

## Gaussian primes in almost all narrow sectors

by

BINGRONG HUANG (Jinan), JIANYA LIU (Jinan)  
and ZEEV RUDNICK (Tel Aviv)

**1. Introduction.** Let  $\mathfrak{p}$  be a prime ideal in the ring  $\mathbb{Z}[i]$  of Gaussian integers. Assuming the norm  $N(\mathfrak{p}) = p = 1 \pmod{4}$  is a split prime, we can write  $p = a^2 + b^2$  with  $\alpha = a + ib$  a generator of  $\mathfrak{p}$ , unique up to multiplication by one of the units  $\pm 1, \pm i$ . This gives us an angle  $\theta_{\mathfrak{p}}$  such that  $\alpha = a + ib = \sqrt{p} e^{i\theta_{\mathfrak{p}}}$ , which is unique if we fix it to lie in  $[0, \pi/2)$ .

Hecke [6] proved that the angles  $\theta_{\mathfrak{p}}$  are equidistributed in  $[0, \pi/2)$  as  $\mathfrak{p}$  varies over prime ideals of  $\mathbb{Z}[i]$ . This means that if we fix a subinterval  $I \subseteq [0, \pi/2)$ , then as  $X \rightarrow \infty$ ,

$$(1.1) \quad \#\{\mathfrak{p} \subset \mathbb{Z}[i] : X < N(\mathfrak{p}) \leq 2X, \theta_{\mathfrak{p}} \in I\} \sim \frac{|I|}{\pi/2} \frac{X}{\log X}.$$

Recall that by the Prime Ideal Theorem,  $\#\{\mathfrak{p} \subset \mathbb{Z}[i] : X < N(\mathfrak{p}) \leq 2X\} \sim X/\log X$ .

Kubilius [8] and others studied the existence of prime angles in narrow sectors. Ricci [10] proved that (1.1) remains valid for any interval  $I \subset [0, \pi/2)$  of length  $|I| > X^{-3/10+\varepsilon}$ . By a sieve method, Harman and Lewis [5] proved the existence of prime angles in somewhat narrower sectors without an asymptotic expression. Assuming the Generalized Riemann Hypothesis (GRH), it is known that (1.1) is valid for any interval  $I \subset [0, \pi/2)$  of length  $|I| > X^{-1/2+\varepsilon}$ . It is important to point out that one cannot do better because of the existence of “forbidden regions”, for instance the interval  $(0, 1/\sqrt{X})$  does not contain any prime angle of norm less than  $X$  (in fact angles for any such integer ideals). This is a striking difference from the well studied problem of (rational) primes in short intervals, where one expects that any interval  $[x, x + y]$  will contain  $\sim y/\log x$  primes as soon as  $y \gg x^\varepsilon$ , for any

---

2010 *Mathematics Subject Classification*: Primary 11R44; Secondary 11N05.

*Key words and phrases*: angle, almost all, Gaussian prime, narrow sector, variance.

Received 31 March 2019; revised 18 June 2019.

Published online 13 January 2020.

$\varepsilon > 0$ , while similarly to our problem, the Riemann Hypothesis only gives the existence in intervals of length  $y \gg x^{1/2+o(1)}$ .

Instead of asking about all short sectors, one can ask about the existence of prime angles in “typical” sectors, that is, in almost all short sectors. Assuming GRH, Parzanchevski and Sarnak [9] and Rudnick and Waxman [11] showed that almost all sectors contain a prime angle, in fact the asymptotic formula (1.1) (at least in a smooth form) remains valid for almost all  $I$  if  $|I| > X^{-1+\varepsilon}$ , which is the most we can expect (up to log factors) since the number of prime ideals with norm about  $X$  is roughly  $X/\log X$ , hence almost all sectors shorter than  $1/X$  will contain no prime angles  $\theta_{\mathfrak{p}}$  with  $N(\mathfrak{p}) \approx X$ . Rudnick and Waxman [11] gave a precise conjecture about the asymptotic behavior of the number variance, supported by a theorem for a function field analogue of the problem.

In the present paper, we prove an unconditional result on Gaussian primes in almost all narrow sectors. We will say that a property holds for *almost all* narrow sectors  $I = (\beta, \beta + \gamma]$  (where  $\gamma \approx X^{-\rho}$ ,  $0 < \rho < 1$ , and  $X < N(\mathfrak{p}) \leq 2X$  with  $X \rightarrow \infty$ ) if it holds for all  $\beta \in [0, \pi/2]$  apart from an exceptional set of measure  $o(1)$ .

**THEOREM 1.** *Let  $0 \leq \rho < 3/5$ . Then (1.1) holds for almost all short sectors  $I \subset [0, \pi/2)$  of length  $|I| > X^{-\rho}$ .*

Our result is a consequence of an upper bound on the number variance estimated using zero-density theorems.

We recall that under RH, Selberg [13] showed (in 1943) that almost all intervals  $(x, x + \phi(x) \log^2 x]$  contain primes for any function  $\phi(x)$  tending to infinity. See [12], [7], [3, Chap. 9]) for unconditional results.

The problem of Gaussian primes in small balls is similar in flavor to that of primes in short intervals, and was studied by Coleman [1] who established individual results, both on GRH and unconditionally (see also [4]), and a result about almost all balls.

We briefly discuss in Section 4 the analogous problems for real quadratic fields.

## 2. Preliminaries

**2.1. Hecke characters and their  $L$ -functions.** For a non-zero ideal  $\mathfrak{a} = (\alpha) \subseteq \mathbb{Z}[i]$ , with generator  $\alpha$ , Hecke defined characters  $\Xi_k(\alpha) = (\alpha/\bar{\alpha})^{2k}$ ,  $k \in \mathbb{Z}$ , which give well defined functions on the ideals of  $\mathbb{Z}[i]$ . In terms of the angles associated to ideals, we have  $e^{i4k\theta_{\mathfrak{a}}} = \Xi_k(\mathfrak{a})$ .

To each such character Hecke [6] associated its  $L$ -function

$$L(s, \Xi_k) := \sum_{0 \neq \mathfrak{a} \subseteq \mathbb{Z}[i]} \frac{\Xi_k(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p} \text{ prime}} (1 - \Xi_k(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}, \quad \Re(s) > 1.$$

Note that  $L(s, \Xi_k) = L(s, \Xi_{-k})$ . Hecke showed that if  $k \neq 0$ , these functions have an analytic continuation to the entire complex plane, and satisfy a functional equation:

$$(2.1) \quad \xi(s, k) := \pi^{-(s+2|k|)} \Gamma(s+2|k|) L(s, \Xi_k) = \xi(1-s, k).$$

We denote  $L_\infty(s, k) := \pi^{-(s+2|k|)} \Gamma(s+2|k|)$ .

**2.2. The zero-free region.** We will need a “non-standard” zero-free region for  $L(s, \Xi_k)$ :

**THEOREM 2** (Kubilius [8]). *For  $k > 0$  and  $V := \sqrt{(T+2)^2 + (2k)^2}$ , if  $\rho_{k,n} = \beta_{k,n} + i\gamma_{k,n}$  is a zero of  $L(s, \Xi_k)$ , then*

$$1 - \beta_{k,n} \gg \frac{1}{(\log V \log \log V)^{3/4}}, \quad |\gamma_{k,n}| < T.$$

**2.3. The zero-density estimate.** We now introduce a zero-density theorem. We set

$$N(\sigma; T, K) := \#\{\rho_{k,n} = \beta_{k,n} + i\gamma_{k,n} : 0 < k \leq K, |\gamma_{k,n}| < T, \beta_{k,n} \geq \sigma\}.$$

In his thesis (1976), Ricci [10] showed

**THEOREM 3** ([10]). *For  $\sigma \geq 1/2$ ,  $K, T \geq 2$ , and  $T = o(K)$ , we have*

$$N(\sigma; T, K) \ll TK^{\frac{10}{3}(1-\sigma)} (\log K)^B$$

for some  $B > 0$ .

**3. The number variance.** We wish to get an unconditional result on angles of Gaussian primes in almost all narrow sectors. We recall some definitions: Pick  $f \in C_c^\infty(\mathbb{R})$ , which is even and real-valued, and for  $K \gg 1$  define

$$(3.1) \quad F_K(\theta) := \sum_{j \in \mathbb{Z}} f\left(\frac{K}{\pi/2} \left(\theta - j \frac{\pi}{2}\right)\right).$$

Let  $\Phi \in C_c^\infty(0, \infty)$  and set

$$\psi_{K,X}(\theta) := \sum_{\mathfrak{a}} \Phi\left(\frac{N(\mathfrak{a})}{X}\right) \Lambda(\mathfrak{a}) F_K(\theta_{\mathfrak{a}} - \theta),$$

with the sum over all powers of prime ideals, where the von Mangoldt function  $\Lambda(\mathfrak{a})$  is  $\log N(\mathfrak{p})$  if  $\mathfrak{a} = \mathfrak{p}^r$  is a power of a prime ideal  $\mathfrak{p}$ , and is zero otherwise.

By [11, Lemma 3.1], the expected value of  $\psi_{K,X}(\theta)$  is

$$\mathbb{E}(\psi_{K,X}) = \langle \psi_{K,X} \rangle \sim \frac{X}{K} \int_{-\infty}^{\infty} f(x) dx \int_0^{\infty} \Phi(u) du.$$

Here for any function  $H$  of  $\theta$ , we define the mean value

$$\langle H \rangle := \frac{1}{\pi/2} \int_0^{\pi/2} H(\theta) d\theta.$$

We wish to study the number variance

$$\text{Var}(\psi_{K,X}) = \frac{1}{\pi/2} \int_0^{\pi/2} |\psi_{K,X}(\theta) - \mathbb{E}(\psi_{K,X})|^2 d\theta.$$

**THEOREM 4.** *If  $K = X^\tau$  with  $\tau < 3/5$ , then for some  $A > 0$ ,*

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll (\log X)^{-A} \quad \text{as } X \rightarrow \infty.$$

Consequently, in this range, for almost all  $\theta$ , there is a prime ideal  $\mathfrak{p}$  with  $N(\mathfrak{p}) \asymp X$  such that  $|\theta - \theta_{\mathfrak{p}}| < 1/K$ .

**3.1. The proof of Theorem 4.** We have [11, Corollary 4.4] <sup>(1)</sup>

$$\begin{aligned} & \psi_{K,X}(\theta) - \langle \psi_{K,X} \rangle \\ &= - \sum_{k \neq 0} e^{-i4k\theta} \frac{1}{K} \widehat{f}(k/K) \left( \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} + O\left(\frac{X^{1/2} \log K}{(\log X)^{100}}\right) \right), \end{aligned}$$

with the inner sum over all non-trivial zeros  $\rho_{k,n} = \beta_{k,n} + i\gamma_{k,n}$  of  $L(s, \Xi_k)$ .

Computing the mean square over  $\theta$  and dividing by the square of the expected value, namely by  $(X/K)^2$ , gives

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll \frac{1}{X^2} \sum_{k \neq 0} |\widehat{f}(k/K)|^2 \left| \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} + O\left(\frac{X^{1/2} \log K}{(\log X)^{100}}\right) \right|^2.$$

Below we will see that the contribution of the first term is  $O((\log X)^{-A})$ . The contribution of the second term  $O\left(\frac{X^{1/2} \log K}{(\log X)^{100}}\right)^2$  is  $O(X^{-\varepsilon})$  if  $K < X^{1-\varepsilon}$ , so that modulo this fact we have

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll \frac{1}{X^2} \sum_{k \neq 0} |\widehat{f}(k/K)|^2 \left| \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} \right|^2 + X^{-\varepsilon}.$$

Next, we note that up to a negligible error, we can truncate the sum over  $k$  to  $0 < k < K^{1+\delta}$ , for any fixed  $0 < \delta < 1$ , where we take  $\delta > 0$  such that

$$K < X^{\frac{3}{5}(1-2\delta)}.$$

---

<sup>(1)</sup> There it is formulated assuming GRH.

Indeed, since  $\widehat{f}$  is rapidly decaying, using the trivial bound  $\Re \rho_{k,n} \leq 1$  we see that the tail of the sum is bounded by

$$\frac{1}{X^2} \sum_{k > K^{1+\delta}} |\widehat{f}(k/K)|^2 \left| \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} \right|^2 \ll \sum_{k > K^{1+\delta}} |\widehat{f}(k/K)|^2 \left( \sum_n |\tilde{\Phi}(\rho_{k,n})| \right)^2.$$

For each  $k \neq 0$ , we use the rapid decay of the Mellin transform  $|\tilde{\Phi}(\rho_{k,n})| \ll (1 + |\gamma_{k,n}|)^{-100}$  and the bound

$$\#\{n : T \leq |\gamma_{k,n}| < T + 1\} \ll \log(2|k|(T + 2))$$

to estimate the sum over the zeros by

$$\sum_n |\tilde{\Phi}(\rho_{k,n})| \leq \sum_{T=0}^{\infty} \log(2|k|) \frac{\log(T + 2)}{(1 + T)^{100}} \ll \log(2|k|).$$

Hence the tail of the sum is bounded by

$$\begin{aligned} \frac{1}{X^2} \sum_{k > K^{1+\delta}} |\widehat{f}(k/K)|^2 \left| \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} \right|^2 \\ \ll \sum_{k > K^{1+\delta}} |\widehat{f}(k/K)|^2 (\log k)^2 \ll K^{-100} \end{aligned}$$

on using  $|\widehat{f}(y)| \ll |y|^{-100/\delta}$  for  $|y| > 1$ . Therefore

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll \frac{1}{X^2} \sum_{0 < k < K^{1+\delta}} |\widehat{f}(k/K)|^2 \left| \sum_n \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} \right|^2 + X^{-\varepsilon}.$$

We may also restrict the sum to zeros with  $|\gamma_{k,n}| < K^\varepsilon$ , at the cost of another negligible error, again by using the rapid decay of  $\tilde{\Phi}$ . Thus we have

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll W + X^{-\varepsilon}$$

where

$$W := \frac{1}{X^2} \sum_{0 < k < K^{1+\delta}} |\widehat{f}(k/K)|^2 \left| \sum_{n: |\gamma_{k,n}| < K^\varepsilon} \tilde{\Phi}(\rho_{k,n}) X^{\rho_{k,n}} \right|^2.$$

By the Cauchy–Schwarz inequality,

$$W \leq \frac{1}{X^2} \sum_{0 < k < K^{1+\delta}} |\widehat{f}(k/K)|^2 \sum_{n: |\gamma_{k,n}| < K^\varepsilon} |\tilde{\Phi}(\rho_{k,n})| \sum_{n: |\gamma_{k,n}| < K^\varepsilon} |\tilde{\Phi}(\rho_{k,n})| X^{2\beta_{k,n}}.$$

Using the rapid decay of  $\tilde{\Phi}$  and the logarithmic density of zeros gives

$$\sum_{n: |\gamma_{k,n}| < K^\varepsilon} |\tilde{\Phi}(\rho_{k,n})| \ll \log K$$

so that

$$W \ll \frac{\log K}{X^2} \sum_{0 < k < K^{1+\delta}} |\widehat{f}(k/K)|^2 \sum_{n: |\gamma_{k,n}| < K^\varepsilon} |\tilde{\Phi}(\rho_{k,n})| X^{2\beta_{k,n}}.$$

Let

$$M = (\log K)^{9/10}, \quad \Delta = \lceil 2/\delta \rceil,$$

so that for  $K \gg 1$ , there are no zeros of  $L(s, \Xi_k)$  with real part  $\beta_{k,n} > 1 - \Delta/M$ , by Theorem 2. We use a dyadic decomposition to bound  $W$ :

$$\begin{aligned} W &\ll \frac{\log K}{X^2} \sum_{0 < k < K^{1+\delta}} \sum_{m=0}^{M-\Delta} \sum_{0 \leq T < K^\varepsilon} \sum_{\substack{\frac{1}{2} + \frac{m}{2M} \leq \beta_{k,n} < \frac{1}{2} + \frac{m+1}{2M} \\ T \leq |\gamma_{k,n}| < T+1}} |\tilde{\Phi}(\rho_{k,n})| X^{2\beta_{k,n}} \\ &\ll \frac{\log K}{X^2} \sum_{0 < k < K^{1+\delta}} \sum_{m=0}^{M-\Delta} \sum_{0 \leq T < K^\varepsilon} \sum_{\substack{\frac{1}{2} + \frac{m}{2M} \leq \beta_{k,n} < \frac{1}{2} + \frac{m+1}{2M} \\ T \leq |\gamma_{k,n}| < T+1}} X^{1 + \frac{m+1}{M}} \frac{1}{(T+1)^{100}} \\ &\ll \frac{\log K}{X^2} \sum_{m=0}^{M-\Delta} X^{1 + \frac{m+1}{M}} \sum_{0 \leq T < K^\varepsilon} \frac{1}{(T+1)^{100}} \left( \sum_{0 < k < K^{1+\delta}} \sum_{\substack{\frac{1}{2} + \frac{m}{2M} \leq \beta_{k,n} < \frac{1}{2} + \frac{m+1}{2M} \\ T \leq |\gamma_{k,n}| < T+1}} 1 \right) \\ &\leq \frac{X^{1/M} \log K}{X} \sum_{m=0}^{M-\Delta} X^{m/M} \sum_{0 \leq T < K^\varepsilon} \frac{1}{(T+1)^{100}} N\left(\frac{1}{2} + \frac{m}{2M}; T, K^{1+\delta}\right). \end{aligned}$$

Now use the zero-density theorem (Theorem 3) to bound, for  $T < K^\varepsilon$ ,

$$N\left(\frac{1}{2} + \frac{m}{2M}; T, K^{1+\delta}\right) \ll TK^{(1+\delta)\frac{10}{3}(\frac{1}{2} - \frac{m}{2M})} (\log K)^B.$$

Hence

$$\begin{aligned} W &\ll \frac{X^{1/M} (\log K)^{B+1}}{X} \sum_{m=0}^{M-\Delta} X^{m/M} \sum_{1 \leq T < K^\varepsilon} \frac{T}{T^{100}} K^{(1+\delta)\frac{10}{3}(\frac{1}{2} - \frac{m}{2M})} \\ &\ll \frac{X^{1/M} (\log K)^{B+1}}{X} \sum_{m=0}^{M-\Delta} X^{m/M} K^{(1+\delta)\frac{10}{3}(\frac{1}{2} - \frac{m}{2M})} \\ &\ll \frac{K^{(1+\delta)\frac{5}{3}}}{X} X^{1/M} (\log K)^{B+1} \sum_{m=0}^{M-\Delta} \left( \frac{X}{K^{(1+\delta)\frac{5}{3}}} \right)^{m/M}. \end{aligned}$$

Since we assume that  $K < X^{\frac{3}{5}(1-2\delta)}$  (here  $\delta > 0$  is arbitrarily small), so that

$$\frac{X}{K^{(1+\delta)\frac{5}{3}}} > X^{1-(1+\delta)\frac{5}{3}(1-2\delta)\frac{3}{5}} = X^{\delta+2\delta^2} > X^\delta,$$

we have

$$\sum_{m=0}^{M-\Delta} \left( \frac{X}{K^{(1+\delta)\frac{5}{3}}} \right)^{m/M} \ll \left( \frac{X}{K^{(1+\delta)\frac{5}{3}}} \right)^{1-\Delta/M},$$

so that we obtain

$$W \ll X^{1/M} (\log K)^{B+1} \left( \frac{X}{K^{(1+\delta)\frac{5}{3}}} \right)^{-\Delta/M} \ll \frac{(\log K)^{B+1}}{X^{\frac{\delta\Delta-1}{M}}}.$$

Since we took  $\Delta = \lceil 2/\delta \rceil$ , we have  $\frac{\delta\Delta-1}{M} \geq \frac{1}{M}$ , so that

$$\begin{aligned} W &\ll \frac{(\log K)^{B+1}}{X^{1/M}} \ll (\log K)^{B+1} \exp(-\log X / (\log K)^{9/10}) \\ &\ll \exp(-\tfrac{1}{2}(\log X)^{1/10}) = O((\log X)^{-A}). \end{aligned}$$

Thus we see that if  $K < X^{\frac{3}{5}(1-2\delta)}$ , then

$$\frac{\text{Var}(\psi_{K,X})}{(\mathbb{E}(\psi_{K,X}))^2} \ll (\log X)^{-A}.$$

This completes the proof of Theorem 4.

**3.2. Primes vs. prime powers.** Now set

$$\psi_{K,X}^{\text{prime}}(\theta) := \sum_{\mathfrak{p}} \Phi\left(\frac{N(\mathfrak{p})}{X}\right) F_K(\theta_{\mathfrak{p}} - \theta) \log N(\mathfrak{p}),$$

with the sum over all prime ideals. Let

$$\text{Var}(\psi_{K,X}^{\text{prime}}) = \frac{1}{\pi/2} \int_0^{\pi/2} |\psi_{K,X}^{\text{prime}}(\theta) - \mathbb{E}(\psi_{K,X}^{\text{prime}})|^2 d\theta.$$

**COROLLARY 5.** *If  $K = X^\tau$  with  $\tau < 3/5$ , then*

$$\frac{\text{Var}(\psi_{K,X}^{\text{prime}})}{(\mathbb{E}(\psi_{K,X}^{\text{prime}}))^2} \ll (\log X)^{-A} \quad \text{as } X \rightarrow \infty.$$

*Proof.* By [11, Lemma 3.1] again, the mean value of  $\psi_{K,X}^{\text{prime}}(\theta)$  is

$$(3.2) \quad \mathbb{E}(\psi_{K,X}^{\text{prime}}) = \langle \psi_{K,X}^{\text{prime}} \rangle \sim \langle \psi_{K,X} \rangle \sim \frac{X}{K} \int_{-\infty}^{\infty} f(x) dx \int_0^{\infty} \Phi(u) du.$$

Moreover,

$$|\langle \psi_{K,X}^{\text{prime}} \rangle - \langle \psi_{K,X} \rangle| \ll X^{1/2}/K.$$

And by the prime ideal theorem, we have

$$\sum_{\mathfrak{a}} \Phi\left(\frac{N(\mathfrak{a})}{X}\right) \Xi_k(\mathfrak{a}) \Lambda(\mathfrak{a}) = \sum_{\mathfrak{p}} \Phi\left(\frac{N(\mathfrak{p})}{X}\right) \Xi_k(\mathfrak{p}) \log N(\mathfrak{p}) + O(X^{1/2}).$$

Then by the same proof as in [11, Lemma 4.8], we have

$$\langle |\psi_{K,X} - \psi_{K,X}^{\text{prime}}|^2 \rangle \ll X/K.$$

Using the triangle inequality as in [11, §4.4], together with the above estimates and Theorem 4, we complete the proof of our corollary. ■

**3.3. Proof of Theorem 1.** By Chebyshev's inequality and Corollary 5, we find that

$$(3.3) \quad \psi_{K,X}^{\text{prime}}(\beta) \sim \mathbb{E}(\psi_{K,X}^{\text{prime}}) \sim \frac{X}{K} \int_{-\infty}^{\infty} f(x) dx \int_0^{\infty} \Phi(u) du$$

for almost all  $\beta \in [0, \pi/2)$ .

Theorem 1 follows by removing the smoothing. Let  $f_{\varepsilon}^{+}(y)$  be a smooth function depending on  $\varepsilon$  and satisfying  $f_{\varepsilon}^{+}(x) = 1$  if  $x \in [0, 1]$ ,  $f_{\varepsilon}^{+}(x) \in [0, 1]$  if  $x \in [-\varepsilon, 0] \cup [1, 1 + \varepsilon]$ , and  $f_{\varepsilon}^{+}(x) = 0$  otherwise. Here  $\varepsilon$  is a small positive constant. Define  $F_{K,\varepsilon}^{+}(\theta)$  as in (3.1), that is,

$$F_{K,\varepsilon}^{+}(\theta) = \sum_{j \in \mathbb{Z}} f_{\varepsilon}^{+} \left( \frac{K}{\pi/2} \left( \theta - j \frac{\pi}{2} \right) \right).$$

Similarly, we can choose  $\Phi_{\varepsilon}^{+}$  to be a smooth function depending on  $\varepsilon$  and satisfying  $\Phi_{\varepsilon}^{+}(u) = 1$  if  $u \in [1, 2]$ ,  $\Phi_{\varepsilon}^{+}(u) \in [0, 1]$  if  $u \in [1 - \varepsilon, 1] \cup [2, 2 + \varepsilon]$ , and  $\Phi_{\varepsilon}^{+}(u) = 0$  otherwise. Let

$$\mathcal{A} := \{\mathfrak{p} \subseteq \mathbb{Z}[i] : \theta_{\mathfrak{p}} \in I := [\beta, \beta + \pi/(2K)], X < N(\mathfrak{p}) \leq 2X\}.$$

Note that for  $X > 2^{1/\varepsilon}$ ,

$$\begin{aligned} \sum_{\mathfrak{p} \in \mathcal{A}} 1 &\leq \sum_{\mathfrak{p}} \Phi_{\varepsilon}^{+} \left( \frac{N(\mathfrak{p})}{X} \right) F_{K,\varepsilon}^{+}(\theta_{\mathfrak{p}} - \beta) \\ &\leq \frac{1 + \varepsilon}{\log X} \sum_{\mathfrak{p}} \Phi_{\varepsilon}^{+} \left( \frac{N(\mathfrak{p})}{X} \right) F_{K,\varepsilon}^{+}(\theta_{\mathfrak{p}} - \beta) \log N(\mathfrak{p}). \end{aligned}$$

For  $\beta \in [0, \pi/2)$  satisfying (3.3) with  $f = f_{\varepsilon}^{+}$  and  $\Phi = \Phi_{\varepsilon}^{+}$ , we have

$$\sum_{\mathfrak{p} \in \mathcal{A}} 1 \leq (1 + 2\varepsilon)^4 \frac{X}{K \log X} = (1 + 2\varepsilon)^4 \frac{|I|}{\pi/2} \frac{X}{\log X}$$

for  $X > X(\varepsilon)$  sufficiently large.

Similarly, we can obtain a lower bound of  $\sum_{\mathfrak{p} \in \mathcal{A}} 1$  by another smooth counting, so that for  $\beta \in [0, \pi/2)$  satisfying (3.3) with  $f = f_{\varepsilon}^{-}$  and  $\Phi = \Phi_{\varepsilon}^{-}$ , we have

$$\sum_{\mathfrak{p} \in \mathcal{A}} 1 \geq (1 - 2\varepsilon)^4 \frac{|I|}{\pi/2} \frac{X}{\log X}$$

for  $X > X(\varepsilon)$ . Taking the limit  $\varepsilon \rightarrow 0$ , we obtain an asymptotic formula for  $\sum_{\mathfrak{p} \in \mathcal{A}} 1$  for almost all  $\beta$ . This completes the proof of Theorem 1.



**4. Prime angles for real quadratic fields.** One can also study questions about angles related to the representation of primes as norms in a real quadratic field, similar to those for Gaussian primes.

Let  $E$  be a real quadratic field, and  $\varepsilon = \varepsilon_E > 1$  the fundamental unit, so that the group of units is  $\{\pm\varepsilon^n : n \in \mathbb{Z}\}$ . For simplicity, we will assume that the fundamental unit has negative norm, and that the (narrow) class number of  $E$  is 1, so that all ideals are principal (Gauss conjectured that this occurs infinitely often). Given a prime  $p$  which splits (completely) in  $E$ , to any solution of  $\text{Norm}_{E/\mathbb{Q}}(\alpha) = p$ , that is, to any generator of an ideal  $\mathfrak{p} \mid (p)$ , we associate an angle variable

$$t(\mathfrak{p}) \in \mathbb{R}/(2 \log \varepsilon)\mathbb{Z} \simeq S^1$$

by

$$\exp\left(\frac{i\pi t(\mathfrak{p})}{\log \varepsilon}\right) = \left| \frac{\alpha}{\tilde{\alpha}} \right|^{i\pi/\log \varepsilon}$$

(where  $\alpha \mapsto \tilde{\alpha}$  is the Galois involution of  $E$ ), which is independent of the choice of generator of the ideal  $\mathfrak{p}$ . Note that for the Galois conjugate ideal we have  $t(\tilde{\mathfrak{p}}) = -t(\mathfrak{p}) \bmod 2 \log \varepsilon$ .

For example, take  $E = \mathbb{Q}(\sqrt{2})$ . The ring of integers is  $O_E = \mathbb{Z}[\sqrt{2}]$ , the fundamental unit  $\varepsilon = 1 + \sqrt{2}$  has negative norm, and the class number (both wide and narrow) is 1. The split primes are those satisfying  $p \equiv \pm 1 \pmod{8}$ . For every such prime, we can represent both  $\pm p$  as a norm, that is, solve  $a^2 - 2b^2 = \pm p$ . The corresponding angle parameters describe the relative size of the solution coordinates  $a$  and  $b$ .

Hecke [6] showed that as  $p$  varies over split primes, the corresponding angle parameters become uniformly distributed in  $\mathbb{R}/(2 \log \varepsilon)\mathbb{Z} \simeq S^1$ .

Using ideas similar to those for the case of Gaussian primes, one can show results analogous to Theorem 1. The non-standard zero-free region for the corresponding  $L$ -functions is due to Coleman [2]. The zero-density theorem needed can be proved along the lines of Theorem 3.

**Acknowledgments.** The research of B.H. and Z.R. was supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n<sup>o</sup> 786758); the research of J.L. was supported by the National Science Foundation of China under Grant 11531008, the Ministry of Education of China under Grant IRT16R43, and the Taishan Scholar Project of Shandong Province.

## References

- [1] M. D. Coleman, *The distribution of points at which binary quadratic forms are prime*, Proc. London Math. Soc. (3) 61 (1990), 433–456.

- [2] M. D. Coleman, *A zero-free region for the Hecke L-functions*, *Mathematika* 37 (1990), 287–304.
- [3] G. Harman, *Prime-Detecting Sieves*, London Math. Soc. Monogr. Ser. 33, Princeton Univ. Press, Princeton, NJ, 2007.
- [4] G. Harman, A. Kumchev and P. A. Lewis, *The distribution of prime ideals of imaginary quadratic fields*, *Trans. Amer. Math. Soc.* 356 (2004), 599–620.
- [5] G. Harman and P. Lewis, *Gaussian primes in narrow sectors*, *Mathematika* 48 (2001), 119–135.
- [6] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. I, II*, *Math. Z.* 1 (1918), 357–376, 6 (1920), 11–51.
- [7] C. Jia, *Almost all short intervals containing prime numbers*, *Acta Arith.* 76 (1996), 21–84.
- [8] J. Kubilius, *On a problem in the  $n$ -dimensional analytic theory of numbers*, *Vilniaus Valst. Univ. Mokslo Darbai. Mat. Fiz. Chem. Mokslu Ser.* 4 (1955), 5–43 (in Lithuanian).
- [9] O. Parzanchevski and P. Sarnak, *Super-golden-gates for  $PU(2)$* , *Adv. Math.* 327 (2018), 869–901.
- [10] S. J. Ricci, *Local distribution of primes*, Ph.D. thesis, Univ. of Michigan, 1976.
- [11] Z. Rudnick and E. Waxman, *Angles of Gaussian primes*, *Israel J. Math.* 232 (2019), 159–199.
- [12] B. Saffari and R. C. Vaughan, *On the fractional parts of  $x/n$  and related sequences. II*, *Ann. Inst. Fourier (Grenoble)* 27 (1977), no. 2, 1–30.
- [13] A. Selberg, *On the normal density of primes in small intervals, and the difference between consecutive primes*, *Arch. Math. Naturvid.* 47 (1943), 87–105.

Bingrong Huang (corresponding author)  
Data Science Institute and School of Mathematics  
Shandong University  
Jinan, Shandong 250100, China  
E-mail: brhuang@sdu.edu.cn

Jianya Liu  
School of Mathematics  
Shandong University  
Jinan, Shandong 250100, China  
E-mail: jyliu@sdu.edu.cn

Ze'ev Rudnick  
School of Mathematical Sciences  
Tel Aviv University  
Tel Aviv, Israel  
E-mail: rudnick@tauex.tau.ac.il