists a dense sequence $(x_k)_{k \ge 0}$ in X such that $(x_k)_{k \ge 0} \in G$.*

Let X be a separable Banach space and $(x_k)_{k \ge 0}$ a dense sequence in X. Thus $X = \overline{(x_k)_{k \ge 0}}$ has K-BAP for some K iff X has K-Lipschitz BAP by [9, Theorem (5.3)], iff there exists a sequence of K-Lipschitz maps Φ_n satisfying (b-1)–(b-4), iff $(x_k)_{k \ge 0} \in G$ by Claim 1.

By Theorem 1.1 the functions f_n are continuous. Hence, the set

$$E_{K,l,t,N,M,\sigma} = \{X \in \mathcal{SE} : (f_n(X))_{n=1}^{\infty} \text{ satisfies (a-1)–(a-3)}\}$$

is closed in $\mathcal{SE}$. Therefore, the set

$$E = \bigcup_{K} \bigcap_{t,l} \bigcup_{N} \bigcap_{M} \bigcup_{\sigma \in \mathbb{Q}^{N \times M}} E_{K,l,t,N,M,\sigma}$$

is a Σ^0_6-set in $\mathcal{SE}$. But by Claim 2, $E = \{X \in \mathcal{SE} : X \text{ has BAP}\}$. ■

Acknowledgements. I would like to thank Prof. Gilles Godefroy for his suggestions and encouragement, and the referee for a number of helpful suggestions for improvement in the article.

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