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Conley index continuation for some classes of RFDEs on manifolds

by

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Abstract. We establish Conley index continuation results for manifold-valued retarded functional differential equations which are close to ordinary differential equations.

1. Introduction. To understand the problem addressed in this paper, let us consider the following delay differential equation:

$$(E_\varepsilon) \qquad \dot{x}(t) = g(x(t)) + \varepsilon h(x(t), x(t-r)).$$

Here, $g\colon \mathbb{R}^m \to \mathbb{R}^m$ and $h\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, for some $m \in \mathbb{N}$, are appropriate functions, $r > 0$ is a delay and $\varepsilon > 0$ is a small parameter. The delay equation (E_ε) is a special case of a retarded functional differential equation (RFDE) and therefore, by well known results, it defines a (one-sided) local semiflow π_ε on the infinite-dimensional phase space $C([-r,0],\mathbb{R}^m)$. We want to study π_ε from the point of view of Conley index theory.

As $\varepsilon \to 0$, equation (E_ε) converges to the following limit equation:

$$(E) \qquad \dot{x}(t) = g(x(t)).$$

Equation (E) is a (delay-free) RFDE too, so it again defines a local semiflow on the phase space $C([-r,0],\mathbb{R}^m)$. However, it is more convenient to view (E) as an ordinary differential equation (ODE), since then equation (E) defines a (two-sided) local flow π on the *finite-dimensional* phase space $\mathbb{R}^m$ and so Conley indices may be easier to compute than in the infinite-dimensional space $C([-r,0],\mathbb{R}^m)$.

In this situation we expect a Conley index continuation result to hold: compact isolated invariant sets K of the *finite-dimensional* local flow π 'continue' to compact isolated invariant sets K_ε of the *infinite-dimensional* local

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semiflow π_ε with the same Conley index. In particular, if the index of K is nonzero, then the index of K_ε is also nonzero, so $K_\varepsilon \neq \emptyset$ and thus there is a full bounded solution of the RFDE (E_ε), a result that may not be easy to obtain directly.

Moreover, we also expect continuation results for Morse decompositions and the so-called homology index braids to hold. Then from the knowledge of homology index braids of π we may deduce the existence and multiplicity of full bounded solutions of equation (E_ε) with some special properties.

In order to give a more precise meaning to the term 'continue', we must first define an appropriate embedding Θ from the phase space $X = \mathbb{R}^m$ of the flow π to the phase space $C([-r, 0], X)$ of the semiflows π_ε. Then, given an isolating neighborhood N of the set K relative to π, we look for isolating neighborhoods of K_ε relative to π_ε in the form of thin closed tubes $[N]^\eta$ (η being the thickness of the tube) around the set $\Theta(N)$. We then conjecture that, for small ε, the Conley index $h(\pi, N)$ is equal to the Conley index $h(\pi_\varepsilon, [N]^\eta)$, and similarly for homology index braids.

Let us remark that the natural embedding which assigns to each point $u \in X$ the constant function $\varphi \equiv u$ does not work.

An embedding which does work in our case is given by the assignment $u \mapsto \varphi$ with $\varphi(s) = u\pi s$ for $s \in [-r, 0]$. Using this embedding we can state and prove the above mentioned conjectures for general retarded functional differential equations which are 'close' to ordinary differential equations. Moreover, the continuation results obtained (see Theorems 3.3 and 3.6 below) hold not only for the flat case $X = \mathbb{R}^m$, but also for quite general curved phase spaces, namely, for $X = \mathcal{M}$, where $\mathcal{M}$ is an arbitrary finite-dimensional smooth Riemannian manifold. (For simplicity of presentation, we assume that $\mathcal{M}$ is a differentiable submanifold of $\mathbb{R}^m$, but the given Riemannian structure on $\mathcal{M}$ can be arbitrary.)

Our main results, Theorem 3.3, Proposition 3.4 and Theorem 3.6, are stated in Section 3 and proved in Section 4. Theorem 3.3 and Proposition 3.4 can be used to establish multiplicity results for full bounded solutions of some manifold-valued RFDEs (cf. Example 3.5). The proofs of Theorems 3.3 and 3.6 are accomplished by an application of some abstract tubular continuation principles established in the recent papers [4, 5, 6].

2. Preliminaries. In this section we collect some background material on differentiable manifolds and both ordinary and retarded functional differential equations. For general comprehensive treatment of RFDEs on $\mathbb{R}^m$, see the book [9]. The theory of RFDEs on manifolds was initiated in [11]. Further developments and references are contained in the book [8]. The book [10] is an early monograph on the global theory of RFDEs on manifolds. Some recent results on RFDEs on manifolds are contained in the papers [1, 2].

Let us now explain some notation used in this paper.

First of all, whenever (X, d_X) is a metric space and S is a subset of X, we denote by the same symbol d_X the restriction of d_X to $S \times S$.

Throughout this paper, we fix numbers $r \in [0, \infty[$ and k, $m \in \mathbb{N}$ with $m \geq k$. We define $I = [-r, 0]$ and we fix an m-dimensional Euclidean space E with the scalar product $\langle \cdot, \cdot \rangle_E$. Let $|\cdot|_E$, $|x|_E = \langle x, x \rangle^{1/2}$, be the corresponding Euclidean norm and d_E, $d_E(x, y) = |x - y|_E$, the corresponding metric.

If S is any set, $t \in \mathbb{R}$ and $x \colon J \to S$ is any map, where $[-r + t, t] \subset J \subset \mathbb{R}$, then $x_t \colon I \to S$ is the function defined as $x_t(s) = x(t + s)$ for $s \in I$.

If Y is a topological space, then $C(I, Y)$ will, as usual, denote the set of all continuous functions from I to Y. The set $C(I, Y)$ is endowed with the compact-open topology. If the topology of Y is induced by a metric d on Y, then the compact-open topology on $C(I, Y)$ is induced by the metric $d^\diamond$ on $C(I, Y)$ given by

$$d^\diamond(\varphi, \psi) = \sup_{s \in I} d(\varphi(s), \psi(s)), \qquad \varphi, \psi \in C(I, Y).$$

If Y is a linear space and the metric d is induced by a norm $|\cdot|$ on Y, then the metric $d^\diamond$ is induced by the norm $|\cdot|^\diamond$ on $C(I, Y)$ given by

$$|\varphi|^\diamond = \sup_{s \in I} |\varphi(s)|, \qquad \varphi \in C(I, Y).$$

Now let W be an *arbitrary* subset of E, $g \colon W \to E$ be an *arbitrary* function and $A_1, A_2 \in]0, \infty]$ be *arbitrary*. As usual, *a solution of the ordinary differential equation*

$$\dot{y} = g(y) \tag{2.1}$$

on $]-A_1, A_2[$ is a differentiable function $x \colon]-A_1, A_2[\to E$ such that $x(t) \in W$ and $x'(t) = g(x(t))$ for all $t \in]-A_1, A_2[$.

Let Ω be an *arbitrary* subset of $C(I, E)$, $f \colon \Omega \to E$ be an *arbitrary* function and $A \in]0, \infty]$ be *arbitrary*. Consider the following *retarded functional differential equation*:

$$\dot{y} = f(y_t). \tag{2.2}$$

A *solution* of (2.2) on $[0, A[$ is a continuous function $x \colon [-r, A[\to E$ such that $x|_{[0,A[}$ is differentiable, $x_t \in \Omega$ and $(x|_{[0,A[})'(t) = f(x_t)$ for all $t \in [0, A[$. If $\varphi \in \Omega$ and $x_0 = \varphi$ (which just means that $\varphi = x|_I$), then we say that x is a solution *through* φ.

Remark 2.1. Given an open set Ω in $C(I, E)$ and a map $h \colon \Omega \to E$, two conditions are used in the literature to guarantee the existence and uniqueness of solutions of the RFDE $\dot{y} = h(y_t)$ through an initial segment: local Lipschitzianity of h and Lipschitzianity of h on compact subsets of Ω. While the former condition is generally perceived as stronger, in fact they are equivalent. This is actually true in general metric spaces. In fact, the following result is easily proved:

Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ be a map. The following conditions are equivalent:

(a) *$f\colon (X, d_X) \to (Y, d_Y)$ is locally Lipschitzian, i.e. every point $a \in X$ has a neighborhood U_a in (X, d_X) such that the restriction $f|_{U_a}\colon (U_a, d_X) \to (Y, d_Y)$ is L-Lipschitzian for some $L \in [0, \infty[$.*

(b) *$f\colon (X, d_X) \to (Y, d_Y)$ is Lipschitzian on compact subsets of X, i.e. for each compact set K in (X, d_X), the restriction $f|_K\colon (K, d_X) \to (Y, d_Y)$ is L-Lipschitzian for some $L \in [0, \infty[$.*

We will tacitly use this remark in the proofs below while referring to known results on RFDEs.

Throughout this paper we fix a smooth k-dimensional submanifold $\mathcal{M}$ of E. The manifold topology $\mathbb{S}_{\mathcal{M}}$ of $\mathcal{M}$ is just the relative topology of E on $\mathcal{M}$, so the topology $\mathbb{S}_{\mathcal{M}}$ is induced by the metric d_E.

As is well known, there are various definitions of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$, all leading to isomorphic vector bundles. Here we define $T\mathcal{M} = \bigcup_{u\in\mathcal{M}}(\{u\} \times T_u\mathcal{M})$, where $T_u\mathcal{M}$ is the set of all vectors $v \in E$ for which there is an $\varepsilon \in {]0, \infty[}$ and a smooth map $\gamma\colon {]{-\varepsilon}, \varepsilon[} \to E$ with $\gamma(]{-\varepsilon}, \varepsilon[) \subset \mathcal{M}$, $\gamma(0) = u$ and $\gamma'(0) = v$.

Remark 2.2. (1) If $\mathcal{M} = E$ or, more generally, if $\mathcal{M}$ is open in E then $\mathcal{M}$ is a smooth submanifold of E with $k = m$. In this case $T_u\mathcal{M} = E$ for all $u \in \mathcal{M}$.

(2) If $m \geq 1$ and $\mathcal{M} = \mathbb{S}^k = \{\, u \in E \mid \langle u, u\rangle_E = 1 \,\}$ then $\mathcal{M}$ is a smooth submanifold of E with $k = m - 1$. In this case, $T_u\mathcal{M} = T_u\mathbb{S}^k = \{\, v \in E \mid \langle u, v\rangle_E = 0 \,\}$ for all $u \in \mathcal{M}$.

Finally, we assume in this paper that $\mathcal{M}$ is connected relative to $\mathbb{S}_{\mathcal{M}}$ and a smooth family $\mathfrak{g} = (\mathfrak{g}_u\colon T_u\mathcal{M} \times T_u\mathcal{M} \to \mathbb{R})_{u\in\mathcal{M}}$ of scalar products is given such that $(\mathcal{M}, \mathfrak{g})$ is a Riemannian manifold. We do not assume that $\mathfrak{g}$ is related in any way to the Euclidean scalar product $\langle\cdot,\cdot\rangle_E$ introduced above.

Given a smooth curve $\gamma\colon [a, b] \to \mathcal{M}$, the *length* $\ell_{\mathfrak{g}}(\gamma)$ of γ is defined as

$$\ell_{\mathfrak{g}}(\gamma) := \int_a^b \mathfrak{g}_{\gamma(\tau)}(\gamma'(\tau), \gamma'(\tau))^{1/2}\, d\tau$$

with the obvious extension to piecewise smooth (p.s.) curves. Now the *Riemann distance* $d_{\mathfrak{g}}\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is defined, for $u, v \in \mathcal{M}$, by

$$d_{\mathfrak{g}}(u, v) = \inf\{\ell_{\mathfrak{g}}(\gamma) \mid \gamma\colon [0, 1] \to \mathcal{M} \text{ is p.s., } \gamma(0) = u \text{ and } \gamma(1) = v\}.$$

It is known (cf. [7]) that $d_{\mathfrak{g}}$ is defined (as $\mathcal{M}$ is connected relative to $\mathbb{S}_{\mathcal{M}}$)

and $d_{\mathfrak{g}}$ is a metric on $\mathcal{M}$. The metrics $d_{\mathfrak{g}}$ and d_E are *equivalent on* $\mathcal{M}$, i.e. they define the same topology on $\mathcal{M}$ (namely, $\mathbb{S}_{\mathcal{M}}$).

The tubular neighborhood theorem implies the existence of an open set U in E with $\mathcal{M} \subset U$ and a smooth map $\rho\colon U \to \mathcal{M}$ with

$$\rho(u) = u, \qquad u \in \mathcal{M}, \tag{2.3}$$

and

$$\langle x - \rho(x), T_{\rho(x)}\mathcal{M}\rangle_E = 0, \qquad x \in U. \tag{2.4}$$

Note that the continuity of the map $\rho\colon U \to \mathcal{M}$ implies that the Nemytskiĭ operator

$$\widehat{\rho}\colon C(I,U) \to C(I,\mathcal{M}), \qquad \widehat{\rho}(\varphi) = \rho \circ \varphi, \text{ is continuous.} \tag{2.5}$$

In the next proposition $f\colon C(I,\mathcal{M}) \subset C(I,E) \to E$ is an arbitrary function satisfying the tangency condition

$$f(\varphi) \in T_{\varphi(0)}\mathcal{M} \quad \text{for all } \varphi \in C(I,\mathcal{M}). \tag{2.6}$$

The *ρ-extension* of f is the map $\widetilde{f}\colon C(I,U) \subset C(I,E) \to E$ defined by $\widetilde{f}(\widetilde{\varphi}) = f(\rho \circ \widetilde{\varphi})$ for $\widetilde{\varphi} \in C(I,U)$. (This simply means that $\widetilde{f} = f \circ \widehat{\rho}$.)

We have the following known result.

Proposition 2.3 (cf. [1, 11]). *Consider the RFDEs*

$$\dot{y}(t) = f(y_t) \tag{2.7}$$

and

$$\dot{y}(t) = \widetilde{f}(y_t). \tag{2.8}$$

The following hold:

(a) *$\widetilde{f}$ is an extension of f.*
(b) *If $A \in\,]0,\infty]$ and $x\colon [-r,A[\, \to E$ with $x([-r,A[) \subset \mathcal{M}$ is a solution of (2.7) then x is a solution of (2.8).*
(c) *Conversely, if $A \in\,]0,\infty]$ and $\widetilde{x}\colon [-r,A[\, \to E$ is a solution of (2.8) and $\widetilde{x}([-r,0]) \subset \mathcal{M}$, then $\widetilde{x}([-r,A[) \subset \mathcal{M}$ and $\widetilde{x}$ is a solution of (2.7).*
(d) *The map $f\colon (C(I,\mathcal{M}), d_E^{\diamond}) \to (E, d_E)$ is Lipschitzian on compact subsets of $C(I,\mathcal{M})$ if and only if the map $\widetilde{f}\colon (C(I,U), d_E^{\diamond}) \to (E, d_E)$ is Lipschitzian on compact subsets of $C(I,U)$.*

In this case, through each $\widetilde{\varphi} \in C(I,U)$ there is a unique solution

$$x(\widetilde{\varphi},\widetilde{f})\colon [-r, A(\widetilde{\varphi},\widetilde{f})[\, \to E, \qquad t \mapsto x(\widetilde{\varphi},\widetilde{f})(t),$$

of (2.8) defined on a maximal interval $[-r, A(\widetilde{\varphi},\widetilde{f})[$ with $A(\widetilde{\varphi},\widetilde{f}) \in\,]0,\infty]$, and through each $\varphi \in C(I,\mathcal{M})$ there is a unique solution

$$x(\varphi,f)\colon [-r, A(\varphi,f)[\, \to E, \qquad t \mapsto x(\varphi,f)(t),$$

of (2.7) defined on a maximal interval $[-r, A(\varphi,f)[$ with $A(\varphi,f) \in\,]0,\infty]$. We have $A(\varphi,f) = A(\varphi,\widetilde{f})$ and $x(\varphi,f) = x(\varphi,\widetilde{f})$ for $\varphi \in C(I,\mathcal{M})$.

Defining $\widetilde{\varphi}\pi_{\widetilde{f}}t=(x(\widetilde{\varphi},\widetilde{f}))_t$ *for* $t\in[0,A(\widetilde{\varphi},\widetilde{f})[$ *(resp.* $\varphi\pi_f t=(x(\varphi,f))_t$ *for* $t\in[0,A(\varphi,f)[$*) we obtain a local semiflow* $\pi_{\widetilde{f}}$ *on* $C(I,U)$ *(resp. a local semiflow* π_f *on* $C(I,\mathcal{M})$*).*

Whenever $\varphi\in C(I,\mathcal{M})$*, then* $\varphi\pi_f t$ *is defined if and only if* $\varphi\pi_{\widetilde{f}}t$ *is defined, and then* $\varphi\pi_f t=\varphi\pi_{\widetilde{f}}t$*.*

Proof. Part (a) follows from (2.3). Part (b) follows from (a).

For (c), we use an argument from [1]. If $\widetilde{x}$ is as in (c), define $x=\rho\circ\widetilde{x}$. Consider the function $\sigma(t)=(1/2)|\widetilde{x}(t)-x(t)|_E^2$, $t\in[0,A[$. Then for $t\in[0,A[$, $\sigma'(t)$ exists and

$$\sigma'(t)=\langle\widetilde{x}(t)-x(t),f(\rho\circ\widetilde{x}_t)-x'(t)\rangle_E=\langle\widetilde{x}(t)-x(t),f(x_t)-x'(t)\rangle_E.$$

Now, by our assumption on f, we have $f(x_t)\in T_{x(t)}\mathcal{M}$, and as $x([0,A[)\subset\mathcal{M}$, we have $x'(t)\in T_{x(t)}\mathcal{M}$. Using (2.4) we obtain $\sigma'(t)\equiv 0$, and since $\widetilde{x}([-r,0])\subset\mathcal{M}$, we have $\sigma(0)=0$, so $\sigma(t)\equiv 0$. Thus $\widetilde{x}([0,A[)\subset\mathcal{M}$ and so $\widetilde{x}([-r,A[)\subset\mathcal{M}$, which means that $\widetilde{f}(\widetilde{x}_t)=f(\widetilde{x}_t)$ for all $t\in[0,A[$. Thus $\widetilde{x}$ is a solution of (2.7). This proves (c).

To prove the equivalence in (d), assume first that $f\colon(C(I,\mathcal{M}),d_E^\diamond)\to(E,d_E)$ is Lipschitzian on compact subsets of $C(I,\mathcal{M})$ and let $\widetilde{S}$ be an arbitrary compact subset of $C(I,U)$. Then, by (2.5), $S:=\widehat{\rho}(\widetilde{S})$ is compact in $C(I,\mathcal{M})$, so by our assumption on f, there is an $L_1\in[0,\infty[$ such that

$$|f(\varphi_1)-f(\varphi_2)|_E\le L_1\sup_{s\in I}|\varphi_1(s)-\varphi_2(s)|_E,\quad \varphi_1,\varphi_2\in S. \tag{2.9}$$

Let $\widetilde{K}$ be the set of all points $\widetilde{\varphi}(s)$, where $\widetilde{\varphi}\in\widetilde{S}$ and $s\in I$. Then $\widetilde{K}$ is compact in E and $\widetilde{K}\subset U$. Thus the smoothness of ρ as a map from U to E implies that there is an $L_2\in[0,\infty[$ such that $|\rho(x_1)-\rho(x_2)|_E\le L_2|x_1-x_2|_E$ for $x_1,x_2\in\widetilde{K}$. Thus, for arbitrary $\widetilde{\varphi}_1,\widetilde{\varphi}_2\in\widetilde{S}$ we have

$$\begin{aligned}|\widetilde{f}(\widetilde{\varphi}_1)-\widetilde{f}(\widetilde{\varphi}_2)|_E&=|f(\rho\circ\widetilde{\varphi}_1)-f(\rho\circ\widetilde{\varphi}_2)|_E\le L_1\sup_{s\in I}|\rho(\widetilde{\varphi}_1(s))-\rho(\widetilde{\varphi}_2(s)|_E\\&\le L_1L_2\sup_{s\in I}|\widetilde{\varphi}_1(s)-\widetilde{\varphi}_2(s)|_E.\end{aligned}$$

Hence $\widetilde{f}\colon(C(I,U),d_E^\diamond)\to(E,d_E)$ is Lipschitzian on compact subsets of $C(I,U)$. The converse statement is obvious.

Now, if $\widetilde{f}\colon(C(I,U),d_E^\diamond)\to(E,d_E)$ is Lipschitzian on compact subsets of $C(I,U)$, then by classical results (cf. [9]) since $\Omega:=C(I,U)$ is open in $C(I,E)$, through every $\widetilde{\varphi}\in\Omega$ there is a unique solution

$$x(\widetilde{\varphi},\widetilde{f})\colon[-r,A(\widetilde{\varphi},\widetilde{f})[\to E,\quad t\mapsto x(\widetilde{\varphi},\widetilde{f})(t), \tag{2.10}$$

of (2.8) defined on a maximal interval $[-r,A(\widetilde{\varphi},\widetilde{f})[$, with $A(\widetilde{\varphi},\widetilde{f})\in]0,\infty]$. Defining $\widetilde{\varphi}\pi_{\widetilde{f}}t=(x(\widetilde{\varphi},\widetilde{f}))_t$ for $t\in[0,A(\widetilde{\varphi},\widetilde{f})[$ we obtain a local semiflow $\pi_{\widetilde{f}}$ on Ω. Now (b) and (c) complete the proof of (d). ∎

In the next proposition, whose proof is analogous to that of Proposition 2.3, let $g\colon \mathcal{M} \subset E \to E$ be an arbitrary function satisfying the tangency condition

$$g(u) \in T_u\mathcal{M} \quad \text{for all } u \in \mathcal{M}. \tag{2.11}$$

The *ρ-extension* of g is the map $\widetilde{g}\colon U \subset E \to E$ defined by $\widetilde{g}(\widetilde{u}) = g(\rho(\widetilde{u}))$ for $\widetilde{u} \in U$. (This simply means that $\widetilde{g} = g \circ \rho$.)

PROPOSITION 2.4. *Consider the ODEs*

$$\dot{y}(t) = g(y) \tag{2.12}$$

and

$$\dot{y}(t) = \widetilde{g}(y). \tag{2.13}$$

The following properties hold:

(a) $\widetilde{g}$ *is an extension of* g.
(b) *If* $A_1, A_2 \in \,]0,\infty]$ *and* $x\colon\,]-A_1, A_2[\, \to E$ *with* $x(]-A_1, A_2[) \subset \mathcal{M}$ *is a solution of* (2.12) *then* x *is a solution of* (2.13).
(c) *Conversely, if* $A_1, A_2 \in \,]0,\infty]$ *and* $\widetilde{x}\colon\,]-A_1, A_2[\, \to E$ *is a solution of* (2.13) *and* $\widetilde{x}(0) \in \mathcal{M}$, *then* $\widetilde{x}(]-A_1, A_2[) \subset \mathcal{M}$ *and* $\widetilde{x}$ *is a solution of* (2.12).
(d) *The map* $g\colon (\mathcal{M}, d_E) \to (E, d_E)$ *is Lipschitzian on compact subsets of* $\mathcal{M}$ *if and only if the map* $\widetilde{g}\colon (U, d_E) \to (E, d_E)$ *is Lipschitzian on compact subsets of* U.

In this case, through each $\widetilde{u} \in U$ *there is a unique solution*

$$x(\widetilde{u}, \widetilde{g})\colon\,]-A_1(\widetilde{u}, \widetilde{g}), A_2(\widetilde{u}, \widetilde{g})[\, \to E, \qquad t \mapsto x(\widetilde{u}, \widetilde{g})(t),$$

of (2.13) *defined on a maximal interval* $]-A_1(\widetilde{u}, \widetilde{g}), A_2(\widetilde{u}, \widetilde{g})[$ *with* $A_1(\widetilde{u}, \widetilde{g})$ *and* $A_2(\widetilde{u}, \widetilde{g})$ *in* $]0,\infty]$, *and through each* $u \in \mathcal{M}$ *there is a unique solution*

$$x(u, g)\colon\,]-A_1(u, g), A_2(u, g)[\, \to E, \qquad t \mapsto x(u, g)(t),$$

of (2.12) *defined on a maximal interval* $]-A_1(u, g), A_2(u, g)[$ *with* $A_1(u, g)$ *and* $A_2(u, g)$ *in* $]0,\infty]$.

We have $A_1(u, g) = A_1(u, \widetilde{g}), A_2(u, g) = A_2(u, \widetilde{g})$ *and* $x(u, g) = x(u, \widetilde{g})$ *for* $u \in \mathcal{M}$.

Defining $\widetilde{u}\pi_{\widetilde{g}}t = x(\widetilde{u}, \widetilde{g})(t)$ *for* $t \in \,]-A_1(\widetilde{u}, \widetilde{g}), A_2(\widetilde{u}, \widetilde{g})[$ (*resp.* $u\pi_g t = x(u, g)(t)$ *for* $t \in \,]-A_1(u, g), A_2(u, g)[$) *we obtain a local flow* $\pi_{\widetilde{g}}$ *on* U (*resp. a local flow* π_g *on* $\mathcal{M}$).

Whenever $u \in \mathcal{M}$, *then* $u\pi_g t$ *is defined if and only if* $u\pi_{\widetilde{g}}t$ *is defined, and then* $u\pi_g t = u\pi_{\widetilde{g}}t$. ■

Let us finally recall a few definitions from Conley index theory (cf. [12]). Let (Y, d_Y) be a metric space, B be a subset of Y, Π be a local semiflow on Y and $(\Pi_n)_n$ be a sequence of local semiflows on Y.

If $J \subset \mathbb{R}$ is an interval and $\sigma\colon J \to Y$, then we say that σ is *a solution of* Π if for all $t \in J$ and $s \in [0,\infty[$ with $t+s \in J$ it follows that $\sigma(t)\Pi s$ is defined and equals $\sigma(t+s)$.

We say that Π *does not explode in* B if whenever $y \in Y$ is such that $y\Pi t \in B$ for all $t \in [0,\infty[$ for which $y\Pi t$ is defined, it follows that $y\Pi t$ is defined for all $t \in [0,\infty[$. We say that B *is* $(\Pi_n)_n$*-admissible* if whenever $(y_n)_n$ is a sequence in Y, $(t_n)_n$ is a sequence in $[0,\infty[$ with $t_n \to \infty$ as $n \to \infty$ and for each $n \in \mathbb{N}$, $y_n\Pi_n t_n$ is defined and $y_n\Pi_n[0,t_n] \subset B$, then the endpoint sequence $(y_n\Pi_n t_n)_n$ has a subsequence converging in (Y, d_Y) to an element of B. If $\Pi_n \equiv \Pi$ and B is $(\Pi_n)_n$-admissible, then we say that B is Π*-admissible*. If B is Π-admissible and Π does not explode in B, then we say that B *is strongly* Π*-admissible*.

In this paper we freely use the notation introduced in [4]. In particular, if B is a strongly Π-admissible isolating neighborhood of an isolated Π-invariant set S, then we denote the Conley index of the pair (Π, S) by $h(\Pi, S)$ and sometimes by $h(\Pi, B)$.

Remark 2.5. Let Π_1 and Π_2 be local semiflows on Y and let $\widetilde{Y}$ be open in Y. We say that Π_1 and Π_2 *agree* on $\widetilde{Y}$ if they have the same solutions lying in $\widetilde{Y}$, i.e. if for every interval $J \subset \mathbb{R}$ and every function $\sigma\colon J \to \widetilde{Y}$, σ is a solution of Π_1 if and only if σ is a solution of Π_2.

The constructions of Conley index theory are 'local' to arbitrary neighborhoods of invariant sets. In particular, if Π_1 and Π_2 agree on the open set $\widetilde{Y}$, then a subset S of $\widetilde{Y}$ is invariant relative to Π_1 if and only if S is invariant relative to Π_2. Moreover, given a partially ordered set $(P, \prec)$, a family $(S_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to Π_1 if and only if $(S_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to Π_2. Furthermore, the Conley index $h(\Pi_1, S)$ is defined if and only if the Conley index $h(\Pi_2, S)$ is defined, and then these Conley indices are the same. Finally, the homology index braid of $(\Pi_1, S, (S_p)_{p\in P})$ is defined if and only if the homology index braid of $(\Pi_2, S, (S_p)_{p\in P})$ is defined, and then these homology index braids are the same.

Now let Π be a local flow on the manifold $\mathcal{M}$. Then we denote by U_π the set of all $u \in \mathcal{M}$ such that $u\pi(-r)$ is defined. Moreover, we define the maps

$$\Theta\colon U_\pi \to C(I, \mathcal{M}), \qquad u \mapsto \varphi, \qquad \text{where } \varphi(s) = u\pi s \text{ for all } s \in I, \tag{2.14}$$

and

$$p\colon C(I, \mathcal{M}) \to \mathcal{M}, \qquad p(\varphi) = \varphi(0). \tag{2.15}$$

We also require the following definition:

DEFINITION 2.6. Let V be an arbitrary subset of U_π. For each $\beta \in]0,\infty[$, we define the *closed β-tube* $[V]^\beta$ in $C(I,\mathcal{M})$ as the closure in $C(I,\mathcal{M})$ of the set of all points φ in $C(I,\mathcal{M})$ such that

$$\varphi(0) \in V \quad \text{and} \quad \sup_{s\in I} d_{\mathfrak{g}}(\varphi(s), \varphi(0)\pi s) < \beta.$$

3. Main results. In this section we will state our main results. Throughout this section, unless explicitly stated to the contrary, we assume the following hypotheses:

($P1$) *(Λ, d_Λ) is a metric (parameter) space, λ_0 is a distinguished point of Λ and Λ_0 is a neighborhood of λ_0 in (Λ, d_Λ).*

($P2$) *$(f_\lambda)_{\lambda\in\Lambda_0\setminus\{\lambda_0\}}$ is a family of functions from $C(I,\mathcal{M})$ to E such that, for each $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$, $f_\lambda(\varphi) \in T_{\varphi(0)}\mathcal{M}$ for all $\varphi \in C(I,\mathcal{M})$ and such that $f_\lambda\colon (C(I,\mathcal{M}), d_E^\diamond) \to (E, d_E)$ is Lipschitzian on compact subsets of $C(I,\mathcal{M})$, thus generating the local semiflow $\pi_\lambda = \pi_{f_\lambda}$ on $C(I,\mathcal{M})$.*

($P3$) *$g\colon \mathcal{M} \to E$ is a map with $g(u) \in T_u\mathcal{M}$ for $u \in \mathcal{M}$ and such that $g\colon (\mathcal{M}, d_E) \to (E, d_E)$ is Lipschitzian on compact subsets of $\mathcal{M}$, thus generating the local (two-sided) flow $\pi = \pi_g$ on $\mathcal{M}$.*

($P4$) *Whenever S is a compact subset of $C(I,\mathcal{M})$, then*

$$\sup_{\varphi\in S} |f_\lambda(\varphi) - g(\varphi(0))|_E \to 0 \qquad \textit{as } \lambda \to \lambda_0 \textit{ in } (\Lambda, d_\Lambda).$$

($P5$) *Whenever B is a bounded subset of $C(I,\mathcal{M}) \subset C(I,E)$ with respect to the norm $|\cdot|_E^\diamond$, and such that the set $\{\varphi(s) \mid \varphi \in B,\ s \in I\}$ is relatively compact in $\mathcal{M}$, then*

$$\sup_{\lambda\in\Lambda_0\setminus\{\lambda_0\}} \sup_{\varphi\in B} |f_\lambda(\varphi)|_E < \infty.$$

We will give two illustrations of the above hypotheses:

REMARK 3.1. (1) Suppose $\mathcal{M}$ is open in E. Set $\Lambda = [0,\infty[$, $\Lambda_0 = [0,1[$ and $\lambda_0 = 0$. We write ε instead of λ. Let $g\colon \mathcal{M} \to E$ be Lipschitzian on compact subsets of $\mathcal{M}$ and $h\colon C(I,\mathcal{M}) \to E$ be Lipschitzian on compact subsets of $C(I,\mathcal{M})$ and bounded on $C(I,\mathcal{M})$. Define $f_\varepsilon\colon C(I,\mathcal{M}) \to E$ by

$$\varphi \mapsto g(\varphi(0)) + \varepsilon h(\varphi) \qquad \text{for } \varphi \in C(I,\mathcal{M}) \text{ and } \varepsilon \in]0,1[.$$

Then hypotheses ($P1$)–($P5$) are satisfied.

In fact, the tangency conditions (2.6) (for $f = f_\varepsilon$) and (2.11) (for g) are satisfied by Remark 2.2(1). The Lipschitzianity of f_ε on compact subsets of $C(I,\mathcal{M})$ follows from the Lipschitzianity of g and h on compact subsets of $\mathcal{M}$. Everything else is clear.

The corresponding differential equations read as follows:

$$\dot{x}(t) = g(x(t)) + \varepsilon h(x_t),$$
$$\dot{u}(t) = g(u(t)).$$

(2) Suppose $m = 2$ and let $\mathcal{M} = \mathbb{S}^1$, so $k = 1$. Set $\Lambda = \mathbb{R} \times [0, \infty[$, $\Lambda_0 = \,]{-1}, 1[\, \times [0, r]$ and $\lambda_0 = (\lambda_{0,1}, \lambda_{0,2}) = (0, 0)$. We write $\lambda = (\lambda_1, \lambda_2)$ for $\lambda \in \Lambda$. Let $h\colon \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be Lipschitzian on compact subsets of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. Define $g\colon \mathcal{M} \to E$ by

$$u \mapsto (-u_2 \cdot h(u, 0, u), u_1 \cdot h(u, 0, u)) \quad \text{for } u = (u_1, u_2) \in \mathcal{M},$$

and $f_\lambda\colon C(I, \mathcal{M}) \to E$ by

$$\varphi \mapsto \big(-\varphi_2(0) \cdot h(\varphi(0), \lambda_1 \varphi(-r), \varphi(-\lambda_2)), \varphi_1(0) \cdot h(\varphi(0), \lambda_1 \varphi(-r), \varphi(-\lambda_2))\big)$$

for $\varphi = (\varphi_1, \varphi_2) \in C(I, \mathcal{M})$ and $\lambda = (\lambda_1, \lambda_2) \in \,]{-1}, 1[\, \times [0, r]$ with $\lambda \neq (0, 0)$. Then hypotheses $(P1)$–$(P5)$ are satisfied. In fact, the tangency conditions (2.6) (for $f = f_\lambda$) and (2.11) (for g) are satisfied by Remark 2.2(2). The Lipschitzianity of f_λ on compact subsets of $C(I, \mathcal{M})$ and the Lipschitzianity of g on compact subsets of $\mathcal{M}$ follow from the Lipschitzianity assumption on h and the ensuing continuity of h. The proof of hypothesis $(P4)$ uses compactness of S in an essential way and it is left to the reader. Hypothesis $(P5)$ is easy to prove.

The corresponding differential equations read as follows:

$$\begin{cases} \dot{x}_1(t) = -x_2(t) \cdot h(x(t), \lambda_1 x(t - r), x(t - \lambda_2)), \\ \dot{x}_2(t) = x_1(t) \cdot h(x(t), \lambda_1 x(t - r), x(t - \lambda_2)), \end{cases}$$

where $x(t) = (x_1(t), x_2(t))$, and

$$\begin{cases} \dot{u}_1(t) = -u_2(t) \cdot h(u(t), 0, u(t)), \\ \dot{u}_2(t) = u_1(t) \cdot h(u(t), 0, u(t)), \end{cases}$$

where $u(t) = (u_1(t), u_2(t))$.

We have the following simple result:

LEMMA 3.2. *Let K_0 be an arbitrary compact isolated invariant set relative to π. Then there is a compact isolating neighborhood N of K_0 such that $N \subset U_\pi$.*

Proof. By the definition of a local flow, the set U_π is open in $\mathcal{M}$. Moreover, by the definition of an invariant set relative to π, we have $K_0 \subset U_\pi$. Choosing any isolating neighborhood N_0 of K_0 relative to π and (by local compactness of $\mathcal{M}$) any open set V_1 in $\mathcal{M}$ with $K_0 \subset V_1 \subset N_1 := \operatorname{cl}_{\mathcal{M}} V_1 \subset U_\pi$ and N_1 compact, and setting $N := N_0 \cap N_1$ we see that N is a compact isolating neighborhood of K_0 relative to π with $N \subset U_\pi$. ∎

We can now state the following Conley index continuation principle:

THEOREM 3.3. *Let K_0 be an arbitrary compact isolated invariant set relative to π and N be a compact isolating neighborhood of K_0 with $N \subset U_\pi$. Then there exists a $\beta = \beta(N) \in]0, \infty[$ with the property that for every $\eta \in]0, \beta[$ there exists a neighborhood $\Lambda^{c} = \Lambda^{c}(\eta)$ of λ_0 such that for every $\lambda \in \Lambda^{c} \setminus \{\lambda_0\}$ the set $[N]^\eta$ is a strongly admissible isolating neighborhood relative to π_λ and $h(\pi_\lambda, [N]^\eta) = h(\pi, N)$, i.e.*

$$h(\pi_\lambda, K_{\lambda,\eta}) = h(\pi, K_0), \tag{3.1}$$

where $K_{\lambda,\eta} := \mathrm{Inv}_{\pi_\lambda}([N]^\eta)$. The family $(K_{\lambda,\eta})_{\lambda \in \Lambda^{c}}$, where $K_{\lambda_0,\eta} = K_0$, is upper semicontinuous at $\lambda = \lambda_0$ in the following sense:

$$\sup_{\varphi \in K_{\lambda,\eta}} \inf_{u \in K_0} \sup_{s \in I} d_{\mathfrak{g}}(\varphi(s), u\pi s) \to 0 \qquad \text{as } \lambda \to \lambda_0.$$

We also state a result which is of interest in its own right.

PROPOSITION 3.4. *Let H denote an arbitrary unreduced* (*co*)*homology theory. Assume that $\mathcal{M}$ is compact. Then the following statements hold:*

(a) *If the map g is as in* ($P3$) *above, then the whole set $\mathcal{M}$ is an isolated invariant set with respect to π_g and the* (*co*)*homology Conley index $Hh(\pi_g, \mathcal{M})$ is* (*the isomorphism class of*) $H(\mathcal{M})$.
(b) *Suppose that the map $f\colon (C(I, \mathcal{M}), d_E^\diamond) \to (E, d_E)$ is Lipschitzian on compact subsets of $C(I, \mathcal{M})$, $f(\varphi) \in T_{\varphi(0)}\mathcal{M}$ for all $\varphi \in C(I, \mathcal{M})$ and $f(B)$ is bounded in E, where $B = C(I, \mathcal{M})$. Let S_f be the set of all $\varphi \in C(I, \mathcal{M})$ for which there is a solution $\sigma\colon \mathbb{R} \to C(I, \mathcal{M})$ such that $\sigma(0) = \varphi$. The set S_f is an isolated invariant set with respect to π_f and the* (*co*)*homology Conley index $Hh(\pi_f, S_f)$ is* (*the isomorphism class of*) $H(\mathcal{M})$.

The two results just stated can be used to establish multiplicity for full bounded solutions of some manifold-valued RFDEs. This is illustrated by the following example on a two-sphere.

EXAMPLE 3.5. Let $\mathcal{M} = \mathbb{S}^2$. Let $K_{0,j}$, $j \in \{1, 2, 3\}$, be pairwise disjoint isolated invariant sets relative to π_g such that $G_j := H_2 h(\pi_g, K_{0,j}) \neq \{0\}$, $j \in \{1, 2\}$ and $h(\pi_g, K_{0,3}) \neq \overline{0}$. The reader may think of the well known examples where the North Pole N and the South Pole S are totally unstable equilibria while the Equator E is an asymptotically stable invariant set. Then $K_{0,1} = \{\mathrm{N}\}$, $K_{0,2} = \{\mathrm{S}\}$ and $K_{0,3} = \mathrm{E}$.

By Theorem 3.3 there is a small neighborhood Λ^{c} of λ_0 and, for each λ in $\Lambda^{c} \setminus \{\lambda_0\}$, there are pairwise disjoint isolated invariant sets $S_{\lambda,j}$, $j \in \{1, 2, 3\}$, relative to $\pi_\lambda = \pi_{f_\lambda}$ such that $h(\pi_\lambda, S_{\lambda,j}) = h(\pi_g, K_{0,j})$, $j \in \{1, 2, 3\}$. In particular,

$$S_{\lambda,j} \neq \emptyset, \qquad \lambda \in \Lambda^{c} \setminus \{\lambda_0\},\ j \in \{1, 2, 3\}. \tag{3.2}$$

Fix $\lambda \in \Lambda^{\mathrm{c}} \setminus \{\lambda_0\}$ and set $S_\lambda := S_{f_\lambda}$ (cf. Proposition 3.4). We cannot have $S_\lambda = \bigcup_{j=1}^{3} S_{\lambda,j}$, since otherwise, for each $q \in \mathbb{Z}$, the group $H_q h(\pi_\lambda, S_\lambda)$ would be the direct sum of the groups $H_q h(\pi_\lambda, S_{\lambda,j})$, $j \in \{1,2,3\}$. For $q = 2$ we have $H_q h(\pi_\lambda, S_\lambda) \cong \mathbb{Z}$ by Proposition 3.4 and so

$$\mathbb{Z} \cong G_1 \oplus G_2 \oplus H_2 h(\pi_\lambda, S_{\lambda,3}).$$

However, this is a contradiction as G_1 and G_2 are nontrivial groups. Since $\bigcup_{j=1}^{3} S_{\lambda,j} \subset S_\lambda$, it follows that $S_\lambda \setminus \bigcup_{j=1}^{3} S_{\lambda,j} \neq \emptyset$. The invariance of the sets S_λ and $S_{\lambda,j}$ together with (3.2) now imply the existence of (full) solutions $\sigma_j \colon \mathbb{R} \to C(I, \mathcal{M})$, $j \in \{1,2,3,4\}$, of π_λ such that $\sigma_j(\mathbb{R}) \subset S_{\lambda,j}$ for j in $\{1,2,3\}$ and $\sigma_4(\mathbb{R}) \not\subset \bigcup_{j=1}^{3} S_{\lambda,j}$. We have thus proved the existence of four distinct full solutions of π_λ.

In addition to the above Conley index continuation result we also have the following homology index braid continuation principle:

THEOREM 3.6. *Let K_0, N and $\beta = \beta(N)$ be as in Theorem 3.3. Fix an $\eta \in \,]0, \beta[$ and let $\Lambda^{\mathrm{c}} = \Lambda^{\mathrm{c}}(\eta)$ be as in that theorem. Let $(P, \prec)$ be a finite poset. Moreover, suppose that $(M_{p,0})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_0 = \mathrm{Inv}_\pi(N)$ relative to π. For each $p \in P$, let $\Upsilon_p \subset N$ be closed in X and such that $M_{p,0} = \mathrm{Inv}_\pi(\Upsilon_p) \subset \mathrm{Int}\,\Upsilon_p$. For $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ and $p \in P$ set $M_{p,\lambda,\eta} := \mathrm{Inv}_{\pi_\lambda}([\Upsilon_p]^\eta)$.*

Then there is a neighborhood $\widetilde{\Lambda}$ of λ_0 with $\widetilde{\Lambda} \subset \Lambda^{\mathrm{c}}$ such that for all $\lambda \in \widetilde{\Lambda} \setminus \{\lambda_0\}$, $K_{\lambda,\eta} \subset \mathrm{Int}_\lambda [N]^\eta$, the family $(M_{p,\lambda,\eta})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_{\lambda,\eta}$ relative to π_λ and the homology index braids of $(\pi, K_0, (M_{p,0})_{p\in P})$ and $(\pi_\lambda, K_{\lambda,\eta}, (M_{p,\lambda,\eta})_{p\in P})$, $\lambda \in \widetilde{\Lambda} \setminus \{\lambda_0\}$, are isomorphic. For each $p \in P$, the family $(M_{p,\lambda})_{\lambda\in\widetilde{\Lambda}}$, where $M_{p,\lambda_0} := M_{p,0}$, is upper semicontinuous at $\lambda = \lambda_0$ in the following sense:

$$\sup_{\varphi \in M_{p,\lambda,\eta}} \inf_{u \in M_{p,0}} \sup_{s\in I} d_{\mathfrak{g}}(\varphi(s), u\pi s) \to 0 \quad \text{as } \lambda \to \lambda_0.$$

4. Proofs of the main results. The first part of this section is devoted to the proof of Theorems 3.3 and 3.6. Recall that we assume hypotheses $(P1)$–$(P5)$ stated at the beginning of section 3. Let K_0 be an arbitrary compact isolated invariant set relative to π and N be a compact isolating neighborhood of K_0 such that $N \subset U_\pi$.

(4.1) First assume that N is nonempty. Let N' be the set of all points $v \in \mathcal{M}$ such that there is a $u \in N$ and an $s \in I$ with $v = u\pi s$. It follows that N' is compact in $\mathcal{M}$. Since $\mathcal{M}$ is locally compact, there is an open set U' and a compact set N'' with $N' \subset U' \subset N''$.

Therefore there is a smooth nonnegative function $\alpha \colon \mathcal{M} \to \mathbb{R}$ with compact support and such that $\alpha(u) = 1$ for all $u \in \mathrm{cl}_{\mathcal{M}}\, U'$.

For each $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$, define the function $\widehat{f}_\lambda\colon C(I,\mathcal{M}) \to E$ by $\widehat{f}_\lambda(\varphi) = \alpha(\varphi(0)) \cdot f_\lambda(\varphi)$. Since α is globally bounded and globally Lipschitzian and, on compact subsets of $C(I,\mathcal{M})$, f_λ is bounded and Lipschitzian, it follows that $\widehat{f}_\lambda$ is Lipschitzian on compact subsets of $C(I,\mathcal{M})$ and $\widehat{f}_\lambda(\varphi) \in T_{\varphi(0)}\mathcal{M}$ for $\varphi \in C(I,\mathcal{M})$, so $\widehat{f}_\lambda$ generates a local semiflow $\widehat{\pi}_\lambda = \pi_{\widehat{f}_\lambda}$ on $C(I,\mathcal{M})$.

Define also the function $\widehat{g}\colon \mathcal{M} \to E$, by $\widehat{g}(u) = \alpha(u) \cdot g(u)$. Then $\widehat{g}(u) \in T_u\mathcal{M}$ for $u \in \mathcal{M}$ and $\widehat{g}$ again is Lipschitzian on compact subsets of $\mathcal{M}$, so it generates a local flow $\widehat{\pi} = \pi_{\widehat{g}}$ on E.

Thus the obvious analogues of hypotheses $(P2)$ and $(P3)$ are satisfied. Since α is bounded, the obvious analogues of hypotheses $(P4)$ and $(P5)$ are also satisfied for $\widehat{g}$ and the family $(\widehat{f}_\lambda)_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}}$.

We claim that $\widehat{\pi}$ is a global flow, i.e. $u\widehat{\pi}s$ is defined for all $u \in \mathcal{M}$ and all $s \in \mathbb{R}$. Indeed, let S be a compact subset of $\mathcal{M}$ such that $\alpha(u) = 0$ for all $u \in \mathcal{M} \setminus S$. It follows that each $u \in \mathcal{M} \setminus S$ is an equilibrium of $\widehat{\pi}$, so $u\widehat{\pi}s$ is defined and equals u for each $s \in \mathbb{R}$. If $u \in S$ then there is no $s \in \mathbb{R}$ with $u\widehat{\pi}s$ defined and $v := u\widehat{\pi}s \in \mathcal{M} \setminus S$, since otherwise v would be an equilibrium of $\widehat{\pi}$ and so $v\widehat{\pi}(-s) = v \in \mathcal{M} \setminus S$. However, by the local group property, $(u\widehat{\pi}s)\widehat{\pi}(-s)$ being defined implies that $u = u\widehat{\pi}(s + (-s)) = v\widehat{\pi}(-s) = v$, so $u \in \mathcal{M} \setminus S$, a contradiction. Thus $u\widehat{\pi}s \in S$, as long as it is defined. Since S is compact, $\widehat{\pi}$ and its reverse local flow do not explode in S, so $u\widehat{\pi}s$ is defined for all $s \in \mathbb{R}$, as claimed.

Now the functions g and $\widehat{g}$ coincide on the open subset U' of $\mathcal{M}$. Since $N' \subset U'$, it follows that

$$u\pi s = u\widehat{\pi}s, \qquad u \in N,\ s \in I. \tag{4.2}$$

Moreover, the local (semi)flows π and $\widehat{\pi}$ agree on U'. Furthermore, for each $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$, the functions f_λ and $\widehat{f}_\lambda$ coincide on the open subset $W' = p^{-1}(U')$ of $C(I,\mathcal{M})$, so the local semiflows π_λ and $\widehat{\pi}_\lambda$ agree on W'. Now (4.2) implies that for any subset V of N (like $V = N$ or $V = \Upsilon_p$) the tube $[V]^\eta$ defined by means of π is equal to the tube $[V]^\eta$ defined by means of $\widehat{\pi}$. Moreover, $[V]^\eta \subset W'$. Consequently, using Remark 2.5, we see that the statements of Theorems 3.3 and 3.6 are equivalent to the corresponding statements obtained by replacing $(\pi_\lambda)_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}}$ and π by $(\widehat{\pi}_\lambda)_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}}$ and $\widehat{\pi}$. Therefore, we may assume without loss of generality that

(C) *π is a global flow.*

The proof is accomplished by checking the assumptions of [4, Theorem 3.9]. Set

$$(X,d) = (\mathcal{M}, d_{\mathfrak{g}}), \quad (Z_\lambda, \Gamma_\lambda) = (C(I,\mathcal{M}), d_{\mathfrak{g}}^{\diamond}) \quad \text{for all } \lambda \in \Lambda_0 \setminus \{\lambda_0\}. \tag{4.3}$$

Since π is a global flow, it follows that $U_\pi = X$ and so the domain of the map Θ defined as in (2.14) is X. Since $Z_\lambda \equiv C(I,\mathcal{M})$ we may now define, for $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$,

$$p_\lambda\colon Z_\lambda \to X, \quad p_\lambda = p, \quad \text{and} \quad \Theta_\lambda\colon X \to Z_\lambda, \quad \Theta_\lambda = \Theta. \tag{4.4}$$

It is clear that p_λ and Θ_λ are continuous. We have $(p_\lambda \circ \Theta_\lambda)u = (p \circ \Theta)u = (\Theta u)(0) = u\pi 0 = u$ for $u \in X$, so $p_\lambda \circ \Theta_\lambda = \mathrm{Id}_X$. Therefore, [4, Hypothesis (H0)] is satisfied.

Let $(\lambda_n)_n$ be a sequence in $\Lambda_0 \setminus \{\lambda_0\}$ converging in Λ to λ_0 as $n \to \infty$, $(\varphi_n)_n$ be a sequence such that $\varphi_n \in Z_{\lambda_n} = C(I, \mathcal{M})$ for every $n \in \mathbb{N}$, and $u_0 \in X$ be such that $\Gamma_{\lambda_n}(\varphi_n, \Theta_{\lambda_n} u_0) \to 0$ as $n \to \infty$, i.e. $d^\diamond_{\mathfrak{g}}(\varphi_n, \Theta u_0) \to 0$ as $n \to \infty$. Hence, $d_{\mathfrak{g}}(\varphi_n(0), (\Theta u_0)(0)) \to 0$ as $n \to \infty$ and so $d(p_{\lambda_n}(\varphi_n), u_0) = d_{\mathfrak{g}}(\varphi_n(0), (\Theta u_0)(0)) \to 0$ as $n \to \infty$. This proves [4, Hypothesis (H1)].

Now let $(\lambda_n)_n$ be a sequence in $\Lambda_0 \setminus \{\lambda_0\}$ converging in Λ to λ_0 as $n \to \infty$ and $(u_n)_n$ be a sequence in X converging to $u_0 \in X$ as $n \to \infty$. The continuity of π implies that $\Gamma_{\lambda_n}(\Theta_{\lambda_n} u_n, \Theta_{\lambda_n} u_0) = d^\diamond_{\mathfrak{g}}(\Theta u_n, \Theta u_0) = \sup_{s \in I} d_{\mathfrak{g}}(u_n \pi s, u_0 \pi s) \to 0$ as $n \to \infty$. This implies that [4, Hypothesis (H2)] is satisfied. It is clear that [4, Hypothesis (H3)] holds.

Now note that the definition of a closed β-tube $[V]^\beta$ given in Definition 2.6 coincides, in the present special case, with the abstract notion of a closed β-tube $[V]^{\lambda,\beta}$ as defined in [4]. In the present case the set $[V]^\beta$ is independent of λ but, in some of the arguments to follow, we keep the notation '$[V]^{\lambda,\beta}$' for easier comparison with [4].

Therefore, as in [4], given an arbitrary subset M of X, $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ and $\beta \in]0, \infty[$, we have $\Theta_\lambda(M) \subset [M]^{\lambda,\beta}$. Let $\Theta_{\lambda,\beta,M}\colon M \to [M]^{\lambda,\beta}$ be the corresponding restricted map. If M is closed in X, then $p_\lambda([M]^{\lambda,\beta}) \subset M$. Let $p_{\lambda,\beta,M}\colon [M]^{\lambda,\beta} \to M$ be the corresponding restricted map.

Thus, whenever N_1 and N_2 are closed subsets of X, then using [4, Lemma 2.1] we obtain well-defined quotient maps

$$\overline{\Theta}_{\lambda,\beta,N_1,N_2}\colon (N_1/N_2, [N_2]) \to ([N_1]^{\lambda,\beta}/[N_2]^{\lambda,\beta}, [[N_2]^{\lambda,\beta}])$$

and

$$\overline{p}_{\lambda,\beta,N_1,N_2}\colon ([N_1]^{\lambda,\beta}/[N_2]^{\lambda,\beta}, [[N_2]^{\lambda,\beta}]) \to (N_1/N_2, [N_2]).$$

In view of [4, Hypothesis (H0)], [4, Lemma 2.1] also implies that

$$\overline{p}_{\lambda,\beta,N_1,N_2} \circ \overline{\Theta}_{\lambda,\beta,N_1,N_2} = \mathrm{Id}_{N_1/N_2,[N_2]}\,.$$

Moreover, it follows from [4, Lemma 3.5] that if $[\overline{\Theta}_{\lambda,\beta,N_1,N_2}]$ is an isomorphism in $\mathcal{HT}^*$, then $[\overline{p}_{\lambda,\beta,N_1,N_2}]$ is its inverse. In particular, $[\overline{p}_{\lambda,\beta,N_1,N_2}]$ is an isomorphism in $\mathcal{HT}^*$. Here as in [12] we denote by $\mathcal{T}^*$ the category of pointed spaces and by $\mathcal{HT}^*$ the corresponding homotopy category.

We can now state the following result which is of interest in its own right.

Lemma 4.1. *There is a $\beta = \beta(N) \in]0, \infty[$ such that for every $\eta \in]0, \beta]$ and every pair (N_1, N_2) of closed subsets of N the morphism $[\overline{\Theta}_{\lambda,\eta,N_1,N_2}]$ is* *an isomorphism in $\mathcal{HT}^*$. Moreover, $\varphi(s) \in N''$ for all $\varphi \in [N]^{\lambda,\beta}$ and all $s \in I$, where N'' is as in* (4.1).

Proof. For $\delta \in]0,\infty[$ let A_δ be the set of all $(p,q) \in N'' \times N''$ with $d_{\mathfrak{g}}(p,q) < \delta$.

By a basic result of Riemannian geometry (cf. [7, Corollary I.6.3]), there is a $\delta = \delta(N'') \in]0,\infty[$ and a map $\Phi\colon A_\delta \to C([0,1],\mathcal{M})$ such that for each $(p,q) \in A_\delta$, the curve $\Phi(p,q)\colon [0,1] \to \mathcal{M}$ is the unique geodesic in $(\mathcal{M},\mathfrak{g})$ with length $\ell_{\mathfrak{g}}(\Phi(p,q)) < \delta$ and such that $\Phi(p,q)(0) = p$ and $\Phi(p,q)(1) = q$. The map Φ is continuous and minimizes the Riemann distance, i.e.

$$\ell_{\mathfrak{g}}(\Phi(p,q)) = d_{\mathfrak{g}}(p,q), \qquad (p,q) \in A_\delta. \tag{4.5}$$

Notice that if $p \in N''$, then $(p,p) \in A_\delta$ and since the constant function $\gamma\colon [0,1] \to \mathcal{M}$, $\gamma(\tau) \equiv p$ is a geodesic with $\gamma(0) = \gamma(1) = p$ and $\ell_{\mathfrak{g}}(\gamma) < \delta$, it follows by uniqueness that

$$\Phi(p,p)(\tau) = p, \qquad p \in N'',\ \tau \in [0,1]. \tag{4.6}$$

Furthermore, let $\delta' := \inf\{d_{\mathfrak{g}}(p,q) \mid p \in N',\ q \in \mathcal{M} \setminus U'\}$. Since N' is compact, we have $\delta' > 0$. Set $\beta := (1/2)\min(\delta,\delta') > 0$ and let $\eta \in]0,\beta]$ be arbitrary. If $\varphi \in [N]^{\lambda,\eta}$, then $\varphi(0) \in N$ and for each $s \in I$, $d_{\mathfrak{g}}(\varphi(s),\varphi(0)\pi s) \le \eta \le \beta$, so, since $\varphi(0)\pi s \in N'$ and $\eta < \delta'$, we have $\varphi(s) \in U'$ and thus $(\varphi(s),\varphi(0)\pi s) \in N'' \times N''$. For $\eta = \beta$ this establishes the second assertion of the lemma. We also have $\eta < \delta$ and so $(\varphi(s),\varphi(0)\pi s) \in A_\delta$. Consequently, for each $\varphi \in [N]^{\lambda,\eta}$ and each $\tau \in [0,1]$, we have a well defined function $F(\varphi,\tau)\colon I \to \mathcal{M}$ given by

$$F(\varphi,\tau)(s) = \Phi(\varphi(s),\varphi(0)\pi s)(\tau).$$

It follows that $F(\varphi,\tau) \in C(I,\mathcal{M})$ and the induced map $F\colon [N]^{\lambda,\eta} \times [0,1] \to C(I,\mathcal{M})$, $(\varphi,\tau) \mapsto F(\varphi,\tau)$, is continuous.

We claim that for each closed subset B of N, the function F maps $[B]^{\lambda,\eta} \times [0,1]$ into $[B]^{\lambda,\eta}$. To wit, let $\varphi \in [B]^{\lambda,\eta}$ and $\tau \in [0,1]$ be arbitrary. Then, by definition, there is a sequence $(\varphi_n)_n$ in $C(I,\mathcal{M})$ with $d^\diamond_{\mathfrak{g}}(\varphi_n,\varphi) \to 0$ as $n \to \infty$ and such that, for each $n \in \mathbb{N}$, $\varphi_n(0) \in B$ and $d^\diamond_{\mathfrak{g}}(\varphi_n,\Theta_\lambda(\varphi_n(0))) < \eta$. Let $\widetilde{\varphi} = F(\varphi,\tau)$ and $\widetilde{\varphi}_n = F(\varphi_n,\tau)$, $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Then, in view of (4.6),

$$\widetilde{\varphi}_n(0) = \Phi(\varphi_n(0),\varphi_n(0)\pi 0)(\tau) = \Phi(\varphi_n(0),\varphi_n(0))(\tau) = \varphi_n(0),$$

so

$$\widetilde{\varphi}_n(0) = \varphi_n(0). \tag{4.7}$$

In particular,

$$\widetilde{\varphi}_n(0) \in B. \tag{4.8}$$

Now fix $s \in I$ and let $\gamma := \Phi(\varphi_n(s),\varphi_n(0)\pi s)$. Then $\gamma(\tau) = \widetilde{\varphi}_n(s)$ and $\gamma(1) = \varphi_n(0)\pi s = \widetilde{\varphi}_n(0)\pi s$ (by (4.7)). Thus $\widetilde{\gamma} := \gamma|_{[\tau,1]}$ is a smooth path from $\widetilde{\varphi}_n(s)$ to $\widetilde{\varphi}_n(0)\pi s$ and so, by the definition of $d_{\mathfrak{g}}$ and an application of (4.5),

$$d_{\mathfrak{g}}(\widetilde{\varphi}_n(s), \widetilde{\varphi}_n(0)\pi s) \leq \ell_{\mathfrak{g}}(\widetilde{\gamma}) \leq \ell_{\mathfrak{g}}(\gamma) = d_{\mathfrak{g}}(\varphi_n(s), \varphi_n(0)\pi s).$$

This implies that $d_{\mathfrak{g}}^{\diamond}(\widetilde{\varphi}_n, \Theta_\lambda(\widetilde{\varphi}_n(0))) \leq d_{\mathfrak{g}}^{\diamond}(\varphi_n, \Theta_\lambda(\varphi_n(0))) < \eta$, so

$$d_{\mathfrak{g}}^{\diamond}(\widetilde{\varphi}_n, \Theta_\lambda(\widetilde{\varphi}_n(0))) < \eta. \tag{4.9}$$

Now the continuity of F and the validity of (4.8) and (4.9) for each $n \in \mathbb{N}$ imply that $\widetilde{\varphi} \in [B]^{\lambda,\eta}$, and this proves our claim. The claim implies, in particular, that F restricts to a continuous map $H\colon [N_1]^{\lambda,\eta} \times [0,1] \to [N_1]^{\lambda,\eta}$ such that $H(([N_1]^{\lambda,\eta} \cap [N_2]^{\lambda,\eta}) \times [0,1]) \subset [N_2]^{\lambda,\eta}$. Now by [4, Lemma 2.1] and the definition of F we see that H induces a (base-point preserving) homotopy

$$\overline{H}\colon ([N_1]^{\lambda,\beta}/[N_2]^{\lambda,\beta}, [[N_2]^{\lambda,\beta}]) \times [0,1] \to ([N_1]^{\lambda,\beta}/[N_2]^{\lambda,\beta}, [[N_2]^{\lambda,\beta}])$$

from $\mathrm{Id}_{[N_1]^{\lambda,\beta}/[N_2]^{\lambda,\beta},[[N_2]^{\lambda,\beta}]}$ to $\overline{\Theta}_{\lambda,\beta,N_1,N_2} \circ \overline{p}_{\lambda,\beta,N_1,N_2}$. ∎

Now let $f\colon C(I,\mathcal{M}) \to E$ be defined by

$$f(\varphi) = g(\varphi(0)) \quad \text{for } \varphi \in C(I,\mathcal{M}). \tag{4.10}$$

Since $g\colon (\mathcal{M}, d_E) \to (E, d_E)$ is Lipschitzian on compact subsets of $\mathcal{M}$, it follows that f is Lipschitzian on compact subsets of $C(I,\mathcal{M})$. Hence, f generates a (local) semiflow π_f on $C(I,\mathcal{M})$.

LEMMA 4.2. *For every $\varphi \in C(I,\mathcal{M})$, the* (*unique*) *solution $x(\varphi, f)$ of the equation*

$$\dot{y}(t) = f(y_t) \tag{4.11}$$

with $y_0 = \varphi$ is defined on $[-r,\infty[$ and $x(\varphi, f) = \widetilde{x}$, where $\widetilde{x}\colon [-r,\infty[\to E$ is defined by

$$\widetilde{x}(t) := \begin{cases} \varphi(t), & t \in [-r,0], \\ v\pi t, & t \in [0,\infty[, \end{cases}$$

where $v = \varphi(0)$.

Proof. From its definition, the function $\widetilde{x}$ is continuous and $\widetilde{x}_t$ is defined for each $t \in [0,\infty[$. Moreover, the definition of π implies that $x := \widetilde{x}|_{[0,\infty[}$ is differentiable and $\dot{x}(t) = g(x(t))$ for each $t \in [0,\infty[$, so $(\widetilde{x}|_{[0,\infty[})'(t)$ exists and

$$(\widetilde{x}|_{[0,\infty[})'(t) = \dot{x}(t) = g(x(t)) = g(\widetilde{x}(t)) = g(\widetilde{x}_t(0)) = f(\widetilde{x}_t).$$

Thus $\widetilde{x}$ is a solution of (4.11) and since clearly $\widetilde{x}_0 = \varphi$, it follows that $\widetilde{x} = x(\varphi, f)$. ∎

LEMMA 4.3. *The family $(\pi_\lambda)_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}}$ converges to π as $\lambda \to \lambda_0$ in the sense of* [4, Definition 3.6].

Proof. We must prove the following statement:

Whenever $(\lambda_n)_n$ is a sequence in $\Lambda_0 \setminus \{\lambda_0\}$ and $(t_n)_n$ is a sequence in $[0,\infty[$ with $\lambda_n \to \lambda_0$ and $t_n \to t_0$ as $n \to \infty$ for some $t_0 \in [0,\infty[$, and whenever $u_0 \in X$ is such that $u_0 \pi t_0$ is defined and $(\varphi_n)_n$ is a sequence with $\varphi_n \in Z_{\lambda_n}$ for each $n \in \mathbb{N}$ and such that $\Gamma_{\lambda_n}(\varphi_n, \Theta_{\lambda_n} u_0) \to 0$ as $n \to \infty$, then $\varphi_n \pi_{\lambda_n} t_n$ is defined for all $n \in \mathbb{N}$ large enough and $\Gamma_{\lambda_n}(\varphi_n \pi_{\lambda_n} t_n, \Theta_{\lambda_n}(u_0 \pi t_0))$ $\to 0$ as $n \to \infty$.

Let $(\lambda_n)_n$ be a sequence in $\Lambda_0 \setminus \{\lambda_0\}$ and $(t_n)_n$ be a sequence in $[0,\infty[$ with $\lambda_n \to \lambda_0$, $t_n \to t_0$ as $n \to \infty$, for some $t_0 \in [0,\infty[$. Furthermore, let $u_0 \in X$ and $(\varphi_n)_n$ be a sequence with $\varphi_n \in Z_{\lambda_n}$ for each $n \in \mathbb{N}$ such that $\Gamma_{\lambda_n}(\varphi_n, \Theta_{\lambda_n} u_0) \to 0$ as $n \to \infty$, i.e.

$$d^\diamond_{\mathfrak{g}}(\varphi_n, \Theta u_0) \to 0 \quad \text{as } n \to \infty. \tag{4.12}$$

(Note that we do not have to assume that $u_0 \pi t_0$ is defined since π is a global flow by (C).)

We must prove that

$$\varphi_n \pi_{\lambda_n} t_n \text{ is defined for all } n \in \mathbb{N} \text{ large enough} \tag{4.13}$$

and

$$\Gamma_{\lambda_n}(\varphi_n \pi_{\lambda_n} t_n, \Theta_{\lambda_n}(u_0 \pi t_0)) \to 0 \quad \text{as } n \to \infty. \tag{4.14}$$

Note that assumption $(P4)$ can be rephrased as follows: whenever S is a compact subset of $C(I, \mathcal{M})$, then $\sup_{\varphi \in S} |f_\lambda(\varphi) - f(\varphi)|_E \to 0$ as $\lambda \to \lambda_0$ in (Λ, d_Λ).

We actually claim that whenever $\widetilde{S}$ is a compact subset of $C(I, U)$, then

$$\sup_{\widetilde{\varphi} \in \widetilde{S}} |\widetilde{f}_\lambda(\widetilde{\varphi}) - \widetilde{f}(\widetilde{\varphi})|_E \to 0 \quad \text{as } \lambda \to \lambda_0 \text{ in } (\Lambda, d_\Lambda),$$

where $\widetilde{f}_\lambda = f_\lambda \circ \widehat{\rho}$ and $\widetilde{f} = f \circ \widehat{\rho}$ (cf. Proposition 2.3).

To prove our claim, let $\widetilde{S}$ be an arbitrary compact subset of $C(I, U)$ and set $S = \widehat{\rho}(\widetilde{S})$. Then S is compact in $C(I, \mathcal{M})$. If $\widetilde{\varphi} \in \widetilde{S}$, then $\psi := \widehat{\rho}(\widetilde{\varphi}) \in S$ and $\widetilde{f}_\lambda(\widetilde{\varphi}) - \widetilde{f}(\widetilde{\varphi}) = f_\lambda(\psi) - f(\psi)$, so $\sup_{\widetilde{\varphi} \in \widetilde{S}} |\widetilde{f}_\lambda(\widetilde{\varphi}) - \widetilde{f}(\widetilde{\varphi})|_E \le \sup_{\varphi \in S} |f_\lambda(\varphi) - f(\varphi)|_E \to 0$ as $\lambda \to \lambda_0$. This proves the claim.

Now the definition of Z_λ together with formula (4.12) and the equivalence of the metrics $d^\diamond_{\mathfrak{g}}$ and $d^\diamond_E$ on $C(I, \mathcal{M})$ imply that $(\varphi_n)_n$ is a sequence in $C(I, \mathcal{M}) \subset C(I, U)$ such that $|\varphi_n - \varphi_0|^\diamond_E = d^\diamond_E(\varphi_n, \varphi_0) \to 0$ as $n \to \infty$, where $\varphi_0 = \Theta u_0$. Thus the above claim and classical results on continuous dependence of solutions of RFDEs on initial values and parameters (cf. [9], see also [12, Theorem I.2.3]) imply convergence of the sequence $(\pi_{\widetilde{f}_{\lambda_n}})_n$ to $\pi_{\widetilde{f}}$ in the sense of [12, Definition I.2.2]. This implies that there is an $n_0 \in \mathbb{N}$

such that $\varphi_n \pi_{\widetilde{f}_{\lambda_n}} t_n$ is defined for all $n \geq n_0$, and

$$|\varphi_n \pi_{\widetilde{f}_{\lambda_n}} t_n - \varphi_0 \pi_{\widetilde{f}} t_0|_E^\diamond \to 0 \quad \text{as } n \to \infty. \tag{4.15}$$

Now, $\varphi_n \in C(I, \mathcal{M})$ for all $n \in \mathbb{N} \cup \{0\}$, so Proposition 2.3 implies that

$$\varphi_n \pi_{\widetilde{f}_{\lambda_n}} t_n = \varphi_n \pi_{f_{\lambda_n}} t_n = \varphi_n \pi_{\lambda_n} t_n, \quad n \in \mathbb{N},$$

and $\varphi_0 \pi_{\widetilde{f}} t_0 = \varphi_0 \pi_f t_0$, and so statement (4.13) follows and, by (4.15),

$$|\varphi_n \pi_{f_{\lambda_n}} t_n - \varphi_0 \pi_f t_0|_E^\diamond \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

Now, Lemma 4.2 and the fact that $\varphi_0(0) = u_0$ imply that $x(\varphi_0, f)(t)$ is defined and $x(\varphi_0, f)(t) = u_0 \pi t$ for all $t \in [0, \infty[$. By the definition of φ_0 it follows that $x(\varphi_0, f)(t) = u_0 \pi t$ for $t \in [-r, \infty[$. In particular, $\varphi_0 \pi_f t_0$ is defined, and for $s \in I$,

$$(\varphi_0 \pi_f t_0)(s) = x(\varphi_0, f)(t_0 + s) = u_0 \pi (t_0 + s) = (u_0 \pi t_0) \pi s = \Theta(u_0 \pi t_0)(s),$$

and so

$$\varphi_0 \pi_f t_0 = \Theta(u_0 \pi t_0). \tag{4.17}$$

Thus, by (4.16) and (4.17),

$$d_E^\diamond(\varphi_n \pi_{\lambda_n} t_n, \Theta(u_0 \pi t_0)) = |\varphi_n \pi_{\lambda_n} t_n - \Theta(u_0 \pi t_0)|_E^\diamond \to 0 \quad \text{as } n \to \infty.$$

Therefore, by the equivalence of the metrics $d_{\mathfrak{g}}^\diamond$ and $d_E^\diamond$ on $C(I, \mathcal{M})$,

$$\Gamma_{\lambda_n}(\varphi_n \pi_{\lambda_n} t_n, \Theta_{\lambda_n}(u_0 \pi t_0)) = d_{\mathfrak{g}}^\diamond(\varphi_n \pi_{\lambda_n} t_n, \Theta(u_0 \pi t_0)) \to 0 \quad \text{as } n \to \infty,$$

so that statement (4.14) follows. ∎

LEMMA 4.4. *Let $\beta = \beta(N) \in]0, \infty[$ be as in Lemma 4.1. Then the set N is strongly admissible with respect to β, π and $(p_\lambda, \Theta_\lambda, \pi_\lambda)_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}}$ in the sense of* [4, Definition 3.7].

Proof. Set $B := [N]^{\lambda,\beta}$. Note again that B does not depend on λ. Let N'' be as in (4.1). Since N'' is compact in $\mathcal{M}$ and the inclusion induced map $\iota\colon \mathcal{M} \to E$ is continuous, it follows that $N'' = \iota(N'')$ is compact in E, so N'' is closed in E and bounded with respect to the norm $|\cdot|_E$. Let $L = \sup_{u \in N''} |u|_E < \infty$. For each $\varphi \in B$ and all $s \in I$ we have $\varphi(s) \in N''$, so $|\varphi|_E^\diamond \leq L$. Thus

$$\sup_{\varphi \in B} |\varphi|_E^\diamond \leq L < \infty. \tag{4.18}$$

Moreover, $\{\varphi(s) \mid \varphi \in B,\, s \in I\} \subset N''$. Since N'' is compact, it follows that $\{\varphi(s) \mid \varphi \in B,\, s \in I\}$ is relatively compact in $\mathcal{M}$. Proposition 2.3 implies that $\widetilde{f}_\lambda(\varphi) = (f_\lambda \circ \widehat{\rho})(\varphi) = f_\lambda(\varphi)$ for all $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ and all $\varphi \in C(I, \mathcal{M})$. As $B \subset C(I, \mathcal{M})$ we deduce from assumption $(P5)$ that

$$L' := \sup_{\lambda \in \Lambda_0 \setminus \{\lambda_0\}} \sup_{\varphi \in B} |\widetilde{f}_\lambda(\varphi)|_E < \infty. \tag{4.19}$$

We claim that

$$B \text{ is closed in } C(I,E). \tag{4.20}$$

To wit, let $(\varphi_n)_n$ be a sequence in B which converges in $C(I,E)$ to a $\varphi \in C(I,E)$. Then for each $s \in I$, the sequence $(\varphi_n(s))_n$ lies in N'' and converges to $\varphi(s)$ in E. It follows that $\varphi(s) \in N'' \subset \mathcal{M}$, so $\varphi(I) \subset \mathcal{M}$. Thus $\varphi \in C(I,\mathcal{M})$ and so $(\varphi_n)_n$ converges to φ in $C(I,\mathcal{M})$. Since B is closed in $C(I,\mathcal{M})$ we have $\varphi \in B$. This shows that B is closed in $C(I,E)$ and completes the proof of our claim.

We will now check the three conditions of [4, Definition 3.7]:

(1) *N is strongly π-admissible.*

In fact, since N is compact, N is π-admissible directly from the definition and π does not explode in N by general results on local semiflows (cf. [3, Lemma 1.10]). This proves (1).

(2) *For each $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ the set $B = [N]^{\lambda,\beta}$ is strongly π_λ-admissible.*

To prove (2), first note that for each $\lambda \in \Lambda \setminus \{\lambda_0\}$, statements (4.18)–(4.20) and known results (cf. [12, Theorems I.2.3 and I.4.2]), imply that

$$B \text{ is strongly } \pi_{\widetilde{f}_\lambda}\text{-admissible.} \tag{4.21}$$

We first claim that π_λ does not explode in B. In fact, let $\varphi \in C(I,\mathcal{M})$ be arbitrary such that $\varphi\pi_\lambda t = \varphi\pi_{f_\lambda} t \in B$ as long as defined. By Proposition 2.3, this means that $\varphi\pi_{f_\lambda}[0, A(\varphi, f_\lambda)[\subset B$. Since $A(\varphi, f_\lambda) = A(\varphi, \widetilde{f}_\lambda)$ and $\varphi\pi_\lambda t = \varphi\pi_{\widetilde{f}_\lambda} t$ for $t \in [0, A(\varphi, \widetilde{f}_\lambda)[$, we obtain $\varphi\pi_{\widetilde{f}_\lambda}[0, A(\varphi, \widetilde{f}_\lambda)[\subset B$. Thus by (4.21), $A(\varphi, \widetilde{f}_\lambda) = \infty$, so $A(\varphi, f_\lambda) = \infty$. This proves the first claim.

We also claim that B is π_λ-admissible. In fact, let $(t_n)_n$ be a sequence in $[0,\infty[$ with $t_n \to \infty$ as $n \to \infty$, and $(\varphi_n)_n$ be a sequence in $Z_\lambda = C(I,\mathcal{M})$ such that $\varphi_n\pi_\lambda t_n$ is defined and $\varphi_n\pi_\lambda[0,t_n] \subset B$ for each $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$ and $t \in [0,t_n]$, $\varphi_n\pi_{\widetilde{f}_\lambda} t = \varphi_n\pi_\lambda t$ is defined and $\varphi_n\pi_{\widetilde{f}_\lambda}[0,t_n] \subset B$. By (4.21), there exist a $\varphi \in C(I,E)$ and a subsequence $(t_n^1, \varphi_n^1)_n$ of $(t_n, \varphi_n)_n$ such that $|\varphi_n^1\pi_{\widetilde{f}_\lambda} t_n^1 - \varphi|_E^\diamond \to 0$ as $n \to \infty$. Since B is closed in $C(I,E)$, it follows that $\varphi \in B \subset C(I,\mathcal{M})$. Therefore, $|\varphi_n^1\pi_\lambda t_n^1 - \varphi|_E^\diamond \to 0$ as $n \to \infty$ and $\varphi \in B$, so the endpoint sequence $(\varphi_n\pi_\lambda t_n)_n$ has a subsequence converging in $(C(I,\mathcal{M}), d_E^\diamond)$, and hence in $(C(I,\mathcal{M}), d_{\mathfrak{g}}^\diamond)$, to an element of B. This proves our second claim. Thus, indeed, $B = [N]^{\lambda,\beta}$ is strongly π_λ-admissible.

(3) *Whenever $(\lambda_n)_n$ is a sequence in $\Lambda_0 \setminus \{\lambda_0\}$, $(t_n)_n$ is a sequence in $[0,\infty[$ with $\lambda_n \to \lambda_0$, $t_n \to \infty$ as $n \to \infty$ and $(\varphi_n)_n$ is a sequence such that $\varphi_n \in Z_{\lambda_n}$, $\varphi_n\pi_{\lambda_n} t_n$ is defined and $\varphi_n\pi_{\lambda_n}[0,t_n] \subset [N]^{\lambda_n,\beta} = B$ for each $n \in \mathbb{N}$, then there exist a $u_0 \in N$ and a subsequence $(\lambda_n^1, t_n^1, \varphi_n^1)_n$ of the sequence $(\lambda_n, t_n, \varphi_n)_n$ such that $\Gamma_{\lambda_n^1}(\varphi_n^1\pi_{\lambda_n^1} t_n^1, \Theta_{\lambda_n^1} u_0) \to 0$ as $n \to \infty$.*

In fact, let $(\lambda_n)_n$, $(t_n)_n$ and $(\varphi_n)_n$ be as in the premise of (3). By (4.18)–(4.20) and known results (cf. [12, Theorem I.4.2]), the set B is $(\pi_{\widetilde{f}_{\lambda_n}})_n$-admissible. Since $t_n - r \to \infty$ as $n \to \infty$, it follows that there exist a $\varphi \in C(I, E)$ and a subsequence $(\lambda_n^1, t_n^1, \varphi_n^1)_n$ of $(\lambda_n, t_n, \varphi_n)_n$ such that

$$(4.22) \qquad |\varphi_n^1 \pi_{\widetilde{f}_{\lambda_n^1}}(t_n^1 - r) - \varphi|_E^\diamond \to 0 \quad \text{as } n \to \infty.$$

Since $\varphi\pi_{\widetilde{f}} r$ is defined by Lemma 4.2 and Proposition 2.3, it follows, as in the proof of Lemma 4.3, that

$$(4.23) \qquad |\varphi_n^1 \pi_{\widetilde{f}_{\lambda_n^1}} t_n^1 - \varphi\pi_{\widetilde{f}} r|_E^\diamond \to 0 \quad \text{as } n \to \infty.$$

Moreover, (4.22) implies that $\varphi \in B \subset C(I, \mathcal{M})$, so $\varphi\pi_{\widetilde{f}} r = \varphi\pi_f r$ and so, by (4.22),

$$(4.24) \qquad |\varphi_n^1 \pi_{f_{\lambda_n^1}} t_n^1 - \varphi\pi_f r|_E^\diamond \to 0 \quad \text{as } n \to \infty.$$

Let $x = x(\varphi, f)$, $v_0 = \varphi(0)$ and $u_0 = v_0\pi r$. An application of Lemma 4.2 shows that $x(t) = v_0\pi t$ for all $t \in [0, r]$. Hence for all $s \in I$ we have $x_r(s) = x(r+s) = v_0\pi(r+s) = (v_0\pi r)\pi s = (\Theta u_0)(s)$, so $\varphi\pi_f r = x_r = \Theta u_0$. Since $\Theta u_0 = \varphi\pi_f r \in B$ by (4.24), it follows from the definition of B that $p(\Theta u_0) \in N$, so $u_0 \in N$. Altogether we obtain $d_E^\diamond(\varphi_n^1\pi_{\lambda_n^1} t_n^1, \Theta u_0) = |\varphi_n^1 \pi_{f_{\lambda_n^1}} t_n^1 - \Theta u_0|_E^\diamond \to 0$ as $n \to \infty$. Consequently, by the equivalence of the metrics $d_E^\diamond$ and $d_{\mathfrak{g}}^\diamond$ on $C(I, \mathcal{M})$, $\Gamma_{\lambda_n^1}(\varphi_n^1\pi_{\lambda_n^1} t_n^1, \Theta_{\lambda_n^1} u_0) = d_{\mathfrak{g}}^\diamond(\varphi_n^1\pi_{\lambda_n^1} t_n^1, \Theta u_0) \to 0$ as $n \to \infty$. This proves (3) and completes the proof. ∎

Proof of Theorem 3.3. If N is a nonempty set, then, using Lemmas 4.3, 4.4 and 4.1 we obtain Theorem 3.3 by an application of [4, Theorem 3.9 and Corollary 3.10]. If $N = \emptyset$, then the results trivially hold. ∎

Proof of Theorem 3.6. This follows from [5, Corollaries 3.21 and 3.22] and [6, Theorem 2.6]. ∎

We end this section with

Proof of Proposition 3.4. (a) As $\mathcal{M}$ is compact, the flow π on $\mathcal{M}$ is global, the set $\mathcal{M}$ is an isolated π-invariant set in $\mathcal{M}$, $N = \mathcal{M}$ is a compact isolating neighborhood of $\mathcal{M}$ and $(N_1, N_2) = (\mathcal{M}, \emptyset)$ is an index pair in N.

Thus the (co)homology Conley index of $(\pi_g, \mathcal{M})$ is given by (the isomorphism class of) $H(N_1/N_2, \{[N_2]\})$. Now, by definition,

$$H(N_1/N_2, \{[N_2]\}) = H(\mathcal{M} \,\dot{\cup}\, \{p\}, \{p\}),$$

where p is any point not in $\mathcal{M}$ and $\mathcal{M} \,\dot{\cup}\, \{p\}$ carries the sum topology. Hence the set $\{p\}$ is both open and closed in $\mathcal{M} \,\dot{\cup}\, \{p\}$ and so the excision axiom implies that $H(\mathcal{M} \,\dot{\cup}\, \{p\}, \{p\}) = H(\mathcal{M}, \emptyset) = H(\mathcal{M})$, so

$$(4.25) \qquad H(h(\pi_g, \mathcal{M})) = H(\mathcal{M} \,\dot{\cup}\, \{p\}, \{p\}) \cong H(\mathcal{M}).$$

(b) We first prove that

$$H(B) \cong H(\mathcal{M}). \tag{4.26}$$

In fact, we have the immersion $\alpha\colon \mathcal{M} \to B = C(I, \mathcal{M})$, $u \mapsto \varphi_u \equiv u$, and the projection $p\colon B \to \mathcal{M}$, $\varphi \mapsto u = \varphi(0)$. The composite map $\alpha \circ p$ is homotopic to Id_B by means of the homotopy $F\colon [0,1] \times B \to B$, $F(\tau, \varphi) = \psi$, where $\psi(s) = \varphi((1-\tau)s)$ for $s \in I = [-r, 0]$. The homotopy axiom thus implies that $H(\alpha) \circ H(\beta) = \mathrm{Id}_{H(B)}$. Now $\beta \circ \alpha = \mathrm{Id}_{\mathcal{M}}$, so $H(\beta) \circ H(\alpha) = \mathrm{Id}_{H(\mathcal{M})}$. Therefore α induces an isomorphism $H(B)$ onto $H(\mathcal{M})$, proving (4.26).

Now our assumptions imply that B is closed and bounded in $C(I, E)$ and $f(B)$ is bounded in E. Therefore, proceeding exactly as in the proof of Lemma 4.16(2) we show that B is strongly π_f-admissible. Clearly, B is an isolating neighborhood of S_f and the pair $(B, \emptyset)$ is an index pair in B. Thus, choosing any point p not in B and proceeding as in the proof of (a), we see that

$$Hh(\pi_f, S_f) = H(B \,\dot{\cup}\, \{p\}, \{p\}) \cong H(B). \tag{4.27}$$

This, together with (4.26) shows that $Hh(\pi_f, S_f) \cong H(\mathcal{M})$. This completes the proof. ∎

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