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Parametrized Measuring and Club Guessing

by

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Abstract. We introduce *Strong Measuring*, a maximal strengthening of J. T. Moore's Measuring principle, which asserts that every collection of fewer than continuum many closed bounded subsets of ω_1 is measured by some club subset of ω_1. The consistency of Strong Measuring with the negation of CH is shown, solving an open problem from Asperó and Mota's 2017 preprint on Measuring. Specifically, we prove that Strong Measuring follows from MRP together with Martin's Axiom for σ-centered forcings, as well as from BPFA. We also consider strong versions of Measuring in the absence of the Axiom of Choice.

Club guessing principles at ω_1 are well-studied natural weakenings of Jensen's $\diamondsuit$ principle. Presented in a general form, they assert the existence of a sequence $\vec{C} = \langle c_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, where each c_α is a club of α, such that $\vec{C}$ *guesses* clubs of ω_1 in some suitable sense. $\vec{C}$ guessing a club D of ω_1 usually means that there is some (equivalently, stationarily many) $\delta \in D$ such that $c_\delta \cap D$ is a suitably large subset of c_δ; for example, we could require that $c_\delta \subseteq D$, in which case the resulting statement is called *club guessing*, or that $c_\delta \cap D$ is cofinal in δ, in which case we call the resulting statement *very weak club guessing*.

Unlike the case of their versions at cardinals higher than ω_1, for which there are non-trivial positive ZFC theorems (see, for example, [14]), club guessing principles at ω_1 are independent of ZFC. On the one hand, all of these principles obviously follow from $\diamondsuit$, and hence they hold in L, and they can always be forced by countably closed forcing. On the other hand, classical forcing axioms at the level of ω_1, such as the Proper Forcing Axiom (PFA), imply the failure of even the weakest of these principles. It should nevertheless be noted that Martin's Axiom + $\neg\mathsf{CH}$ is compatible with Club

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Guessing. This is because Martin's Axiom can always be forced by a c.c.c. forcing, and the fact that every club of ω_1 in a generic extension via a c.c.c. forcing contains a club of ω_1 from the ground model implies that a club guessing sequence from the ground model remains club guessing in the extension. (On the other hand, this is of course not the case for $\diamondsuit$ since the negation of CH violates $\diamondsuit$.)

Measuring is a particularly strong failure of Club Guessing introduced by J. T. Moore [8]. Let X and Y be countable subsets of ω_1 with the same supremum δ. We say that X *measures* Y if there exists $\beta < \delta$ such that $X \setminus \beta$ is either contained in Y, or disjoint from it. *Measuring* is the statement that for any sequence $\langle c_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, where each c_α is a closed subset of α, there exists a club $D \subseteq \omega_1$ such that for all limit points $\delta \in D$ of D, $D \cap \delta$ measures c_δ.

Measuring can be viewed as a strong negation of Club Guessing since, as is easy to see, it implies the failure of Very Weak Club Guessing. Measuring follows from the Mapping Reflection Principle (MRP), and therefore from PFA, and it can be forced over any model of ZFC.

From Measuring as a vantage point, one can attempt to consider even stronger failures of Club Guessing. In this vein, the following parametrized family of strengthenings of Measuring was considered in [2].

Definition. For a cardinal κ, let $\mathsf{Measuring}_{<\kappa}$ denote the statement that whenever $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ is a sequence such that each $\mathcal{C}_\alpha$ is a family of fewer than κ many closed subsets of α, there exists a club $D \subseteq \omega_1$ such that for every limit point δ of D and every $c \in \mathcal{C}_\delta$, $D \cap \delta$ measures c. For a cardinal λ, let $\mathsf{Measuring}_\lambda$ denote $\mathsf{Measuring}_{<\lambda^+}$.

In the situation given by the above definition, we say that D *measures* $\vec{\mathcal{C}}$. We also define *Strong Measuring* to be the statement $\mathsf{Measuring}_{<2^\omega}$.

In the present article we contribute to the body of information on Measuring and related strong failures of Club Guessing (see also [8], [3], [5], [6], and [2]). One of the questions left unresolved in [2] is whether $\mathsf{Measuring}_{\omega_1}$ is consistent at all. Answering this question was the motivation for the work in the present article. Our main result is that Strong Measuring + $\neg\mathsf{CH}$ is consistent. In fact, this statement follows from MRP + Martin's Axiom for the class of σ-centered posets, and also from BPFA (1). We also show the failure, in ZFC, of $\mathsf{Measuring}_\kappa$, where κ is among some of the classical cardinal characteristics of the continuum. Finally, we consider very strong versions of Measuring in contexts in which the Axiom of Choice fails.

(1) We can also prove the consistency of Strong Measuring with the continuum being arbitrarily large. This result will appear in a sequel to the present article.

1. Background. We review some background material and notation which is needed for understanding the paper. Let $\mathfrak{c}$ denote the cardinality of the continuum 2^ω. A set $S \subseteq [\omega]^\omega$ is a *splitting family* if for any infinite set $x \subseteq \omega$, there exists $A \in S$ such that A *splits* x in the sense that both $x \cap A$ and $x \setminus A$ are infinite. The *splitting number* $\mathfrak{s}$ is the least cardinality of some splitting family. Given functions $f, g : \omega \to \omega$, we say that g dominates f if $f(n) < g(n)$ for all $n < \omega$. We say that g eventually dominates f if there is some $m < \omega$ such that $f(n) < g(n)$ for all $n > m$. A family $B \subseteq \omega^\omega$ is *bounded* if there exists a function $g \in \omega^\omega$ which eventually dominates every member of B, and otherwise it is *unbounded*. The *bounding number* $\mathfrak{b}$ is the least cardinality of some unbounded family. Both cardinal characteristics $\mathfrak{s}$ and $\mathfrak{b}$ are uncountable.

Let $\mathbb{P}$ be a forcing poset. A set $X \subseteq \mathbb{P}$ is *centered* if every finite subset of X has a lower bound. We say that $\mathbb{P}$ is *σ-centered* if it is a union of countably many centered sets. *Martin's Axiom for σ-centered forcings* (MA(σ-centered)) is the statement that for any σ-centered forcing $\mathbb{P}$ and any collection of fewer than $\mathfrak{c}$ many dense subsets of $\mathbb{P}$, there exists a filter on $\mathbb{P}$ which meets each dense set in the collection. More generally, define $\mathfrak{m}$(σ-centered) to be the least cardinality of a collection of dense subsets of some σ-centered forcing poset for which there does not exist a filter which meets each dense set in the collection. Note that MA(σ-centered) is equivalent to the statement that $\mathfrak{m}$(σ-centered) equals $\mathfrak{c}$.

The *Bounded Proper Forcing Axiom* (BPFA) is the statement that whenever $\mathbb{P}$ is a proper forcing and $\langle A_i : i < \omega_1 \rangle$ is a sequence of maximal antichains of $\mathbb{P}$ each of size at most ω_1, then there exists a filter on $\mathbb{P}$ which meets each A_i [9]. We note that BPFA implies $\mathfrak{c} = \omega_2$ [12, Section 5]. It easily follows that BPFA implies Martin's Axiom, and in particular implies MA(σ-centered). The forcing axiom BPFA is equivalent to the statement that for any proper forcing poset $\mathbb{P}$ and any Σ_1 statement Φ with a parameter from $H(\omega_2)$, if Φ holds in a generic extension by $\mathbb{P}$, then Φ holds in the ground model [7].

An *open stationary set mapping* for an uncountable set X and regular cardinal $\theta > \omega_1$ is a function Σ whose domain is the collection of all countable elementary substructures M of $H(\theta)$ with $X \in M$, such that for all such M, $\Sigma(M)$ is an open, M-stationary subset of $[X]^\omega$. By *open* we mean open in the Ellentuck topology on $[X]^\omega$, and *M-stationary* means meeting every club subset of $[X]^\omega$ which is a member of M (see [12] for the complete details). In this article, we are only concerned with these ideas in the simplest case that $X = \omega_1$ and $\Sigma(M) \subseteq \omega_1$ for each $M \in \mathrm{dom}(\Sigma)$. In this case, being open is equivalent to being open in the topology on ω_1 with basis the collection of all open intervals of ordinals, and being M-stationary is equivalent to meeting every club subset of ω_1 in M.

For an open stationary set mapping Σ for X and θ, a *Σ-reflecting sequence* is an $\in$-increasing and continuous sequence $\langle M_i : i < \omega_1\rangle$ of countable elementary substructures of $H(\theta)$ containing X as a member, such that for all limit ordinals $\delta < \omega_1$, there exists $\beta < \delta$ such that for all $\beta \le \xi < \delta$, $M_\xi \cap X \in \Sigma(M_\delta)$. The *Mapping Reflection Principle* (MRP) is the statement that for any open stationary set mapping Σ, there exists a Σ-reflecting sequence. We will use the fact that for any open stationary set mapping Σ, there exists a proper forcing which adds a Σ-reflecting sequence [12, Section 3]. Consequently, MRP follows from PFA.

2. Parametrized Measuring and Club Guessing. Let X and Y be countable subsets of ω_1 with the same supremum δ. We say that X *measures* Y if there exists $\beta < \delta$ such that $X \setminus \beta$ is either contained in, or disjoint from, Y. *Measuring* is the statement that for any sequence $\langle c_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, where each c_α is a closed and cofinal subset of α, there exists a club $D \subseteq \omega_1$ such that for all limit points α of D, $D \cap \alpha$ measures c_α.

The next two results are due to J. T. Moore [8].

THEOREM 2.1. MRP *implies Measuring.*

THEOREM 2.2. BPFA *implies Measuring.*

We now describe parametrized forms of measuring which were introduced in [2]. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ be a sequence such that each $\mathcal{C}_\alpha$ is a collection of closed and cofinal subsets of α. A club $D \subseteq \omega_1$ is said to *measure* $\vec{\mathcal{C}}$ if for all $\alpha \in \lim(D)$ and all $c \in \mathcal{C}_\alpha$, $D \cap \alpha$ measures c.

DEFINITION 2.3. For a cardinal κ, let $\mathsf{Measuring}_{<\kappa}$ denote the statement that whenever $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ is a sequence such that each $\mathcal{C}_\alpha$ is a collection of fewer than κ many closed and cofinal subsets of α, then there exists a club $D \subseteq \omega_1$ which measures $\vec{\mathcal{C}}$. For a cardinal λ, let $\mathsf{Measuring}_\lambda$ denote $\mathsf{Measuring}_{<\lambda^+}$.

Observe that the principle Measuring is the same as $\mathsf{Measuring}_1$. If $\kappa < \lambda$, then clearly $\mathsf{Measuring}_{<\lambda}$ implies $\mathsf{Measuring}_{<\kappa}$. It is easy to see that $\mathsf{Measuring}_{\mathfrak{c}}$ is false.

DEFINITION 2.4. *Strong Measuring* is the statement that $\mathsf{Measuring}_{<\mathfrak{c}}$ holds.

Since the intersection of countably many clubs in ω_1 is club, Measuring easily implies $\mathsf{Measuring}_\omega$. In particular, Measuring together with CH implies Strong Measuring. We will prove in Section 3 the consistency of Strong Measuring together with $\neg$CH. We also observe at the end of that section that Measuring does not imply $\mathsf{Measuring}_{\omega_1}$.

PROPOSITION 2.5 ([2]). $\mathsf{Measuring}_{\mathfrak{s}}$ *is false.*

Proof. Fix a splitting family S of cardinality $\mathfrak{s}$. For each limit ordinal $\alpha < \omega_1$, fix a function $f_\alpha : \omega \to \alpha$ which is increasing and cofinal in α. For each $A \in S$, let $c_{\alpha,A} = \bigcup\{(f_\alpha(n), f_\alpha(n+1)] : n \in A\}$, which is clearly closed and cofinal in α. Let $\mathcal{C}_\alpha := \{c_{\alpha,A} : A \in S\}$. Then $\vec{\mathcal{C}} := \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ is a sequence such that for each α, $\mathcal{C}_\alpha$ is a collection of at most $\mathfrak{s}$ many closed and cofinal subsets of α.

Let $D \subseteq \omega_1$ be a club. Fix $\alpha \in \lim(D)$. We will show that there exists a member of $\mathcal{C}_\alpha$ which $D \cap \alpha$ does not measure. Define $x := \{n < \omega : D \cap (f_\alpha(n), f_\alpha(n+1)] \neq \emptyset\}$. Since $\alpha \in \lim(D)$, x is infinite. As S is a splitting family, we can fix $A \in S$ which splits x. So both $x \cap A$ and $x \setminus A$ are infinite. We claim that $D \cap \alpha$ does not measure $c_{\alpha,A}$.

Suppose for a contradiction that for some $\beta < \alpha$, $(D \cap \alpha) \setminus \beta$ is either a subset of, or disjoint from, $c_{\alpha,A}$. Since $A \cap x$ is infinite, we can fix $n \in A \cap x$ such that $f_\alpha(n) > \beta$. Then $n \in x$ implies that $D \cap (f_\alpha(n), f_\alpha(n+1)] \neq \emptyset$, and $n \in A$ implies that $(f_\alpha(n), f_\alpha(n+1)] \subseteq c_{\alpha,A}$. It follows that $(D \cap \alpha) \setminus \beta$ meets $c_{\alpha,A}$. By the choice of β, this implies that $(D \cap \alpha) \setminus \beta$ is a subset of $c_{\alpha,A}$. But $x \setminus A$ is also infinite, so we can fix $m \in x \setminus A$ such that $f_\alpha(m) > \beta$. Then $m \in x$ implies that $D \cap (f_\alpha(m), f_\alpha(m+1)] \neq \emptyset$, and $m \notin A$ implies that $(f_\alpha(m), f_\alpha(m+1)]$ is disjoint from $c_{\alpha,A}$. Thus, there is a member of $(D \cap \alpha) \setminus \beta$ which is not in $c_{\alpha,A}$, which is a contradiction. ∎

We will prove later in this section that $\mathsf{Measuring}_{\mathfrak{b}}$ is also false.

We now turn to parametrized club guessing. We recall some standard definitions. Consider a sequence $\vec{L} = \langle L_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, where each L_α is a cofinal subset of α with order type ω (that is, a *ladder system*). We say that $\vec{L}$ is a *club guessing sequence*, *weak club guessing sequence*, or *very weak club guessing sequence* if for every club $D \subseteq \omega_1$, there exists a limit ordinal $\alpha < \omega_1$ such that:

(1) $L_\alpha \subseteq D$,
(2) $L_\alpha \setminus D$ is finite, or
(3) $L_\alpha \cap D$ is infinite,

respectively. We say that *Club Guessing*, *Weak Club Guessing*, or *Very Weak Club Guessing* holds if there exists a club guessing sequence, a weak club guessing sequence, or a very weak club guessing sequence, respectively. It is well known that Measuring implies the failure of Very Weak Club Guessing (see Proposition 2.8 below).

Definition 2.6. Let $\vec{\mathcal{L}} = \langle \mathcal{L}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ be a sequence where each $\mathcal{L}_\alpha$ is a non-empty collection of cofinal subsets of α with order type ω. The sequence $\vec{\mathcal{L}}$ is said to be a *club guessing sequence*, *weak club guessing sequence*, or *very weak club guessing sequence* if for every club $D \subseteq \omega_1$, there exists a limit ordinal $\alpha < \omega_1$ and some $L \in \mathcal{L}_\alpha$ such that:

(1) $L \subseteq D$,
(2) $L \setminus D$ is finite, or
(3) $L \cap D$ is infinite,

respectively.

Definition 2.7. For a cardinal κ, let $\mathsf{CG}_{<\kappa}$, $\mathsf{WCG}_{<\kappa}$, and $\mathsf{VWCG}_{<\kappa}$ be the statements that there exists a club guessing sequence, weak club guessing sequence, or very weak club guessing sequence $\langle \mathcal{L}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, respectively, such that for each α, $|\mathcal{L}_\alpha| < \kappa$. Let CG_κ, WCG_κ, and VWCG_κ denote the statements $\mathsf{CG}_{<\kappa^+}$, $\mathsf{WCG}_{<\kappa^+}$, and $\mathsf{VWCG}_{<\kappa^+}$, respectively.

Clearly, if $\kappa < \lambda$, then $\mathsf{CG}_{<\kappa}$ implies $\mathsf{CG}_{<\lambda}$, and similarly with WCG and VWCG. Observe that Club Guessing, Weak Club Guessing, and Very Weak Club Guessing are equivalent to CG_1, WCG_1, and VWCG_1, respectively. Obviously, $\mathsf{CG}_\mathfrak{c}$ is true. The weakest forms of club guessing principles which are not provable in ZFC are when the index is $< \mathfrak{c}$.

Proposition 2.8. *For any cardinal $\kappa \geq 2$,* $\mathsf{Measuring}_{<\kappa}$ *implies the failure of* $\mathsf{VWCG}_{<\kappa}$.

Proof. Suppose for a contradiction that $\mathsf{Measuring}_{<\kappa}$ and $\mathsf{VWCG}_{<\kappa}$ both hold. Fix a very weak club guessing sequence $\vec{\mathcal{L}} = \langle \mathcal{L}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ such that each $\mathcal{L}_\alpha$ has cardinality less than κ. Observe that for each α, every member of $\mathcal{L}_\alpha$ is vacuously a closed subset of α since it has order type ω.

By $\mathsf{Measuring}_{<\kappa}$, there exists a club $D \subseteq \omega_1$ which measures $\vec{\mathcal{L}}$. Let E be the club set of indecomposable limit ordinals $\alpha > \omega$ in $\lim(D)$ such that $\mathrm{ot}(D \cap \alpha) = \alpha$. Since $\vec{\mathcal{L}}$ is a very weak club guessing sequence, there exists a limit ordinal α and $L \in \mathcal{L}_\alpha$ such that $L \cap E$ is infinite. In particular, α is a limit point of E, and hence of D.

Since D measures $\vec{\mathcal{L}}$ and $L \in \mathcal{L}_\alpha$, $D \cap \alpha$ measures L. So we can fix $\beta < \alpha$ such that $(D \cap \alpha) \setminus \beta$ is either a subset of L, or disjoint from it. Now $L \cap E$, and hence $L \cap D$, is infinite. As L has order type ω, this implies that $L \cap D$ is cofinal in α. By the choice of β, $(D \cap \alpha) \setminus \beta$ must be a subset of L. But since $\alpha \in E$, $\mathrm{ot}(D \cap \alpha) = \alpha$ and α is indecomposable, which implies that $\mathrm{ot}((D \cap \alpha) \setminus \beta) = \alpha$. As $\alpha > \omega$, this is impossible since $(D \cap \alpha) \setminus \beta$ is a subset of L and L has order type ω. ■

In particular, since Strong Measuring is consistent, so is the failure of $\mathsf{VWCG}_{<\mathfrak{c}}$. (The consistency of $\neg\mathsf{VWCG}_{<\mathfrak{c}}$ together with $\mathfrak{c}$ arbitrarily large was previously shown in [4].)

Proposition 2.9 (Hrušák, [5]). $\mathsf{VWCG}_\mathfrak{b}$ *is true.*

Proof. Fix an unbounded family $\{r_\alpha : \alpha < \mathfrak{b}\}$ in ω^ω. For each limit ordinal $\delta < \omega_1$, fix a cofinal subset C_δ of δ with order type ω and a bijection

$h_\delta : \omega \to \delta$. Let $C_\delta(n)$ denote the nth member of C_δ for all $n < \omega$. For all limit ordinals $\delta < \omega_1$ and $\alpha < \mathfrak{b}$, define

$$A^\alpha_\delta := C_\delta \cup \bigcup \{h_\delta[r_\alpha(n)] \setminus C_\delta(n) : n < \omega\}.$$

It is easy to check that for all δ and α, A^α_δ has order type ω and $\sup(A^\alpha_\delta) = \delta$. Given a club $C \subseteq \omega_1$, let δ be a limit point of C and let $g_{C,\delta} : \omega \to \omega$ be the function given by

$$g_{C,\delta}(n) = \min\{m < \omega : h_\delta(m) \in C \setminus C_\delta(n)\}.$$

Now let $\alpha < \mathfrak{b}$ be such that $r_\alpha(n) > g_{C,\delta}(n)$ for infinitely many n. It then follows that $|A^\alpha_\delta \cap C| = \omega$. ■

By Propositions 2.8 and 2.9, the following is immediate.

Corollary 2.10. $\mathsf{Measuring}_\mathfrak{b}$ *is false.*

An obvious question is whether the parametrized versions of club guessing are actually the same as the usual ones. We conclude this section by showing that they are not.

Recall that a forcing poset $\mathbb{P}$ is ω^ω*-bounding* if every function in $\omega^\omega \cap V^\mathbb{P}$ is dominated by a function in $\omega^\omega \cap V$.

Lemma 2.11 (Hrušák). *Assume that* VWCG *fails. Let* $\mathbb{P}$ *be any* ω_1*-c.c.,* ω^ω*-bounding forcing. Then* $\mathbb{P}$ *forces that* VWCG *fails.*

Proof. Since $\mathbb{P}$ is ω_1-c.c. and ω^ω-bounding, a standard argument shows that whenever $p \in \mathbb{P}$ and p forces that $\dot{b} \in \omega^\omega$, then there exists a function $b^* \in \omega^\omega$ such that p forces that b^* dominates $\dot{b}$.

Let us show that whenever $p \in \mathbb{P}$, $\delta < \omega_1$, and p forces that $\dot{X}$ is a cofinal subset of δ of order type ω, then there exists a set Y with order type ω such that p forces that $\dot{X} \subseteq Y$. To see this, fix a bijection $f : \omega \to \delta$ and a strictly increasing sequence $\langle \alpha_n : n < \omega \rangle$ cofinal in α with $\alpha_0 = 0$. We claim that there exists a $\mathbb{P}$-name $\dot{b}$ for a function from ω to ω such that p forces that for all $n < \omega$, $\dot{b}(n)$ is the least $m < \omega$ such that $\dot{X} \cap [\alpha_n, \alpha_{n+1}) \subseteq f[m]$. This is true since p forces that $\dot{X}$ has order type ω and hence that $\dot{X} \cap [\alpha_n, \alpha_{n+1})$ is finite for all $n < \omega$. Fix a function $b^* : \omega \to \omega$ such that p forces that b^* dominates $\dot{b}$. Now let

$$Y := \bigcup \{f[b^*(n)] \cap [\alpha_n, \alpha_{n+1}) : n < \omega\}.$$

It is easy to check that Y has order type ω and p forces that $\dot{X} \subseteq Y$.

Now we are ready to prove the lemma. So suppose that $p \in \mathbb{P}$ forces that $\langle \dot{X}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ is a very weak club guessing sequence. By the previous paragraph, for each limit ordinal $\alpha < \omega_1$ we can fix a cofinal subset Y_α of α with order type ω such that p forces that $\dot{X}_\alpha \subseteq Y_\alpha$. We claim that $\langle Y_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ is a very weak club guessing sequence in the ground model, which completes the proof. So consider a club $C \subseteq \omega_1$. Then C is still a club in $V^{\mathbb{P}}$. Fix $q \leq p$ and a limit ordinal $\alpha < \omega_1$ such that q forces that $\dot{X}_\alpha \cap C$ is infinite. Then clearly q forces that $Y_\alpha \cap C$ is infinite, so in fact, $Y_\alpha \cap C$ is infinite. ∎

PROPOSITION 2.12. *It is consistent that* $\neg\mathsf{VWCG}$ *and* CG_{ω_1} *both hold.*

Proof. Let V be a model in which CH holds and VWCG fails. Such a model was shown to exist by Shelah [13]. Let $\mathbb{P}$ be an ω_1-c.c., ω^ω-bounding forcing poset which adds at least ω_2 many reals; for example, random real forcing with product measure is such a forcing. We claim that in $V^{\mathbb{P}}$, CG_{ω_1} holds but VWCG fails. By Lemma 2.11, VWCG is false in $V^{\mathbb{P}}$. In V, define $\vec{\mathcal{L}} = \langle \mathcal{L}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ by letting $\mathcal{L}_\alpha$ be the collection of all cofinal subsets of α with order type ω. Since CH holds, the cardinality of each $\mathcal{L}_\alpha$ is ω_1. If C is a club subset of ω_1 in $V^{\mathbb{P}}$, then since $\mathbb{P}$ is ω_1-c.c., there is a club $D \subseteq \omega_1$ in V such that $D \subseteq C$. In V, fix $d \subseteq D$ with order type ω, and let $\alpha := \sup(d)$. Then $d \in \mathcal{L}_\alpha$ and $d \subseteq C$. Thus, $\vec{\mathcal{L}}$ witnesses that CG_{ω_1} holds in $V^{\mathbb{P}}$. ∎

3. The consistency of Strong Measuring and $\neg\mathsf{CH}$. As we previously mentioned, Measuring is equivalent to $\mathsf{Measuring}_\omega$, and therefore under CH, Measuring is equivalent to Strong Measuring. In this section we establish the consistency of Strong Measuring with the negation of CH. More precisely, we will prove that MRP together with MA (σ-centered) implies Strong Measuring, and BPFA implies Strong Measuring. Recall that both MRP and BPFA imply that $\mathfrak{c} = \omega_2$ (see [12]).

A set M is *suitable* if for some regular cardinal $\theta > \omega_1$, M is a countable elementary substructure of $H(\theta)$. We will follow the conventions introduced in Section 1 that the properties "open" and "M-stationary" refer to open and M-stationary subsets of ω_1 (where ω_1 is considered as a subspace of $[\omega_1]^\omega$).

PROPOSITION 3.1. *Assume that M is suitable. Let $\delta := M \cap \omega_1$. Suppose that $\mathcal{Y}$ is a collection of open subsets of δ such that for any finite set $a \subseteq \mathcal{Y}$, $\bigcap a$ is M-stationary. Then there exists a σ-centered forcing $\mathbb{P}$ and a collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of size at most $|\mathcal{Y}| + \omega$ such that whenever G is a filter on $\mathbb{P}$ in some outer model W of V with $\omega_1^V = \omega_1^W$ which meets each member of $\mathcal{D}$, then there exists a set $z \subseteq \delta$ in W which is open, M-stationary, and such that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in δ.*

Proof. Define a forcing poset $\mathbb{P}$ to consist of conditions which are pairs (x, a), where x is an open and bounded subset of δ in M and a is a finite subset of $\mathcal{Y}$. Let $(y, b) \leq (x, a)$ if y is an end-extension of x, $a \subseteq b$, and $y \setminus x \subseteq \bigcap a$.

Since M is countable, there are only countably many possibilities for the first component of a condition. If $(x, a_0), \ldots, (x, a_n)$ are finitely many conditions with the same first component, then clearly $(x, a_0 \cup \cdots \cup a_n)$ is a condition in $\mathbb{P}$ which is below each of the conditions $(x, a_0), \ldots, (x, a_n)$. It follows that $\mathbb{P}$ is σ-centered.

For each $X \in \mathcal{Y}$, let D_X denote the set of conditions (x, a) such that $X \in a$. Observe that D_X is dense. For every club C of ω_1 which is a member of M, let E_C denote the set of conditions (x, a) such that $x \cap C$ is non-empty. We claim that E_C is dense. Let (x, a) be a condition. Since $\bigcap a$ is M-stationary and $\lim(C) \setminus (\sup(x) + 1)$ is a club subset of ω_1 in M, we can find a limit ordinal α in $C \cap \bigcap a$ which is in the interval $(\sup(x), \delta)$. Since $\alpha \in \bigcap a$ and $\bigcap a$ is open, we can find $\beta < \gamma < \delta$ such that $\alpha \in (\beta, \gamma) \subseteq \bigcap a$. As $\sup(x) + 1 < \alpha$, without loss of generality $\sup(x) < \beta$. By elementarity, the interval $b := (\beta, \gamma)$ is in M. It follows that $(x \cup b, a)$ is a condition, $x \cup b$ end-extends x, and $(x \cup b) \setminus x = b \subseteq \bigcap a$. Thus, $(x \cup b, a) \leq (x, a)$, and since $\alpha \in C$, $(x \cup b, a) \in E_C$.

Let $\mathcal{D}$ denote the collection of all dense sets of the form D_X where $X \in \mathcal{Y}$, or E_C where C is a club subset of ω_1 belonging to M. Then $|\mathcal{D}| \leq |\mathcal{Y}| + \omega$. Let G be a filter on $\mathbb{P}$ in some outer model W with $\omega_1^V = \omega_1^W$ which meets each dense set in $\mathcal{D}$. Define $z := \bigcup\{x : \exists a \ (x, a) \in G\}$. Note that since z is a union of open sets, it is open (using the fact that being open is absolute between V and W). For each club $C \subseteq \omega_1$ which lies in M, there exists a condition (x, a) which belongs to $G \cap E_C$, and thus $x \cap C \neq \emptyset$. Therefore, $z \cap C \neq \emptyset$. Hence, z is M-stationary.

It remains to show that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in δ. Consider $X \in \mathcal{Y}$. Then we can fix $(x, a) \in G \cap D_X$, which means that $X \in a$. Now the definition of the ordering on $\mathbb{P}$ together with the fact that G is a filter easily implies that $z \setminus x \subseteq X$. Therefore, $z \setminus X \subseteq x$, and hence $z \setminus X$ is bounded in δ. ∎

COROLLARY 3.2. *Assume that M is suitable. Let $\delta := M \cap \omega_1$. Suppose that $\mathcal{Y}$ is a collection of less than $\mathfrak{m}(\sigma$-centered$)$ many open subsets of δ such that for any finite set $a \subseteq \mathcal{Y}$, $\bigcap a$ is M-stationary. Then there exists a set $z \subseteq \delta$ which is open, M-stationary, and such that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in δ.*

Proof. Fix a σ-centered forcing $\mathbb{P}$ and a collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of size at most $|\mathcal{Y}| + \omega$ as described in Proposition 3.1. Since $\mathfrak{m}(\sigma\text{-centered})$ is uncountable, $|\mathcal{D}| < \mathfrak{m}(\sigma\text{-centered})$. Hence, there exists a filter G on $\mathbb{P}$ which meets each dense set in $\mathcal{D}$. By Proposition 3.1, there exists a set $z \subseteq \delta$ which is open, M-stationary, and such that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in δ. ∎

PROPOSITION 3.3. *Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ be a sequence such that each $\mathcal{C}_\alpha$ is a collection of less than $\mathfrak{m}(\sigma\text{-centered})$ many closed and cofinal subsets of α. Then there exists an open stationary set mapping Σ such that,* *if W is any outer model with the same ω_1 in which there exists a Σ-reflecting sequence, then there exists in W a club subset of ω_1 which measures $\vec{\mathcal{C}}$.*

Proof. For each limit ordinal $\alpha < \omega_1$, let $\mathcal{D}_\alpha := \{\alpha \setminus c : c \in \mathcal{C}_\alpha\}$. Observe that each $\mathcal{D}_\alpha$ is a collection of fewer than $\mathfrak{m}(\sigma\text{-centered})$ many open subsets of α.

We will define Σ to have domain the collection of all countable elementary substructures M of $H(\omega_2)$. Consider such an M; we will define $\Sigma(M)$. Note that M is suitable. Let $\delta := M \cap \omega_1$. We consider two cases.

Case 1: There does not exist a member of $\mathcal{D}_\delta$ which is M-stationary. Define $\Sigma(M) := \delta$, which is clearly open and M-stationary.

Case 2: There is some member of $\mathcal{D}_\delta$ which is M-stationary. A straightforward application of Zorn's lemma implies that there exists a non-empty set $\mathcal{Y}_M \subseteq \mathcal{D}_\delta$ such that for any $a \in [\mathcal{Y}_M]^{<\omega}$, $\bigcap a$ is M-stationary, and moreover $\mathcal{Y}_M$ is a maximal subset of $\mathcal{D}_\delta$ with this property. Since $\mathcal{Y}_M \subseteq \mathcal{D}_\delta$, $|\mathcal{Y}_M| < \mathfrak{m}(\sigma\text{-centered})$. So the collection $\mathcal{Y}_M$ satisfies the assumptions of Corollary 3.2. It follows that there exists a set $z_M \subseteq \delta$ which is open, M-stationary, and such that for all $X \in \mathcal{Y}_M$, $z_M \setminus X$ is bounded in δ. Now define $\Sigma(M) := z_M$.

This completes the definition of Σ. Consider an outer model W of V with the same ω_1, and assume that in W there exists a Σ-reflecting sequence $\langle M_\delta : \delta < \omega_1\rangle$. Let $\alpha_\delta := M_\delta \cap \omega_1$ for all $\delta < \omega_1$. Let D be the club set of $\delta < \omega_1$ such that $\alpha_\delta = \delta$. We claim that D measures $\vec{\mathcal{C}}$.

Consider $\delta \in \lim(D)$. Then $\delta = \alpha_\delta = M_\delta \cap \omega_1$. Let $M := M_\delta$. We first claim that if $c \in \mathcal{C}_\delta$ and $\delta \setminus c$ is not M-stationary, then for some $\beta < \delta$, $(D \cap \delta) \setminus \beta \subseteq c$. Fix a club subset E of ω_1 in M which is disjoint from $\delta \setminus c$. By the continuity of the Σ-reflecting sequence, there exists $\beta < \delta$ such that $E \in M_\beta$. We claim that $(D \cap \delta) \setminus \beta \subseteq c$. Let $\xi \in (D \cap \delta) \setminus \beta$. Then $E \in M_\xi$, and hence by elementarity, $\xi = M_\xi \cap \omega_1 \in E$. Since E is disjoint from $\delta \setminus c$, $\xi \in c$.

We split the argument according to the two cases in the definition of $\Sigma(M)$. In the first case, there does not exist a member of $\mathcal{D}_\delta$ which is M-stationary. Consider $c \in \mathcal{C}_\delta$. Then $\delta \setminus c$ is not M-stationary. By the previous paragraph, there exists $\beta < \delta$ such that $(D \cap \delta) \setminus \beta \subseteq c$.

In the second case, there exists a member of $\mathcal{D}_\delta$ which is M-stationary. Consider $c \in \mathcal{C}_\delta$. Then $X := \delta \setminus c \in \mathcal{D}_\delta$. We consider two possibilities.

First, assume that X is in $\mathcal{Y}_M$. By the choice of $\mathcal{Y}_M$ and z_M, we know that $z_M \setminus X$ is bounded in δ. So fix $\beta_0 < \delta$ such that $z_M \setminus \beta_0 \subseteq X$. By the definition of a Σ-reflecting sequence, there exists $\beta_1 < \delta$ such that for all $\beta_1 \le \xi < \delta$, $M_\xi \cap \omega_1 \in \Sigma(M) = z_M$. Let $\beta := \max\{\beta_1, \beta_2\}$. Consider $\xi \in (D \cap \delta) \setminus \beta$. Then $\xi \ge \beta_1$ implies that $\xi = M_\xi \cap \omega_1 \in z_M$. So $\xi \in z_M \setminus \beta_0 \subseteq X = \delta \setminus c$.

Now assume that X is not in $\mathcal{Y}_M$. By the maximality of $\mathcal{Y}_M$, there exists a set $a \in [\mathcal{Y}_M]^{<\omega}$ such that $X \cap \bigcap a$ is not M-stationary. Fix a club E in M which is disjoint from $X \cap \bigcap a$. By the continuity of the Σ-reflecting sequence, there exists $\beta < \delta$ such that $E \in M_\beta$. Consider $\xi \in (D \cap \delta) \setminus \beta$. Then $E \in M_\xi$, which implies that $\xi = M_\xi \cap \omega_1 \in E$. Thus, ξ is not in $X \cap \bigcap a$. On the other hand, letting $a = \{X_0, \ldots, X_n\}$, for each $i \leq n$ the previous paragraph implies that there exists $\beta_i < \delta$ such that $(D \cap \delta) \setminus \beta_i \subseteq X_i$. Let β^* be an ordinal in δ which is larger than β and β_i for all $i \leq n$. Consider $\xi \in (D \cap \delta) \setminus \beta^*$. Then by the choice of β, $\xi \notin X \cap \bigcap a$. By the choice of the β_i's, $\xi \in \bigcap a$. Therefore, $\xi \notin X = \delta \setminus c$, which means that $\xi \in c$. Thus, $(D \cap \delta) \setminus \beta^* \subseteq c$. ∎

Corollary 3.4. *Assume* MRP *and* MA(σ-*centered*). *Then Strong Measuring holds.*

Proof. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ be a sequence such that each $\mathcal{C}_\alpha$ is a collection of fewer than $\mathfrak{c}$ many closed and cofinal subsets of α. We claim that there exists a club subset of ω_1 which measures $\vec{\mathcal{C}}$. By MA(σ-centered), $\mathfrak{m}$(σ-centered) equals $\mathfrak{c}$. So each $\mathcal{C}_\alpha$ has size less than $\mathfrak{m}$(σ-centered).

By Proposition 3.3, there exists an open stationary set mapping Σ such that if W is any outer model with the same ω_1 in which there exists a Σ-reflecting sequence, then there exists in W a club subset of ω_1 which measures $\vec{\mathcal{C}}$. Applying MRP, there exists a Σ-reflecting sequence in V. Thus, in V there exists a club subset of ω_1 which measures $\vec{\mathcal{C}}$. ∎

Corollary 3.5. *Assume* BPFA. *Then Strong Measuring holds.*

Proof. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$ be a sequence such that each $\mathcal{C}_\alpha$ is a collection of fewer than $\mathfrak{c} = \omega_2$ many closed and cofinal subsets of α. We claim that there exists a club subset of ω_1 which measures $\vec{\mathcal{C}}$. Since $\mathfrak{c} = \omega_2$, $\vec{\mathcal{C}}$ is a member of $H(\omega_2)$. Thus, the existence of a club subset of ω_2 which measures $\vec{\mathcal{C}}$ is expressible as a Σ_1 statement involving a parameter in $H(\omega_2)$. By BPFA, it suffices to show that there exists a proper forcing which forces that such a club exists.

Now BPFA implies Martin's Axiom, and in particular, that $\mathfrak{m}$(σ-centered) is equal to $\mathfrak{c}$. So each $\mathcal{C}_\alpha$ has size less than $\mathfrak{m}$(σ-centered). By Proposition 3.3, there exists an open stationary set mapping Σ such that, if W is any outer model with the same ω_1 in which there exists a Σ-reflecting sequence, then there exists in W a club subset of ω_1 which measures $\vec{\mathcal{C}}$. By [12, Section 3], there exists a proper forcing $\mathbb{P}$ which adds a Σ-reflecting sequence, so in $V^{\mathbb{P}}$ there is a club subset of ω_1 which measures $\vec{\mathcal{C}}$. ∎

We now sketch a proof that MRP alone does not imply Strong Measuring. In particular, Measuring does not imply Strong Measuring. Start with a model of CH in which there exists a supercompact cardinal κ. Construct a forcing iteration $\mathbb{P}$ in the standard way to obtain a model of MRP. To do this, fix a Laver function $f : \kappa \to V_\kappa$. Then define a countable support forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \kappa, \beta < \kappa \rangle$ as follows. Given $\mathbb{P}_\alpha$, consider $f(\alpha)$. If $f(\alpha)$ happens to be a $\mathbb{P}_\alpha$-name for some open stationary set mapping, then let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a proper forcing which adds an $f(\alpha)$-reflecting sequence. Otherwise let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{Col}(\omega_1, \omega_2)$. Now define $\mathbb{P} := \mathbb{P}_\kappa$. Arguments similar to those in the standard construction of a model of PFA can be used to show that $\mathbb{P}$ forces MRP.

The forcing for adding a Σ-reflecting sequence for a given open stationary set mapping does not add reals [12, Section 3]. In particular, it is vacuously ω^ω-bounding. The property of being proper and ω^ω-bounding is preserved under countable support forcing iterations [1, Theorem 3.5], so $\mathbb{P}$ is also ω^ω-bounding. In particular, $V \cap \omega^\omega$ is an unbounded family in $V^{\mathbb{P}}$, and it has size ω_1 since CH holds in V. It follows that the bounding number $\mathfrak{b}$ is equal to ω_1. But by Corollary 2.9, $\mathsf{Measuring}_{\mathfrak{b}}$ is false. So $\mathbb{P}$ forces that $\mathsf{Measuring}_{\omega_1}$ is false. As $\mathfrak{c} = \omega_2$ in $V^{\mathbb{P}}$, Strong Measuring fails in $V^{\mathbb{P}}$.

We also note that Strong Measuring plus $\mathfrak{c} = \omega_2$ is consistent with the existence of an ω_1-Suslin tree. Namely, both the forcing for adding a Σ-reflecting sequence for a given open stationary set mapping Σ, as well as any σ-centered forcing, preserve Suslin trees [11]. And the property of being proper and preserving a given Suslin tree is preserved under countable support forcings iterations [10]. So starting with a model in which there exists an ω_1-Suslin tree S and a supercompact cardinal κ, we can iterate forcing similar to the argument in the preceding paragraphs to produce a model of $\mathsf{MA}(\sigma\text{-centered})$ plus MRP in which S is an ω_1-Suslin tree. By Corollary 3.4, Strong Measuring holds in that model.

4. Measuring without the Axiom of Choice. Another natural way to strengthen $\mathsf{Measuring}$ is to allow, in the sequence to be measured, not just closed sets, but also sets of higher Borel complexity. This line of strengthenings of $\mathsf{Measuring}$ was also considered in [2]. For completeness, we are including here the corresponding observations.

The version of $\mathsf{Measuring}$ where one considers sequences $\vec{X} = \langle X_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$, with each X_α an open subset of α in the order topology, is of course equivalent to $\mathsf{Measuring}$. A natural next step would therefore be to consider sequences in which each X_α is a countable union of closed sets. This is obviously the same as allowing each X_α to be an arbitrary subset of α. Let us call the corresponding statement $\mathsf{Measuring}^*$:

Definition 4.1. $\mathsf{Measuring}^*$ holds if and only if for every sequence $\vec{X} = \langle X_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim} \rangle$, if $X_\alpha \subseteq \alpha$ for each α, then there is some club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of D, $D \cap \delta$ measures X_δ.

It is easy to see that $\mathsf{Measuring}^*$ is false in ZFC. In fact, given a stationary and co-stationary $S \subseteq \omega_1$, there is no club of ω_1 measuring $\vec{X} = \langle S \cap \alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$. The reason is that if D is any club of ω_1, then both $D \cap S \cap \delta$ and $(D \cap \delta) \setminus S$ are cofinal subsets of δ for each δ in the club of limit points in ω_1 of both $D \cap S$ and $D \setminus S$.

The status of $\mathsf{Measuring}^*$ is more interesting in the absence of the Axiom of Choice. Let $\mathcal{C}_{\omega_1} = \{X \subseteq \omega_1 : C \subseteq X \text{ for some club } C \text{ of } \omega_1\}$.

OBSERVATION 4.2. (ZF + "$\mathcal{C}_{\omega_1}$ *is a normal filter on* ω_1") *Suppose* $\vec{X} = \langle X_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ *is such that:*

(1) $X_\delta \subseteq \delta$ *for each* δ.
(2) *For each club* $C \subseteq \omega_1$,

   (a) *there is some* $\delta \in C$ *such that* $C \cap X_\delta \neq \emptyset$, *and*
   (b) *there is some* $\delta \in C$ *such that* $(C \cap \delta) \setminus X_\delta \neq \emptyset$.

Then there is a stationary and co-stationary subset of ω_1 *definable from* $\vec{X}$.

Proof. We have two possible cases:

CASE 1: For all $\alpha < \omega_1$, either

- $W^0_\alpha = \{\delta < \omega_1 : \alpha \notin X_\delta\}$ is in $\mathcal{C}_{\omega_1}$, or
- $W^1_\alpha = \{\delta < \omega_1 : \alpha \in X_\delta\}$ is in $\mathcal{C}_{\omega_1}$.

For each $\alpha < \omega_1$, let W_α be W^ϵ_α for the unique $\epsilon \in \{0,1\}$ such that $W^\epsilon_\alpha \in \mathcal{C}_{\omega_1}$, and let $W^* = \Delta_{\alpha<\omega_1} W_\alpha \in \mathcal{C}_{\omega_1}$. Then $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*. It then follows, by (2), that $S = \bigcup_{\delta \in W^*} X_\delta$, which of course is definable from $\vec{C}$, is a stationary and co-stationary subset of ω_1. Indeed, suppose $C \subseteq \omega_1$ is a club, and let us fix a club $D \subseteq W^*$. Then there is some $\delta \in C \cap D$ and some $\alpha \in C \cap D \cap X_\delta$. But then $\alpha \in S$ since $\delta \in W^*$ and $\alpha \in W^* \cap X_\delta$. There is also some $\delta \in C \cap D$ and some $\alpha \in C \cap D$ such that $\alpha \notin X_\delta$, which implies that $\alpha \notin S$ by a symmetrical argument, using the fact that $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*.

CASE 2: There is some $\alpha < \omega_1$ such that both W^0_α and W^1_α are stationary subsets of ω_1. But now we can let S be W^0_α, where α is the first such that W^0_α is stationary and co-stationary. ∎

It is worth comparing Observation 4.2 with Solovay's classic result that an ω_1-sequence of pairwise disjoint stationary subsets of ω_1 is definable from any given ladder system on ω_1 (working in the same theory).

COROLLARY 4.3. (ZF + "$\mathcal{C}_{\omega_1}$ *is a normal filter on* ω_1") *The following are equivalent:*

(1) $\mathcal{C}_{\omega_1}$ *is an ultrafilter on* ω_1.
(2) $\mathsf{Measuring}^*$.

(3) *For every sequence* $\langle X_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$*, if* $X_\alpha \subseteq \alpha$ *for each* α*, then there is a club* $C \subseteq \omega_1$ *such that either*

- $C \cap \delta \subseteq X_\delta$ *for every* $\delta \in C$*, or*
- $C \cap X_\delta = \emptyset$ *for every* $\delta \in C$*.*

Proof. (3) trivially implies (2), and by Observation 4.2, (1) implies (3). Finally, to see that (2) implies (1), note that the argument right after the definition of Measuring* uses only ZF together with the regularity of ω_1 and the negation of (1). ∎

In particular, the strong form of Measuring* given by (3) in Corollary 4.3 follows from ZF together with the Axiom of Determinacy.

We finish this digression into set theory without the Axiom of Choice by observing that any attempt to parametrize Measuring*, in the same vein as we did with Measuring, gives rise to principles vacuously equivalent to Measuring* itself, at least when the parametrization is done with the alephs ([2]).

Specifically, given an aleph κ, let us define $\mathsf{Measuring}^*_\kappa$ as the statement that for every sequence $\langle \mathcal{X}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$, if each $\mathcal{X}_\alpha$ is a set of cardinality at most κ consisting of subsets of α, then there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of D, $D \cap \delta$ measures X for all $X \in \mathcal{X}_\delta$. Then $\mathsf{Measuring}^*_\omega$ is clearly equivalent to Measuring* under ZF together with the normality of $\mathcal{C}_{\omega_1}$ and the Axiom of Choice for countable families of subsets of ω_1 (which of course follows from the Axiom of Choice for countable families of sets of reals, and therefore also from ZF + AD). On the other hand, working in ZF + "$\mathcal{C}_{\omega_1}$ is a normal filter on ω_1", we see that $\mathsf{Measuring}^*_{\omega_1}$ follows vacuously from Measuring* simply because under Measuring* there is no sequence $\langle \mathcal{X}_\alpha : \alpha \in \omega_1 \cap \mathrm{Lim}\rangle$ as in the definition of $\mathsf{Measuring}^*_{\omega_1}$ and such that $|\mathcal{X}_\alpha| = \omega_1$ for some α; indeed, Measuring* implies, over this base theory, that $\mathcal{C}_{\omega_1}$ is an ultrafilter (Corollary 4.3), and if $\mathcal{C}_{\omega_1}$ is an ultrafilter then there is no ω_1-sequence of distinct reals, whereas the existence of a family of size ω_1 consisting of subsets of some fixed countable ordinal clearly implies that there is such a sequence.

We conclude the article with two natural questions.

Question 4.4. *Is* $\mathsf{Measuring}_{\mathsf{p}}$ *false?*

Question 4.5. *Are Measuring and Strong Measuring equivalent statements assuming Martin's Axiom?*

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