

On Galois groups and PAC substructures

by

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Abstract. (1) We show that for an arbitrary stable theory T , a group G is profinite if and only if G occurs as the Galois group of some Galois extension inside a monster model of T .

(2) We prove that any PAC substructure of the monster model of T has projective absolute Galois group.

(3) Moreover, any projective profinite group G is isomorphic to the absolute Galois group of a definably closed substructure P of the monster model. If T is ω -stable, then P can be chosen to be PAC.

(4) Finally, we provide a description of some Galois groups of existentially closed substructures with G -action in terms of the universal Frattini cover. Such structures might be understood as a new source of examples of PAC structures.

1. Introduction. Our general goal for this and subsequent papers is to develop Galois theory of *pseudo-algebraically closed* (PAC) substructures in the stable context. What this in fact means is that we want to generalize the Galois theory of PAC fields. Due to the Elementary Equivalence Theorem [9, Theorem 20.3.3], first order theory of a PAC field is determined by the isomorphism class of its absolute Galois group, hence its model-theoretic properties are coded by Galois theory. The question was: is this phenomenon true only for PAC substructures in the case of fields or maybe it is true for PAC substructures in the case of models of any stable theory and so it is somehow related to stability? A partial answer is provided by [21], where the author shows that if the absolute Galois group of a PAC substructure (in the stable context) is small, then this PAC substructure is simple. Results from [21] motivated us to study the problem and in subsequent papers we give an answer by achieving a desired generalization of the Elementary Equivalence

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Theorem (in [7]) and by establishing a link between some notion of independence in the absolute Galois group of a PAC substructure and the Kim independence in this PAC substructure (see [12]).

The aforementioned research is a part of an even more general picture in the ongoing studies on neo-stability. Namely, we would like to “break” stability in a controlled manner. If we start with a stable theory and consider an existentially closed substructure (of some monster model of this stable theory) equipped with a group action of a given group G (as in [10], see also Section 5), then the theory of this substructure tends to be (almost-)simple non-stable (see [10]). This means that there is a way to “break” stability, but can this process be somehow controlled? Yes, and there are two reasons for that. First, a general rule is that existentially closed substructures are PAC substructures, so their model-theoretic properties are coded by their absolute Galois groups. Second, in Section 5 we provide a description of Galois groups related to existentially closed substructures with a group action of a given group G , so we can control the above mentioned absolute Galois groups and so also these PAC substructures.

This paper contains results which are natural generalizations of facts known in the classical Galois theory and which play an auxiliary role in the above described research. Suppose that T is a stable complete \mathcal{L} -theory with quantifier elimination and elimination of imaginaries (possibly many-sorted), \mathfrak{C} is a monster model of T , and G is a group smaller than the saturation of \mathfrak{C} . In this paper, we are interested in the following two properties related to profinite groups:

- (P $_T$) G is profinite iff there exists a Galois extension $A \subseteq B$ in \mathfrak{C} such that $G \cong \text{Aut}_{\mathcal{L}}(B/A)$.
- (PP $_T$) A profinite G is projective iff $G \cong \mathcal{G}(P)$ for some PAC substructure P of \mathfrak{C} .

(Here $\mathcal{G}(P)$ is the absolute Galois group of P .) By well known facts, properties P $_T$ and PP $_T$ hold for $T = \text{ACF}$ (e.g. Fact 2.8 and [9, Corollaries 1.3.4 and 23.1.3]). We ask here whether other stable theories enjoy properties P $_T$ and PP $_T$. In Corollary 3.3 we show that the property P $_T$ holds for any stable theory T and its proof is not hard. On the other hand, we encounter difficulties in proving property PP $_T$ for any stable theory T (with elimination of quantifiers and elimination of imaginaries). Theorem 4.9 shows that any projective profinite group G is isomorphic to the absolute Galois group of some (definably closed) substructure P . However, to state that this substructure P is PAC, additional assumptions on T are needed (the “moreover” part of Theorem 4.9). More precisely, we need to assume that any type over a small $A \subseteq \mathfrak{C}$ has only finitely many extensions over $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$. This additional assumption is related to Lemma 4.5 (i.e. a generalization of the Ax–Roquette

Theorem), which is used in the proof of the “moreover” part of Theorem 4.9. It turns out that, in contrast to the theory of fields, an algebraic extension of a PAC substructure is not necessarily a PAC substructure (see Remark 4.6), and that was the main obstacle in proving that any stable theory T enjoys property PP_T .

The central result of this paper is Theorem 4.4 which says that the absolute Galois group of a PAC substructure is projective. This was already known for fields and also in the case of strongly minimal theories [13, Lemma 1.17]). We use the notion of *regularity/stationarity* to enhance the proof of [13, Lemma 1.17] to the case of stable theories.

Section 5 uses results of the previous sections to describe Galois groups of substructures of our stable monster model \mathfrak{C} which are equipped with a group action by automorphisms. We proved in [10] that existentially closed substructures of \mathfrak{C} equipped with a group action are PAC substructures, similarly to invariants of the group action. Therefore considering existentially closed substructures with a group action provides a new class of PAC substructures for any stable theory T . It was natural to ask what we can say about absolute Galois groups of such PAC substructures, and so get more information about their model-theoretic properties. Some statements in Section 5 generalize theorems from [24], which was restricted to the theory of fields with a group action. We explain the details in Section 5.

Studying PAC substructures was, and still is, interesting since the notion of a PAC substructure generalizes the notion of a PAC field and the theory of PAC fields is an important and pleasing part of the theory of fields. Some crucial results were achieved in the case of strongly minimal theories (in [13]) and then bounded PAC structures were described in the stable context as structures with simple theory ([19] and [21]). Among other model-theoretical results on PAC fields (e.g. [2]–[6], [8]) there are many results promising a very interesting generalization to the level of PAC substructures. For example, Nick Ramsey in his doctoral dissertation [22] shows that a PAC field is $NSOP_1$ if and only if its absolute Galois group is $NSOP_1$ (as a special many-sorted structure, see also [12]).

2. Basics

2.1. Preliminaries and conventions. If A and B are two sequences, then AB denotes their concatenation, i.e. $A \frown B$. If A and B are considered only as sets, then AB denotes $A \cup B$. Finally, if H is a group and A is a set, then the orbit of A under an action of H will be denoted by $H \cdot A$ or (if it will not lead to confusion) by HA .

We assume that theories in this paper are theories with *infinite models*, and (*sub*)*structures* we are interested in are *also infinite* (the referee cor-

rectly noticed that Fact 2.13 does not work for finite substructures, hence we emphasize in statements relying on Fact 2.13 that we are dealing with infinite structures). More precisely, by “infinite substructure” we understand a substructure whose intersection with some sort is an infinite set. Let N and N' be \mathcal{L} -structures and let E be a subset of N . We use $\langle E \rangle_{\mathcal{L}}$ to denote the \mathcal{L} -substructure of N generated by E . Moreover, $\text{acl}_{\mathcal{L}}^N(E)$ denotes the algebraic closure of E in N in the sense of the language \mathcal{L} and the \mathcal{L} -theory $\text{Th}(N)$ (similarly for $\text{dcl}_{\mathcal{L}}^N(E)$ and $\text{tp}_{\mathcal{L}}^N(a/E)$).

We fix (once for all) an \mathcal{L} -structure \mathfrak{C} which is κ -saturated and κ -strongly homogeneous and set $T := \text{Th}_{\mathcal{L}}(\mathfrak{C})$. In other words, \mathfrak{C} is a monster model for the complete theory T . At some point, we will start to assume that T is stable and has quantifier elimination and elimination of imaginaries. These are rather technical assumptions if T is stable, since we are able to force every stable theory T to have quantifier elimination and elimination of imaginaries by passing to T^{eq} and then taking the Morleyization $(T^{\text{eq}})^m$ of the theory T^{eq} .

A group G is considered in different places of the paper, but we *always assume that* $|G| < \kappa$.

We use the following convention throughout: “Facts” are results recalled from previous papers, while results named in other way are generalizations or new results.

2.2. Old definitions and facts. In this subsection, we provide material which is mostly well known. The standard reference for the content related to model-theoretic Galois theory (from Definition 2.4 to Definition 2.9) is Poizat’s [20]. Another option can be [17], which is a very good introduction to the subject (however, it does not cover the case of infinite Galois extensions, which is covered by [10]).

Material related to stability (from Fact 2.11 to Fact 2.17) is almost standard and can be found in most textbooks on stability theory (e.g. [25]). For the reader’s convenience, we refer mainly to [10], where all notions and facts listed below appear in the notation we are using. We omit proofs, but indicate the corresponding facts in [10], so the interested reader can easily find proofs.

Let us note here that our approach to stationarity (which was introduced in [14] and [15]), i.e. stationarity treated as a generalization of regularity from the algebra of fields, led to easier proofs of our results (e.g. Theorem 4.4 or auxiliary results like Proposition 3.1). From our point of view, it is interesting how stationarity is involved in the model-theoretic generalization of techniques related to the tensor product (see e.g. Fact 2.14).

The notion of a regular extension (Definition 2.1) is equivalent to the notion of stationarity given in [15, Definition 5.17] and considered in [14],

but stationarity predated regularity. In our work on structures with group action ([10]), we gave the definition of a regular extension simply by generalizing this notion from the algebra of fields to arbitrary structures. After [10] was posted on the arXiv, Silvain Rideau informed us that our definition of regularity coincides with the definition of stationarity from [15] and [14], and we had not been aware of that. Although we just gave a new name to a known notion, we would like to keep this new name, since in our opinion it is more suitable as it points out the connection with one of the basic notions from algebra, and so it is easier for us to translate ideas from algebra to stability.

DEFINITION 2.1.

- (1) Let $E \subseteq A$ be small subsets of \mathfrak{C} . We say that $E \subseteq A$ is \mathcal{L} -regular (or just regular) if

$$\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(A) \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(E) = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(E).$$

- (2) Let N be a small \mathcal{L} -substructure of \mathfrak{C} . We say that N is *pseudo-algebraically closed* (PAC) if for every small \mathcal{L} -substructure N' of \mathfrak{C} , which is an \mathcal{L} -regular extension of N , we have $N \preceq_1 N'$ (i.e. N is existentially closed in N').

FACT 2.2 ([10, Remark 3.2]).

- (1) *The regularity condition is invariant under the action of automorphisms.*
- (2) *If E is algebraically closed, then $E \subseteq A$ is regular for any small A .*
- (3) *If $E \subseteq A$ is regular and $E \subseteq A' \subseteq A$, then $E \subseteq A'$ is regular.*
- (4) *If $E \subseteq A$ and $A \subseteq B$ are regular, then $E \subseteq B$ is regular.*
- (5) *Let P be a small \mathcal{L} -substructure of \mathfrak{C} . Then there exists a small \mathcal{L} -substructure P^* of \mathfrak{C} such that $P \subseteq P^*$ is regular and P^* is PAC.*

From this point we assume that T allows one to *eliminate quantifiers*. Note that quantifier elimination in T implies that for a small PAC substructure $P \subseteq \mathfrak{C}$ we have $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(P) = P$.

FACT 2.3 ([10, Lemma 3.3]). *If for some small \mathcal{L} -substructures $P \subseteq N$ of \mathfrak{C} we have $P \preceq_1 N$, then $P \subseteq N$ is regular.*

DEFINITION 2.4.

- (1) Assume that $A \subseteq C$ are \mathcal{L} -substructures of \mathfrak{C} . We say that C is *normal over A* (or that $A \subseteq C$ is a *normal extension*) if $\text{Aut}_{\mathcal{L}}(\mathfrak{C}/A) \cdot C \subseteq C$. (Note that if C is small and $A \subseteq C$ is normal, then necessarily $C \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$.)
- (2) Assume that $A \subseteq C \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$ are small \mathcal{L} -substructures of \mathfrak{C} such that $A = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(A)$, $C = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(C)$ and C is normal over A . Then we say that $A \subseteq C$ is a *Galois extension*.

We recall several facts about Galois extensions, which will be used in the rest of the paper. These facts are standard (see [20], [17], or [10]).

FACT 2.5. *Let A, B and C be small \mathcal{L} -substructures of \mathfrak{C} such that $A \subseteq B \subseteq C \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$, and C and B are normal over A . Then*

$$1 \rightarrow \text{Aut}_{\mathcal{L}}(C/B) \xrightarrow{\cong} \text{Aut}_{\mathcal{L}}(C/A) \xrightarrow{\text{l}_B} \text{Aut}_{\mathcal{L}}(B/A) \rightarrow 1$$

is an exact sequence and hence $\text{Aut}_{\mathcal{L}}(C/B) \trianglelefteq \text{Aut}_{\mathcal{L}}(C/A)$.

FACT 2.6. *Assume that $A \subseteq C$ is a Galois extension and $A \subseteq B = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(B) \subseteq C$. The extension $A \subseteq B$ is Galois if and only if*

$$\text{Aut}_{\mathcal{L}}(C/B) \trianglelefteq \text{Aut}_{\mathcal{L}}(C/A).$$

From now we assume that T additionally admits *elimination of imaginaries*.

FACT 2.7 (The Galois correspondence). *Let $A \subseteq C$ be a Galois extension, introduce*

$$\begin{aligned} \mathcal{B} &:= \{B \mid A \subseteq B = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(B) \subseteq C\}, \\ \mathcal{H} &:= \{H \mid H \trianglelefteq \text{Aut}_{\mathcal{L}}(C/A) \text{ is closed}\}. \end{aligned}$$

Then $\alpha(B) := \text{Aut}_{\mathcal{L}}(C/B)$ is a mapping from \mathcal{B} to \mathcal{H} , $\beta(H) := C^H$ is a mapping from \mathcal{H} to \mathcal{B} , and

$$\alpha \circ \beta = \text{id}, \quad \beta \circ \alpha = \text{id}.$$

FACT 2.8. *If $A \subseteq C$ is a Galois extension, then $\text{Aut}_{\mathcal{L}}(C/A)$ is a profinite group.*

DEFINITION 2.9. For a small subset A of \mathfrak{C} we define the *absolute Galois group* of A :

$$\mathcal{G}(A) := \text{Aut}_{\mathcal{L}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)/\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(A)).$$

The following lemma is a generalization of [10, Lemma 3.24]. The original proof still works for items (1)–(3).

LEMMA 2.10. *Assume that N is a small definably closed \mathcal{L} -substructure of \mathfrak{C} equipped with a G -action $(\tau_g)_{g \in G}$ (i.e. a group action $(\tau_g)_{g \in G}$ of the group G by elements of $\text{Aut}_{\mathcal{L}}(N)$). Let $i : G \rightarrow \text{Aut}_{\mathcal{L}}(N/N^G)$ be given by $i(g) := \tau_g$ and let N^G denote the invariants of that group action on N .*

- (1) *If for every $b \in N$ the orbit $G \cdot b$ is definable, then $N^G \subseteq N$ is normal.*
- (2) *If $N \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N^G)$, then $N^G \subseteq N$ is a Galois extension.*
- (3) *If G is finite, then $N \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N^G)$, hence also the conclusion of (2) follows.*
- (4) (Artin's theorem) *If G is finite and the G -action $(\tau_g)_{g \in G}$ is faithful, then $i : G \cong \text{Aut}_{\mathcal{L}}(N/N^G)$.*

- (5) Assume that G is profinite, the G -action $(\tau_g)_{g \in G}$ is faithful and for every $m \in N$ the stabilizer $\text{Stab}(m) = \{g \in G \mid \tau_g(m) = m\}$ is an open subgroup of G . Then $N^G \subseteq N$ is Galois and $i : G \cong \text{Aut}_{\mathcal{L}}(N/N^G)$ (as profinite groups).

Proof. We only need to prove (4) and (5). We start with (4). Since G acts faithfully, i is an embedding. By (3) we know that $N^G \subseteq N$ is Galois. By Fact 2.8, the group $\text{Aut}_{\mathcal{L}}(N/N^G)$ is profinite. Therefore $i(G)$, which is finite, is a closed subgroup. By Fact 2.7, we have $i(G) = \text{Aut}(N/N^{i(G)})$, and (since $N^G = N^{i(G)}$) it follows that $\text{Aut}_{\mathcal{L}}(N/N^G) = i(G) \cong G$.

Our proof of (5) is based on the proof of [9, Lemma 1.3.2], which is the same result, but for the theory $T = \text{ACF}$. Our assumptions ensure that $i : G \rightarrow \text{Aut}_{\mathcal{L}}(N/N^G)$ is an embedding of groups. We need to show that i is continuous and onto, and that $N^G \subseteq N$ is Galois.

Let $m_1, \dots, m_n \in N$. The subgroup $H = \text{Stab}(m_1) \cap \dots \cap \text{Stab}(m_n)$ is open in G , hence \mathcal{N} , equal to the intersection of all conjugates of H in G , is open (and therefore closed and of finite index). Consider the G/\mathcal{N} -action on $N_0 := \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N^G, G \cdot m_1, \dots, G \cdot m_n)$ given by $g(a) = i(g)|_{N_0}(a)$. It is faithful and $N_0^{G/\mathcal{N}} = N^G$. Note that G/\mathcal{N} is finite, hence $N^G \subseteq N_0$ is Galois and $G/\mathcal{N} \cong \text{Aut}_{\mathcal{L}}(N_0/N^G)$, where the isomorphism is given by $g\mathcal{N} \mapsto i(g)|_{N_0}$.

Since each $m \in N$ belongs to some N_0 , we conclude that $N^G \subseteq N$ is normal and $N \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N^G)$. Obviously $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N^G) = N^G$, hence $N^G \subseteq N$ is Galois.

Now, we will show that i is an isomorphism of profinite groups. Let $\{N_\alpha \mid \alpha < \beta\}$ be the set of all finite Galois extensions of N^G which are of the form of N_0 . The collection of finite groups $\text{Aut}_{\mathcal{L}}(N_\alpha/N^G)$ with restriction maps forms an inverse system, so we can speak about its limit, $(\lim_{\leftarrow \alpha} \text{Aut}_{\mathcal{L}}(N_\alpha/N^G), \pi_\alpha)$. Note that

$$f := \lim_{\leftarrow \alpha} f_\alpha : \text{Aut}_{\mathcal{L}}(N/N^G) \rightarrow \lim_{\leftarrow \alpha} \text{Aut}_{\mathcal{L}}(N_\alpha/N^G),$$

where $f_\alpha : \text{Aut}_{\mathcal{L}}(N/N^G) \rightarrow \text{Aut}_{\mathcal{L}}(N_\alpha/N^G)$, $f_\alpha(\sigma) := \sigma|_{N_\alpha}$, is a continuous isomorphism of groups (it is onto by [23, Corollary 1.1.6], and it is one-to-one because the family $\{N_\alpha\}$ covers the whole N), hence it is also a homeomorphism and an isomorphism of profinite groups.

On the other hand,

$$h := \lim_{\leftarrow \alpha} h_\alpha : G \rightarrow \lim_{\leftarrow \alpha} \text{Aut}_{\mathcal{L}}(N_\alpha/N^G),$$

where $h_\alpha : G \rightarrow \text{Aut}(N_\alpha/N^G)$, $h_\alpha(g) := i(g)|_{N_\alpha}$, is (by [23, Corollary 1.1.6] and the previous part of this proof) a continuous epimorphism of groups. Hence $f^{-1}h : G \rightarrow \text{Aut}_{\mathcal{L}}(N/N^G)$ is a continuous epimorphism of groups. To finish the proof, observe that $(f^{-1}h)(g) = i(g)$ for each $g \in G$. ■

Now, we add one more assumption, but stronger: T is stable. The following facts are basic properties of regularity/stationarity and all have their algebraic counterparts in [1] and [16]. The only new thing is the quite simple Lemma 2.15 where we bind together regularity/stationarity and surjectivity of a restriction map at the level of absolute Galois groups, which will be helpful later.

FACT 2.11 ([10, Fact 3.33]). *Assume that $E, A \subseteq \mathfrak{C}$, A is \mathcal{L} -regular over E , $f_1, f_2 \in \text{Aut}_{\mathcal{L}}(\mathfrak{C})$, and $f_1|_E = f_2|_E$. Then there exists $h \in \text{Aut}_{\mathcal{L}}(\mathfrak{C})$ such that $h|_A = f_1|_A$ and $h|_{\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(E)} = f_2|_{\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(E)}$.*

FACT 2.12 ([10, Lemma 3.35]). *For a small set $E \subseteq \mathfrak{C}$ and a complete type p over E ,*

$$\begin{aligned} p \text{ is stationary} &\iff (\forall A_0 \models p)(E \subseteq EA_0 \text{ is } \mathcal{L}\text{-regular}) \\ &\iff (\exists A_0 \models p)(E \subseteq EA_0 \text{ is } \mathcal{L}\text{-regular}). \end{aligned}$$

FACT 2.13 ([10, Corollary 3.36]). *For every small (but infinite) \mathcal{L} -substructure N of \mathfrak{C} and every $n < \omega$, there exists a non-algebraic stationary type over N in n variables.*

FACT 2.14 ([10, Corollary 3.38]). *Assume that $E, A, B \subseteq \mathfrak{C}$, A is \mathcal{L} -regular over E , $f_1, f_2 \in \text{Aut}_{\mathcal{L}}(\mathfrak{C})$, and $f_1|_E = f_2|_E$. If $A \downarrow_E^{\mathfrak{C}} B$ and $f_1(A) \downarrow_{f_1(E)}^{\mathfrak{C}} f_2(B)$, then there exists $h \in \text{Aut}_{\mathcal{L}}(\mathfrak{C})$ such that $h|_A = f_1|_A$ and $h|_B = f_2|_B$.*

LEMMA 2.15. *Assume that $N \subseteq N'$ are small subsets of \mathfrak{C} . The set N' is regular over N if and only if the restriction map $\mathcal{G}(N') \rightarrow \mathcal{G}(N)$ is onto.*

Proof. If N' is regular over N , then $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N) \subseteq \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N')$ is a regular extension and we can use Fact 2.14 to show the surjectivity.

Let $n \in \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N') \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)$ and let $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/N)$. The map $f|_{\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)}$ belongs to $\text{Aut}_{\mathcal{L}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)/N) = \mathcal{G}(N)$ and therefore it is the restriction of some $\tilde{f} \in \mathcal{G}(N')$. Because $n \in \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N')$, we have $f(n) = \tilde{f}(n) = n$, thus $n \in \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N)$. ■

FACT 2.16 ([10, Lemma 3.39]). *Assume that $E \subseteq A$ is \mathcal{L} -regular, $E \subseteq B$ and $B \downarrow_E^{\mathfrak{C}} A$. Then $B \subseteq BA$ is \mathcal{L} -regular.*

FACT 2.17 ([10, Corollary 3.40]). *Assume that $E \subseteq A$ and $E \subseteq B$ are \mathcal{L} -regular, and $B \downarrow_E^{\mathfrak{C}} A$. Then $E \subseteq BA$ is \mathcal{L} -regular.*

3. Profinite group as Galois group. Now, we will make use of Lemma 2.10 to show that every profinite group is isomorphic to some Galois group present in our stable structure \mathfrak{C} . The following proposition is a straightforward generalization of [9, Proposition 1.3.3].

PROPOSITION 3.1. *Assume that $N_0 \subseteq N$ is a Galois extension of small (but infinite) substructures of \mathfrak{C} and there is an epimorphism $\alpha : G \rightarrow \text{Aut}_{\mathcal{L}}(N/N_0)$ of profinite groups. Then there exist small substructures M_0, M of \mathfrak{C} such that $N_0 \subseteq M_0$ and $N \subseteq M$ are regular, $M_0 \subseteq M$ is Galois, and there is an isomorphism $\beta : G \rightarrow \text{Aut}_{\mathcal{L}}(M/M_0)$ such that*

$$\begin{array}{ccc} G & \xrightarrow{\beta} & \text{Aut}_{\mathcal{L}}(M/M_0) \\ & \searrow \alpha & \downarrow |_N \\ & & \text{Aut}_{\mathcal{L}}(N/N_0) \end{array}$$

is commuting.

Proof. Let X denote the disjoint union of all quotient groups G/\mathcal{N} , where \mathcal{N} is an open normal subgroup of G . Assume that X is ordered in some way, say $X = \{x_\lambda \mid \lambda < \lambda'\}$.

Consider a non-algebraic stationary type $p(x) \in S(N_0)$ (Fact 2.13) and a Morley sequence in $p(x)$ indexed by X , $\bar{b} = (b_{g\mathcal{N}})_{g\mathcal{N} \in X}$.

We define a G -action on the substructure $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N_0\bar{b})$ in the following way:

$$g' \cdot b_{g\mathcal{N}} := b_{g'g\mathcal{N}}$$

(similarly to [10, proof of Proposition 3.57], we inductively prove that there exists an automorphism of \mathfrak{C} satisfying the above line).

By Fact 2.17, transfinite induction shows that $N_0 \subseteq \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N_0\bar{b})$ is regular. Since N is algebraic over N_0 , we can use Fact 2.11 to extend the above defined G -action on $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N_0\bar{b})$, and the G -action on N given by

$$g \cdot m = \alpha(g)(m),$$

where $g \in G$ and $m \in N$, to a G -action on $M := \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N\bar{b})$, say $\beta : G \rightarrow \text{Aut}_{\mathcal{L}}(M)$. By Fact 2.16, it follows that $N \subseteq M$ is regular.

Let M_0 denote M^G . Note that $M_0 \cap N = N_0$. To see this take $m \in M_0 \cap N$. Because $m \in M_0$, it follows that $m = g \cdot m = \alpha(g)(m)$ for each $g \in G$. Since α is onto, we obtain $m = f(m)$ for each $f \in \text{Aut}_{\mathcal{L}}(N/N_0)$, hence $m \in N_0$.

To show that $N_0 \subseteq M_0$ is regular, recall that $N \subseteq M$ is regular:

$$M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N) = N.$$

Intersecting both sides with M_0 , we get

$$M_0 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N_0) = M_0 \cap N = N_0.$$

Note that $\text{Stab}(b_{g\mathcal{N}}) = \mathcal{N}$, which is an open subgroup of G , and for $m \in N$ we have $\text{Stab}(m) = \alpha^{-1}(\{f \in \text{Aut}_{\mathcal{L}}(N/N_0) \mid f(a) = a\})$, which is also an open subgroup of G . We see that the action of group G on M is faithful. Before we can use Lemma 2.10(5), we need to check whether for every $m \in M$ the stabilizer $\text{Stab}(m)$ is an open subgroup of G .

Because $m \in M$, there exists an \mathcal{L} -formula ψ such that for some $a_1, \dots, a_n \in N$ and some $g_1\mathcal{N}_1, \dots, g_{n'}\mathcal{N}_{n'} \in X$ we have

$$\psi(a_1, \dots, a_n, b_{g_1\mathcal{N}_1}, \dots, b_{g_{n'}\mathcal{N}_{n'}}, \mathfrak{C}) = \{m\}.$$

Therefore $\text{Stab}(m)$ contains the open subgroup

$$\text{Stab}(a_1) \cap \dots \cap \text{Stab}(a_n) \cap \text{Stab}(b_{g_1\mathcal{N}_1}) \cap \dots \cap \text{Stab}(b_{g_{n'}\mathcal{N}_{n'}}),$$

hence $\text{Stab}(m)$ is an open subgroup of G . By Lemma 2.10(5) it follows that $M_0 \subseteq M$ is Galois and $\beta : G \rightarrow \text{Aut}_{\mathcal{L}}(M/M_0)$ is an isomorphism of profinite groups. The last thing we need to check is that $\beta(g)|_N = \alpha(g)$ for any $g \in G$, but this follows from the construction of the G -action on M . ■

COROLLARY 3.2. *For every profinite group G there exists a Galois extension $M_0 \subseteq M$ of small substructures of \mathfrak{C} such that $G \cong \text{Aut}_{\mathcal{L}}(M/M_0)$.*

COROLLARY 3.3. *A group G is profinite if and only if there exists a Galois extension $M_0 \subseteq M$ of small substructures of \mathfrak{C} such that $G \cong \text{Aut}_{\mathcal{L}}(M/M_0)$.*

Proof. Use Fact 2.8 and Corollary 3.2. ■

Note that our definition of a PAC substructure implies that a PAC substructure is definably closed, which corresponds to being a perfect field in the case of the theory ACF, so one could wonder whether in the case of the theory ACF, projective profinite groups correspond to absolute Galois groups of PAC fields or *perfect* PAC fields. In fact, they correspond to perfect PAC fields (see [9, Corollary 23.1.2]).

4. Projective profinite groups

4.1. PAC has projective absolute Galois group. We start with a simple remark which helps to better understand the property of being a PAC substructure.

REMARK 4.1. A small substructure $P \subseteq \mathfrak{C}$ is PAC if and only if every stationary type over P is finitely satisfiable in P .

The above remark might be used as an alternative definition of being a PAC substructure. For more details about other possible definitions the reader may consult [10, Subsection 3.1].

We note here an easy fact, which can be understood as saying that “sometimes” (see Lemma 4.5) being a PAC substructure is being “one step before being a model” (it is enough to take the algebraic closure—if algebraic extensions preserve being PAC).

FACT 4.2 ([19, Corollary 3.10]). *Assume that P is a small PAC substructure of \mathfrak{C} such that $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P) = P$. Then $P \preceq \mathfrak{C}$.*

Proof. Since $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P) = P$, every extension of P is regular. Therefore the Tarski–Vaught test implies that $P \preceq M$ for some small $M \preceq \mathfrak{C}$. ■

The following example arose during discussions between Alex Kruckman and Nick Ramsey, and between Ludomir Newelski and myself. The example shows that elimination of imaginaries is an important assumption for our purposes. To avoid that inconvenience, one might modify the definition of regularity as proposed in [10, Remark 3.2(2)]. In this example we do not assume elimination of imaginaries for T .

EXAMPLE 4.3. Consider a language \mathcal{L} consisting of only one relation R and a theory T stating that R is an equivalence relation and that R has only classes of size 3. Note that T is ω -stable and has quantifier elimination. Let us choose some monster model $\mathfrak{C} \models T$. We want to construct a PAC substructure of \mathfrak{C} .

To do this, consider two countable and disjoint families of equivalence classes of R , say A and B . A substructure P consists of

- all elements from every equivalence class belonging to A ,
- exactly one element from each equivalence class belonging to B .

If $P \subseteq N$ is regular, then the intersection of N with any equivalence class belonging to B contains only one element, which we already chose for P . It follows that P is existentially closed in N and therefore P is PAC.

Note that adding to P only one element from every equivalence class belonging to B will not produce a PAC substructure. To see this let $N := P \cup \{a, b, c\}$, where $\{a, b, c\} \cap P = \emptyset$. Then $P \subseteq N$ is regular, but N satisfies the sentence stating that there exist three different elements x, y and z such that $R(x, y)$ and $R(y, z)$.

The absolute Galois group of P is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\omega$:

$$\mathcal{G}(P) \cong (\mathbb{Z}/2\mathbb{Z})^\omega.$$

By [9, Corollary 22.7.11], $\mathcal{G}(P)$ cannot be projective.

Now, we generalize [13, Lemma 1.17] (which states that the absolute Galois group of a PAC substructure—in the strongly minimal context—is projective) to our (stable) context. Of course, we still assume that T is stable and has quantifier elimination and elimination of imaginaries.

THEOREM 4.4. *If a small (but infinite) N is PAC, then $\mathcal{G}(N)$ is projective.*

Proof. Assume that for some finite groups A and B we have continuous epimorphisms $\rho : \mathcal{G}(N) \rightarrow A$ and $\alpha : B \rightarrow A$. We will find

$$\gamma : \mathcal{G}(N) \rightarrow B$$

such that $\rho = \alpha\gamma$,

$$\begin{array}{ccc} \mathcal{G}(N) & \xrightarrow{\rho} & A \\ & \searrow \gamma & \uparrow \alpha \\ & & B \end{array}$$

$$\underline{A \rightsquigarrow \text{Aut}_{\mathcal{L}}(L/N)}$$

Because ρ is continuous, $\ker \rho$ is a closed subgroup of $\mathcal{G}(N)$. The Galois correspondence (Fact 2.7) implies that $L := \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)^{\ker \rho}$ is definably closed and

$$\ker \rho = \text{Aut}_{\mathcal{L}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)/L).$$

By Fact 2.6, $N \subseteq L$ is Galois. Thus, by Fact 2.5, $A \cong \text{Aut}_{\mathcal{L}}(L/N)$. Therefore, without loss of generality, we can assume that $A = \text{Aut}_{\mathcal{L}}(L/N)$ and $\rho = |L$,

$$\begin{array}{ccc} \mathcal{G}(N) & \xrightarrow{|L} & \text{Aut}_{\mathcal{L}}(L/N) \\ & & \uparrow \alpha \\ & & B \end{array}$$

$$\underline{B \rightsquigarrow \text{Aut}_{\mathcal{L}}(M/M^B)}$$

By Fact 2.13, there exists a non-algebraic stationary type over N (in the sense of \mathfrak{C}), say $p(x)$. Take elements from a Morley sequence in $p(x)$, say $b_1, \dots, b_n \in \mathfrak{C}$, where $n := |B|$, such that

$$L \downarrow_N^{\mathfrak{C}} b_1 \dots b_n.$$

By multiple use of Fact 2.16, we obtain a sequence of regular extensions

$$N \subseteq Nb_1 \subseteq Nb_1b_2 \subseteq \dots \subseteq Nb_1 \dots b_n,$$

hence, by Fact 2.2(4), $N \subseteq Nb_1 \dots b_n$ is also regular. Stability implies that the tuple $b_1 \dots b_n$ is N -indiscernible as a set, hence for every $\sigma \in S_n$ (a permutation of n elements) there exists $h_\sigma \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/N)$ such that $h_\sigma(b_i) = b_{\sigma(i)}$. Without loss of generality, $B \leq S_n$. Note that B acts on L :

$$g(m) = \alpha(g)(m),$$

where $g \in B$ and $m \in L$. Moreover, B acts on $b_1 \dots b_n$ by h_σ , where $\sigma \in B$. Fact 2.14 allows us to extend both of these actions of B to an action on $M := \text{dcl}_{\mathcal{L}}(Lb_1 \dots b_n)$. Now, we will treat B as a subgroup of $\text{Aut}_{\mathcal{L}}(M/M^B)$.

Since B is finite, Lemma 2.10(3) implies that $M^B \subseteq M$ is Galois and $M \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^B)$. Moreover, B as a finite group is a closed subgroup of $\text{Aut}_{\mathcal{L}}(M/M^B)$. Therefore, the Galois correspondence (Fact 2.7) and

$$M^B = M^{\text{Aut}_{\mathcal{L}}(M/M^B)}$$

imply that $B = \text{Aut}_{\mathcal{L}}(M/M^B)$. Again, without loss of generality, we change the set-up:

$$\begin{array}{ccc} \mathcal{G}(N) & \xrightarrow{\quad |L \quad} & \text{Aut}_{\mathcal{L}}(L/N) \\ & & \uparrow |L \\ & & \text{Aut}_{\mathcal{L}}(M/M^B) \end{array}$$

$N \subseteq M^B$ is regular

(A similar argument is used in the proof of Proposition 3.1.) Now, we will show that $M^B \cap L = N$. Of course $N \subseteq M^B \cap L$. Let $m \in M^B \cap L$, i.e. $m \in L$ and $m = \sigma(m) = \alpha(\sigma)(m)$ for all $\sigma \in B$. Because α is an epimorphism, it follows that for each $f \in \text{Aut}_{\mathcal{L}}(L/N)$ we have $f(m) = m$, hence $m \in N$.

Recall that $L \downarrow_N^{\mathfrak{C}} b_1 \dots b_n$, $M = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(Lb_1 \dots b_n)$, and $N \subseteq Nb_1 \dots b_n$ is regular. Fact 2.16 implies that also $L \subseteq M$ is regular. We have

$$\begin{aligned} M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(L) &= L, & \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(L) &= \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N), \\ M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N) &= L, \\ M^B \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N) &= M^B \cap L = N, \end{aligned}$$

i.e. $N \subseteq M^B$ is regular.

$N \rightsquigarrow N'$ which contains a copy of M^B

Let $\bar{c} \subseteq \mathfrak{C}$ be such that $M^B = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N\bar{c})$ (\bar{c} can be an enumeration of M^B). Moreover, we introduce a set of $\mathcal{L} \cup \{N\}$ -formulas,

$$q(\bar{x}) := \text{qftp}_{\mathcal{L}}^{\mathfrak{C}}(\bar{c}/N) \cup \{\bar{x}_0 \subseteq N \mid \bar{x}_0 \subseteq \bar{x}\}.$$

Take a small $D \preceq \mathfrak{C}$ such that $N \subseteq D$.

CLAIM. $\text{Th}_{\mathcal{L} \cup \{N\}}(D) \cup q(\bar{x})$ is consistent.

Proof of the claim. Since N is PAC and $N \subseteq M^B$ is regular, we have $N \preceq_1 M^B$. Therefore if $\varphi(n, \bar{x}_0) \in \text{qftp}_{\mathcal{L}}^{\mathfrak{C}}(\bar{c}/N)$, then $M^B \models (\exists \bar{x}_0)(\varphi(n, \bar{x}_0))$ and so also $N \models (\exists \bar{x}_0)(\varphi(n, \bar{x}_0))$ and $(D, N) \models (\exists \bar{x}_0)(\bar{x}_0 \subseteq N \wedge \varphi(n, \bar{x}_0))$.

Consider $(D', N') \succeq (D, N)$ which is $|N|^+$ -saturated. Without loss of generality, $D \preceq D' \preceq \mathfrak{C}$. Note that $N' \succeq N$ and $\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N') = N'$.

There exists $\bar{c}' \subseteq N'$ such that $\bar{c}' \models \text{qftp}_{\mathcal{L}}^{\mathfrak{C}}(\bar{c}/N)$. Quantifier elimination in T implies that $\bar{c}' \models \text{tp}_{\mathcal{L}}^{\mathfrak{C}}(\bar{c}/N)$, thus there exists $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/N)$ such that $f(\bar{c}) = \bar{c}'$. Since $N \subseteq N\bar{c}$ is regular, Fact 2.11 allows us to assume that $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N))$.

Note that

$$f(M^B) = f(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N\bar{c})) = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(Nf(\bar{c})) = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N\bar{c}') \subseteq \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N') = N'$$

and

$$f(M) \subseteq f(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^B)) = \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(f(M^B)) \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N').$$

Note that $\text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))/N')$, as determined by values on $\bar{a}f(\bar{b})$, is finite and if there is no $h \in \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))/N')$ such that

$$h(a_1 \dots a_m f(b_1) \dots f(b_n)) = a_{\sigma(1)} \dots a_{\sigma(m)} f(b_{\sigma'(1)}) \dots f(b_{\sigma'(n)}),$$

then there is no such h in $\text{Aut}_{\mathcal{L}}(\mathfrak{C}/N')$ (since $N' \subseteq \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))$ is normal) and hence

$$(*) \quad a_1 \dots a_m f(b_1) \dots f(b_n) \not\equiv_{N'} a_{\sigma(1)} \dots a_{\sigma(m)} f(b_{\sigma'(1)}) \dots f(b_{\sigma'(n)}).$$

We choose a formula $\psi_{\sigma, \sigma'}$ such that

$$\begin{aligned} & \models \psi_{\sigma, \sigma'}(d, a_1, \dots, a_m, f(b_1), \dots, f(b_n)), \\ & \models \neg \psi_{\sigma, \sigma'}(d, a_{\sigma(1)}, \dots, a_{\sigma(m)}, f(b_{\sigma'(1)}), \dots, f(b_{\sigma'(n)})). \end{aligned}$$

Now, we introduce an \mathcal{L} -formula $\theta(d, \bar{a})$ given by

$$\begin{aligned} & (\exists y_1, \dots, y_n) \left(\bigwedge_{k < i_2} \varphi_{i_1}(d, y_k) \wedge (\forall y) \left(\varphi_{i_1}(d, y) \rightarrow \bigvee_{k < j_2} y = y_k \right) \right. \\ & \quad \vdots \\ & \quad \wedge \bigwedge_{i_s \leq k \leq n} \varphi_{i_s}(d, y_k) \wedge (\forall y) \left(\varphi_{i_s}(d, y) \rightarrow \bigvee_{i_s \leq k \leq n} y = y_k \right) \\ & \quad \wedge \bigwedge_{(\sigma, \sigma') \text{ as in } (*)} \psi_{\sigma, \sigma'}(d, a_1, \dots, a_m, y_1, \dots, y_n) \\ & \quad \left. \wedge \neg \psi_{\sigma, \sigma'}(d, a_{\sigma(1)}, \dots, a_{\sigma(m)}, y_{\sigma'(1)}, \dots, y_{\sigma'(n)}) \right). \end{aligned}$$

We have

$$\begin{aligned} (D', N') & \models \theta(d, \bar{a}), \\ (D', N') & \models (\exists z)(z \subseteq N' \wedge \theta(z, \bar{a})), \\ (D, N) & \models (\exists z)(z \subseteq N \wedge \theta(z, \bar{a})). \end{aligned}$$

Let $d' \subseteq N$ be such that $D \models \theta(d' \bar{a})$ and let $b'_1, \dots, b'_n \subseteq D$ witness existence of y_1, \dots, y_n for $\theta(d', \bar{a})$ in D . We see that $\bar{b}' \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N)$ and

$$\text{Aut}_{\mathcal{L}}(\mathfrak{C}/N') \cdot \bar{b}' \subseteq \text{Aut}_{\mathcal{L}}(\mathfrak{C}/N) \cdot \bar{b}' = \bar{b}'.$$

Therefore $N' \subseteq \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')$ is Galois and there is a restriction map

$$\text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N), N')/N') \rightarrow \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')/N').$$

Final diagram

The last map we need is

$$\Delta : \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')/N') \rightarrow \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))/N')$$

and we define it as follows. Let $h \in \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')/N')$ and

$$h(a_1 \dots a_m b'_1 \dots b'_n) = a_{\sigma(1)} \dots a_{\sigma(m)} b'_{\sigma'(1)} \dots b'_{\sigma'(n)}.$$

There exists $\tilde{h} \in \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))/N')$ such that

$$\tilde{h}(a_1 \dots a_m f(b_1) \dots f(b_n)) = a_{\sigma(1)} \dots a_{\sigma(m)} f(b_{\sigma'(1)}) \dots f(b_{\sigma(n)})$$

(otherwise (σ, σ') would satisfy $(*)$, but then

$$\begin{aligned} &\models \psi_{\sigma, \sigma'}(d', a_1, \dots, a_m, b'_1, \dots, b'_n), \\ &\models \neg \psi_{\sigma, \sigma'}(d', a_{\sigma(1)}, \dots, a_{\sigma(m)}, b'_{\sigma'(1)}, \dots, b'_{\sigma'(n)}), \end{aligned}$$

which contradicts the existence of h). We set $\Delta(h) := \tilde{h}$.

Now, we put all the above steps together:

$$\begin{array}{ccc} \mathcal{G}(N) & \xrightarrow{\quad |L \quad} & \text{Aut}_{\mathcal{L}}(L/N) \\ H^{-1} \downarrow \cong & & \uparrow |L \\ \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N), N')/N') & & \text{Aut}_{\mathcal{L}}(M/M^B) \\ & & \cong \uparrow F^{-1} \\ & & \text{Aut}_{\mathcal{L}}(f(M)/f(M^B)) \\ & \xleftarrow{\quad \models \quad} & \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(L, f(\bar{b}))/f(M^B)) \\ & & \downarrow |_{\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(L, f(\bar{b}))} \\ \downarrow |_{\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')} & & \\ \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, \bar{b}')/N') & \xrightarrow{\quad \Delta \quad} & \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', \bar{a}, f(\bar{b}))/N') \xleftarrow{\quad \models \quad} \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N', L, f(\bar{b}))/N') \end{array}$$

The above diagram commutes, since the “long path” does not do anything with values of automorphisms on $L = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(N\bar{a})$. ■

4.2. Projective profinite group as absolute Galois group. In this subsection, we show that the property PP_T holds for a subclass of the class of stable theories (namely, for ω -stable theories). The only issue not allowing us to extend our result over all stable theories is the fact that sometimes the algebraic closure of a PAC substructure is not a PAC substructure, which seems rather strange if we remember that “PAC” stands for “pseudo-algebraically closed”. However, many interesting stable theories satisfy a simplified version of the main assumption of Lemma 4.5:

a type over A has only finitely many extensions over $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$,

which holds for any type in e.g. any ω -stable theory. Because we did not achieve property PP_T for arbitrary stable T , we consider a weaker version of the left-to-right implication in the property PP_T :

if a profinite G is projective then $G \cong \mathcal{G}(P)$ for some definably closed substructure P of \mathfrak{C} ,

which holds for any stable theory T (with quantifier elimination and elimination of imaginaries)—see Theorem 4.9. Since the absolute Galois group of a PAC substructure is projective for any stable T , the right-to-left implication in PP_T is true for any stable T .

The following lemma is a simple modification of [19, Proposition 3.9] (related to our alternative definition of a PAC substructure), which generalizes a well known fact about PAC fields: any algebraic extension of a PAC field is a PAC field (Ax–Roquette Theorem, see [9, Corollary 11.2.5]). The proof of our slight modification is based on the original proof in [19, Proposition 3.9], but for the reader’s convenience, instead of listing all the small differences, we provide the whole proof.

LEMMA 4.5. *Let P be a small PAC substructure of \mathfrak{C} , and let $P \subseteq Q = \text{dcl}_{\mathfrak{L}}^{\mathfrak{C}}(Q) \subseteq Q' \subseteq \text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)$, where Q and Q' are substructures of \mathfrak{C} such that $P \subseteq Q'$ is normal (e.g. $Q' = \text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)$). If any type over P has only finitely many (non-forking) extensions over Q' , then Q is PAC.*

Proof. We want to show that if a type $p(x)$ over Q is stationary, then it is finitely satisfiable in Q (as in Remark 4.1). Assume that $\varphi(m_0, x) \in p(x)$.

There are only finitely many distinct extensions of $p|_P$ over Q' , say $p_1, \dots, p_n \in S(Q')$. We assume that $p_1 \supseteq p$, so p_1 is stationary. Since $P \subseteq Q'$ is normal, the type p_i is stationary for every $i \leq n$. Therefore there are only finitely many distinct extensions of type $p|_P$ over $\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)$, abusing notation: $p_1, \dots, p_n \in S(\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P))$.

Consider

$$\tilde{p} := \bigotimes_{i \leq n} p_i \in S(\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)),$$

some $d_1 \dots d_n \models \tilde{p}$ and the code d' for the set $\{d_1, \dots, d_n\}$ (we are using here the fact that there are only finitely many extensions).

CLAIM. $\text{tp}_{\mathfrak{L}}^{\mathfrak{C}}(d'/P)$ is stationary.

Proof of the claim. It is enough to show that there is only one extension of the type $\text{tp}_{\mathfrak{L}}^{\mathfrak{C}}(d'/P)$ over $\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)$. Let $\phi(c, y) \in \text{tp}_{\mathfrak{L}}^{\mathfrak{C}}(d'/\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P))$, i.e.

$$\mathfrak{C} \models \phi(c, d'),$$

and let $f \in \text{Aut}_{\mathfrak{L}}(\mathfrak{C}/P)$. Since $\{d_1, \dots, d_n\}$ is P -independent, it follows that $\{f(d_1), \dots, f(d_n)\}$ is also P -independent and so $\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P)$ -independent. Note that there exists some permutation $\sigma \in S_n$ such that $f(d_{\sigma(i)}) \models p_i$, hence

$$f(d_{\sigma(1)}) \dots f(d_{\sigma(n)}) \models \tilde{p}.$$

There exists $h \in \text{Aut}_{\mathfrak{L}}(\mathfrak{C}/\text{acl}_{\mathfrak{L}}^{\mathfrak{C}}(P))$ such that

$$f(d_{\sigma(1)}) \dots f(d_{\sigma(n)}) = h(d_1) \dots h(d_n).$$

Therefore $h^{-1}f(d_{\sigma(i)}) = d_i$ for each $i \leq n$, so $h^{-1}f(d') = d'$. We have

$$\mathfrak{C} \models \phi(h^{-1}f(c), h^{-1}f(d')),$$

$$\mathfrak{C} \models \phi(h^{-1}f(c), d'),$$

but since $c \in \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$, we have $h^{-1}f(c) = f(c)$ and the previous line can be written as

$$\mathfrak{C} \models \phi(f(c), d').$$

Because $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/P)$ was arbitrary, the proof of the claim is finished.

Note that $d_1 \in \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(Q, d')$. To see this, take any $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/Qd')$. Since $f(d') = d'$, it follows that $f(d_1) = d_i$ for some $i \leq n$. We have $\text{tp}_{\mathcal{L}}^{\mathfrak{C}}(d_i/Q) = \text{tp}_{\mathcal{L}}^{\mathfrak{C}}(d_1/Q) = p$, which is stationary. Therefore p_i and p_1 are non-forking extensions of a stationary type and so $p_i = p_1$ and $d_i = d_1$.

There exists an \mathcal{L} -formula θ such that $\theta(q_0, d', \mathfrak{C}) = \{d_1\}$. We have

$$\begin{aligned} (\exists x) \left(\bigvee_{f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/P)} \theta(f(q_0), y, x) \wedge (\exists! x') (\theta(f(q_0), y, x')) \wedge \varphi(m_0, x) \right) \\ \in \text{tp}_{\mathcal{L}}^{\mathfrak{C}}(d'/P). \end{aligned}$$

Since P is PAC, and $\text{tp}_{\mathcal{L}}^{\mathfrak{C}}(d'/P)$ is stationary, there exist $a' \in P$, $b \in \mathfrak{C}$ and $f \in \text{Aut}_{\mathcal{L}}(\mathfrak{C}/P)$ such that

$$\begin{aligned} \mathfrak{C} \models \theta(f(q_0), a', b) \wedge (\exists! x') (\theta(f(q_0), a', x')) \wedge \varphi(m_0, b), \\ \mathfrak{C} \models \theta(q_0, a', f^{-1}(b)) \wedge (\exists! x') (\theta(q_0, a', x')) \wedge \varphi(m_0, f^{-1}(b)). \end{aligned}$$

It follows that $f^{-1}(b) \in \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(q_0, a') \subseteq Q$ and $\mathfrak{C} \models \varphi(m_0, f^{-1}(b))$, which ends the proof. ■

REMARK 4.6. One could ask about possible generalizations of the above lemma. Section 5 in [19] provides an example of a superstable theory T and a bounded PAC substructure P of \mathfrak{C} such that $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$ is not an elementary substructure (recall that an algebraically closed PAC substructure is an elementary substructure). Therefore, it seems that there is no natural generalization of Lemma 4.5.

COROLLARY 4.7. *If a small $P \subseteq \mathfrak{C}$ is PAC and there are only finitely many extensions over $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$ of every type over P , then $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$ is PAC and $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P) \preceq \mathfrak{C}$.*

QUESTION 4.8. Assume that P is PAC.

- (1) What are the obstacles to $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$ being PAC?
- (2) Assume moreover that $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$ is PAC. Does every type over P have only finitely many extensions over $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(P)$?

The following proposition generalizes [9, Theorem 23.1.1].

THEOREM 4.9. *Assume that $N_0 \subseteq N$ is a Galois extension of small (but infinite) substructures of \mathfrak{C} and assume that there is an epimorphism $\alpha : G \rightarrow \text{Aut}_{\mathcal{L}}(N/N_0)$ of profinite groups, and G is projective. Then there exists a definably closed substructure $P \supseteq N_0$ of \mathfrak{C} and an isomorphism*

$\gamma : G \rightarrow \mathcal{G}(P)$ of profinite groups such that

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & \mathcal{G}(P) \\ & \searrow \alpha & \downarrow |_N \\ & & \text{Aut}_{\mathcal{L}}(N/N_0) \end{array}$$

is commuting. Moreover, if any type over A has only finitely many extensions over $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(A)$, then P is PAC.

Proof. By Proposition 3.1, there exist regular extensions $N_0 \subseteq M_0$ and $N \subseteq M$ such that $M_0 \subseteq M$ is Galois, and $\beta : G \cong \text{Aut}_{\mathcal{L}}(M/M_0)$ such that

$$\begin{array}{ccc} \text{Aut}_{\mathcal{L}}(M/M_0) & & \\ \beta \uparrow & \searrow |_N & \\ G & \xrightarrow{\alpha} & \text{Aut}_{\mathcal{L}}(N/N_0) \end{array}$$

is commuting. By [10, Proposition 3.6], there is a PAC substructure M'_0 such that $M_0 \subseteq M'_0$ is regular. Because $M_0 \subseteq M'_0$ is regular and $\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_0) = \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M)$, it follows that $M'_0 \cap M = M_0$. Hence the restriction map

$$\text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0) \xrightarrow{|_M} \text{Aut}_{\mathcal{L}}(M/M_0)$$

is an isomorphism. Let $w : \text{Aut}_{\mathcal{L}}(M/M_0) \rightarrow \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0)$ be the inverse of the restriction map $|_M$. Hence $\text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0) \cong G$ is projective, and so the restriction map

$$\mathcal{G}(M'_0) \xrightarrow{|_{\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)}} \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0)$$

has a section $i : \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0) \rightarrow \mathcal{G}(M'_0)$,

$$\begin{array}{ccc} & \xrightarrow{|_{\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)}} & \\ \mathcal{G}(M'_0) & \xleftarrow{i} & \text{Aut}_{\mathcal{L}}(\text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M, M'_0)/M'_0) \\ & & \uparrow w \quad \downarrow |_M \\ & & \text{Aut}_{\mathcal{L}}(M/M_0) \\ & & \beta \uparrow \quad \searrow |_N \\ & & G \xrightarrow{\alpha} \text{Aut}_{\mathcal{L}}(N/N_0) \end{array}$$

We set $\gamma := i \circ w \circ \beta$ (note that $\gamma : G \rightarrow \mathcal{G}(M'_0)$ is a continuous embedding) and

$$P := \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M'_0)^{\gamma(G)}$$

and note that $\mathcal{G}(P) = \gamma(G)$ and

$$\begin{array}{ccc} G & \xrightarrow{\gamma} & \mathcal{G}(P) \\ & \searrow \alpha & \downarrow \downarrow N \\ & & \text{Aut}_{\mathcal{L}}(N/N_0) \end{array}$$

is commuting. The “moreover” part follows from Corollary 4.7. ■

COROLLARY 4.10. *Assume that T is ω -stable. Then property PP_T holds, i.e. a profinite group G is projective if and only if G is isomorphic to the absolute Galois group of some PAC substructure of \mathfrak{C} .*

QUESTION 4.11. What is the biggest class of stable theories T for which property PP_T holds?

5. G -actions on substructures. From now on, let us assume that G is *finitely generated*, and T is a stable \mathcal{L} -theory, which has quantifier elimination and elimination of imaginaries. We denote by \hat{G} the profinite completion of the group G .

We are interested in (absolute Galois groups of) substructures of a monster model \mathfrak{C} of the theory T , which are equipped with a group action of the group G . Note that a small \mathcal{L} -structure M can be embedded as a (not necessarily elementary) substructure into \mathfrak{C} if and only if $M \models T_{\forall}$, where T_{\forall} is the theory of the class of small \mathcal{L} -substructures of \mathfrak{C} (i.e. the universal part of the theory T). We introduce a new language

$$\mathcal{L}^G := \mathcal{L} \cup \{\sigma_g\}_{g \in G},$$

where each σ_g is a unary function symbol (for simplicity it will denote also the interpretation of σ_g in an \mathcal{L}^G -structure $(M, \bar{\sigma})$). An \mathcal{L}^G -structure $(M, \bar{\sigma})$ is a model of $(T_{\forall})_G$ if and only if

- $M \models T_{\forall}$,
- $\sigma_g \in \text{Aut}_{\mathcal{L}}(M)$ for each $g \in G$,
- $G \ni g \mapsto \sigma_g \in \text{Aut}_{\mathcal{L}}(M)$ is a homomorphism of groups.

Assume for the rest of the paper that $(M, \bar{\sigma})$ is an existentially closed model of the theory $(T_{\forall})_G$ (i.e. existentially closed among all small substructures of \mathfrak{C} equipped with an action of G). Note that M^G (the substructure of invariants) must be infinite in this scenario.

We are interested in describing $\mathcal{G}(M)$. It turns out that it is good to start with the description of $\mathcal{G}(M^G)$ (it is easier and might be used in the desired description of $\mathcal{G}(M)$). The idea behind the next results is the following: the action of G on M depends only on the action of G on $M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)$, the relative algebraic closure of the invariants. The following proposition, which

is a generalization of [24, Theorem 4], partially expresses this idea by embedding G into the group of automorphisms of the relative algebraic closure of the invariants. Let \mathcal{A} denote the profinite group $\text{Aut}_{\mathcal{L}}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)/M^G)$.

PROPOSITION 5.1. *In the situation described above we have $\mathcal{A} \cong \hat{G}$. More precisely, if $(M, \bar{\sigma})$ is a small existentially closed model of $(T_{\forall})_G$, then*

$$\text{Aut}_{\mathcal{L}}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)/M^G) \cong \hat{G}.$$

Proof. By [10, Proposition 3.31], \mathcal{A} is finitely generated. Corollary 3.2.8 in [23] says that two finitely generated groups have isomorphic profinite completions if and only if they have the same finite quotients:

$$\hat{\mathcal{A}} \cong \hat{G} \quad \text{iff} \quad \text{Im}(\mathcal{A}) = \text{Im}(G).$$

We need to evoke a significant theorem [18, Theorem 1.1]: the profinite completion of a finitely generated profinite group is equal to this group (equivalently: its subgroups of finite index are open). Hence $\mathcal{A} = \hat{\mathcal{A}}$. Moreover, every homomorphism $\alpha : \mathcal{A} \rightarrow H$, where H is a finite group with discrete topology, is continuous and therefore $\text{Im}(\mathcal{A}) \subseteq \text{Im}(G)$. We need to show that $\text{Im}(\mathcal{A}) \supseteq \text{Im}(G)$.

Let $\pi : G \rightarrow H$ be a homomorphism onto a finite group H and let $m := |H|$. Our goal is to prove the existence of a surjective group homomorphism $\mathcal{A} \twoheadrightarrow H$.

Take $\bar{b} = (b_1, \dots, b_m)$, where b_i are pairwise different elements of a Morley sequence in some stationary type $p(x)$ over M^G (which exists by Fact 2.13). Without loss of generality, we assume that $M \downarrow_{M^G}^{\mathfrak{C}} \bar{b}$. We treat H as a subgroup of S_m and since $\{b_1, \dots, b_m\}$ is M^G -indiscernible, H acts on $N := \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M^G \bar{b})$. Moreover, G acts on N by $g \cdot a = \pi(g)(a)$ for $a \in N$. By Fact 2.14 the action of G on M and the action of G on N extend simultaneously to an action of G on $M' := \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(M \bar{b})$ (as in the proof of Theorem 4.4). Note that, by Lemma 2.10, $N^H \subseteq N$ is Galois (in particular $N \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N^H)$). Since π is onto, it follows that $M^G \subseteq N^H \subseteq (M')^G$, thus $(M, \bar{\sigma}) \subseteq (M', \bar{\sigma}')$, where $\bar{\sigma}'$ is the above defined action of G on M' . Existential closedness of $(M, \bar{\sigma})$ implies that $(M, \bar{\sigma}) \preceq_1 (M, \bar{\sigma}')$.

Fix some $|M|^+$ -strongly homogeneous $D \preceq \mathfrak{C}$ such that $M \subseteq D$ and $D \downarrow_M^{\mathfrak{C}} M'$. Every σ_g extends to an element of $\text{Aut}_{\mathcal{L}}(D)$, and for simplicity we denote such an extension also by σ_g . Since $D \downarrow_M^{\mathfrak{C}} M'$ and $M \subseteq M'$ is regular (by Fact 2.3, $M \preceq_1 M'$ implies regularity), we can simultaneously extend $\sigma_g : D \rightarrow D$ and $\sigma'_g : M' \rightarrow M'$ to an element of $\text{Aut}_{\mathcal{L}}(\mathfrak{C})$, which for simplicity we denote by σ'_g . Moreover, regularity of $M \subseteq M'$ implies that $M' \cap D = M$ and so $(M')^G \cap D = M^G$.

We introduce a new language, $\mathcal{L}^D := \mathcal{L}^G \cup \{M, M^G\}$. Consider the following extension of \mathcal{L}^D -structures:

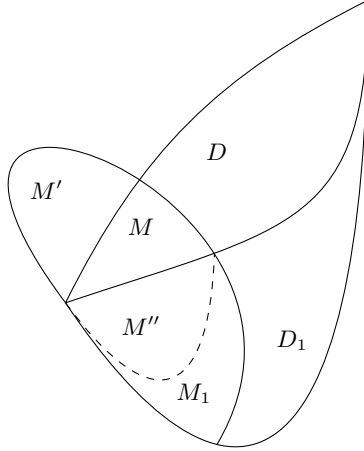
$$(D, \bar{\sigma}', M, M^G) \subseteq (\mathfrak{C}, \bar{\sigma}', M', (M')^G).$$

Since G is finitely generated, M^G as a predicate is definable by an $\mathcal{L}^G \cup \{M\}$ -formula, and we could skip M^G and $(M')^G$ in the above extension and forget about the symbol M^G in the definition of the language \mathcal{L}^D , but we want to keep things more transparent.

CLAIM. *The type $\text{qftp}_{\mathcal{L}^D}^{\mathfrak{C}}(M'/M)$ is consistent with $\text{Th}_{\mathcal{L}^D}(D)$.*

Proof of the claim. The claim follows from the definability of M^G and $(M')^G$ in the \mathcal{L}^G -structures $(M, \bar{\sigma})$ and $(M', \bar{\sigma}')$, and from $(M, \bar{\sigma}) \preceq_1 (M', \bar{\sigma}')$.

Let $(D, \bar{\sigma}, M, M^G) \preceq (D_1, \bar{\sigma}_1, M_1, M_1^G)$ be such that $D_1 \preceq \mathfrak{C}$ and assume $(D_1, \bar{\sigma}_1, M_1, M_1^G)$ realizes $\text{qftp}_{\mathcal{L}^D}^{\mathfrak{C}}(M'/M)$. Moreover, let $M'' \subseteq D_1$ be a realization of $\text{qftp}_{\mathcal{L}^D}^{\mathfrak{C}}(M'/M)$:



There exists an \mathcal{L}^G -isomorphism (over M)

$$h : (M, \bar{\sigma}) \rightarrow (M'', \bar{\sigma}_1),$$

which is a restriction of some $\hat{h} \in \text{Aut}_{\mathcal{L}}(\mathfrak{C})$ for simplicity denoted also by h . We have

$$(h^{-1}(D), \bar{\sigma}^{h^{-1}}, M, M^G) \preceq (h^{-1}(D_1), \bar{\sigma}_1^{h^{-1}}, h^{-1}(M_1), (h^{-1}(M_1))^G),$$

$M' = h^{-1}(M'') \subseteq h^{-1}(M_1)$, and since $h : M' \rightarrow M''$ is an \mathcal{L}^G -isomorphism, $\sigma_{1,g}^{h^{-1}}$ extends σ'_g for each $g \in G$. Observe that $N \subseteq M' \subseteq h^{-1}(M_1)$ and $N^{\hat{H}} \subseteq (M')^G \subseteq (h^{-1}(M_1))^G$.

So far we have obtained

$$\begin{aligned} D \preceq D_2 \preceq \mathfrak{C}, \quad (D, M) \preceq (D_2, M_2), \quad (D, M^G) \preceq (D_2, M_2^G), \\ (M, \bar{\sigma}) \preceq (M_2, \bar{\sigma}_2), \quad N \subseteq M_2, \quad N^H \subseteq M_2^G, \end{aligned}$$

where $D_2 := h^{-1}(D_1)$, $M_2 := h^{-1}(M_1)$ and $\bar{\sigma}_2 := \bar{\sigma}_1^{h^{-1}}$.

By [10, Lemma 3.55], M^G is bounded. Proposition 2.5 in [19] (and its proof) implies that the restriction map

$$R : \mathcal{G}(M_2^G) \rightarrow \mathcal{G}(M^G)$$

is an isomorphism. Consider one more restriction map:

$$r : \mathcal{G}(M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)) \rightarrow \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)).$$

It is not hard to see that $M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G) \subseteq M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)$ is regular. Thus Lemma 2.15 implies that r is onto. By [19, Proposition 2.5],

$$\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G) = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G), M_2^G) = \text{dcl}_{\mathcal{L}}^{\mathfrak{C}}(\text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G), M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)),$$

hence r is injective.

We have the following diagram:

$$\begin{array}{ccc} \mathcal{G}(M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)) & \xrightarrow{r} & \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)) \\ \subseteq \downarrow & & \subseteq \downarrow \\ \mathcal{G}(M_2^G) & \xrightarrow{R} & \mathcal{G}(M^G) \\ \downarrow & & \downarrow \\ \text{Aut}_{\mathcal{L}}(M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)/M_2^G) & \xrightarrow{\cong R'} & \text{Aut}_{\mathcal{L}}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)/M^G) \end{array}$$

and R' is an isomorphism. Therefore we will show that $H \in \text{Im}(\mathcal{A})$ if we show that

$$H \in \text{Im}(\text{Aut}_{\mathcal{L}}(M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)/M_2^G)).$$

Since $N \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(N^H) \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)$, $N \subseteq M_2$ and $N^H \subseteq M_2^G$ we have the restriction map

$$\pi' : \text{Aut}_{\mathcal{L}}(M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)/M_2^G) \rightarrow \text{Aut}_{\mathcal{L}}(N/N^H) = H.$$

We are done if we prove that π' is onto. Let $h \in \text{Aut}_{\mathcal{L}}(N/N^H)$. Then $h = \pi(g)$ for some $g \in G$. Thus

$$h = \pi(g) = \sigma'_g|_N = \sigma_{2,g}|_N = (\sigma_{2,g}|_{M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)})|_N = \pi'(\sigma_{2,g}|_{M_2 \cap \text{acl}_{\mathcal{L}}^{\mathfrak{C}}(M_2^G)}). \quad \blacksquare$$

The following corollary is related to Corollary 3.2 (since G is isomorphic to the Galois group of some Galois extension) and to Corollary 4.10 (since P in Corollary 5.2 is PAC). The devil's in the detail: in Corollary 4.10 we assume that T is ω -stable and get the absolute Galois group of a PAC substructure, while in Corollary 5.2 we assume that G is finitely generated, but

get only the Galois group of some Galois extension (as in Corollary 3.2) of a PAC substructure.

COROLLARY 5.2. *For every finitely generated profinite group G , there exist a bounded PAC substructure P of \mathfrak{C} and a Galois extension $P \subseteq N$ such that*

$$\mathrm{Aut}_{\mathcal{L}}(N/P) \cong G.$$

Proof. Use [18, Proposition 5.1 and Theorem 1.1]. ■

Now, we will link some Galois groups of $(M, \bar{\sigma})$ and the group G , which was partially established in [10, Corollary 3.47]. To do this, the notion of a *Frattini cover* is needed.

DEFINITION 5.3. Let H, H' be profinite groups and $\pi : H \rightarrow H'$ a continuous epimorphism. The mapping π is called a *Frattini cover* if for each closed subgroup H_0 of H , the condition $\pi(H_0) = H'$ implies that $H_0 = H$.

COROLLARY 5.4. *The restriction map*

$$\Xi : \mathcal{G}(M^G) \rightarrow \mathrm{Aut}_{\mathcal{L}}(M \cap \mathrm{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)/M^G)$$

is a Frattini cover.

A Frattini cover is *universal* if its domain is a projective profinite group (and then it is the smallest projective cover); the universal Frattini cover of a group H will be denoted by $\mathrm{Fratt}(H) \rightarrow H$. By [10, Propositions 3.52 and 3.57], M^G and M are PAC substructures of \mathfrak{C} . Hence Theorem 4.4 shows that Ξ is in fact the universal Frattini cover.

COROLLARY 5.5. *The restriction map Ξ is the universal Frattini cover.*

Moreover, Proposition 5.1 allows us to place \hat{G} in the following short exact sequence:

$$\mathcal{G}(M \cap \mathrm{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)) \rightarrow \mathcal{G}(M^G) \rightarrow \mathrm{Aut}_{\mathcal{L}}(M \cap \mathrm{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)/M^G) \cong \hat{G}.$$

So we conclude:

COROLLARY 5.6.

- (1) $\mathcal{G}(M^G) \cong \mathrm{Fratt}(\hat{G})$.
- (2) $\mathcal{G}(M \cap \mathrm{acl}_{\mathcal{L}}^{\mathfrak{C}}(M^G)) \cong \ker(\mathrm{Fratt}(\hat{G}) \rightarrow \hat{G})$.

The above corollary was known in the case of fields (see [24], [11]). Actually [24, Theorem 6] states even more:

$$\mathcal{G}(K) \cong \ker(\mathrm{Fratt}(\hat{G}) \rightarrow \hat{G}),$$

where $(K, \bar{\sigma})$ is an existentially closed field with an action of G . Unfortunately, the proof in [24] has essential gaps and so this very reasonable statement cannot be considered to be already proven.

Now, we will discuss some aspects of [24, proof of Theorem 6] in our stable context. Obviously, we need to examine the restriction map

$$\Theta : \mathcal{G}(M) \rightarrow \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G)).$$

Since $M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G) \subseteq M$ is regular, Lemma 2.15 ensures us that Θ is onto. It is one-to-one if and only if

$$(*) \quad \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M) = \text{dcl}_{\mathcal{L}}^{\mathfrak{e}}(M, \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G)).$$

The incorrect proof of [24, Theorem 6] uses the fact that $M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G)$ is PAC, which is true in the case of fields, but there are no reasons for that in the general framework (since ACF is ω -stable, there are only finitely many non-forking extensions of any type as required in Lemma 4.5, but this is not the case as follows from Remark 4.6). Therefore we suggest a slight modification of the statement of [24, Theorem 6]:

CONJECTURE 5.7. The map $\Theta : \mathcal{G}(M) \rightarrow \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G))$ is a universal Frattini cover.

Note that Conjecture 5.7 implies [24, Theorem 6]:

REMARK 5.8. If the map $\Theta : \mathcal{G}(M) \rightarrow \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G))$ is the universal Frattini cover and $\mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G))$ is projective then $\mathcal{G}(M) \cong \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G))$.

The algebraic and model-theoretic structure of a PAC field is controlled by its absolute Galois group (see e.g. [9, Theorem 20.3.3]). The same remains true for arbitrary PAC substructures (embedded in an ambient stable monster, see [7]). Conjecture 5.7 gives us a way to produce PAC substructures of a monster model of our chosen stable theory T , whose absolute Galois groups can be “calculated” (as the kernel of the universal Frattini cover of the profinite completion of a finitely generated group G).

In particular, since we start with G finitely generated, we see that $\hat{G} = G$ and that $\mathcal{G}(M^G) \cong \text{Fratt}(G)$ is finitely generated (by [9, Lemma 22.6.2]). Therefore M^G is a bounded PAC substructure and one could expect that M^G is simple (as in [21]). However, to use [21], one needs to verify whether being PAC is a first order property [21, Definition 2.7 and Section 3] or to show that the class of existentially closed substructures with G -action is elementary [10, Theorem 4.40]).

In the case of finite G , we have $\hat{G} = G$ and $M \subseteq \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G)$ (by Lemma 2.10(3)). Hence

$$\mathcal{G}(M) = \mathcal{G}(M \cap \text{acl}_{\mathcal{L}}^{\mathfrak{e}}(M^G)) \cong \ker(\text{Fratt}(G) \rightarrow G)$$

i.e. we obtain a PAC substructure M whose absolute Galois group $\mathcal{G}(M)$ is known (compare to main results of [7]).

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References

- [1] N. Bourbaki, *Algebra II, Chapters 4–7*, Springer, 1990.
- [2] Z. Chatzidakis, *Properties of forking in ω -free pseudo-algebraically closed fields*, J. Symbolic Logic 67 (2002), 957–996.
- [3] Z. Chatzidakis and E. Hrushovski, *Perfect pseudo-algebraically closed fields are algebraically bounded*, J. Algebra 271 (2004), 627–637.
- [4] Z. Chatzidakis and A. Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic 95 (1998), 71–92.
- [5] G. Cherlin, L. van den Dries, and A. Macintyre, *The elementary theory of regularly closed fields*, <http://sites.math.rutgers.edu/~cherlin/Preprint/CDM2.pdf>.
- [6] G. Cherlin, L. van den Dries, and A. Macintyre, *Decidability and undecidability theorems for pac-fields*, Bull. Amer. Math. Soc. (N.S.) 4 (1981), 101–104.
- [7] J. Dobrowolski, D. M. Hoffmann, and J. Lee, *Elementary equivalence theorem for pac structures*, submitted; arXiv:1811.08510.
- [8] Yu. Ershov, *Regularly closed fields*, Soviet Math. Doklady 21 (1980), 510–512.
- [9] M. D. Fried and M. Jarden, *Field Arithmetic*, Springer, 2008.
- [10] D. M. Hoffmann, *Model theoretic dynamics in Galois fashion*, Ann. Pure Appl. Logic 170 (2019), 755–804.
- [11] D. M. Hoffmann and P. Kowalski, *Existentially closed fields with finite group actions*, J. Math. Logic 18 (2018), no. 1, art. 1850003, 26 pp.
- [12] D. M. Hoffmann and J. Lee, *Co-theory of sorted profinite groups for PAC structures*, arXiv:1905.09748 (2019).
- [13] E. Hrushovski, *Pseudo-Finite Fields and Related Structures*, Quad. Mat. 11, Aracne, Roma, 2002.
- [14] E. Hrushovski, *On finite imaginaries*, in: Logic Colloquium 2006, Lecture Notes in Logic 32, Assoc. Symbolic Logic, Chicago, IL, 2009, 195–212.
- [15] E. Hrushovski, B. Martin, S. Rideau, and R. Cluckers, *Definable equivalence relations and zeta functions of groups*, arXiv:math/0701011v5 (2017).
- [16] S. Lang, *Algebra*, Grad. Texts in Math. 211, Springer, New York, 2002.
- [17] A. Medvedev and R. Takloo-Bighash, *An invitation to model-theoretic Galois theory*, Bull. Symbolic Logic 16 (2010), 261–269.
- [18] N. Nikolov and D. Segal, *On finitely generated profinite groups, I: strong completeness and uniform bounds*, Ann. of Math. 165 (2007), 171–238.
- [19] A. Pillay and D. Polkowska, *On PAC and bounded substructures of a stable structure*, J. Symbolic Logic 71 (2006), 460–472.
- [20] B. Poizat, *Une théorie de Galois imaginaire*, J. Symbolic Logic 48 (1983), 1151–1170.
- [21] N. M. Polkowska, O.P., *On simplicity of bounded pseudoalgebraically closed structures*, J. Math. Logic 7 (2007), 173–193.

- [22] S. N. Ramsey, *Independence, Amalgamation, and Trees*, PhD thesis, UC Berkeley, 2018.
- [23] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer, New York, 2000.
- [24] N. Sjögren, *The model theory of fields with a group action*, Research Reports in Mathematics 7, Department of Mathematics, Stockholm Univ., 2005.
- [25] K. Tent and M. Ziegler, *A Course in Model Theory*, Lecture Notes in Logic 40, Cambridge Univ. Press, 2012.

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