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Statistically characterized subgroups of the circle

by

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Abstract. In this paper, we introduce a new version of characterized subgroups of the circle group $\mathbb{T}$ that we call *statistically characterized subgroups* (briefly, *s-characterized subgroups*). We primarily investigate these subgroups for arithmetic sequences of integers and show that these subgroups are essentially different and strictly larger in size than the much investigated class of characterized subgroups, having cardinality $\mathfrak{c}$ but remaining nontrivial (i.e. different from $\mathbb{T}$) though remaining topologically nice.

1. Introduction. Throughout $\mathbb{R}, \mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ will stand for the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers respectively. The first three are equipped with their usual abelian group structure; we denote by $\mathbb{T}$ the circle group $\mathbb{R}/\mathbb{Z}$ in additive notation.

The motivation to study the so called *characterized subgroups* can be traced back to the distribution of sequences of multiples of a given real number modulo 1. The classical case when these sequences of real numbers are uniformly distributed modulo 1 is considered in the Appendix. The interest in the case when these sequences are small, actually *null sequences*, stems from harmonic analysis where A-sets (short for Arbault sets) were introduced in [1], in particular, for trigonometric series (see also [13, 14, 29]).

Occasionally, the interest for the same type of sets comes also from a completely different direction. The notion of characterized subgroups has evolved over the years as a generalization of the notion of *torsion subgroup* (recall that an element x of an abelian group is torsion if there exists $k \in \mathbb{N}$

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such that $kx = 0$). An element x of an abelian topological group G is ([10], see also [36, 38]):

(i) *topologically torsion* if $n!x \to 0$;
(ii) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.

It is obvious that any p-torsion element is topologically p-torsion. Armacost [2] defined the subgroups

$$X_p = \{x \in X : p^n x \to 0\} \quad \text{and} \quad X! = \{x \in X : n!x \to 0\}$$

of an abelian topological group X, and started their investigation (see Example 1.2).

We now start by recalling the definition of a *characterized subgroup* of $\mathbb{T}$.

DEFINITION 1.1. Let (a_n) be a sequence of integers. The subgroup

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a *characterized* (by (a_n)) *subgroup* of $\mathbb{T}$.

Even though the notion was inspired by the various (earlier) instances mentioned above, the term *characterized* appeared much later, coined in [7].

EXAMPLE 1.2. (a) Let p be a prime. For the sequence (a_n), defined by $a_n = p^n$ for every n, obviously $t_{(p^n)}(\mathbb{T})$ contains the Prüfer group $\mathbb{Z}(p^\infty)$. Armacost [2] proved that $t_{(p^n)}(\mathbb{T})$ simply coincides with $\mathbb{Z}(p^\infty)$.

(b) Armacost [2] posed the problem of describing the group $\mathbb{T}! = t_{(n!)}(\mathbb{T})$. It was resolved independently and almost simultaneously in [26, Chap. 4] and by J.-P. Borel [9].

Historically some of the most interesting cases studied (like the two types of subgroups considered by Armacost) are particular cases of characterized groups characterized by arithmetic sequences. Precisely, a sequence (a_n) of positive integers is an *arithmetic sequence* if

$$1 = a_0 < a_1 < a_2 < \cdots \quad \text{and} \quad a_n \mid a_{n+1} \quad \text{for every } n \in \mathbb{N}.$$

In [18, 26] an element x in an abelian topological group G is called an *$\underline{a}$-torsion element* if $a_n x \to 0$. Borel [8] studied $\underline{a}$-torsion elements (null sequences modulo 1) in $\mathbb{R}$. The first results on describing the $\underline{a}$-torsion elements of $\mathbb{T}$ can be found in [26, Chap. 4], further progress was reported in [16], and the final solution was given in [22].

Coming back to characterized subgroups, in general, it has been observed already by Eggleston [28] (see also [3]) that the asymptotic behavior of the sequence $q_n := a_n/a_{n-1}$ of ratios has a strong impact on the size of $t_{(a_n)}(\mathbb{T})$:

(E1) $t_{(a_n)}(\mathbb{T})$ is countable if (q_n) is bounded,
(E2) $|t_{(a_n)}(\mathbb{T})| = \mathfrak{c}$ if $q_n \to \infty$.

Bíró, Deshouillers and Sós [7] established the important fact that every countable subgroup of $\mathbb{T}$ is characterized. The whole history concerning these investigations along with relevant references can be found in [26, 16, 3] (see also the surveys [18, 17] on characterized subgroups of $\mathbb{T}$). Characterized subgroups of compact abelian groups were introduced in [18, 25] and studied later by Hart and Kunen [33, 34], as well as by the first named author with coauthors [23, 24]. For surveys on characterized subgroups of (locally) compact groups see [19, 21, 32]. For the application of characterized subgroups to the problem of building group topologies with or without convergent sequences, see Remark 6.8.

Although the correspondence $(a_n) \mapsto t_{(a_n)}(\mathbb{T})$ is decreasing (with respect to inclusion), in many cases (as in Example 1.2) the subgroup $t_{(a_n)}(\mathbb{T})$ is rather small, even if the sequence (a_n) is not too dense (in the above example, it is a geometric progression, so has exponential growth). This suggests that asking $a_n x \to 0$ is maybe somewhat too restrictive. Motivated by this observation, we intend to modify the above definition as follows.

For $m, n \in \mathbb{N}$ and $m \leq n$, let $[m, n]$ denote the set $\{m, m+1, m+2, \ldots, n\}$. By $|A|$ we denote the cardinality of a set A. The *lower* and *upper natural densities* of $A \subset \mathbb{N}$ are defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the *natural density* of A exists and denote it by $d(A)$. As usual,

$$\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$$

denotes the ideal of "natural density zero" sets and $\mathcal{I}_d^*$ is the dual filter, i.e. $\mathcal{I}_d^* = \{A \subset \mathbb{N} : d(A) = 1\}$.

Let us now recall the notion of statistical convergence in the sense of [41, 30, 39, 31] (see also [11, 12, 37] for applications to number theory and analysis).

Definition 1.3. A sequence of real numbers (x_n) is said to *converge* to a real number x_0 *statistically* if $d(\{n \in \mathbb{N} : |x_n - x_0| \geq \varepsilon\}) = 0$ for any $\varepsilon > 0$.

It was proved in [39] that $x_n \to x_0$ statistically precisely when there exists a subset A of $\mathbb{N}$ of asymptotic density 0 such that $\lim_{n\in\mathbb{N}\setminus A} x_n = x_0$. Over the years, the notion of statistical convergence has been studied in metric spaces using the metric instead of the modulus and then has been extended to general topological spaces using open neighborhoods [27]. Over the last three decades a lot of work has been done on the notion of statistical convergence primarily because it extends the notion of usual convergence very naturally preserving many of the basic properties but at the same time including more sequences under its purview.

It seems very natural to use the notion of statistical convergence to relax the condition $a_n x \to 0$, i.e. we will consider the situation when $a_n x \to 0$ statistically (rephrasing Definition 1.3, in our context, this means that for every $\varepsilon > 0$ there exists a subset A of $\mathbb{N}$ of asymptotic density 0, such that $\|a_n x\| < \varepsilon$ for every $n \notin A$, where $\|x\|$ denotes the usual norm in $\mathbb{T}$ defined by the length of the shortest arc connecting x with 0).

Using this notion we can introduce our main definition:

Definition 1.4. For a sequence of integers (a_n) the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\} \tag{1.1}$$

of $\mathbb{T}$ is called a *statistically characterized* (briefly, an *s-characterized*) (by (a_n)) *subgroup* of $\mathbb{T}$.

The following result justifies the investigation of this new notion of s-characterized subgroups as it is established that, though in general larger in size, these subgroups are still essentially topologically nice.

Theorem A. *For any sequence of integers (a_n), $t^s_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.*

This result seems reasonable enough for further investigation of the notion of s-characterized subgroups. However, in order to really justify that the theory of s-characterized subgroups of $\mathbb{T}$ is worthy of further studies and its investigation may not follow from the existing literature on characterized subgroups, we will present instances of sequences (a_n) with non-coinciding $t^s_{(a_n)}(\mathbb{T})$ and $t_{(a_n)}(\mathbb{T})$ (for which $t^s_{(a_n)}(\mathbb{T}) \neq \mathbb{T}$). Theorem C below provides a general result in this direction for all arithmetic sequences. Throughout the article, our main focus is to investigate this new notion for arithmetic sequences.

Our next theorem leads us to a general assertion about the size of s-characterized subgroups for arithmetic sequences which asserts one of the main differences of this new notion from the existing notion of characterized subgroups.

Theorem B. *Let (a_n) be an arithmetic sequence. Then $|t^s_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.*

This theorem reveals a substantial difference between characterized and s-characterized subgroups by "breaking" Eggleston's dichotomy (E1)–(E2) for a very large class of s-characterized subgroups. As a consequence, the new subgroup $t^s_{(a_n)}(\mathbb{T})$ always differs from the subgroup $t_{(a_n)}(\mathbb{T})$ for arithmetic sequences:

Theorem C. *For any arithmetic sequence (a_n), $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.*

Unlike Theorem A, both Theorems B and C rely on *arithmetic* sequences. We are not aware if these theorems remain valid if we relax this condition (see Question 6.3).

The paper is organized as follows. At the end of the introduction we provide a list of the notation and terminology used throughout the paper. In particular, we recall a well-known fact providing a canonical representation (1.2) of elements $x \in \mathbb{T}$ with respect to a given arithmetic sequence (a_n), essentially used in the paper. In §2 we collect the topological properties of s-characterized subgroups, in particular, the proof of Theorem A.

Since the area of characterized subgroups is rather rich with results, notions and techniques accumulated in the last 6–7 decades, we felt that a gradual approach for the exposition of its "statistical counterpart" can be much better supported by the reader. That is why in §3 we focus on the special case of the arithmetic sequence (2^n), developing the main ideas and getting a proof of Theorems B and C in this easier setting. Only in §4, do we pass to the general case of an arithmetic sequence (a_n) and here we prove Theorems B and C in full generality. Here we provide also various sufficient conditions for $x \in t^s_{(a_n)}(\mathbb{T})$, while §5 provides sufficient conditions for $x \notin t^s_{(a_n)}(\mathbb{T})$, where $x \in \mathbb{T}$. The final §6 contains connections of characterized subgroups (and their "statistical counterpart") to two relevant fields: number theory (via Weyl's uniform distribution theorem) and topology (construction of group topologies with non-trivial convergent sequences). Here we also collect some final comments and open problems and questions.

1.1. Notation and terminology. For a positive natural number m we denote by $\mathbb{Z}(m)$ the set of solutions of the equation $mx = 0$ in $\mathbb{T}$. This is a cyclic group of order m. Also, for $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the greatest integer less than x, and by $\{x\}$ the fractional part of x.

Let (a_n) be an arithmetic sequence. In this case the ratio, defined by $q_n = a_n/a_{n-1}$ for $n > 0$ (so that $q_1 := a_1$), is an integer; since an arithmetic sequence is strictly increasing, we have $q_n \geq 2$ for all $n \geq 1$. For arithmetic sequences, the fact that any $x \in \mathbb{T}$ can be represented canonically has been used below time and again. So here we recapitulate the result once. Here and throughout, we identify $\mathbb{T}$ with $[0, 1)$.

Lemma 1.5 ([26]). *For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, there exists a unique sequence $(c_n)_{n=1}^{\infty}$ of integers such that $c_n \in [0, q_n - 1]$ for all n and*

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n}, \tag{1.2}$$

where $c_n < q_n - 1$ for infinitely many n.

We define the support of x by

$$\operatorname*{supp}_{(a_n)}(x) = \{n \in \mathbb{N} : c_n \neq 0\}.$$

When no confusion is possible, we simply write $\mathrm{supp}(x)$.

2. Some topological properties of s-characterized subgroups. We start with the proof of Theorem A.

Proof of Theorem A. We have to prove that $t^s_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

First we show that $t^s_{(a_n)}(\mathbb{T})$ is indeed a subgroup of the circle group $\mathbb{T}$. Since $t^s_{(a_n)}(\mathbb{T})$ obviously contains $t_{(a_n)}(\mathbb{T})$, which is known to be a subgroup, it suffices to check that $x-y \in t^s_{(a_n)}(\mathbb{T})$ whenever $x, y \in t^s_{(a_n)}(\mathbb{T})$. Indeed, for $\varepsilon > 0$ we get $M, N \in \mathcal{I}_d$ such that for all $n \notin M$, $\|a_n x\| < \varepsilon/2$ and for all $n \notin N$, $\|a_n y\| < \varepsilon/2$. So for all $n \notin M \cup N$,

$$\|a_n(x-y)\| = \|a_n x - a_n y\| \le \|a_n x\| + \|a_n y\| < \varepsilon.$$

Since $d(M \cup N) = 0$, $(a_n(x-y))$ again statistically converges to 0, so $x-y \in t^s_{(a_n)}(\mathbb{T})$.

For $k, n \in \mathbb{N}$ consider the open subset

$$O_{n,k} := \left\{ x \in \mathbb{T} : \|a_n x\| > \frac{1}{k} \right\}$$

of $\mathbb{T}$. Clearly, $x \in t_{(a_n)}(\mathbb{T})$ if and only if for every fixed positive $k \in \mathbb{N}$, $x \in O_{n,k}$ may occur for only finitely many $n \in \mathbb{N}$, i.e.,

$$t_{(a_n)}(\mathbb{T}) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x \in \mathbb{T} : \|a_n x\| \le \frac{1}{k} \right\} = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathbb{T} \setminus O_{n,k}.$$

Since $\mathbb{T} \setminus O_{n,k}$ is a closed subset of $\mathbb{T}$ for all $k, n \in \mathbb{N}$, this proves that $t_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ set and so a Borel subgroup of $\mathbb{T}$.

The proof for $t^s_{(a_n)}(\mathbb{T})$ will use the same approach, so the sets $O_{n,k}$ will prominently appear below. According to (1.1), $x \in t^s_{(a_n)}(\mathbb{T})$ if and only if $a_n x \to 0$ statistically in $\mathbb{T}$, so by the definition of statistical convergence,

$$\begin{aligned} t^s_{(a_n)}(\mathbb{T}) &= \{x \in \mathbb{T} : (\forall k \in \mathbb{N})\ d(\{n : x \in O_{n,k}\}) = 0\} \\ &= \bigcap_{k=1}^{\infty} \{x \in \mathbb{T} : d(\{n : x \in O_{n,k}\}) = 0\}. \end{aligned}$$

Hence,

$$\begin{aligned} t^s_{(a_n)}(\mathbb{T}) &= \bigcap_{k=1}^{\infty} \left\{ x \in \mathbb{T} : \lim_{m\to\infty} \frac{|\{i \in \mathbb{N} : x \in O_{i,k}\} \cap [1,m]|}{m} = 0 \right\} \\ &= \bigcap_{k=1}^{\infty} \left\{ x \in \mathbb{T} : (\forall j \in \mathbb{N})(\exists m \in \mathbb{N})\ q_n \le \frac{1}{j} \text{ for all } n \ge m \right\}, \end{aligned}$$

where $q_n = |\{i \in \mathbb{N} : x \in O_{i,k}\} \cap [1,n]|/n$. To rewrite this equality more efficiently, let $\mathfrak{J}_{n,j} = \{J \subseteq [0,n] : |J| > n/j\}$ for positive $j, n \in \mathbb{N}$. Then it is easy to realize that the set

$$B_{k,j,n} = \left\{ x \in \mathbb{T} : \frac{|\{i \in \mathbb{N} : x \in O_{i,k}\} \cap [1,n]|}{n} \leq \frac{1}{j} \right\}$$

satisfies

$$B_{k,j,n} = \{x \in \mathbb{T} : \forall J \in \mathfrak{J}_{n,j}\ \exists i \in J\ x \notin O_{i,k}\} = \bigcap_{J \in \mathfrak{J}_{n,j}} \bigcup_{i \in J} \mathbb{T} \setminus O_{n,k}.$$

Since the sets J and $\mathfrak{J}_{n,j}$ are finite and $\mathbb{T} \setminus O_{n,k}$ is a closed set of $\mathbb{T}$ for every $n, k \in \mathbb{N}$, we deduce that $B_{k,j,n}$ is closed in $\mathbb{T}$ for every fixed triple k, j and n. Since the above equalities obviously give

$$t^s_{(a_n)}(\mathbb{T}) = \bigcap_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} B_{k,j,n},$$

we deduce that $t^s_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ set and so a Borel subgroup of $\mathbb{T}$. ∎

Since $t^s_{(a_n)}(\mathbb{T})$ is a Borel set of $\mathbb{T}$, it is measurable with respect to the Haar measure μ of $\mathbb{T}$. More precisely, $\mu(t^s_{(a_n)}(\mathbb{T})) = 1$ when $t^s_{(a_n)}(\mathbb{T}) = \mathbb{T}$. Otherwise, when $t^s_{(a_n)}(\mathbb{T}) \neq \mathbb{T}$, $\mu(t^s_{(a_n)}(\mathbb{T})) = 0$ since in this case the subgroup $t^s_{(a_n)}(\mathbb{T})$ will have infinite index (as $\mathbb{T}/t^s_{(a_n)}(\mathbb{T})$ is divisible as a quotient of the divisible group $\mathbb{T}$) and $\mu(\mathbb{T}) = 1$.

By Theorem A, the subgroup $t^s_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ set. In general, it may not be complete with respect to the usual norm $\|\cdot\|$ prevalent in $\mathbb{T}$ as one can see by taking any proper infinite s-characterized subgroup (for example $t^s_{(2^n)}(\mathbb{T})$) which is dense, so non-closed, hence cannot be complete.

In [6] it was shown that the characterized subgroup $t_{(a_n)}(\mathbb{T})$ is Polishable with respect to the finer Polish topology generated by the metric d_1 on $\mathbb{T}$ given by

$$d_1(x,y) = \|x - y\| + \max_{n \in \mathbb{N}} \|a_n(x-y)\| \quad \text{for } x, y \in \mathbb{T}.$$

Proceeding in a similar way, we can give a suitable metric in $\mathbb{T}$ and consequently endow $\mathbb{T}$ with a finer topology so that the subgroup $t^s_{(a_n)}(\mathbb{T})$ becomes closed in $\mathbb{T}$ and hence complete with respect to that metric.

PROPOSITION 2.1. *There is a metric in $\mathbb{T}$ with respect to which the subgroup $t^s_{(a_n)}(\mathbb{T})$ is closed in $\mathbb{T}$.*

Proof. Let $\delta : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be defined by

$$\delta(x,y) = \sup_{n \in \mathbb{N}} \{\|x-y\|, \|a_n(x-y)\|\} \quad \text{for } x, y \in \mathbb{T}.$$

It is easy to check that δ is a metric on $\mathbb{T}$. Let x be a limit point of $t^s_{(a_n)}(\mathbb{T})$ with respect to the topology induced by δ. We show that $x \in t^s_{(a_n)}(\mathbb{T})$. Let $\varepsilon > 0$. Then there exists $x_\varepsilon \in t^s_{(a_n)}(\mathbb{T})$ such that $\|a_n(x - x_\varepsilon)\| < \varepsilon/2$ for all $n \in \mathbb{N}$. Again $x_\varepsilon \in t^s_{(a_n)}(\mathbb{T})$ implies that there exists $B \in \mathcal{I}_d^*$ such that $\|a_n x_\varepsilon\| < \varepsilon/2$ for all $n \in B$. So for all $n \in B$ we get

$$\|a_n x\| \le \|a_n(x - x_\varepsilon)\| + \|a_n x_\varepsilon\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that $x \in t^s_{(a_n)}(\mathbb{T})$, and the closedness of $t^s_{(a_n)}(\mathbb{T})$ follows immediately. ∎

COROLLARY 2.2. *There is a finer topology on the subgroup $t^s_{(a_n)}(\mathbb{T})$ which is completely metrizable.*

Proof. Let us recall from [20] and [23] that the metric δ on $\mathbb{T}$ is complete. Now in view of Proposition 2.1, $t^s_{(a_n)}(\mathbb{T})$ is closed in $\mathbb{T}$ with respect to the topology induced by δ, and hence complete with respect to δ. ∎

Although we cannot say anything about its separability in this case, using Proposition 2.1 with [40, Theorem 3.2.4] we can conclude that $t^s_{(a_n)}(\mathbb{T})$ is Polishable.

THEOREM 2.3 ([40]). *Let X be a Polish space. Then for every Borel set B in X there is a finer Polish topology τ_B on X such that B is closed in X with respect to τ_B.*

So our next observation is the following.

PROPOSITION 2.4. *For any sequence (a_n) of natural numbers, $t^s_{(a_n)}(\mathbb{T})$ is Polishable.*

Proof. $\mathbb{T}$ is Polish with respect to the usual topology, and $t^s_{(a_n)}(\mathbb{T}) =: B$ is Borel in $\mathbb{T}$, by Theorem A. By Theorem 2.3 there is a finer Polish topology τ_B on $\mathbb{T}$ such that B is closed in $\mathbb{T}$ with respect to τ_B, making B complete and hence Polish with respect to τ_B. ∎

As $t^s_{(a_n)}(\mathbb{T})$ is Borel, we now focus on the Borel complexity of the subgroup $t^s_{(a_n)}(\mathbb{T})$. We already know that $t^s_{(a_n)}(\mathbb{T})$ is $F_{\sigma\delta}$. We may ask the converse, i.e. whether every $F_{\sigma\delta}$ subgroup in $\mathbb{T}$ can be written as $t^s_{(a_n)}(\mathbb{T})$ for some suitably chosen (a_n). We answer it negatively using the results obtained in [6] that there is a subgroup G of $\mathbb{T}$ generated by an uncountable Kronecker set K which is F_σ but not Polishable, and hence not s-characterizable.

PROPOSITION 2.5 ([6]). *Suppose K is an uncountable compact subset of $\mathbb{T}$ with respect to the usual topology on $\mathbb{T}$ and K is independent over $\mathbb{Z}$. Let G be the subgroup of $\mathbb{T}$ generated by K and let d be a metric defined on G such that (G, d) is a Polish group. Then the inclusion map $i : (G, d) \to \mathbb{T}$ given by $i(g) = g$ for all $g \in G$ is not continuous.*

We use the above result to obtain the next one.

PROPOSITION 2.6. *There is a subgroup G of $\mathbb{T}$ which is F_σ but not Polishable, hence not s-characterizable.*

Proof. Let us take the same G as in Proposition 2.5. Assume that $G = t^s_{(a_n)}(\mathbb{T})$ for some (a_n). Then Proposition 2.1 tells us that there is a finer topology generated by a metric δ on G such that (G, δ) is Polish. So all the conditions of Proposition 2.5 are satisfied, which implies that the inclusion map $i : (G, \delta) \to \mathbb{T}$ given by $i(g) = g$ for all $g \in G$ is not continuous. But this is not the case as the topology generated by δ is finer than the usual topology. So G cannot be written as $t^s_{(a_n)}(\mathbb{T})$ for any (a_n). ∎

Now we discuss when the s-characterized subgroups $t^s_{(a_n)}(\mathbb{T})$ are G_δ sets. It was shown in [20] that the G_δ subgroups of $\mathbb{T}$ coincide with the closed ones (in fact this is true in all compact metrizable groups [20, Proposition 2.4]). Since the proper closed subgroups of $\mathbb{T}$ are precisely the finite cyclic ones, they can be characterized. This shows in particular that all G_δ-subgroups of $\mathbb{T}$ can be s-characterized.

Following [20], for a sequence of integers (a_n) (that can be viewed also as characters of $\mathbb{T}$), one can define

$$K_{(a_n)}(\mathbb{T}) = \bigcap_{n \in \mathbb{N}} \mathbb{Z}(a_n).$$

Our next observation is a counterpart of [20, Lemma 2.2]:

PROPOSITION 2.7. *$K_{(a_n)}(\mathbb{T}) \subset t^s_{(a_n)}(\mathbb{T})$ and $t^s_{(a_n)}(\mathbb{T})/K_{(a_n)}(\mathbb{T})$ is an s-characterized subgroup of $\mathbb{T}/K_{(a_n)}(\mathbb{T}) \cong \mathbb{T}$.*

Proof. Clearly $K_{(a_n)}(\mathbb{T}) \subset t_{(a_n)}(\mathbb{T}) \subset t^s_{(a_n)}(\mathbb{T})$. For the remaining part, we proceed exactly in the same way as in [20, proof of Lemma 2.2]. ∎

3. The s-characterized subgroup by the arithmetic sequence $a_n = 2^n$. As mentioned earlier, the relevance of the newly obtained subgroups $t^s_{(a_n)}(\mathbb{T})$ depends on

(i) whether $t^s_{(a_n)}(\mathbb{T})$ actually becomes the whole circle group $\mathbb{T}$ and
(ii) whether as subgroups of $\mathbb{T}$, they are really "new" compared to the already studied characterized subgroups $t_{(a_n)}(\mathbb{T})$.

The study of question (i) is relatively easy, as it is known that $t_{(a_n)}(\mathbb{T}) = \mathbb{T}$ precisely when $a_n = 0$ for almost all n [3, 26]. Using this fact one can conclude that $t^s_{(a_n)}(\mathbb{T}) = \mathbb{T}$ precisely when $d(\{n : a_n \neq 0\}) = 0$. Since no arithmetic sequence (a_n) satisfies $d(\{n : a_n \neq 0\}) = 0$, we deduce

FACT 3.1. $t^s_{(a_n)}(\mathbb{T}) \neq \mathbb{T}$ *for every arithmetic sequence* (a_n).

Question (ii) is far more complicated and seems worth studying. We thoroughly investigate this problem for general arithmetic sequences.

As the general case seems quite complicated, we begin with a special case providing a basic example and then step by step generalize the idea.

Our main results will be presented in Theorems B and C. Apparently, the easiest relevant example can be obtained with $a_n = 2^n$ for every n.

Note that $t_{(2^n)}(\mathbb{T})$ is simply the Prüfer group $\mathbb{Z}(2^\infty)$. So it remains to check that $t^s_{(2^n)}(\mathbb{T})$ contains an element x that does not belong to $\mathbb{Z}(2^\infty)$. Clearly, $x \in \mathbb{Z}(2^\infty)$ precisely when $\operatorname{supp}(x)$ is finite (see [22]). Note that, for $a_n = 2^n$, c_n can only be 0 or 1. Below we observe that one of the easiest way to determine an element which belongs to $t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$ is to construct a suitable support. However, to do that, we first present a more precise construction which will be carried out later in a more general situation.

Proposition 3.2. *The subgroup $t^s_{(2^n)}(\mathbb{T})$ is not contained in $t_{(2^n)}(\mathbb{T})$.*

Proof. Choose $x \in \mathbb{T}$ with

$$\operatorname*{supp}_{(2^n)}(x) = \bigcup_{n=1}^{\infty} [(2n)^2, (2n+1)^2]. \tag{3.1}$$

We show that $x \in t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$. To check that $x \in t^s_{(2^n)}(\mathbb{T})$, pick an $m \in \mathbb{N}$ and define a subset A of $\mathbb{N}$ as follows: First let $B_n := [(2n)^2, (2n+1)^2]$ for brevity. Then choose n_0 such that $4n_0 + 1 = (2n_0 + 1)^2 - (2n_0)^2 > m$. Now let

$A_1 := [(2n_0+2)^2 - m, (2n_0+2)^2 - 1], \quad A'_1 := [(2n_0+3)^2 - m, (2n_0+3)^2 - 1].$

Similarly, let

$$\begin{aligned} A_k &:= [(2(n_0 + k))^2 - m, (2(n_0 + k))^2 - 1], \\ A'_k &:= [(2(n_0 + k) + 1)^2 - m, (2(n_0 + k) + 1)^2 - 1]. \end{aligned}$$

Finally, put $A = \bigcup_k (A_k \cup A'_k)$.

Let us see first that $d(A) = 0$. Indeed, $|A_k| = |A'_k| = m - 1$, so

$$\begin{aligned} \overline{d}(A) &= \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n} \\ &\le \max\left\{ \lim_{k\to\infty} \frac{2mk}{(2(n_0 + k) + 1)^2}, \lim_{k\to\infty} \frac{2mk - m}{(2(n_0 + k))^2} \right\} = 0. \end{aligned}$$

We claim that $\|2^n x\| < 1/2^m$ for all $n \in \mathbb{N} \setminus A$.

As $n \in \mathbb{N} \setminus A$, by the choice of A and the definition of B_n, we deduce that either

(a) $n \in [(2r)^2, (2r+1)^2]$ for some $r \in \mathbb{N}$, and $n+1, \ldots, n+m \in [(2r)^2, (2r+1)^2]$, or

(b) $n \in [(2r + 1)^2, (2(r + 1))^2]$ for some $r \in \mathbb{N}$, and $n + 1, \ldots, n + m \in [(2r + 1)^2, (2(r + 1))^2]$.

In both cases we have $c_{n+1} = \cdots = c_{n+m}$. In case (b) this leads to $c_{n+1} = \cdots = c_{n+m} = 0$, so

$$2^n x = \frac{c_{n+1}}{2} + \frac{c_{n+2}}{2^2} + \cdots + \frac{c_{n+m}}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \cdots = \frac{c_{n+m+1}}{2^{m+1}} + \cdots .$$

Therefore, $\|2^n x\| < 1/2^m$. In case (a) this leads to $c_{n+1} = \dots = c_{n+m} = 1$, so

$$2^n x = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \dots = 1 - \frac{1}{2^m} + \frac{c_{n+m+1}}{2^{m+1}} + \dots .$$

Therefore, we obtain again $\|2^n x\| < 1/2^m$. Since $m \in \mathbb{N}$ was chosen arbitrarily, we deduce that $2^n x$ statistically converges to 0 in $\mathbb{T}$, i.e., $x \in t^s_{(2^n)}(\mathbb{T})$.

According to [22], $x \notin t_{(2^n)}(\mathbb{T})$ since $\mathrm{supp}(x)$ is infinite. ∎

However, we can say more: the newly obtained subgroup $t^s_{(2^n)}(\mathbb{T})$ contains "many more" elements compared to $t_{(2^n)}(\mathbb{T})$. We prove that in Proposition 3.5; we need the following two lemmas.

We start with the observation that the element $x \in \mathbb{T}$ in Proposition 3.2 can be replaced by a more generally defined element of $\mathbb{T}$. To explain the choice we note that for every x as in (3.1) and such that $x \notin \mathbb{Z}(2^\infty)$, the support can be represented as a disjoint union $\bigcup_n B_n$ of infinitely many consecutive intervals. Let us define

$$\text{(3.2)}\qquad \mathbb{I} = \Big\{ \bigcup_{n=1}^{\infty} B_n : B_n = [b_n, d_n],\ b_{n+1} > d_n + 1\ \forall n; \\ \lim_{n\to\infty} |d_n - b_n| = \infty = \lim_{n\to\infty} |b_{n+1} - d_n| \Big\}.$$

In Proposition 3.2 we used the following specific member of $\mathbb{I}$:

$$\text{(3.3)}\qquad B = \bigcup_{n=1}^{\infty} B_n \in \mathbb{I}, \quad \text{where } B_n := [(2n)^2, (2n+1)^2].$$

Since $d(B) = 1/2 > 0$, this shows that $\mathbb{I} \not\subseteq \mathcal{I}_d$, and obviously $\mathbb{I} \not\supseteq \mathcal{I}_d$. Here we prove a result concerning both $\mathbb{I}$ and $\mathcal{I}_d$ that will be frequently used.

Lemma 3.3. *$|\mathbb{I}| = |\mathcal{I}_d| = \mathfrak{c}$.*

Proof. Let B be as in (3.3). Fix a sequence $\xi = (z_i) \in \{0,1\}^{\mathbb{N}}$ and define $B^\xi := \bigcup_{k=1}^{\infty} B_{2k+z_k}$. In other words, B^ξ is obtained by taking at each stage k either B_{2k} of B_{2k+1} depending on the choice imposed by ξ. As obviously $B^\xi \neq B^\eta$ for distinct $\xi, \eta \in \{0,1\}^{\mathbb{N}}$, this provides an injective map

$$\{0,1\}^{\mathbb{N}} \ni \xi \mapsto B^\xi \in \mathbb{I}.$$

Since $|\{0,1\}^{\mathbb{N}}| = \mathfrak{c}$, we are done.

A similar proof works for $\mathcal{I}_d$. It suffices to take $C = \bigcup_{n=1}^{\infty} C_n \in \mathcal{I}_d$ with $C_n := [(2n)^2, (2n)^2 + 1]$ and define $C^\xi := \bigcup_{k=1}^{\infty} C_{2k+z_k}$ for every $\xi = (z_i) \in \{0,1\}^{\mathbb{N}}$. The map $\{0,1\}^{\mathbb{N}} \ni \xi \mapsto C^\xi \in \mathbb{I}$ is injective, so $|\mathcal{I}_d| = \mathfrak{c}$. ∎

Let us note that the element $x \in \mathbb{T}$ in Proposition 3.2 has the property $\mathrm{supp}(x) \in \mathbb{I}$. Now we see that the argument works with any element x of $\mathbb{T}$ with $\mathrm{supp}(x) \in \mathbb{I}$.

Lemma 3.4. *Let* $x \in \mathbb{T}$ *be such that* $\operatorname{supp}_{(2^n)}(x) \in \mathbb{I}$. *Then* $x \in t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$.

Proof. The fact that $\operatorname{supp}(x) \in \mathbb{I}$ implies that $\operatorname{supp}(x)$ is infinite. Hence $x \notin t_{(2^n)}(\mathbb{T})$.

Let us define $l_n := |d_n - b_n|$ and $g_n := |b_{n+1} - d_n|$ for all $n \in \mathbb{N}$. As $\operatorname{supp}(x) \in \mathbb{I}$, for every $m \in \mathbb{N}$ we can choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ one has $l_n > m$ and $g_n > m$. Also let $G_n := [d_n, b_{n+1}]$. Then for every $n \geq n_0$ we choose two subintervals $A_n \subseteq B_n$ and $A'_n \subseteq G_n$ such that

(a) the lengths of all such intervals A_n, A'_n are m;
(b) the right end point of A_n coincides with the right end point of B_n, namely d_n, and the right end point of A'_n coincides with the right end point of G_n, namely b_{n+1}.

In other words, $A_n = [d_n - m, d_n]$ and $A'_n = [b_{n+1} - m, b_{n+1}]$.

Put $A = \bigcup_{k \geq n_0} (A_k \cup A'_k)$. The proof that this A witnesses the statistical convergence with respect to $\varepsilon = 1/2^m$ follows the lines of the proof of Proposition 3.2. Clearly, $d(A) = 0$ as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n} = \max\left\{\lim_{k\to\infty} \frac{2mk}{d_{n_0+k}}, \lim_{k\to\infty} \frac{2mk - m}{b_{n_0+k}}\right\} = 0,$$

and $\|2^n x\| \leq 1/2^m$ for all $n \in \mathbb{N} \setminus A$ as we can see below.

For $n \in \mathbb{N} \setminus A$, by the choice of A and the definition of $B_n = [b_n, d_n]$, we deduce that either

(a) $n \in B_r$ for some $r \in \mathbb{N}$, and $n+1, \ldots, n+m \in B_r$, or
(b) $n \in G_r$ for some $r \in \mathbb{N}$, and $n+1, \ldots, n+m \in G_r$.

In both cases we have $c_{n+1} = \cdots = c_{n+m}$. The rest of the proof is analogous to the corresponding part of the proof of Proposition 3.2 and so is omitted. ∎

Immediately we arrive at an interesting conclusion, namely that there are $\mathfrak{c}$ many new elements in $t^s_{(2^n)}(\mathbb{T})$ compared to those in $t_{(2^n)}(\mathbb{T})$.

Proposition 3.5. $|t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})| = \mathfrak{c}$.

Proof. In Lemma 3.4 we have shown that $\{x : \operatorname{supp}_{(2^n)}(x) \in \mathbb{I}\} \subset t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})$. Now since $|\{x : \operatorname{supp}_{(2^n)}(x) \in \mathbb{I}\}| = |\mathbb{I}|$, Lemma 3.3 shows that $|\{x : \operatorname{supp}_{(2^n)}(x) \in \mathbb{I}\}| = |\mathbb{I}| = \mathfrak{c}$. That is, $|t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})| \geq \mathfrak{c}$, which gives our result. ∎

Since we know that $t_{(2^n)}(\mathbb{T})$ is countably infinite, an obvious but important consequence of Proposition 3.5 is the following.

Corollary 3.6. $|t^s_{(2^n)}(\mathbb{T})| = \mathfrak{c}$.

Proof. This follows immediately from

$$t^s_{(2^n)}(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T}) \subseteq t^s_{(2^n)}(\mathbb{T}) \subseteq \mathbb{T}. \ ∎$$

By Fact 3.1, $t^s_{(2^n)}(\mathbb{T}) \neq \mathbb{T}$ but it is always good to have at least one concrete example of an element not in $t^s_{(2^n)}(\mathbb{T})$, which is given below.

EXAMPLE 3.7. Consider

$$x = 1/3 = \sum_{n=1}^{\infty} \frac{1}{4^n} \in \mathbb{T}.$$

Clearly $2^n x \in \{1/3, 2/3\}$, so $\|2^n x\| = 1/3$ for all $n \in \mathbb{N}$ and so $x \notin t^s_{(2^n)}(\mathbb{T})$. For a more general result see Proposition 5.1.

4. The general case for arithmetic sequences—proofs of Theorems B and C. In this section, we exploit the whole idea of the previous section for arbitrary arithmetic sequences and we generalize Proposition 3.2 and Corollary 3.6 in this context. This provides proofs of theorems B and C, given at the end of the section.

First we prove a counterpart of Lemma 3.4 which gives a sufficient condition for some x to be in $t^s_{(a_n)}(\mathbb{T})$.

THEOREM 4.1. *Let (a_n) be any arithmetic sequence and let $x \in \mathbb{T}$ be such that* $\operatorname{supp}(x) \in \mathbb{I}$ *and $c_n = q_n - 1$ for all $n \in \operatorname{supp}(x)$. Then $x \in t^s_{(a_n)}(\mathbb{T})$.*

Proof. Let $x = \sum_{n=1}^{\infty} c_n/a_n$ be the canonical representation of $x \in \mathbb{T}$ where $c_1 = 0$, c_n is either 0 or $(q_n - 1)$ for any $n > 1$ and $\{n : c_n = q_n - 1\} = \bigcup_{n=1}^{\infty} [b_n, d_n]$ for b_n's and d_n's defined as in (3.2). We claim that $x \in t^s_{(a_n)}(\mathbb{T}) \setminus t_{(a_n)}(\mathbb{T})$. Indeed, $x \notin t_{(a_n)}(\mathbb{T})$ by [22, Theorem 2.3]. To show that $x \in t^s_{(a_n)}(\mathbb{T})$ we proceed exactly as in the proof of Lemma 3.4. We take an arbitrary $m \in \mathbb{N}$ and get the same $n_0 \in \mathbb{N}$ and $A \subset \mathbb{N}$ with $d(A) = 0$. What is required now is to show that $\lim_{n\to\infty,\, n\in\mathbb{N}\setminus A} \|a_n x\| = 0$. For $n \in \mathbb{N} \setminus A$, by the choice of A and the definition of $B_n = [b_n, d_n]$, we deduce that either

(a) $n \in B_r$ for some $r \in \mathbb{N}$, and $n+1, \dots, n+m \in B_r$, or
(b) $n \in G_r$ for some $r \in \mathbb{N}$, and $n+1, \dots, n+m \in G_r$.

In case (b) this leads to $c_{n+1} = \dots = c_{n+m} = 0$, so

$$\{a_n x\} = \sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} \cdot a_n \leq \sum_{k=n+1+m}^{\infty} \frac{q_k - 1}{a_k} \cdot a_n = \sum_{k=n+1+m}^{\infty} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k} \right) \cdot a_n \leq \frac{a_n}{a_{m+n}}.$$

In case (a) this leads to $c_k = q_k - 1$ for $k = n+1, \dots, n+m$, so

$$\{a_n x\} = \sum_{k=n+1}^{n+m} \frac{q_k - 1}{a_k} \cdot a_n + \sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} \cdot a_n.$$

Now the first sum is

$$\sum_{k=n+1}^{n+m} \frac{q_k - 1}{a_k} \cdot a_n = \sum_{k=n+1}^{n+m} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k}\right) \cdot a_n = 1 - \frac{a_n}{a_{n+m}}$$

and the second is

$$\begin{aligned}\sum_{k=n+1+m}^{\infty} \frac{c_k}{a_k} \cdot a_n &\le \sum_{k=n+1+m}^{\infty} \frac{q_k - 1}{a_k} \cdot a_n \\ &= \sum_{k=n+1+m}^{\infty} \left(\frac{1}{a_{k-1}} - \frac{1}{a_k}\right) \cdot a_n \le \frac{a_n}{a_{m+n}}.\end{aligned}$$

Therefore, $\|a_n x\| \le \frac{a_n}{a_{n+m}} \le \frac{1}{2^m}$. As $m \in \mathbb{N}$ was chosen arbitrarily, we conclude that $\|a_n x\|$ converges statistically to 0. ∎

The condition prescribed in Theorem 4.1 is not necessary, as one can see in the next example.

EXAMPLE 4.2. Let $a_n = 2^n$ and $x \in \mathbb{T}$ be such that $\operatorname{supp}(x) = \{n^2 : n \in \mathbb{N}\}$ i.e. $x = \sum_{n=1}^{\infty} 2^{-n^2}$. So $\operatorname{supp}(x)$ here does not satisfy the condition described in Lemma 4.1. We claim that $x \in t^s_{(2^n)}(\mathbb{T})$.

Indeed, for $l \in \mathbb{N}$ choose n satisfying $n = k^2 - l > (k-1)^2$. Then

$$\{2^n x\} = \frac{1}{2^l} + \sum_{r=1}^{\infty} \frac{2^{k^2-l}}{2^{(k+r)^2}} = \frac{1}{2^l} + \sum_{r=1}^{\infty} \frac{1}{2^{2rk+r^2+l}} = \frac{1}{2^l}\left(1 + \sum_{r=1}^{\infty} \frac{1}{2^{r^2+2rk}}\right).$$

As the terms are decreasing in k, we have

$$\{2^n x\} \le \frac{1}{2^l}\left(1 + \sum_{r=1}^{\infty} \frac{1}{2^{r^2+2r}}\right) = \frac{1}{2^l}(1+M),$$

where $M := \sum_{r=1}^{\infty} 1/2^{r^2+2r} < \infty$. Let $\varepsilon > 0$ be arbitrary. Now choose $l \in \mathbb{N}$, $l > 1$ such that $2^{-l}(1+M) < \varepsilon$. Hence $\{n \in \mathbb{N} : \|2^n x\| \ge \varepsilon\} \subset \bigcup_{i=0}^{l-1} \{n^2 - i : n \in \mathbb{N}\}$. As $d(\{n^2 - i : n \in \mathbb{N}\}) = 0$ for any $i \ge 0$, we conclude that

$$d(\{n \in \mathbb{N} : \|2^n x\| \ge \varepsilon\}) \le d\Big(\bigcup_{i=0}^{l-1} \{n^2 - i : n \in \mathbb{N}\}\Big) = 0,$$

i.e. $x \in t^s_{(2^n)}(\mathbb{T})$.

REMARK 4.3. A closer look at the argument of Example 4.2 allows one to see that it can be extended to an arbitrary arithmetic sequence (a_n) as follows. If for $x \in \mathbb{T}$ the support $\operatorname{supp}(x)$ is finite or $\operatorname{supp}(x) = \{m_1 < m_2 < \cdots\}$ with $\lim_{r\to\infty}(m_r - m_{r-1}) = \infty$, then $x \in t^s_{(a_n)}(\mathbb{T})$.

Indeed, if $\operatorname{supp}(x)$ is finite then $x \in t_{(a_n)}(\mathbb{T}) \subset t^s_{(a_n)}(\mathbb{T})$ by [22, Corollary 3.2]. In the more interesting case when $\operatorname{supp}(x)$ is infinite, consider the canonical representation (1.2) of $x \in \mathbb{T}$ and let $\operatorname{supp}(x) = \{m_1 < m_2 < \cdots\}$

with $\lim_{r\to\infty}(m_r - m_{r-1}) = \infty$. Then for $n \notin \text{supp}(x)$ there exist unique $r, l > 1$ such that $n = m_r - l > m_{r-1}$. Then

$$\{a_n x\} = a_{m_r-l}\sum_{k=r}^{\infty}\frac{c_{m_k}}{a_{m_k}} \le a_{m_r-l}\sum_{k=r}^{\infty}\frac{q_{m_k}-1}{a_{m_k}} = a_{m_r-l}\sum_{k=r}^{\infty}\left(\frac{1}{a_{m_k-1}} - \frac{1}{a_{m_k}}\right)$$
$$\le \frac{a_{m_r-l}}{a_{m_r-1}} \le \frac{1}{2^{l-1}}.$$

Now let $\varepsilon > 0$. Choose $l \in \mathbb{N}$, $l > 1$ such that $1/2^{l-1} < \varepsilon$. Consequently, $\|a_n x\| < 1/2^{l-1} < \varepsilon$ for all $n = m_r - l, m_r - (l+1), \ldots$ and for all r. This gives us $\{n : \|a_n x\| \ge \varepsilon\} \subset \bigcup_{i=0}^{l-1}\{m_r - i : r \in \mathbb{N}\}$, which implies that

$$d(\{n : \|a_n x\| \ge \varepsilon\}) \le d\Big(\bigcup_{i=0}^{l-1}\{m_r - i : r \in \mathbb{N}\}\Big) = 0.$$

Taking cue from Example 4.2 and Remark 4.3, and noting that every subset $A = \{m_1 < m_2 < \cdots\}$ of $\mathbb{N}$ with $\lim_{r\to\infty}(m_r - m_{r-1}) = \infty$ satisfies $d(A) = 0$, we can provide another, stronger and more natural, sufficient condition for $x \in t^s_{(a_n)}(\mathbb{T})$.

THEOREM 4.4. *Let (a_n) be an arithmetic sequence. If $x \in \mathbb{T}$ is such that* $\text{supp}(x) \in \mathcal{I}_d$, *then $x \in t^s_{(a_n)}(\mathbb{T})$.*

Proof. Set $A = \text{supp}(x)$. Then $d(A) = 0$, by hypothesis.

Pick a positive $k \in \mathbb{N}$ and note that $A^* = \bigcup_{i=0}^{k}(A - i) \cap \mathbb{N} \in \mathcal{I}_d$. Hence, it is enough to check that $\|a_n x\| \le 1/k$ for all $n \in \mathbb{N} \setminus A^*$. Note that $n \in \mathbb{N} \setminus A^*$ precisely when $n + i \notin A$ for $i = 0, 1, \ldots, k$. This means that in the canonical representation of x one has $c_n = c_{n+1} = \cdots = c_{n+k} = 0$. Hence,

$$\{a_n x\} = a_n \cdot \sum_{i=n+k+1}^{\infty}\frac{c_i}{a_i} \le a_n \cdot \sum_{i=n+k+1}^{\infty}\frac{q_i - 1}{a_i} = a_n \cdot \sum_{i=n+k+1}^{\infty}\left(\frac{1}{a_{i-1}} - \frac{1}{a_i}\right)$$
$$\le \frac{a_n}{a_{n+k}} \le \frac{1}{2^k} < \frac{1}{k}. \ \blacksquare$$

We shall invert this theorem in Corollary 5.3 for arithmetic sequences with $\lim_n q_n = \infty$. Now we see that the condition $d(A) = 0$ in Theorem 4.4 is not necessary for some $x \in \mathbb{T}$ to be in $t^s_{(a_n)}(\mathbb{T})$.

EXAMPLE 4.5. Let b_n, d_n, $n \in \mathbb{N}$, be such that $b_1 = 1$ and for all n, $d_n - b_n = n^2$ and $b_{n+1} - d_n = n$. Let $B = \bigcup_{n=1}^{\infty}[b_n, d_n]$ and $x = \sum_{n=1}^{\infty} c_n/a_n \in \mathbb{T}$ be such that $c_n = 0$ for all $n \notin B$, whereas $c_n = q_n - 1$ for all $n \in B$, as described in Lemma 4.1. Then applying Lemma 4.1 we can see that $x \in t^s_{(a_n)}(\mathbb{T})$. Now

$$\begin{aligned}\overline{d}(B) &\geq \lim_{n\to\infty} \frac{(d_1-b_1)+(d_2-b_2)+\cdots+(d_n-b_n)}{d_n}\\ &= \lim_{n\to\infty} \frac{(d_1-b_1)+(d_2-b_2)+\cdots+(d_n-b_n)}{b_1+(d_1-b_1)+(b_2-d_1)+\cdots+(b_n-d_{n-1})+(d_n-b_n)}\\ &= \lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{1+(1+2+\cdots+n-1)+(1^2+2^2+\cdots+n^2)} = 1 > 0.\end{aligned}$$

This shows that $d(\mathrm{supp}(x)) \neq 0$, which implies that $\mathrm{supp}(x)$ does not satisfy the criterion of Theorem 4.4 though $x \in t^s_{(a_n)}(\mathbb{T})$.

Proof of Theorem B. Let (a_n) be an arithmetic sequence. We have to prove that $|t^s_{(a_n)}(\mathbb{T})| = \mathfrak{c}$. Clearly $t^s_{(a_n)}(\mathbb{T}) \subset \mathbb{T}$ implies $|t^s_{(a_n)}(\mathbb{T})| \leq |\mathbb{T}| = \mathfrak{c}$.

To prove $|t^s_{(a_n)}(\mathbb{T})| \geq |\mathbb{T}| = \mathfrak{c}$ we use two alternative arguments.

Let $B \in \mathbb{I}$. Define $x_B \in \mathbb{T}$ with $\mathrm{supp}(x_B) = B$ and $c_n = q_n - 1$ for all $n \in B$. According to Lemma 4.1, $x_B \in t^s_{(a_n)}(\mathbb{T})$. Since the map $\mathbb{I} \ni B \mapsto x_B \in t^s_{(a_n)}(\mathbb{T})$ is obviously injective, $|t^s_{(a_n)}(\mathbb{T})| = \mathfrak{c}$, by Lemma 3.3.

The second argument uses the fact that $|\mathcal{I}_d| = \mathfrak{c}$, in view of Proposition 3.3. This provides $\mathfrak{c}$ many elements $\{x_i : i \in I\}$ in $\mathbb{T}$ with distinct supports of density 0. By Theorem 4.4, $x_i \in t^s_{(a_n)}(\mathbb{T})$ for every $i \in I$. ∎

We can immediately infer the following results as corollaries. The first corollary follows from Theorem B.

Corollary 4.6. *Let (a_n) be an arithmetic sequence. If $t_{(a_n)}(\mathbb{T})$ is countable, then $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.*

Corollary 4.7. *Let (a_n) be an arithmetic sequence. If the sequence (q_n) is bounded, then $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$.*

Proof. This follows from Corollary 4.6, since we know that $|t_{(a_n)}(\mathbb{T})| < \mathfrak{c}$ when the sequence (q_n) of ratios is bounded. ∎

As a consequence we have the following example.

Example 4.8. For any prime p, we have $|t^s_{(p^n)}(\mathbb{T})| = \mathfrak{c}$, and consequently $t^s_{(p^n)}(\mathbb{T}) \neq t_{(p^n)}(\mathbb{T})$.

Corollaries 4.6 and 4.7 have much to say about the distinction between characterized subgroups and s-characterized subgroups.

We are now in a position to prove also Theorem C.

Proof of Theorem C. We have to prove that $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$ for any arithmetic sequence (a_n).

If (q_n) is bounded then $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$, by Corollary 4.7.

Assume that (q_n) is not bounded. Then there exists $B \subset \mathbb{N}$ such that $(q_n)_{n\in B}$ diverges to ∞. Passing possibly to a subset of B we can assume additionally that $d(B) = 0$. Take

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \in \mathbb{T} \quad \text{with } \operatorname{supp}(x) = B \text{ and } c_n = \left\lfloor \frac{q_n}{2} \right\rfloor \text{ for all } n \in B.$$

Then $x \in t^s_{(a_n)}(\mathbb{T})$ by Theorem 4.4, while $x \notin t_{(a_n)}(\mathbb{T})$ (by [22, Theorem 2.3]). This proves $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$. ■

5. Sufficient conditions for $x \notin t^s_{(a_n)}(\mathbb{T})$. Finally, we address the natural problem of finding, for an arithmetic sequence (a_n), specific elements of $\mathbb{T}$ which do not belong to $t^s_{(a_n)}(\mathbb{T})$. This seems difficult for general arithmetic sequences, but for two typical forms of arithmetic sequences we can provide some sufficient conditions.

In the next proposition we show that if the sequence (q_n) is bounded, then an element $x \in \mathbb{T}$ whose support has bounded lengths of intervals and gaps, cannot belong to $t^s_{(a_n)}(\mathbb{T})$.

PROPOSITION 5.1. *Let (a_n) be an arithmetic sequence with bounded sequence (q_n). Let $x \in \mathbb{T}$ be such that*

- $\operatorname{supp}(x) = \bigcup_{n=1}^{\infty} [b_n, d_n]$, $b_n, d_n \in \mathbb{N}$, $b_n \le d_n < b_{n+1}$ *for all* $n \in \mathbb{N}$;
- *there exist $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $|d_n - b_n| \le l$ and $|b_{n+1} - d_n| \le l$.*

Then $x \notin t^s_{(a_n)}(\mathbb{T})$.

Proof. Assume that $q_n \le M$ for all $n \in \mathbb{N}$. Furthermore, we can assume that $q_n \ge 2$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ consider the following two cases.

CASE 1: $n + 1 \in [b_k, d_k]$ for some k. Then clearly we have $\{a_n x\} \ge a_n \cdot \frac{1}{a_{n+1}} \ge 1/M$. On the other hand,

$$\begin{aligned}
\{a_n x\} &\le a_n\left(\frac{q_{n+1}-1}{a_{n+1}} + \frac{q_{n+2}-1}{a_{n+2}} + \cdots + \frac{q_{n+l}-1}{a_{n+l}}\right) \\
&\quad + a_n\left(\frac{q_{n+l+2}-1}{a_{n+l+2}} + \cdots + \frac{q_{n+2l+1}-1}{a_{n+2l+1}}\right) + \cdots \\
&= \left(1 - \frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+1}} - \frac{a_n}{a_{n+2}} + \cdots - \frac{a_n}{a_{n+l}}\right) \\
&\quad + a_n\left(\frac{1}{a_{n+l+1}} - \frac{1}{a_{n+l+2}} + \frac{1}{a_{n+l+2}} - \cdots - \frac{1}{a_{n+2l+1}}\right) + \cdots \\
&= \left(1 - \frac{a_n}{a_{n+l}}\right) + a_n\left(\frac{1}{a_{n+l+1}} - \frac{1}{a_{n+2l+1}} + \frac{1}{a_{n+2l+3}} - \frac{1}{a_{n+3l+2}} + \cdots\right)
\end{aligned}$$

$$= 1 - \frac{a_n}{a_{n+l}} + \frac{a_n}{a_{n+l+1}} - a_n\left(\frac{1}{a_{n+2l+1}} - \frac{1}{a_{n+2l+3}} + \frac{1}{a_{n+3l+2}} - \frac{1}{a_{n+3l+4}} + \cdots\right)$$
$$\le 1 - \frac{a_n}{a_{n+l}} + \frac{a_n}{a_{n+l+1}} = 1 - \frac{a_n}{a_{n+l}} + \frac{a_n}{a_{n+l}} \cdot \frac{a_{n+l}}{a_{n+l+1}}$$
$$\le 1 - \frac{a_n}{a_{n+l}} + \frac{1}{2} \cdot \frac{a_n}{a_{n+l}} = 1 - \frac{1}{2} \cdot \frac{a_n}{a_{n+l}} \le 1 - \frac{1}{2} \cdot \frac{1}{M^l}.$$

So we have $\frac{1}{M} \le \{a_n x\} \le 1 - \frac{1}{2} \cdot \frac{1}{M^l}$.

CASE 2: $n+1 \in [d_k, b_{k+1}]$ for some k. Then clearly $\{a_n x\} \ge a_n \cdot \frac{1}{a_{n+l+1}} \ge \frac{1}{M^{l+1}}$. On the other hand,

$$\{a_n x\} \le a_n\left(\frac{q_{n+2}-1}{a_{n+2}} + \frac{q_{n+3}-1}{a_{n+3}} + \cdots\right)$$
$$= a_n\left(\frac{1}{a_{n+1}} - \frac{1}{a_{n+2}} + \frac{1}{a_{n+2}} - \frac{1}{a_{n+3}} + \cdots\right) \le \frac{a_n}{a_{n+1}} \le \frac{1}{2}.$$

Thus $\frac{1}{M^{l+1}} \le \{a_n x\} \le \frac{1}{2}$.

So in both cases the sequence $(\|a_n x\|)$ is bounded from below (as well as from above), which implies that it cannot statistically converge to 0 (because a statistically convergent sequence must have a subsequence converging to the same limit), which implies that $x \notin t^s_{a_n}(\mathbb{T})$. ∎

With $a_n = 2^n$, $b_n = d_n = 2n$ and $l = 2$ we obtain $x = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{3} \notin t^s_{(2^n)}(\mathbb{T})$ in Example 3.7.

One can construct many $x \in \mathbb{T}$ lying outside $t^s_{(a_n)}(\mathbb{T})$ also when q_n diverges to ∞. The following proposition presents a sufficient condition for some $x \in \mathbb{T}$ not to be in $t^s_{(a_n)}(\mathbb{T})$. As a by-product, it allows us to invert Theorem 4.4 for arithmetic sequences (a_n) with (q_n) diverging to ∞. The main idea goes back to [26, Chap. 4a].

PROPOSITION 5.2. *Let (a_n) be an arithmetic sequence with (q_n) diverging to ∞. Let $x \in \mathbb{T}$ be such that $d(\mathrm{supp}(x)) > 0$. If there exist reals r_1, r_2 with $0 < r_1 \le r_2 < 1$ such that*

$$c_n/q_n \in [r_1, r_2] \quad \text{for all } n \in \mathrm{supp}(x),$$

where c_n is defined as in Lemma 1.5, then $x \notin t^s_{(a_n)}(\mathbb{T})$.

Proof. Let (1.2) be the canonical representation of x. Without loss of generality we assume that $1 \notin \mathrm{supp}(x)$. Let

$$A_1 = \{n-1 : n \in \mathrm{supp}(x)\}.$$

Then $d(A_1) > 0$ and for all $n \in A_1$ one has

$$\{a_n x\} = a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{c_k}{a_k} \geq a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{r_1 \cdot q_k}{a_k}$$
$$= a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{r_1}{a_{k-1}} \geq a_n \cdot \frac{r_1}{a_n} = r_1.$$

Now fix $0 < \varepsilon < \min\{r_2, 1 - r_2\}$. Then

$$u := \frac{1-\varepsilon}{1-r_2-\varepsilon} \in (1, \infty),$$

so

$$\frac{r_2 \cdot u}{u-1} < 1 - \varepsilon. \tag{5.1}$$

As $q_n \to \infty$, there exists $n_1 \in \mathbb{N}$ such that $q_n \geq u$ for all $n \geq n_1$. Let $A_2 = A_1 \cap [n_1, \infty)$. So, in view of $\frac{1}{q_n} \leq \frac{1}{u}$ for all $n \geq n_1$ and (5.1), for $n \in A_2$ we get

$$\{a_n x\} = a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{c_k}{a_k} \leq a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{r_2 \cdot q_k}{a_k} = a_n \cdot \sum_{\substack{k\in \mathrm{supp}(x)\\ k>n}} \frac{r_2}{a_{k-1}}$$
$$\leq r_2 \cdot \left(1 + \frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+2}} + \cdots\right) \leq r_2 \cdot \left(1 + \frac{1}{u} + \frac{1}{u^2} + \cdots\right)$$
$$= r_2 \cdot \frac{1}{1-\frac{1}{u}} = \frac{r_2 \cdot u}{u-1} < 1 - \varepsilon.$$

Note that $d(A_2) = d(A_1) > 0$ and for all $n \in A_2$, $\{a_n x\} \in [r_1, 1-\varepsilon]$, which implies that $\|a_n x\|$ cannot converge statistically to 0. Thus $x \notin t^s_{(a_n)}(\mathbb{T})$. ∎

Now we invert Theorem 4.4 for arithmetic sequence with (q_n) diverging to ∞ by proving that for some $B \subseteq \mathbb{N}$ all elements $x \in \mathbb{T}$ with support B belong to $t^s_{(a_n)}(\mathbb{T})$ precisely when $d(B) = 0$. This partially answers Problem 6.10 below.

Corollary 5.3. *Let (a_n) be an arithmetic sequence with (q_n) diverging to ∞. Then for a subset $B \subseteq \mathbb{N}$, there exists $x \in \mathbb{T}$ with $\mathrm{supp}_{(a_n)}(x) = B$ and $x \notin t^s_{(a_n)}(\mathbb{T})$ if and only if $d(B) > 0$.*

Proof. The conjunction of $x \notin t^s_{(a_n)}(\mathbb{T})$ and $\mathrm{supp}_{(a_n)}(x) = B$ implies $d(B) > 0$, by Theorem 4.4. On the other hand, if $d(B) > 0$, then Proposition 5.2 provides an element $x \in \mathbb{T}$ such that $\mathrm{supp}_{(a_n)}(x) = B$ and $x \notin t^s_{(a_n)}(\mathbb{T})$. ∎

6. Connection with Weyl's uniform distribution theorem and final comments

6.1. Weyl's uniform distribution theorem. Recall that a sequence (x_n) of real numbers is said to be *uniformly distributed modulo* 1 if for every

$[a, b] \subseteq [0, 1)$ one has

$$\lim_{n\to\infty} \frac{|\{j : 0 \le j < n,\ \{x_j\} \in [a, b]\}|}{n} = b - a$$

where $\{x_j\}$, as before, is the fractional part of x_j. In his celebrated results proved in 1916, H. Weyl [42] investigated the set

$$W_{\mathbf{u}} = \{x \in [0, 1] : (u_n x) \text{ is uniformly distributed modulo } 1\}$$

where $\mathbf{u} = (u_n) \in \mathbb{Z}^{\mathbb{N}}$. Note that for every $\alpha \in [0, 1] \setminus \mathbb{Q}$, we have $\alpha \notin W_{\mathbf{u}}$ for an appropriate choice of $\mathbf{u}$. Indeed, consider the convergents r_n/u_n of the continued fraction expansion of α; and as $\|u_n\alpha\|_{\mathbb{Z}} \to 0$ (where $\|\cdot\|_{\mathbb{Z}}$ is the distance from the integers), we conclude that $\alpha \notin W_{\mathbf{u}}$. In a really impressive observation, Larcher [35] proved in 1988 that if the continued fraction expansion of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is bounded then

$$\{\beta \in \mathbb{R} : \|u_n\beta\|_{\mathbb{Z}} \to 0\} = \langle\alpha\rangle + \mathbb{Z}, \tag{6.1}$$

the subgroup of $\mathbb{R}$ generated by α modulo 1. Instead of using the fractional part $\{x_j\}$ or working modulo 1, one can conveniently work in the circle group $\mathbb{R} \setminus \mathbb{Z} = \mathbb{T}$, as in the previous sections.

From the observations in Sections 3 and 4, we conclude that if (a_n) is an arithmetic sequence of natural numbers such that the sequence of ratios (q_n) is either bounded or diverges to ∞, then we get nontrivial statistically characterized subgroups $t^s_{(a_n)}(\mathbb{T})$ which are strictly smaller than the whole circle group $\mathbb{T}$ but at the same time (being uncountable) strictly larger than $t_{(a_n)}(\mathbb{T})$.

As for the connection of the new notion with Weyl's result we start by recalling the following.

Example 6.1. Let (a_n) be a strictly increasing sequence of naturals. According to a celebrated theorem of Weyl [42]:

(a) if there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that $a_n = P(n)$, then the set $W_{\mathbf{a}}$ coincides with $\mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$;
(b) in general the set $W_{(a_n)}$ has $\mu(W_{(a_n)}) = 1$.

It is obvious that $W_{(a_n)} \cap t_{(a_n)}(\mathbb{T}) = \emptyset$. Actually, one has the sharper equality

$$W_{(a_n)} \cap t^s_{(a_n)}(\mathbb{T}) = \emptyset. \tag{6.2}$$

Indeed, for $x \in W_{(a_n)}$ the sequence $(a_n x)$ is uniformly distributed in $\mathbb{T}$, hence $a_n x$ cannot statistically converge to 0.

Now (6.2), along with (a), implies that $t^s_{(a_n)}(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$ when $a_n = P(n)$. Actually, $1/k \in t^s_{(a_n)}(\mathbb{T})$ precisely when $\{n \in \mathbb{N} : k \nmid a_n\}$ has asymptotic density 0. In such a case $t^s_{(a_n)}(\mathbb{T})$ contains the cyclic subgroup generated by $1/k$.

Now we show that if $W_{\mathbf{a}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$, then $t_{\mathbf{a}}(\mathbb{T})$ is properly contained in $\mathbb{Q}/\mathbb{Z}$, i.e., there exists $r \in \mathbb{Q}/\mathbb{Z}$ for which $\lim_n \|a_n r\| \neq 0$.

THEOREM 6.2. *Let* $\mathbf{a} = (a_n)$ *be an arithmetic sequence for which* $W_{\mathbf{a}} = \mathbb{T}\setminus\mathbb{Q}/\mathbb{Z}$. *Then* $t_{\mathbf{a}}(\mathbb{T})$ *is properly contained in* $\mathbb{Q}/\mathbb{Z}$. *In particular,* $\mathbb{Q}/\mathbb{Z}$ *cannot be characterized by means of an arithmetic sequence.*

Proof. First of all, $W_{\mathbf{a}} = \mathbb{T}\setminus\mathbb{Q}/\mathbb{Z}$ implies that $t_{(a_n)}(\mathbb{T}) \leq t^s_{(a_n)}(\mathbb{T}) \leq \mathbb{Q}/\mathbb{Z}$. Therefore, from $t^s_{(a_n)}(\mathbb{T}) \neq t_{(a_n)}(\mathbb{T})$, granted by Theorem C, it follows that $t_{\mathbf{a}}(\mathbb{T})$ is properly contained in $\mathbb{Q}/\mathbb{Z}$. ∎

6.2. Final comments and questions. In conclusion we collect some open questions and problems.

QUESTION 6.3. *Do Theorems B and C hold true for arbitrary sequences* (a_n)*?*

A notoriously non-arithmetic sequence is the Fibonacci sequence (f_n), defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. It is known that $t_{(f_n)}(\mathbb{T})$ is infinite cyclic, generated by the Golden ratio [35, 18, 7, 3] (see also (6.1) above).

QUESTION 6.4. *Compute* $t^s_{(f_n)}(\mathbb{T})$. *Is it countable? Is it distinct from* $t_{(f_n)}(\mathbb{T})$*?*

The Fibonacci sequence is recursive. It obviously satisfies the condition $f_{n-1} \mid f_n - f_{n-2}$, so one can consider the most general sequences a_n with this property, which means that

$$a_n = b_{n-1}a_{n-1} + a_{n-2} \tag{6.3}$$

for some sequence (b_n) of naturals. One can consider an even more complicated recursion:

$$a_n = b^{(1)}_{n-1}a_{n-1} + b^{(2)}_{n-1}a_{n-2} + \cdots + b^{(k)}_{n-1}a_{n-k} \tag{6.4}$$

for some k-tuple of sequences $(b^{(j)}_n)$ $(j = 1, \ldots, k)$ of naturals (see [5] for topological torsion related to such sequences). In particular, one may extend the above question to recursive sequences of integers satisfying (6.3) and (6.4):

QUESTION 6.5. *Let* (a_n) *be a recursive sequence as in* (6.3) *or* (6.4). *Compute* $t^s_{(f_n)}(\mathbb{T})$. *When is it countable? Is it distinct from* $t_{(f_n)}(\mathbb{T})$*?*

Another relevant instance of non-arithmetic sequence is the sequence (b_n) defined in [24] (see also [7]) as follows:

$$1, 2, 4, 6, 12, 18, 24, \ldots, n!, 2\cdot n!, 3\cdot n!, \ldots, n\cdot n!, (n+1)!, \ldots.$$

It was proved in [24] that $t_{(b_n)}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$ (recall that $\mathbb{Q}/\mathbb{Z}$ cannot be characterized by means of an arithmetic sequence, in view of Theorem 6.2).

QUESTION 6.6. *Compute* $t^s_{(b_n)}(\mathbb{T})$. *Is it countable? Is it distinct from* $\mathbb{Q}/\mathbb{Z}$*?*

According to Theorem B, $t^s_{(a_n)}(\mathbb{T})$ is always uncountable for an arithmetic sequence (a_n). On the other hand, Example 6.1, along with (6.2), implies that $t^s_{(a_n)}(\mathbb{T})$ is countable (actually, contained in $\mathbb{Q}/\mathbb{Z}$) when there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that $a_n = P(n)$ for almost all n.

QUESTION 6.7. *Are there other instances of countable subgroups $t^s_{(a_n)}(\mathbb{T})$ beyond those described above?*

REMARK 6.8. Here we discuss the application of characterized subgroups to the problem of building group topologies with, and without, convergent sequences, faced by many authors, e.g. Comfort, Raczkowski and Trigos-Arrieta [15]. In the special case of the group $\mathbb{Z}$ one can use (as done in [4, 15]) group topologies of $\mathbb{Z}$ generated by subgroups H of $\mathbb{T}$ as follows: One takes the initial topology T_H on $\mathbb{Z}$ of all characters $\chi_h : \mathbb{Z} \to \mathbb{T}$ defined by $\chi_h(n) = nh$, for $h \in H, n \in Z$. Then T_H is Hausdorff if and only if H is infinite (i.e., dense in $\mathbb{T}$), while T_H is metrizable if and only if H is countably infinite. A sequence (a_n) converges to 0 in $(\mathbb{Z}, T_H)$ precisely when H is contained in $t_{(a_n)}(\mathbb{T})$. Moreover, every precompact topology on $\mathbb{Z}$ making (a_n) a null sequence has this form. Hence, $T_{t_{(a_n)}(\mathbb{T})}$ is the finest precompact topology on $\mathbb{Z}$ making (a_n) a null sequence.

As an open problem we suggest studying the "statistical counterpart" of this problem. Namely, is it true that when a countably infinite subgroup H is contained in $t^s_{(a_n)}(\mathbb{T})$ then the sequence (a_n) *statistically converges* to 0 in $(\mathbb{Z}, T_H)$?

Lemma 4.1 and Theorem 4.4 give sufficient conditions for some $x \in \mathbb{T}$ to be in $t^s_{(a_n)}(\mathbb{T})$ only in terms of $\mathrm{supp}(x)$. On the other hand, Propositions 5.1 and 5.2 present a sufficient condition for $x \notin t^s_{(a_n)}(\mathbb{T})$ (the latter one involves also the coefficients c_n of the canonical representation of x, beyond $\mathrm{supp}(x)$). This motivates the natural question whether a necessary and sufficient condition can be found for $x \in t^s_{(a_n)}(\mathbb{T})$. We prefer to formulate it first for the sequence (2^n):

PROBLEM 6.9. *Describe explicitly the subgroup $t^s_{(2^n)}(\mathbb{T})$ of $\mathbb{T}$, or more generally, $t^s_{(p^n)}(\mathbb{T})$, for a prime p.*

Our choice to isolate the case of the sequence (2^n) is determined by the fact that in this case an element $x \in \mathbb{T}$ is completely determined by its support so the already available results, mentioned above, can be of some help and the characterization will certainly be in terms of the support.

When (a_n) does not (definitely) coincide with (2^n), it may occur that among two elements x and y with the same support, one may have $x \in t^s_{(a_n)}(\mathbb{T})$, while $y \notin t^s_{(a_n)}(\mathbb{T})$, depending on their coefficients (c_n). This is why we expect that the following problem may have a *negative* solution:

Problem 6.10. *Let (a_n) be an arithmetic sequence such that $q_n > 2$ for infinitely many n. Does there exist a characterization of the elements of the subgroup $t^s_{(a_n)}(\mathbb{T})$ only in terms of the support?*

Finally, one is left with the general problem of exploring statistically characterized subgroups of (compact-like) topological abelian groups:

Problem 6.11. *Study the statistically characterized subgroups of compact metrizable abelian groups, following the standard way already used for the characterized subgroups in* [25, 24, 20, 23, 32]. *Are they Borel sets?*

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