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On maximal ideals which are also minimal prime ideals in certain Banach rings

by

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Abstract. We study the existence of a maximal ideal which is also a minimal prime ideal in Banach rings in a wide class containing the Banach algebra $\mathrm{C}_{\mathrm{bd}}(X,k)$ of bounded continuous functions $X \to k$ for a topological space X and a Banach field k with a mild condition, the quotient of $\mathrm{C}_{\mathrm{bd}}(X,k)$ by the closed ideal $\mathrm{C}_0(X,k)$ of functions vanishing at infinity, the bounded direct product $\prod_{\lambda\in\Lambda}\kappa_\lambda$ of a family $\kappa=(\kappa_\lambda)_{\lambda\in\Lambda}$ of Banach fields with a mild condition, and the quotient of $\prod_{\lambda\in\Lambda}\kappa_\lambda$ by the completed direct sum $\widehat{\bigoplus}_{\lambda\in\Lambda}\kappa_\lambda$. We describe the maximal spectrum and the Berkovich spectrum of such Banach rings, and generalise the classical result on the relation between the existence of such a maximal ideal of the Banach $\mathbb{R}$-algebra $\mathrm{C}_{\mathrm{bd}}(\mathbb{N},\mathbb{R})/\mathrm{C}_0(\mathbb{N},\mathbb{R})$ and the existence of a P-point in $\beta\mathbb{N}\setminus\mathbb{N}$.

Contents

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Introduction. We study maximal ideals which are also minimal prime ideals in a wide class of Banach rings. For example, we consider the following Banach rings:

(i) The Banach ring $\mathrm{C}_{\mathrm{bd}}(X, k)$ of bounded continuous functions $X \to k$ for a topological space X and a Banach field k under a mild condition, and the Banach ring $\mathrm{C}_{\partial}(X, k)$ given as the quotient of $\mathrm{C}_{\mathrm{bd}}(X, k)$ by the closed ideal of functions vanishing at infinity.

(ii) The Banach ring $\prod_{\lambda \in \Lambda} \kappa_\lambda$ given as the bounded direct product of a family $\kappa = (\kappa_\lambda)_{\lambda \in \Lambda}$ of Banach fields under a mild condition, and the Banach ring $\partial\kappa$ given as the quotient of $\prod_{\lambda \in \Lambda} \kappa_\lambda$ by the completed direct sum $\widehat{\bigoplus}_{\lambda \in \Lambda} \kappa_\lambda$.

The first Banach ring in (i) appears in both real analysis and p-adic analysis, especially when we study operator theory. The first Banach ring in (ii) appears when we consider the non-Archimedean Gel'fand transform (cf. [Ber90, 1.2.2. Remark (i)]), and hence is itself interesting. The second Banach rings in (i) and (ii) are Banach analogues of the direct product modulo the direct sum. Indeed, the direct product modulo the direct sum of abstract fields is the underlying ring of the bounded direct sum modulo the completed direct sum of the fields equipped with the trivial valuation. They carry information on the asymptotic behaviour of functions or sequences, and frequently appear in various fields.

For a topological space X, an $x \in X$ is said to be a *P-point of* X if the intersection of any non-empty countable set of neighbourhoods of x in X is again a neighbourhood of x in X. For example, the Banach $\mathbb{R}$-algebra of bounded sequences modulo zero sequences in $\mathbb{R}$ is naturally isometrically isomorphic to the Banach $\mathbb{R}$-algebra of continuous functions on $\beta\mathbb{N} \setminus \mathbb{N}$, and hence is used in the study of the existence of a P-point in $\beta\mathbb{N} \setminus \mathbb{N}$ (cf. [Wim82, 6.5. Theorem] and [She98, VI.4.8]). Similarly, the Banach $\mathbb{R}$-algebra of bounded continuous functions modulo functions vanishing at infinity on a locally compact Hausdorff topological space is naturally isometrically isomorphic to the Banach $\mathbb{R}$-algebra of continuous functions on its growth (cf. §1.3), and is used in the study of the existence of a P-point in the growth of a non-pseudo-compact locally compact Hausdorff topological space (cf. [Wal74, 4.34. Theorem]). As a non-commutative example, the bounded operators modulo compact operators on a separable infinite-dimensional Hilbert space form a Banach $\mathbb{C}$-algebra called the Calkin algebra, which plays a central role in Fredholm theory and the Brown–Douglas–Fillmore problem (cf. [PW07, Theorem 2.4] and [Far11, Theorem 1]). The direct product modulo the direct sum of $\mathbb{N}$-copies of $\mathbb{F}_2$ is naturally isomorphic to the boolean algebra of subsets modulo finite subsets of $\mathbb{N}$, whose automorphism group is naturally isomorphic to the automorphism group of $\beta\mathbb{N}\setminus\mathbb{N}$ and hence is classically studied (cf. [She82, 4.5 Theorem]). The direct product modulo the direct sum of $(\mathbb{F}_p)_{p \in \mathbb{P}}$ is used in the study of finite multiple zeta values (cf. [KZ]), and the bounded direct product modulo the completed direct sum of $(\mathbb{Q}_{p,2^{-1}})_{p \in \mathbb{P}}$ is used in the study of weighted finite multiple zeta values (cf. [Ros15]). Here, $\mathbb{P}$ denotes the set of prime numbers, and for a $(p, \epsilon) \in \mathbb{P} \times (0,1)$, $\mathbb{Q}_{p,\epsilon}$ denotes the Banach field $\mathbb{Q}_p$ equipped with the p-adic valuation $|-|_{p,\epsilon}$ uniquely determined by the normalisation condition $|p|_{p,\epsilon} = \epsilon$. Since many of these studies are related to set-theoretic problems, the structures of such mathematical objects are mysterious and complicated.

In order to simultaneously deal with these distinct Banach rings, we will introduce in §2.1 a wide notion of a uniform ring, which is a common generalisation of a uniform algebra, a Banach field under a mild condition, and a uniform Banach ring, and study the Zariski topology on the set of maximal ideals. For a ring R, we denote by $R^\times \subset R$ the subset of invertible elements. We state our main results (cf. Theorems 3.1 and 3.2) restricted to the specific Banach rings in (i) and (ii).

THEOREM 0.1. *Let X be a topological space, and k be a Banach field with $\sup_{c \in k^\times} |c|\,|c^{-1}| < \infty$. Suppose that X is strongly zero-dimensional or k is non-Archimedean. Then for any maximal ideal $m \subset \mathrm{C_{bd}}(X,k)$, the following are equivalent:*

(i) *The maximal ideal m is also a minimal prime ideal.*
(ii) *The maximal ideal m is generated by idempotents.*
(iii) *The topology of $\mathrm{C_{bd}}(X,k)/m$ is discrete, or the ultrafilter of the Boolean algebra of clopen subsets of X associated to m is a P-point of ζX.*

THEOREM 0.2. *Let X be a zero-dimensional Hausdorff topological space, and k be a Banach field with $\sup_{c \in k^\times} |c|\,|c^{-1}| < \infty$. Suppose that X is strongly zero-dimensional or k is non-Archimedean. Then for any maximal ideal $\overline{m} \subset \mathrm{C}_\partial(X,k)$, the following are equivalent:*

(i) *The maximal ideal $\overline{m}$ is also a minimal prime ideal.*
(ii) *The maximal ideal $\overline{m}$ is generated by idempotents.*
(iii) *The topology of $\mathrm{C}_\partial(X,k)/\overline{m}$ is discrete, or the ultrafilter of the Boolean algebra of clopen subsets of X associated to the pull-back of $\overline{m}$ in $\mathrm{C_{bd}}(X,k)$ is a P-point of the closure of $\{F \in \zeta X \mid \bigcap_{U \in F} U = \emptyset\}$ in ζX.*

THEOREM 0.3. *Let $\kappa = (\kappa_\lambda)_{\lambda \in \Lambda}$ be a family of Banach fields with $\sup_{\lambda \in \Lambda} \sup_{c \in \kappa_\lambda^\times} |c|\,|c^{-1}| < \infty$. Then for any maximal ideal $m \subset \prod_{\lambda \in \Lambda} \kappa_\lambda$, the following are equivalent:*

(i) *The maximal ideal m is also a minimal prime ideal.*
(ii) *The maximal ideal m is generated by idempotents.*
(iii) *The topology of $(\prod_{\lambda \in \Lambda} \kappa_\lambda)/m$ is discrete, or the ultrafilter of the Boolean algebra of subsets of Λ associated to m is a P-point of $\zeta\Lambda$.*

THEOREM 0.4. *Let* $\kappa = (\kappa_\lambda)_{\lambda \in \Lambda}$ *be a family of Banach fields with* $\sup_{\lambda \in \Lambda} \sup_{c \in \kappa_\lambda^\times} |c|\,|c^{-1}| < \infty$. *Then for any maximal ideal* $\overline{m} \subset \partial\kappa$, *the following are equivalent:*

(i) *The maximal ideal* $\overline{m}$ *is also a minimal prime ideal.*
(ii) *The maximal ideal* $\overline{m}$ *is generated by idempotents.*
(iii) *The topology of* $(\partial\kappa)/m$ *is discrete, or the ultrafilter of the Boolean algebra of subsets of* Λ *associated to the pull-back of* $\overline{m}$ *in* $\prod_{\lambda \in \Lambda} \kappa_\lambda$ *is a P-point of* $(\zeta\Lambda) \setminus \Lambda$.

Here, for a topological space X, ζX denotes the Banaschewski compactification of X recalled in §1.3. The correspondence between maximal ideals of $\mathrm{C_{bd}}(X,k)$ (resp. $\prod_{\lambda \in \Lambda} \kappa_\lambda$) and ultrafilters of the Boolean algebra of clopen subsets of X (resp. the Boolean algebra of subsets of Λ) is given in §2.2. It is a non-Archimedean analogue of the correspondence between maximal ideals of $\mathrm{C_{bd}}(X,\mathbb{R})$ (resp. $\mathrm{C_{bd}}(\Lambda,\mathbb{R})$) and points of βX (resp. $\beta\Lambda$), and is a generalisation of the descriptions of maximal ideals in [Mih14, §2.2–2.3] and in [Ber90, 1.2.3. Proposition]. We also verify a complete generalisation of [Mih14, §2.2–2.3] and [Ber90, 1.2.3. Proposition] (cf. Theorem 3.23).

Theorem 0.4 helps us understand how the ring structure of the bounded direct product of complete valuation fields depends on the normalisations of the valuations. For example, the existence of a maximal ideal of $\partial(\mathbb{Q}_{p,2^{-1}})_{p \in \mathbb{P}}$ which is also a minimal prime ideal is independent of ZFC, while every maximal ideal of $\partial(\mathbb{Q}_{p,p^{-1}})_{p \in \mathbb{P}}$ is a minimal prime ideal by Theorem 0.2 (cf. Example 3.17).

We explain the background of our main results. The existence of a P-point of a topological space is classically studied in general topology and set theory, and there are several proofs of the unprovability of the non-existence of a P-point in $\beta\mathbb{N} \setminus \mathbb{N}$, which are much easier than that of the unprovability of the existence. We recall classical results to which one of the proofs is deeply related (cf. [Dal00, Propositions 4.1.26 and 4.2.18(i)]).

THEOREM 0.5. *Let* X *be a topological space. Then for any maximal ideal* $m \subset \mathrm{C_{bd}}(X,\mathbb{R})$, *the following are equivalent:*

(i) *The maximal ideal* m *is also a minimal prime ideal.*
(ii) *For any* $f \in m$, *the zero-set of the continuous extension* $\beta X \to \mathbb{R}$ *of* f *is a neighbourhood of the point of* βX *corresponding to* m *in* βX.
(iii) *The point of* βX *corresponding to* m *is a P-point of* βX.

THEOREM 0.6. *Let* X *be a completely regular Hausdorff topological space. Then for any maximal ideal* $\overline{m} \subset \mathrm{C}_\partial(X,\mathbb{R})$, *the following are equivalent:*

(i) *The maximal ideal* $\overline{m}$ *is also a minimal prime ideal.*
(ii) *For any* $\overline{f} \in \overline{m}$, *the zero-set of the restriction to* $\beta X \setminus X$ *of the continuous extension* $\beta X \to \mathbb{R}$ *of a representative* $f \colon X \to \mathbb{R}$ *of* $\overline{f}$ *is a*

neighbourhood of the point of βX *corresponding to* $\{f \in \mathrm{C}_{\mathrm{bd}}(X, \mathbb{R}) \mid f + \mathrm{C}_0(X, \mathbb{R}) \in \overline{m}\}$ *in* $\beta X \setminus X$.

(iii) *The point of* βX *corresponding to* $\{f \in \mathrm{C}_{\mathrm{bd}}(X, \mathbb{R}) \mid f + \mathrm{C}_0(X, \mathbb{R}) \in \overline{m}\}$ *is a P-point of* $\beta X \setminus X$.

Theorems 0.1 and 0.2 are obviously non-Archimedean analogues of Theorems 0.5 and 0.6 respectively. Moreover, Theorem 0.1 is a generalisation of Theorem 0.5 restricted to the case where X is strongly zero-dimensional, because βX is naturally homeomorphic to ζX for such an X. Similarly, Theorem 0.2 is a generalisation of Theorem 0.6 restricted to the same case.

The strategy of the proofs of Theorems 0.1–0.4 is completely similar to that of the proofs of Theorems 0.5 and 0.6 except for the absence of the Gel'fand transform. The Gel'fand theory for bounded continuous functions is invalid here because we deal with Banach fields which are not necessarily isometrically isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and hence we need to compute maximal ideals in an explicit way. In particular, Theorem 0.1 is not reduced to the case where X is compact, while Theorem 0.5 is easily reduced to the case where X is compact through the Gel'fand transform. Similarly, Theorem 0.2 is not reduced to Theorem 0.1, while Theorem 0.6 is reduced to Theorem 0.5 because $\mathrm{C}_\partial(X, \mathbb{R})$ is naturally isometrically isomorphic to $\mathrm{C}_{\mathrm{bd}}(\beta X \setminus X, \mathbb{R})$. Therefore we prove another assertion essentially including Theorems 0.1–0.4 (cf. Lemma 3.7).

We emphasise that Theorems 0.2 and 0.4 do not seem to give a new example of the existence of a P-point, while Theorem 0.6 restricted to the case where X is a non-pseudo-compact locally compact Hausdorff topological space is used in a construction of a P-point in $\beta X \setminus X$ under CH (cf. [Wal74, 4.34. Theorem]). For example, Theorem 0.2 ensures the existence of a P-point in the growth of a non-compact paracompact locally compact Hausdorff topological space under CH, but such a topological space is non-pseudo-compact. At least, Theorems 0.2, 0.4, and 3.2 give new statements independent of ZFC (cf. Corollary 3.19).

1. Preliminaries. We will introduce the notion of a uniform ring on a topological space in §2. Roughly speaking, a uniform ring A is a Banach ring equipped with additional data, and yields another Banach ring ∂A whose ring structure is quite mysterious. In order to state several properties of a uniform ring, we clarify the convention and the terminology, and recall basic notions on Banach rings and topological spaces.

1.1. Convention and terminology. Throughout this paper, we work in ZFC set theory, and X always denotes a topological space. We denote by ω the discrete topological space of natural numbers.

Rings and fields are always assumed to be unital and commutative. For a ring R, we denote by $\mathrm{Spec}(R)$ the set of prime ideals of R equipped with the Zariski topology, and by $\mathrm{Max}(R) \subset \mathrm{Spec}(R)$ the subset of maximal ideals equipped with the relative topology. For a proper ideal $I \subset R$, we denote by $\mathrm{Jac}_R(I) \subset R$ the Jacobson radical $\bigcap\{m \in \mathrm{Max}(R) \mid I \subset m\}$ of I.

For a set Λ, a *family indexed by* Λ means a map from Λ. If a family indexed by Λ assigns a Banach ring to each $\lambda \in \Lambda$, we call it a *family of Banach rings indexed by* Λ. We use a similar convention for other notions. When we refer to sup (resp. inf) of a family of non-negative real numbers indexed by a (possibly empty) set, we always mean its supremum (resp. infimum) to be in $[0, \infty]$. In particular, we have $\sup_{x \in \emptyset} f(x) = 0$ and $\inf_{x \in \emptyset} f(x) = \infty$ for any map f.

1.2. Banach rings. A *Banach ring* is a pair $R = (\mathscr{R}, \|-\|)$ of a non-zero ring $\mathscr{R}$ and a map $\|-\| \colon \mathscr{R} \to [0, \infty)$, $f \mapsto \|f\|$, satisfying $\|f_0 f_1\| \leq \|f_0\| \, \|f_1\|$ for any $(f_0, f_1) \in \mathscr{R}^2$, $\|1\| = 1$, and the map $\mathscr{R}^2 \to [0, \infty)$, $(f_0, f_1) \mapsto \|f_0 - f_1\|$, is a complete metric on $\mathscr{R}$. We call $\mathscr{R}$ the *underlying ring* of R, and $\|-\|$ the *norm* of R.

For a Banach ring R, we often denote by the same symbol R the underlying ring of R equipped with the metric topology as long as there is no ambiguity, and by $\|-\|$ the norm of R unless otherwise specified. We note that we should distinguish Banach rings from their underlying rings when we consider the bounded direct product of Banach rings, which will be introduced later. On the other hand, such a traditional abuse of notation is convenient when we refer to an *element* (resp. *ideal*) of a Banach ring as an element (resp. ideal) of its underlying ring.

We say that R is *non-Archimedean* if $\|f_0 - f_1\| \leq \max\{\|f_0\|, \|f_1\|\}$ for any $(f_0, f_1) \in R^2$, and is *uniform* if $\|f^n\| = \|f\|^n$ for any $(f, n) \in R \times \omega$. The notion of a non-Archimedean Banach ring (resp. a uniform non-Archimedean Banach ring) is a generalisation of a ring (resp. a reduced ring) in the following sense:

EXAMPLE 1.1. Let $\mathscr{R}$ be a ring. Denote by $\|-\|$ the map $\mathscr{R} \to [0, \infty)$ which assigns 0 to 0 and 1 to each $f \in \mathscr{R} \setminus \{0\}$. Put $R = (\mathscr{R}, \|-\|)$. Then R forms a non-Archimedean Banach algebra. Moreover, R is uniform if and only if $\mathscr{R}$ is reduced, i.e. has no nilpotent element.

Let R_0 and R_1 be Banach rings, and $\varphi \colon R_0 \to R_1$ a ring homomorphism. We say that φ is *bounded* if there exists a $C \in (0, \infty)$ such that any $f \in R_0$ satisfies $\|\varphi(f)\| \leq C\|f\|$, is *contractive* if any $f \in R_0$ satisfies $\|\varphi(f)\| \leq \|f\|$, and is *isometric* if any $f \in R_0$ satisfies $\|\varphi(f)\| = \|f\|$.

Let R be a Banach ring, and $I \subset \mathscr{R}$ a closed ideal of the underlying ring $\mathscr{R}$ of R. We denote by $\|-\|_{R,I}$ the map $\mathscr{R}/I \to [0, \infty)$, $\overline{f} \mapsto \inf_{f \in \overline{f}} \|f\|$, called the *quotient norm*, and put $R/I := (\mathscr{R}/I, \|-\|_{R,I})$. Then R/I forms a Banach ring, and the canonical projection $R \twoheadrightarrow R/I$ is a contractive ring homomorphism.

Let R be a family of Banach rings indexed by a set Λ. For each $\lambda \in \Lambda$, we temporarily denote by $\mathscr{R}_\lambda$ the underlying ring of $R(\lambda)$, put $\mathscr{R} := \{f \in \prod_{\lambda \in \Lambda} \mathscr{R}_\lambda \mid \sup_{\lambda \in \Lambda} \|f(\lambda)\| < \infty\}$ and denote by $\|-\|_{\mathrm{bd}}$ the supremum norm $\mathscr{R}_{\mathrm{sup}} \to [0,\infty)$, $f \mapsto \sup_{\lambda \in \Lambda} \|f(\lambda)\|$. We put $\prod_{\lambda \in \Lambda} R(\lambda) := (\mathscr{R}_{\mathrm{bd}}, \|-\|_{\mathrm{sup}})$, and call it the *bounded direct product* of R. Then $\prod_{\lambda \in \Lambda} R(\lambda)$ forms a Banach ring. If $R(\lambda)$ is non-Archimedean (resp. uniform) for any λ, then so is $\prod_{\lambda \in \Lambda} R(\lambda)$.

Let R' be a Banach ring, and φ a family indexed by Λ which assigns to each $\lambda \in \Lambda$ a contractive ring homomorphism $R' \to R(\lambda)$. We denote by $\prod_{\lambda \in \Lambda} \varphi(\lambda)$ the unique contractive ring homomorphism $R' \to \prod_{\lambda \in \Lambda} R(\lambda)$ satisfying $\varphi(f)(\lambda) = \varphi(\lambda)(f)$ for any $(f, \lambda) \in R' \times \Lambda$.

Let I be a family indexed by Λ which assigns to each $\lambda \in \Lambda$ a closed ideal of $R(\lambda)$. We denote by $\widehat{\bigoplus}_{\lambda \in \Lambda} I(\lambda) \subset \prod_{\lambda \in \Lambda} R(\lambda)$ the closed ideal given as the closure of $\bigoplus_{\lambda \in \Lambda} I(\lambda)$, and call it the *completed direct sum* of I. We put $\partial R := (\prod_{\lambda \in \Lambda} R(\lambda))/(\widehat{\bigoplus}_{\lambda \in \Lambda} R(\lambda))$, and call it the *growth* of R.

A *Banach field* is a Banach ring such that the underlying ring k is a field. We always denote by $|-|$ (instead of $\|-\|$) the norm of a Banach field for convenience. A Banach field k is said to be *multiplicative* if $|c_0 c_1| = |c_0|\,|c_1|$ for any $(c_0, c_1) \in k^2$. It is well-known that a Banach field is multiplicative if and only if it is a complete valuation field or is isometrically isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by variants of Ostrowski's theorem (cf. [Neu99, (3.6) Proposition and (3.7) Proposition]) and Gel'fand–Mazur theorem (cf. [Dou72, 2.31 Theorem]). Banach fields which are not necessarily multiplicative are deeply studied in [Ked18], and possibly appear when we consider the quotient of a Banach ring by a maximal ideal. In order to explain the precise meaning of "the quotient by a maximal ideal", we recall the following well-known fact:

Proposition 1.2. *Let R be a Banach ring. Then the subset $R^\times \subset R$ of invertible elements is open, and every maximal ideal of R is closed.*

Proof. The first assertion follows from the fact that the infinite sum $\sum_{n \in \omega} (1-u)^n$ converges to the inverse of u for any $u \in R$ with $\|u - 1\| < 1$. The second assertion immediately follows from the first. ∎

Let R be a Banach ring, and $m \subset R$ a maximal ideal. Then $m \subset R$ is a closed ideal by Proposition 1.2, and hence the quotient Banach ring R/m makes sense. Since m is a maximal ideal, R/m forms a Banach field, which is not necessarily multiplicative.

Let k be a Banach field. A *Banach k-algebra* is a pair $(A, \|-\|)$ of a k-algebra A and a map $\|-\|\colon A \to [0,\infty)$ such that the underlying ring of A forms a Banach ring with respect to $\|-\|$, and the structure ring homomorphism $k \to A$ is bounded. For example, k itself is a trivial example of a Banach k-algebra. We note that k is not necessarily multiplicative in this context, and hence we do not assume as usual that any $(c, f) \in k \times A$ satisfies $\|cf\| = |c| \, \|f\|$. We introduce a non-trivial example of a Banach k-algebra using bounded continuous functions.

We denote by $\mathscr{C}(X, k)$ the k-algebra of continuous functions $X \to k$. An $f \in \mathscr{C}(X, k)$ is said to be *bounded* if $\|f\|_{X,k} := \sup_{x \in X} |f(x)| < \infty$. We denote by $\mathscr{C}_{\mathrm{bd}}(X, k) \subset \mathscr{C}(X, k)$ the k-subalgebra of bounded continuous functions, by $\|-\|_{X,k}$ the supremum norm $\mathscr{C}_{\mathrm{bd}}(X, k) \to [0, \infty)$, $f \mapsto \|f\|_{X,k}$, by $\mathrm{C}_{\mathrm{bd}}(X, k)$ the pair $(\mathscr{C}_{\mathrm{bd}}(X, k), \|-\|_{X,k})$, and by $\mathrm{C}_0(X, k) \subset \mathscr{C}(X, k)$ the subset of continuous functions vanishing at infinity. Then $\mathrm{C}_{\mathrm{bd}}(X, k)$ forms a Banach ring, and $\mathrm{C}_0(X, k)$ forms a closed ideal of $\mathrm{C}_{\mathrm{bd}}(X, k)$. We denote by $\mathrm{C}_{\partial}(X, k)$ the Banach k-algebra $\mathrm{C}_{\mathrm{bd}}(X, k)/\mathrm{C}_0(X, k)$. We note that the equality $\mathrm{C}_0(X, k) = \mathrm{C}_{\mathrm{bd}}(X, k)$ holds if and only if X is compact, in which case the equality $\mathscr{C}_{\mathrm{bd}}(X, k) = \mathscr{C}(X, k)$ also holds. Therefore when X is compact, we simply denote $\mathrm{C}_{\mathrm{bd}}(X, k)$ by $\mathrm{C}(X, k)$.

For a Banach ring R we denote by $\mathscr{M}(R)$ the *Berkovich spectrum* of R, i.e. the compact Hausdorff topological space of bounded multiplicative seminorms on R (cf. [Ber90, §1.1–1.2]). We studied several properties of the compact Hausdorff topological space $\mathscr{M}(\mathrm{C}_{\mathrm{bd}}(X, k))$ in the case where k is a complete valuation field in our previous paper [Mih14]. We will recall our related result in the next subsection.

1.3. The growth of a topological space. We recall two compactifications of a topological space, which are useful for describing maximal ideals of specific Banach rings. For this purpose, we prepare some terminology.

We say that a subset of X is *clopen* if it is closed and open, and a *zero-set* if it is the pre-image of $0 \in \mathbb{R}$ by a continuous function $X \to \mathbb{R}$. We denote by $\mathrm{CO}(X)$ the Boolean algebra of clopen subsets of X, and by $\mathrm{Z}(X)$ the set of zero-sets of X. We say that X is *zero-dimensional* if $\mathrm{CO}(X)$ generates the topology of X, and *strongly zero-dimensional* if X is a completely regular Hausdorff topological space and for any $(Z_0, Z_1) \in \mathrm{Z}(X)^2$ with $Z_0 \cap Z_1 = \emptyset$ there is a $U \in \mathrm{CO}(X)$ with $Z_0 \subset U$ and $Z_1 \subset X \setminus U$ (cf. [Eng89, 6.2.4. Theorem]). Every zero-dimensional topological space is completely regular, and every strongly zero-dimensional topological space is zero-dimensional by [Eng89, 6.2.6. Theorem]. For a Banach field k and a topological space X with $x \in X$, we denote by $m_{X,k,x} \subset \mathrm{C}_{\mathrm{bd}}(X, k)$ the ideal $\{f \in \mathrm{C}_{\mathrm{bd}}(X, k) \mid f(x) = 0\}$, which is maximal because it is the kernel of the surjective k-algebra homomorphism $\mathrm{C}_{\mathrm{bd}}(X, k) \twoheadrightarrow k$, $f \mapsto f(x)$.

First, we denote by βX the Stone–Čech compactification of X (cf. [Eng89, p. 169]), by $\iota_{\beta,X}$ the canonical map $X \to \beta X$, and by $\partial_\beta X \subset \beta X$ the growth of X, i.e. the complement of the image of $\iota_{\beta,X}$. It is well-known that $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,\mathbb{R}))$ equipped with the continuous map $X \to \beta X,\ x \mapsto m_{X,\mathbb{R},x}$, has the universality property of the Stone–Čech compactification of X. For a compact Hausdorff topological space K and a continuous map $\varphi\colon X \to K$, we denote by φ^{β} the unique continuous map $\beta X \to K$ with $\varphi^{\beta} \circ \iota_{\beta,X} = \varphi$.

Next, we denote by ζX the *Banaschewski compactification* of X, i.e. a totally disconnected compact Hausdorff topological space equipped with a continuous map $\iota_{\zeta,X}\colon X \to \zeta X$ such that for any totally disconnected compact Hausdorff topological space K and any continuous map $\varphi\colon X \to K$, there is a unique continuous map $\varphi^{\zeta}\colon \zeta X \to K$ with $\varphi^{\zeta} \circ \iota_{\zeta,X} = \varphi$. By universality, ζX is unique up to unique homeomorphism under X. Moreover, the Stone space of the Boolean algebra $\mathrm{CO}(X)$ equipped with the continuous map $X \to \zeta X,\ x \mapsto \{U \in \mathrm{CO}(X) \mid x \in U\}$, has the universality property of the Banaschewski compactification of X. Therefore we deal with ζX as a shorthand for the Stone space of $\mathrm{CO}(X)$ throughout this paper.

We denote by Cl_X the family of subsets of ζX indexed by $\mathrm{CO}(X)$ which assigns $\{F \in \zeta X \mid U \in F\}$ to each $U \in \mathrm{CO}(X)$. By the definition of the Stone topology, $\{\mathrm{Cl}_X(U) \mid U \in \mathrm{CO}(X)\}$ consists of clopen subsets of ζX and generates the topology of ζX. We denote by Int_X the family of subsets of X indexed by $\mathrm{CO}(\zeta X)$ which assigns $\{x \in X \mid \iota_{\zeta,X}(x) \in \overline{U}\}$ to each $\overline{U} \in \mathrm{CO}(\zeta X)$. By the continuity of $\iota_{\zeta,X}$, $\mathrm{Int}_X(\overline{U})$ is a clopen subset of ζX for any $\overline{U} \in \mathrm{CO}(\zeta X)$. Therefore Cl_X and Int_X can be regarded as maps $\mathrm{CO}(X) \to \mathrm{CO}(\zeta X)$ and $\mathrm{CO}(\zeta X) \to \mathrm{CO}(X)$. We recall some elementary facts about these maps.

Proposition 1.3. *The closure of* $\{\iota_{\zeta,X}(x) \mid x \in U\}$ *in* ζX *coincides with* $\mathrm{Cl}_X(U)$ *for any* $U \in \mathrm{CO}(X)$, *and the map* Int_X *is the inverse of* Cl_X.

We denote by $\partial_{\zeta} X \subset \zeta X$ the complement of the image of $\iota_{X,\zeta}$, and by $\overline{\partial}_{\zeta} X \subset \zeta X$ the closure of $\partial_{\zeta} X$. If X is a zero-dimensional Hausdorff topological space, then $\iota_{X,\zeta}$ is a homeomorphism onto its image (cf. [Mih14, Proposition 4.14]). Therefore if X is a totally disconnected locally compact Hausdorff topological space, then $\iota_{X,\zeta}$ is an open immersion, and hence $\partial_{\zeta} X$ coincides with $\overline{\partial}_{\zeta} X$.

The continuous map $\iota_{\zeta,X}^{\beta}\colon \beta X \to \zeta X$ associated to $\iota_{\zeta,X}$ by the universality of βX is surjective, because βX and ζX are compact Hausdorff topological spaces and the image of $\iota_{\zeta,X}^{\beta} \circ \iota_{\beta,X} = \iota_{\zeta,X}$ is dense in ζX. If X is a strongly zero-dimensional completely regular Hausdorff topological space, then βX is totally disconnected by [Eng89, 6.2.10. Theorem and 6.2.12. Theorem], and hence $\iota_{\zeta,X}^{\beta}$ is a homeomorphism by the universality of ζX. We will show a relation between P-points of $\overline{\partial}_{\zeta} X$ and maximal ideals of certain Banach rings which are also minimal prime ideals.

2. Uniform rings. We introduce the notion of a uniform ring A on X, and construct a Banach ring ∂A associated with A. Our main interest in this paper is to characterise when a given maximal ideal of A or ∂A is also a minimal prime ideal. We will state a criterion in terms of general topology in Theorems 3.1 and 3.2. For this purpose, we study an analogue of Shilov's idempotent theorem. A natural question is whether ∂A admits a maximal ideal which is also a minimal prime ideal or not. Applying Theorem 3.2 to several examples in §2.3, we will construct in §3.2 explicit examples of rings A for which the existence of such a maximal ideal in ∂A is independent of ZFC set theory.

2.1. Definition. A Banach field k is said to be *smooth* if there is a $C \in (0,\infty)$ such that any $c \in k^{\times}$ satisfies $|c|\,|c^{-1}| < C$. A family κ of Banach fields indexed by a set Λ is said to be *smooth* if there is a $C \in (0,\infty)$ such that any $c \in \kappa(\lambda)^{\times}$ with $\lambda \in \Lambda$ satisfies $|c|\,|c^{-1}| < C$. In particular, every smooth family of Banach fields is a family of smooth Banach fields.

Remark 2.1. For any Banach field k, the following are equivalent:

(i) The Banach field k is multiplicative.
(ii) The Banach field k is uniform and any $c \in k^{\times}$ satisfies $|c|\,|c^{-1}| = 1$.
(iii) The Banach field k is uniform and smooth.

Therefore the notion of a smooth Banach field is a generalisation of that of a multiplicative Banach field, and the notion of a smooth family of Banach fields is a generalisation of that of a family of multiplicative Banach fields.

A *uniform ring* on X is a tuple $A = (\mathscr{A}, \kappa, \varepsilon)$ of a Banach ring $\mathscr{A}$, a smooth family κ of Banach fields indexed by X, and an isometric ring homomorphism $\varepsilon\colon \mathscr{A} \hookrightarrow \prod_{x\in X} \kappa(x)$ satisfying the following two axioms:

(A1) The map $|f|_{\varepsilon}\colon X \to [0,\infty)$, $x \mapsto |\varepsilon(f)(x)|$, is continuous for any $f \in \mathscr{A}$.
(A2) Any $f \in \mathscr{A}$ satisfying $\inf_{x\in X} |\varepsilon(f)(x)| > 0$ is invertible.

We often denote by the same symbol A the underlying Banach ring $\mathscr{A}$ as long as there is no ambiguity, by κ_A the underlying family κ of Banach fields, and by ε_A the structure morphism ε of A. We identify each $f \in A$ with $\varepsilon_A(f)$.

The notion of a uniform ring on a topological space is a common generalisation of the three distinct notions of a uniform algebra on a compact Hausdorff topological space, a smooth Banach field, and a uniform Banach ring in the following sense:

Example 2.2. Suppose that X is a dense subset of a compact Hausdorff topological space K, and let $\mathscr{A}$ be a uniform algebra on K, i.e. a closed $\mathbb{C}$-subalgebra of the commutative C^*-algebra $\mathrm{C}(K, \mathbb{C})$ separating the points of K. Denote by ε the map $\mathscr{A} \to \prod_{x \in X} \mathbb{C}$, $f \mapsto (f(x))_{x \in X}$. Then $(\mathscr{A}, (\mathbb{C})_{x \in X}, \varepsilon)$ forms a uniform ring over X. In particular, every uniform algebra on K is canonically regarded as a uniform ring on K.

EXAMPLE 2.3. Let k be a smooth Banach field, and suppose that X is a singleton. Then $(k, (k)_{x \in X}, \mathrm{id}_k)$ forms a uniform ring on X. In particular, every smooth Banach field is canonically regarded as a uniform ring on $\{\emptyset\}$.

EXAMPLE 2.4. Let $\mathscr{A}$ be a uniform Banach ring, and suppose that X is a dense subset of $\mathscr{M}(\mathscr{A})$. Denote by κ the family of complete valuation fields indexed by X which assigns to each $x \in X$ the completed residue field (cf. [Ber90, 1.2.2. Remark (i)]) of $\mathscr{A}$ at x. Then the Gel'fand transform (cf. [Ber90, 1.2.2. Remark (i)]) induces a ring homomorphism $\varepsilon \colon \mathscr{A} \to \prod_{x \in X} \kappa(x)$. By the definition of the topology of $\mathscr{M}(\mathscr{A})$, $|f|_\varepsilon \colon X \to [0, \infty)$ is continuous for any $f \in \mathscr{A}$. Since X is dense in $\mathscr{M}(\mathscr{A})$, [Ber90, 1.3.2. Corollary (ii)] implies that ε is isometric. For any non-invertible element $f \in A$, there is an $x \in \mathscr{M}(\mathscr{A})$ with $|\varepsilon(f)(x)| = 0$ by [Ber90, 1.2.4. Corollary], and hence $\inf_{x \in X} |\varepsilon(f)(x)| = 0$ because X is dense in $\mathscr{M}(\mathscr{A})$. This implies that $(\mathscr{A}, \kappa, \varepsilon)$ forms a uniform ring on X. In particular, every uniform Banach ring is canonically regarded as a uniform ring on its Berkovich spectrum.

Let A be a uniform ring on X. We denote by $\mu_A \colon X \to \mathrm{Spec}(A)$ the map which assigns to each $x \in X$ the kernel of the continuous map $A \to \kappa(x)$, $f \mapsto f(x)$, which is a closed prime ideal because $\kappa(x)$ is Hausdorff.

An $f \in A$ is said to *vanish at infinity* if so does the map $X \to [0, \infty)$, $x \mapsto |f(x)|$. We denote by $\mathrm{Int}(A) \subset A$ the subset of elements vanishing at infinity. Since ε_A is isometric, $\mathrm{Int}(A)$ is a closed ideal of A. We denote by ∂A the quotient Banach ring $A/\mathrm{Int}(A)$, and call it the *growth* of A.

2.2. Non-Archimedean properties. In this subsection, we introduce and study several non-Archimedean properties of a uniform ring A. We say that A is *SIT* if it satisfies Shilov's idempotent theorem in the following sense: For any $U \in \mathrm{CO}(X)$, there is a unique idempotent $e \in A$ with $U = \{x \in X \mid e(x) = 1\}$ and $X \setminus U = \{x \in X \mid e(x) = 0\}$. We denote by $1_{A,U} \in A$ such an e, and call it the *idempotent corresponding to* U. When A is SIT, then we denote by Φ_A the family of subsets of $\mathrm{CO}(X)$ indexed by $\mathrm{Spec}(A)$ which assigns $\{U \in \mathrm{CO}(X) \mid 1_{A,U} \notin \wp\}$ to each $\wp \in \mathrm{Spec}(A)$.

PROPOSITION 2.5. *If A is SIT, then $\Phi_A(\wp)$ is an ultrafilter of the Boolean algebra* $\mathrm{CO}(X)$ *for any* $\wp \in \mathrm{Spec}(A)$.

Proof. Let $\wp \in \mathrm{Spec}(A)$. We have $1_{A,\emptyset} = 0 \in \wp$ by the uniqueness of $1_{A,\emptyset}$, and hence $\emptyset \notin \Phi_A(\wp)$. For any $(U_0, U_1) \in \Phi_A(\wp) \times \mathrm{CO}(X)$ with $U_0 \subset U_1$, we have $1_{A,U_1} 1_{A,U_0} = 1_{A,U_0} \notin \wp$ by the uniqueness of $1_{A,U_0}$, and hence $U_1 \in \Phi_A(\wp)$. For any $(U_0, U_1) \in \Phi_A(\wp)^2$, we have $1_{A,U_0 \cap U_1} = 1_{A,U_0} 1_{A,U_1} \notin \wp$

by the uniqueness of $1_{A,U_0 \cap U_1}$, and hence $U_0 \cap U_1 \in \Phi_A(\wp)$. Therefore $\Phi_A(\wp)$ forms a filter of $\mathrm{CO}(X)$. For any $U \in \mathrm{CO}(X)$, we have $1_{A,U} 1_{A,X \setminus U} = 0$ by the uniqueness of $1_{A,\emptyset}$, and hence $1_{A,U} + 1_{A,X \setminus U} = 1 \notin \wp$ by the uniqueness of $1_{A,X}$, which implies $U \in \Phi_A(\wp)$ or $X \setminus U \in \Phi_A(\wp)$. Thus $\Phi_A(\wp)$ is an ultrafilter of $\mathrm{CO}(X)$. ∎

By Proposition 2.5, Φ_A can be regarded as a map $\mathrm{Spec}(\mathscr{A}) \to \zeta X$.

PROPOSITION 2.6. *If A is SIT, then $\Phi_A \colon \mathrm{Spec}(A) \to \zeta X$ is continuous.*

Proof. We have $\{\wp \in \mathrm{Spec}(A) \mid \Phi_A(\wp) \in \mathrm{Cl}_X(U)\} = \{\wp \in \mathrm{Spec}(A) \mid 1_{A,U} \notin \wp\}$ for any $U \in \mathrm{CO}(X)$, and hence Φ_A is continuous by the definition of the Stone topology. ∎

The continuous map Φ_A is an extension of $\iota_{\zeta,X}$ in the following sense:

PROPOSITION 2.7. *If A is SIT, then $\Phi_A \circ \mu_A$ coincides with $\iota_{\zeta,X}$.*

Proof. For any $x \in X$, we have $(\Phi_A \circ \mu_A)(x) = \{U \in \mathrm{CO}(X) \mid 1_{A,U}(x) \neq 0\} = \iota_{\zeta,X}(x)$. ∎

We say that A is *classical* if μ_A is continuous and $\mu_A(x)$ is a maximal ideal for any $x \in X$, in which case we always regard μ_A as a map $X \to \mathrm{Max}(A)$. We say that A is *geometrically profinite* if A is classical, $\mathrm{Max}(A)$ is a totally disconnected compact Hausdorff topological space, and $\mu_A^{\zeta} \colon \zeta X \to \mathrm{Max}(A)$ is a homeomorphism.

THEOREM 2.8. *If A is geometrically profinite, then A is SIT.*

In order to prove Theorem 2.8, we first give a lemma.

LEMMA 2.9. *Let R be a non-trivial ring such that $\mathrm{Jac}_R(\{0\}) = \{0\}$. If $\mathrm{Max}(R)$ is compact, then for any $U \in \mathrm{CO}(\mathrm{Max}(R))$, there is a unique idempotent $e \in R$ with $U = \{m \in \mathrm{Max}(R) \mid e - 1 \in m\}$ and $\mathrm{Max}(R) \setminus U = \{m \in \mathrm{Max}(R) \mid e \in m\}$.*

Proof. The uniqueness follows from the fact that the difference of such idempotents is contained in $\mathrm{Jac}_R(\{0\})$. Since $\mathrm{Max}(R)$ is compact, so are U and $\mathrm{Max}(R) \setminus U$. Therefore there is an $(f_0, f_1) \in R^2$ such that $U = \{m \in \mathrm{Max}(R) \mid f_0 \notin m\}$ and $\mathrm{Max}(R) \setminus U = \{m \in \mathrm{Max}(R) \mid f_1 \notin m\}$. We have $f_0 f_1 \in \mathrm{Jac}_R(\{0\})$, and hence $f_0 f_1 = 0$. On the other hand, for any $m \in \mathrm{Max}(R)$, we have either $m \in U$ or $m \in \mathrm{Max}(R) \setminus U$, and hence either $(f_0, f_1) \in m \times (R \setminus m)$ or $(f_0, f_1) \in (R \setminus m) \times m$, which implies that $f_0 + f_1 \notin m$. Therefore $f_0 + f_1$ is invertible in R. Put $e := (f_0 + f_1)^{-1} f_0$. We have $e^2 = e$ by $f_0 f_1 = 0$. For any $m \in U$, we have $(f_0, f_1) \in (R \setminus m) \times m$, and hence $e - 1 = (f_0 + f_1)^{-1} f_1 \in m$. For any $m \in \mathrm{Max}(R) \setminus U$, we have $(f_0, f_1) \in m \times (R \setminus m)$, and hence $e = (f_0 + f_1)^{-1} f_0 \in m$. We obtain $U = \{m \in \mathrm{Max}(R) \mid e - 1 \in m\}$ and $\mathrm{Max}(R) \setminus U = \{m \in \mathrm{Max}(R) \mid e \in m\}$. ∎

Proof of Theorem 2.8. Let $U \in \mathrm{CO}(X)$. The uniqueness of the desired idempotent follows from the injectivity of ε_A. Since μ_A^ζ is a homeomorphism, there is a unique idempotent $e \in A$ with $\mathrm{Cl}_X(U) = \{F \in \zeta X \mid e - 1 \in \mu_A^\zeta(F)\}$ and $(\zeta X) \setminus \mathrm{Cl}_X(U) = \{F \in \zeta X \mid e \in \mu_A^\zeta(F)\}$ by Lemma 2.9. As $\mu_A^\zeta \circ \iota_{\zeta,X} = \mu_A$, we obtain $U = \{x \in X \mid \iota_{\zeta,X}(x) \in \mathrm{Cl}_X(U)\} = \{x \in X \mid e(x) = 1\}$ and $X \setminus U = \{x \in X \mid \iota_{\zeta,X}(x) \notin \mathrm{Cl}_X(U)\} = \{x \in X \mid e(x) = 0\}$. ∎

For $(f, F) \in A \times \zeta X$, we put $\|f\|_{A,F} \coloneqq \inf_{U \in F} \sup_{x \in U} |f(x)|$. We say that A is *non-Archimedean* if A is a classical SIT uniform ring on X and for any $(f, F, \epsilon) \in A \times \zeta X \times (0, \infty)$ with $\|f\|_{A,F} > \epsilon$ there exists a $U \in F$ with $\inf_{x \in U} |f(x)| > \epsilon$.

THEOREM 2.10. *If A is non-Archimedean, then A is geometrically profinite.*

In order to prove Theorem 2.10, we prepare several lemmata. We suppose that A is non-Archimedean in the remainder of this subsection. We denote by $\widehat{\mu}_A$ the family of subsets of A indexed by ζX which assigns $\{f \in A \mid \|f\|_{A,F} = 0\}$ to each $F \in \zeta X$.

LEMMA 2.11. *For any $F \in \zeta X$, $\widehat{\mu}_A(F)$ is a maximal ideal of A.*

Proof. We show that $\widehat{\mu}_A(F)$ is a proper ideal of A. We have $\sup_{x \in X} |0(x)| = 0$, and hence $0 \in \widehat{\mu}_A(F)$. As $\emptyset \notin F$, we have $\sup_{x \in U} |1(x)| = 1$ for any $U \in F$, and hence $1 \notin \widehat{\mu}_A(F)$. We show $f_0 + f_1 \in \widehat{\mu}_A(F)$ for any $(f_0, f_1) \in \widehat{\mu}_A(F)^2$. For any $\epsilon \in (0, \infty)$, there exists a $U \in F$ with $\sup_{x \in U} |f_0(x)| < 2^{-1}\epsilon$ and $\sup_{x \in U} |f_1(x)| < 2^{-1}\epsilon$ by the definition of $\widehat{\mu}_A(F)$, and hence we have $\sup_{x \in U} |(f_0 + f_1)(x)| < \epsilon$. This implies that $f_0 + f_1 \in \widehat{\mu}_A(F)$. We show $f_0 f_1 \in \widehat{\mu}_A(F)$ for any $(f_0, f_1) \in A \times \widehat{\mu}_A(F)$. For any $\epsilon \in (0, \infty)$, there exists a $U \in F$ with $\sup_{x \in U} |f_1(x)| < (\max\{1, \|f_0\|\})^{-1}\epsilon$ by the definition of $\widehat{\mu}_A(F)$, and hence we have $\sup_{x \in U} |(f_0 f_1)(x)| < \epsilon$. This implies that $f_0 f_1 \in \widehat{\mu}_A(F)$. Thus $\widehat{\mu}_A(F)$ is a proper ideal of A.

Put $k \coloneqq A/\widehat{\mu}_A(F) \neq \{0\}$. In order to verify that $\widehat{\mu}_A(F)$ is a maximal ideal, it suffices to show that any $\overline{f} \in k \setminus \{0\}$ is invertible. Take a representative $f \in A \setminus \widehat{\mu}_A(f)$ of $\overline{f} \neq 0$. There exists an $\epsilon \in (0, \infty)$ such that any $U \in F$ satisfies $\sup_{x \in U} |f(x)| \geq \epsilon$ because $f \in A \setminus \widehat{\mu}_A(f)$. This implies that $\inf_{U \in F} \sup_{x \in U} |f(x)| \geq \epsilon$, and hence there exists a $U \in F$ with $\inf_{x \in U} |f(x)| > 2^{-1}\epsilon$ because A is non-Archimedean. Put $g \coloneqq 1_{A,U} f + (1 - 1_{A,U}) \in A$. As $\inf_{x \in X} |g(x)| \geq \min\{2^{-1}\epsilon, 1\} > 0$, g is invertible. We obtain $fg^{-1} \in 1 + \widehat{\mu}_A(F)$. This implies that the image of g in k is the inverse of $\overline{f}$. Therefore $\widehat{\mu}_A(F)$ is a maximal ideal. ∎

By Lemma 2.11, $\widehat{\mu}_A$ can be regarded as a map $\zeta X \to \mathrm{Max}(A)$.

LEMMA 2.12. *Every closed prime ideal $\wp \subset A$ satisfies $\wp = \widehat{\mu}_A(\Phi_A(\wp))$ and is maximal.*

Proof. We show $f \in \wp$ for any $f \in \widehat{\mu}_A(\Phi_A(\wp))$. Let $\epsilon \in (0,\infty)$. There is a $U \in \Phi_A(\wp)$ with $\sup_{x \in U} |f(x)| < \epsilon$ since $f \in \widehat{\mu}_A(\Phi(\wp))$. We have $1_{A,U} \notin \wp$ since $U \in \Phi(\wp)$, and hence $1_{A,X \setminus U} \in \wp$ as $1_{A,U} 1_{A,X \setminus U} = 0$. We obtain $1_{A,X \setminus U} f \in \wp$ and $\|f - 1_{A,X \setminus U} f\| < \epsilon$. This implies that $f \in \wp$ by the closedness of $\wp$. We obtain $\widehat{\mu}_A(\Phi(\wp)) \subset \wp$. Since $\wp$ is a proper ideal, it follows from Lemma 2.11 that $\wp$ coincides with $\widehat{\mu}_A(\Phi_A(\wp))$ and is maximal. ∎

LEMMA 2.13. *The map* $\widehat{\mu}_A \colon \zeta X \to \mathrm{Max}(A)$ *is continuous and satisfies* $\widehat{\mu}_A \circ \iota_{\zeta,X} = \mu_A$.

Proof. For any $x \in X$, we have $\Phi_A(\mu_A(x)) = \{U \in \mathrm{CO}(X) \mid 1_{A,U} \notin \mu_A(x)\} = \iota_{\zeta,X}(x)$. This implies that $\widehat{\mu}_A \circ \iota_{\zeta,X} = \mu_A$ by Lemma 2.12. Let $F_0 \in \zeta X$ and $f \in A$ with $f \notin \widehat{\mu}_A(F_0)$. Since $f \notin \widehat{\mu}_A$, there is an $\epsilon \in (0,\infty)$ such that any $U \in F_0$ satisfies $\sup_{x \in U} |f(x)| \geq \epsilon$. By Lemma 2.17, there exists a $U \in F_0$ with $\inf_{x \in U} |f(x)| > 2^{-1}\epsilon$. Then we have $\inf_{x \in X} |(1_{A,U} f + 1_{A,X \setminus U})(x)| > \min\{2^{-1}\epsilon, 1\}$, and hence $1_{A,U} f + 1_{A,X \setminus U}$ is invertible. For any $F \in \mathrm{Cl}_X(U)$, we have $(1_{A,U} f + 1_{A,X \setminus U})^{-1} f \in 1 + \widehat{\mu}_A(F)$, and hence $f \notin \widehat{\mu}_A(F)$. We obtain $\mathrm{Cl}_X(U) \subset \widehat{\mu}_A^{-1}(\{m \in \mathrm{Max}(A) \mid f \in m\})$. Thus $\widehat{\mu}_A$ is continuous. ∎

LEMMA 2.14. *The map* $\widehat{\mu}_A$ *is a homeomorphism, and its inverse is* Φ_A.

Proof. By Proposition 1.2 and Lemma 2.12, we have $\widehat{\mu}_A \circ \Phi_A = \mathrm{id}_{\mathrm{Max}(A)}$. By Proposition 2.6 and Lemma 2.13, Φ_A and $\widehat{\mu}_A$ are continuous. By Proposition 2.7 and Lemma 2.13, we have $\Phi_A \circ \widehat{\mu}_A \circ \iota_{\zeta,X} = \Phi_A \circ \mu_A = \iota_{\zeta,X}$. Since the image of $\iota_{\zeta,X}$ is dense in ζX, this implies that $\Phi_A \circ \widehat{\mu}_A = \mathrm{id}_{\zeta X}$ by the continuity of $\Phi_A \circ \widehat{\mu}_A$. ∎

Proof of Theorem 2.10. By Lemma 2.14, $\widehat{\mu}_A$ is a homeomorphism. Since ζX is a totally disconnected compact Hausdorff topological space, so is $\mathrm{Max}(A)$. By Lemma 2.13, we obtain $\widehat{\mu}_A = \mu_A^{\zeta}$. Thus μ_A^{ζ} is a homeomorphism. ∎

By Theorems 2.8 and 2.10, every non-Archimedean uniform ring on X is SIT, and hence results and conventions for a SIT uniform ring are applicable to it.

2.3. Examples. We give two explicit examples of non-Archimedean uniform rings on X. One is the uniform ring of bounded continuous functions, and the other one is the bounded direct product of uniform rings.

First, we construct the uniform ring of bounded continuous functions. Let k be a smooth Banach field. We denote by $\kappa_{X,k}$ the constant map $X \to \{k\}$, and by $\varepsilon_{X,k}$ the inclusion $\mathrm{C}_{\mathrm{bd}}(X,k) \hookrightarrow \prod_{x \in X} k,\ f \mapsto (f(x))_{x \in X}$. We put $A_{X,k} \coloneqq (\mathrm{C}_{\mathrm{bd}}(X,k), \kappa_{X,k}, \varepsilon_{X,k})$. We say that (X,k) is *non-Archimedean* if either X is a strongly zero-dimensional completely regular Hausdorff topological space, or k is non-Archimedean.

Theorem 2.15. *The tuple $A_{X,k}$ forms a classical uniform ring on X. In addition, if (X,k) is non-Archimedean, then so is $A_{X,k}$.*

Before proving Theorem 2.15, we explain that it is a common partial generalisation of the classical result that $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,\mathbb{R}))$ has the universality property of the Stone–Čech compactification and the result in [Mih14, §2.2–2.3] that $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$ has the universality property of the Banaschewski compactification under the additional assumption that k is a complete valuation field in the following sense:

Remark 2.16. (i) If X is a strongly zero-dimensional completely regular Hausdorff topological space, then $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$ has the universality property of the Stone–Čech compactification by Theorems 2.10 and 2.15, because $\iota^{\beta}_{\zeta,X} \colon \beta X \to \zeta X$ is a homeomorphism.

(ii) If k is non-Archimedean, then $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$ has the universality property of the Banaschewski compactification by Theorems 2.10 and 2.15.

We note that the main interest of [Mih14] is $\mathscr{M}(\mathrm{C}_{\mathrm{bd}}(X,k))$, and $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$ is just used for the comparison between the Berkovich spectrum and the Banaschewski compactification (cf. Remark 3.24(i)).

Proof of the first assertion of Theorem 2.15. By the definition of the norm of $\mathrm{C}_{\mathrm{bd}}(X,k)$, $\varepsilon_{X,k}$ is isometric. For any $f \in \mathrm{C}_{\mathrm{bd}}(X,k)$, $|f|_{\varepsilon_{X,k}}$ coincides with the composite of continuous maps $f \colon X \to k$ and $|-| \colon k \to [0,\infty)$, and hence is continuous. Therefore $A_{X,k}$ satisfies (A1). Also, $A_{X,k}$ satisfies (A2) by the definition of $\mathrm{C}_{\mathrm{bd}}(X,k)$ and the smoothness of k. For any $x \in X$, $\mu_{A_{X,k}}(x)$ is a maximal ideal because it is the kernel of a surjective ring homomorphism $\mathrm{C}_{\mathrm{bd}}(X,k) \twoheadrightarrow k$. We show the continuity of $\mu_{A_{X,k}}$. Let $x_0 \in X$ and $f \in \mathrm{C}_{\mathrm{bd}}(X,k)$ with $f \notin \mu_{A_{X,k}}(x_0)$. Put $\epsilon := 2^{-1} f(x_0)$. By the definition of $\mu_{A_{X,k}}$, we have $\epsilon > 0$. Put $U := \{x \in X \mid |f(x)| > \epsilon\}$. By the continuity of $f \colon X \to k$, U is an open neighbourhood of x_0. For any $x \in U$, we have $|f(x)| \neq 0$ and hence $f \notin \mu_{A_{X,k}}(x)$. This implies the continuity of $\mu_{A_{X,k}}$. ■

In order to prove the second assertion of Theorem 2.15, we prepare several lemmata.

Lemma 2.17. *If (X,k) is non-Archimedean, then for any $(f,F,\epsilon) \in A_{X,k} \times \zeta X \times (0,\infty)$ with $\|f\|_{A_{X,k},F} > \epsilon$, there exists a $U \in F$ with $\inf_{x \in U} |f(x)| > \epsilon$.*

Proof. Put $\epsilon' := 2^{-1}(\epsilon + \|f\|_{A_{X,k},F})$. As $\|f\|_{A_{X,k},F} > \epsilon$, we have $\epsilon < \epsilon' < \|f\|_{A_{X,k},F}$. First, suppose that X is a strongly zero-dimensional completely regular Hausdorff topological space. Then βX is zero-dimensional, and $\iota^{\beta}_{\zeta,X} \colon \beta X \to \zeta X$ is a homeomorphism. Put $x_0 := (\iota^{\beta}_{\zeta,X})^{-1}(F) \in \beta X$.

By the universality of the Stone–Čech compactification, the bounded continuous map $|f|_{\varepsilon_{X,k}}\colon X \to [0,\infty)$ uniquely extends to a continuous map $g\colon \beta X \to [0,\infty)$. If the inequality $g(x_0) < \epsilon'$ held, there would be a $\widehat{U} \in \mathrm{CO}(\beta X)$ with $x_0 \in \widehat{U}$ and $\sup_{x\in\widehat{U}} g(x) < \epsilon'$ by the continuity of g, but this contradicts $\epsilon' < \|f\|_{A_{X,k},F} \leq \sup_{x\in\widehat{U}} g(x)$ by Proposition 1.3. We obtain $g(x_0) \geq \epsilon'$. By the continuity of g, there exists a $\widehat{U} \in \mathrm{CO}(\beta X)$ with $x_0 \in \widehat{U}$ and $\inf_{x\in\widehat{U}} g(x) > \epsilon$. Put $U \coloneqq \{x \in X \mid \iota_{\beta,X}(x) \in \widehat{U}\} \in \mathrm{CO}(X)$. We have $\inf_{x\in U} |f(x)| \geq \inf_{x\in\widehat{U}} g(x) > \epsilon$. Denote by $\overline{U} \subset \zeta X$ the image of $\widehat{U}$ by $\iota^{\beta}_{\zeta,X}$. We obtain $U = \mathrm{Int}_X(\overline{U}) \in F$ from $\iota^{\beta}_{\zeta,X} \circ \iota_{\beta,X} = \iota_{\zeta,X}$, $F = \iota^{\beta}_{\zeta,X}(x_0) \in \overline{U}$, and Proposition 1.3. Next, suppose that k is non-Archimedean. Put $U \coloneqq \{x \in X \mid |f(x)| > \epsilon'\}$. Then we have $\inf_{x\in U} |f(x)| \geq \epsilon'$. Since $|f|_{\varepsilon_{X,k}}$ is continuous, we have $U \in \mathrm{CO}(X)$ as $\{c \in k \mid |c| > \epsilon'\} \in \mathrm{CO}(k)$. As $\sup_{x\in X\setminus U} |f(x)| \leq \epsilon' < \|f\|_{A_{X,k},F}$, we have $X \setminus U \notin F$. Since F is an ultrafilter, we obtain $U \in F$. ∎

Lemma 2.18. *The uniform ring $A_{X,k}$ on X is SIT.*

Proof. The characteristic function of any clopen subset is bounded and continuous. ∎

By Lemma 2.18, results and conventions for a SIT uniform ring in §2.2 are applicable to $A_{X,k}$. We note that Proposition 2.5 (resp. Lemma 2.11) applied to $A_{X,k}$ is a generalisation of the argument of [Mih14, p. 10] (resp. a generalisation of [Mih14, Propositions 2.7 and 2.8]), in which k is supposed to be a complete valuation field.

Proof of the second assertion of Theorem 2.15. We apply Lemma 2.18 to $A_{X,k}$ without specific notice. Suppose that (X,k) is non-Archimedean. By Lemma 2.11, the image of $\widehat{\mu}_{A_{X,k}}$ is contained in $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$. Therefore we regard $\widehat{\mu}_{A_{X,k}}$ as a map $\zeta X \to \mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$. We denote by Φ the restriction of $\Phi_{A_{X,k}}$ to $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$. By Proposition 2.7, we have $\Phi_{A_{X,k}} \circ \mu_{A_{X,k}} = \iota_{\zeta,X}$. By Proposition 2.6 and Lemma 2.13, $\Phi_{A_{X,k}}$ and $\widehat{\mu}_{A_{X,k}}$ are continuous maps satisfying $\Phi_{A_{X,k}} \circ \widehat{\mu}_{A_{X,k}} \circ \iota_{\zeta,X} = \Phi_{A_{X,k}} \circ \mu_{A_{X,k}} = \iota_{\zeta,X}$. Since the image of $\iota_{\zeta,X}$ is dense in ζX, we obtain $\Phi_{A_{X,k}} \circ \widehat{\mu}_{A_{X,k}} = \mathrm{id}_{\zeta X}$. This implies that $\Phi \circ \widehat{\mu}_{A_{X,k}} = \mathrm{id}_{\zeta X}$. By Proposition 1.2 and Lemma 2.12, we also have $\widehat{\mu}_{A_{X,k}} \circ \Phi = \mathrm{id}_{\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))}$. Therefore Φ and $\widehat{\mu}_{A_{X,k}}$ are homeomorphisms which are the inverses of each other. Since ζX is a totally disconnected compact Hausdorff topological space, so is $\mathrm{Max}(\mathrm{C}_{\mathrm{bd}}(X,k))$. By Lemma 2.13 and the universality of Banaschewski compactification, we obtain $\widehat{\mu}_{A_{X,k}} = \mu^{\zeta}_{A_{X,k}}$. ∎

Next, we construct the bounded direct product of uniform rings. Let $\mathscr{U}$ be a family of clopen subsets of X indexed by a set Λ with $X = \bigsqcup_{\lambda\in\Lambda} \mathscr{U}(\lambda)$, and A a family indexed by Λ which assigns a uniform ring on $\mathscr{U}(\lambda)$ to each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we temporarily denote by $\mathscr{A}_\lambda$ the underlying Banach ring of $A(\lambda)$, by $\mathscr{A}$ the bounded direct product $\prod_{\lambda \in \Lambda} \mathscr{A}(\lambda)$, by κ the family of Banach fields indexed by X which assigns $\kappa_{A(\lambda)}(x)$ to each $x \in X$, where λ denotes the unique element of Λ with $x \in \mathscr{U}(\lambda)$, and by ε the composite of $\prod_{\lambda \in \Lambda} \varepsilon_{A(\lambda)} \colon \mathscr{A} \to \prod_{\lambda \in \Lambda} \prod_{x \in X_\lambda} \kappa_{A(\lambda)}(x)$ and the inverse of the isometric ring isomorphism $\prod_{x \in X} \kappa(x) \to \prod_{\lambda \in \Lambda} \prod_{x \in X_\lambda} \kappa_{A(\lambda)}(x)$, $f \mapsto ((f(x))_{x \in \mathscr{U}(\lambda)})_{\lambda \in \Lambda}$. We put $\prod_{\lambda \in \Lambda} A(\lambda) := (\mathscr{A}, \kappa, \varepsilon)$.

THEOREM 2.19. *If κ is smooth, then $\prod_{\lambda \in \Lambda} A(\lambda)$ forms a uniform ring on X. In addition, if $A(\lambda)$ is classical (resp. SIT, non-Archimedean) for any $\lambda \in \Lambda$, then so is $\prod_{\lambda \in \Lambda} A(\lambda)$.*

Proof. Abbreviate $\prod_{\lambda \in \Lambda} A(\lambda)$ to $\mathbb{A}$. For any $f \in \mathscr{A}$, we have $\|f\| = \sup_{x \in X} \|\varepsilon(f)(x)\| = \|\varepsilon(f)\|$. Therefore ε is isometric. We show the continuity of $|f|_\varepsilon$ at x for any $(f, x) \in \mathscr{A} \times X$. Let $x \in X$. Denote by λ the unique element of Λ with $x \in \mathscr{U}(\lambda)$. Since the restriction of $|f|_\varepsilon$ to $\mathscr{U}(\lambda)$ coincides with the continuous map $|f(\lambda)|_{\varepsilon_{A(\lambda)}}$ and $\mathscr{U}(\lambda)$ is an open neighbourhood of x in X, $|f|_\varepsilon$ is continuous at x. Therefore $\mathbb{A}$ satisfies (A1). We show that any $f \in \mathscr{A}$ satisfying $\inf_{x \in X} |\varepsilon(f)(x)| > 0$ is invertible. Put $\epsilon := \inf_{x \in X} |\varepsilon(f)(x)|$. By the smoothness of κ, there is a $C \in (0, \infty)$ such that any $c \in \kappa(\lambda)^\times$ with $\lambda \in \Lambda$ satisfies $|c|\,|c^{-1}| < C$. For any $\lambda \in \Lambda$, $f(\lambda)$ is invertible by $\inf_{x \in \mathscr{U}(\lambda)} |\varepsilon_{A(\lambda)}(f(\lambda))(x)| = \inf_{x \in \mathscr{U}(\lambda)} |\varepsilon(f)(x)| \geq \epsilon$. The family $(f(\lambda)^{-1})_{\lambda \in \Lambda}$ forms an element $g \in \prod_{\lambda \in \Lambda} \mathscr{A}_\lambda$ because $\sup_{\lambda \in \Lambda} \|f(\lambda)^{-1}\| = \sup_{\lambda \in \Lambda} \sup_{x \in \mathscr{U}(\lambda)} \|\varepsilon(f(\lambda))(x)^{-1}\| \leq C\epsilon^{-1}$, and we have $fg = 1$ by the definition of g. Therefore $\mathbb{A}$ satisfies (A2). Thus $\mathbb{A}$ is a uniform ring on X.

First, suppose that $A(\lambda)$ is classical for any $\lambda \in \Lambda$. We show that $m_{\mathbb{A},x}$ is a maximal ideal of $\mathscr{A}$ for any $x \in X$. Denote by λ the unique element of Λ with $x \in \mathscr{U}(\lambda)$. By the definition of κ and ε, $\varepsilon(x)$ coincides with the composite of the canonical projection $\mathscr{A} \twoheadrightarrow \mathscr{A}_\lambda$ and $\varepsilon_{A(\lambda)}(x)$. Therefore $m_{\mathbb{A},x}$ coincides with $\{f \in \mathscr{A} \mid f(\lambda) \in m_{A(\lambda),x}\}$, which is a maximal ideal because the canonical projection is surjective and $m_{A(\lambda),x}$ is a maximal ideal. We show the continuity of $\mu_{\mathbb{A}}$ at any $x \in X$. Let $f \in \mathscr{A}$ with $f \notin \mu_{\mathbb{A}}(x)$. Denote by λ the unique element of Λ with $x \in \mathscr{U}(\lambda)$, and by $U \subset X$ the subset $\{x' \in \mathscr{U}(\lambda) \mid f(\lambda) \notin \mu_{A(\lambda)}(x')\}$, which is open by the continuity of $\mu_{A(\lambda)}$. As $f \notin \mu_{\mathbb{A}}(x)$, we have $\varepsilon_{A(\lambda)}(f(\lambda))(x) = \varepsilon_{\mathbb{A}}(f)(x) \neq 0$, and hence $f(\lambda) \notin \mu_{A(\lambda)}(x)$. Therefore U is an open neighbourhood of x. On the other hand, for any $x' \in U$, we have $\varepsilon_{\mathbb{A}}(x') = \varepsilon_{A(\lambda)}(x') \neq 0$, and hence $f(\lambda) \notin \mu_{\mathbb{A}}(x')$. This implies that $\mu_{\mathbb{A}}$ is continuous at x. Thus $\mathbb{A}$ is classical.

Next, suppose that $A(\lambda)$ is SIT for any $\lambda \in \Lambda$. Let $U \in \mathrm{CO}(X)$. Denote by $e_U \in \mathscr{A}$ the idempotent $(1_{A(\lambda), U \cap \mathscr{U}(\lambda)})_{\lambda \in \Lambda} \in \mathscr{A}$. Then we have $U = \{x \in X \mid \varepsilon(e_U)(x) = 1\}$ and $X \setminus U = \{x \in X \mid \varepsilon(e_U)(x) = 0\}$. On the other hand, let $e \in \mathbb{A}$ be an idempotent satisfying $U = \{x \in X \mid \varepsilon(e)(x) = 1\}$ and $X \setminus U = \{x \in X \mid \varepsilon(e)(x) = 0\}$. Then we have $\varepsilon(e - e_U)(x) = 0$ for any $x \in X$, and hence $e = e_U$ by the injectivity of ε. Thus $\mathbb{A}$ is SIT.

Finally, suppose that $A(\lambda)$ is non-Archimedean for any $\lambda \in \Lambda$. We show that for any $(f, F, \epsilon) \in \mathscr{A} \times \zeta X \times (0, \infty)$ with $\|f\|_{A,F} > \epsilon$ there exists a $U \in F$ with $\inf_{x \in U} |f(x)| > \epsilon$. Put $\epsilon' := 2^{-1}(\|f\|_{A,F} + \epsilon)$. Then we have $\epsilon < \epsilon' < \|f\|_{A,F}$. Let $\lambda \in \Lambda$. Put $F_\lambda := \mathrm{CO}(\mathscr{U}(\lambda)) \cap F$. We have $\|f(\lambda)\|_{A,F_\lambda} \geq \|f\|_{A,F} > \epsilon'$, and hence there exists a $U_\lambda \in F_\lambda$ with $\inf_{x \in U_\lambda} |\varepsilon_{A(\lambda)}(f(\lambda))(x)| > \epsilon'$. Put $U := \bigcup_{\lambda \in \Lambda} U_\lambda$. Since $\mathscr{U}$ gives a disjoint covering of X by clopen subsets, we have $X \setminus U = \bigcup_{\lambda \in \Lambda} (\mathscr{U}(\lambda) \setminus U_\lambda)$. Since U and $X \setminus U$ are unions of open subsets of X, U is a clopen subset of X. We have $\inf_{x \in U} |f(x)| = \inf_{\lambda \in \Lambda} \inf_{x \in U(\lambda)} |\varepsilon_{A(\lambda)}(f(\lambda))| \geq \epsilon'$. ∎

Theorem 2.19 can be regarded as an extension of the result on the description of maximal ideals of the bounded direct product of complete valuation fields in [Ber90, 1.2.3. Proposition] in the following sense:

Corollary 2.20. *For any smooth family κ of Banach fields indexed by a discrete topological space Λ, there is a natural homeomorphism from $\beta\Lambda$ onto* $\mathrm{Max}(\prod_{\lambda \in \Lambda} \kappa(\lambda))$.

We note that the main interest of [Ber90, 1.2.3. Proposition] itself is the Berkovich spectrum of the bounded direct product, and the description of maximal ideals is just used for the comparison between the Berkovich spectrum and the Stone–Čech compactification (cf. Remark 3.24(ii)).

Proof. For any $\lambda \in \Lambda$, $A_{\{\lambda\},\kappa(\lambda)}$ is a non-Archimedean uniform ring on $\{\lambda\}$ by Theorem 2.15. Let $\mathscr{U}$ denote the family of clopen subsets of Λ indexed by Λ which assigns $\{\lambda\}$ to each $\lambda \in \Lambda$, and A the family indexed by Λ which assigns $A_{\{\lambda\},\kappa(\lambda)}$ to each $\lambda \in \Lambda$. Then the underlying Banach ring of $\prod_{\lambda \in \Lambda} A(\lambda)$ coincides with $\prod_{\lambda \in \Lambda} \kappa(\lambda)$. The assertion immediately follows from Theorem 2.19 applied to $X := \Lambda$, $\mathscr{U}$, and A, because $\iota^{\beta}_{\zeta,X} \colon \beta X \to \zeta X$ is a homeomorphism. ∎

3. P-points. Let A be a non-Archimedean uniform ring on X. We give precise statements of our main results, which imply Theorems 0.1–0.4, on the relation of P-points and maximal ideals which are also minimal prime ideals through a description of maximal ideals of specific Banach rings by ultrafilters.

3.1. Main results. Now we restate our main theorems. For a ring R and an ideal $I \subset R$, we denote by $\mathrm{Idem}_R(I) \subset R$ the ideal generated by $\{e \in I \mid e^2 = e\}$, and by $j_{R,I}$ the closed immersion $\mathrm{Max}(R/I) \hookrightarrow \mathrm{Max}(R)$.

THEOREM 3.1. *For any $m \in \mathrm{Max}(A)$, the following are equivalent:*

(i) *The maximal ideal m is also a minimal prime ideal.*
(ii) $\mathrm{Idem}_A(m) = m$.
(iii) *The topology of A/m is discrete, or $\Phi_A(m)$ is a P-point of ζX.*

THEOREM 3.2. *Suppose that X is a zero-dimensional Hausdorff topological space. For any $\overline{m} \in \mathrm{Max}(\partial A)$, the following are equivalent:*

(i) *The maximal ideal $\overline{m}$ is also a minimal prime ideal.*
(ii) $\mathrm{Idem}_{\partial A}(\overline{m}) = \overline{m}$.
(iii) *The topology of $(\partial A)/\overline{m}$ is discrete, or $\Phi_A(j_{A,\mathrm{Int}(A)}(\overline{m}))$ is a P-point of $\overline{\partial}_\zeta X$.*

In order to verify Theorems 3.1 and 3.2 simultaneously, we prepare several lemmata. We apply Lemma 2.14 without specific notice. Let $I \subset A$ be a closed ideal. We temporarily denote by $I^\perp \subset \zeta X$ the subset $\{F \in \zeta X \mid I \subset \widehat{\mu}_A(F)\}$, and by $\overline{A}$ the quotient Banach ring A/I. Then $I^\perp$ coincides with $\{F \in \zeta X \mid \widehat{\mu}_A(F) \in j_{A,I}(\mathrm{Max}(\overline{A}))\}$, and hence is closed by the continuity of $\widehat{\mu}_A$.

LEMMA 3.3. *If $\mathrm{Jac}_A(I) = I$, then $\overline{A}$ is reduced.*

Proof. This follows immediately from $I \subset \bigcap\{\wp \in \mathrm{Spec}(A) \mid I \subset \wp\} \subset \mathrm{Jac}_A(I) = I$. ∎

LEMMA 3.4. *Let $m \in \mathrm{Max}(A)$. Then $\|f + m\|_{A,m} = \|f\|_{A,\Phi_A(m)}$ for any $f \in A$.*

Proof. We show $\|f\|_{A,\Phi_A(m)} \leq \|f'\|$ for any $f' \in f + m$. We have $f - f' \in m = \widehat{\mu}_A(\Phi_A(m))$ and hence $\|f - f'\|_{A,\Phi_A(m)} = 0$. Let $\epsilon \in (0,\infty)$. There exists a $U \in \Phi_A(m)$ with $\sup_{x\in U} |(f' - f)(x)| < \epsilon$ because $\|f - f'\|_{A,\Phi_A(m)} = 0$. Therefore we have $\|f\|_{A,\Phi_A(m)} \leq \sup_{x\in U} |f(x)| \leq \epsilon + \|f'\|$. We obtain $\|f\|_{A,\Phi_A(m)} \leq \|f'\|$. This implies that $\|f\|_{A,\Phi_A(m)} \leq \|f + m\|_{A,m}$. We show $\|f + m\|_{A,m} \leq \sup_{x\in U} |f(x)|$ for any $U \in \Phi_A(m)$. As $1_{A,X\setminus U} \in m$, we have $1_{A,U} f = f - (1_{A,X\setminus U} f) \in f + m$ and hence $\|f + m\|_{A,m} \leq \|1_{A,U} f\| = \sup_{x\in U} |f(x)|$. This implies that $\|f + m\|_{A,m} \leq \|f\|_{A,\Phi_A(m)}$. ∎

LEMMA 3.5. *Let $C \subset \zeta X$ be a closed subset. For any $\overline{V} \in \mathrm{CO}(C)$, there exists a $U \in \mathrm{CO}(X)$ with $\mathrm{Cl}_X U \cap C = \overline{V}$.*

Proof. Since $\overline{V}$ is an open subset of C, there exists an open subset $\overline{U} \subset \zeta X$ with $\overline{U} \cap C = \overline{V}$. Since ζX is zero-dimensional, there is a $\mathscr{U} \subset \mathrm{CO}(\zeta X)$ with $\bigcup_{\overline{W}\in\mathscr{U}} \overline{W} = \overline{U}$. Since ζX is compact and $\overline{V} \subset C \subset \zeta X$ is closed, $\mathscr{U}$ admits a finite subset $\mathscr{U}_0$ with $\overline{V} \subset \bigcup_{\overline{W}\in\mathscr{U}_0} \overline{W} \subset \overline{U}$. We put $U := \mathrm{Int}_X(\bigcup_{\overline{W}\in\mathscr{U}_0} \overline{W})$. By Proposition 1.3, $\mathrm{Cl}_X U \cap C = (\bigcup_{\overline{W}\in\mathscr{U}_0} \overline{W}) \cap C = \overline{V}$ because $\overline{U} \cap C = \overline{V} \subset \bigcup_{\overline{W}\in\mathscr{U}_0} \overline{W} \subset \overline{U}$. ∎

Lemma 3.6. *If* $\mathrm{Jac}_A(I) = I$, *then for any idempotent* $\overline{e} \in \overline{A}$, *there is a* $U \in \mathrm{CO}(X)$ *with* $1_{A,U} + I = \overline{e}$.

Proof. Take a representative $f \in A$ of $\overline{e}$, and put $\overline{V} := \{F \in I^{\perp} \mid f \notin \widehat{\mu}_A(F)\}$. Since $(f^2 - f) + I = \overline{e}^2 - \overline{e} = 0$, we have $f^2 - f \in I$. Therefore for any $m \in \mathrm{Max}(A)$ with $I \subset m$, we have either $f \in m$ or $f - 1 \in m$ as $f^2 - f \in I \subset m$. This implies that $\overline{V}$ is a clopen subset of $I^{\perp}$ with $I^{\perp} \setminus \overline{V} = \{F \in I^{\perp} \mid f - 1 \notin \widehat{\mu}_A(F)\}$ because f and $f - 1$ are coprime with each other. By Lemma 3.5, there is a $U \in \mathrm{CO}(X)$ with $\mathrm{Cl}_X(U) \cap I^{\perp} = \overline{V}$. In order to verify $f - 1_{A,U} \in I$, we show $f - 1_{A,U} \in m$ for any $m \in \mathrm{Max}(A)$ with $I \subset m$. We have $I \subset m = \widehat{\mu}_A(\Phi_A(m))$, and hence $\Phi_A(m) \in I^{\perp}$.

First, suppose $\Phi_A(m) \notin \overline{V}$. By the definition of $\overline{V}$, we have $f \in \widehat{\mu}_A(\Phi_A(m))$. Since $\Phi_A(m) \notin \overline{V} = \mathrm{Cl}_X(U) \cap I^{\perp}$ and $\Phi_A(m) \in I^{\perp}$, we have $U \notin \Phi_A(m)$, and hence $1_{A,U}(\Phi_A(m)) = 0$. This implies that $1_{A,U} \in \widehat{\mu}_A(\Phi_A(m)) = m$. As $f \notin m$ and $1_{A,U} \notin m$, we obtain $f - 1_{A,U} \notin m$. Next, suppose $\Phi_A(m) \in \overline{V}$. By the definition of $\overline{V}$, we have $f \notin \widehat{\mu}_A(\Phi_A(m)) = m$. By the argument above, we have $f - 1 \in m = \widehat{\mu}_A(\Phi_A(m))$. As $\Phi_A(m) \in \overline{V} = \mathrm{Cl}_X(U) \cap I^{\perp}$, we have $U \in \Phi_A(m)$, and hence $1_{A,U}(\Phi_A(m)) = 1$. This implies that $1_{A,U} \notin \widehat{\mu}_A(\Phi_A(m)) = m$. As $(1 - 1_{A,U})1_{A,U} = 0 \in m$, we have $1 - 1_{A,U} \in m$. Since $f - 1 \in m$ and $1 - 1_{A,U} \in m$, we obtain $f - 1_{A,U} = (f-1) + (1 - 1_{A,U}) \in m$. Since $\mathrm{Jac}_A(I) = I$, we obtain $f - 1_{A,U} \in I$. This implies that $1_{A,U} + I = f + I = \overline{e}$. ∎

Lemma 3.7. *Suppose that* I *is a proper ideal with* $\mathrm{Jac}_A(I) = I$. *Then for any* $\overline{m} \in \mathrm{Max}(\overline{A})$, *the following are equivalent:*

(i) *The maximal ideal* $\overline{m}$ *is also a minimal prime ideal.*
(ii) $\mathrm{Idem}_{\overline{A}}(\overline{m}) = \overline{m}$.
(iii) *The topology of* $\overline{A}/\overline{m}$ *is discrete, or* $\Phi_A(j_{A,I}(\overline{m}))$ *is a P-point of* $I^{\perp}$.

Proof. First, suppose (i). We show (ii). As $\mathrm{Idem}_{\overline{A}}(\overline{m}) \subset \overline{m}$, it suffices to show $\overline{m} \subset \mathrm{Idem}_{\overline{A}}(\overline{m})$. Let $\overline{f} \in \overline{m}$. Since $\overline{m}$ is a minimal prime ideal, the multiplicative subset $S \subset \overline{A}$ generated by $\overline{f}$ and $\overline{A} \setminus \overline{m}$ satisfies $0 \in S$. Namely, there is an $(n, \overline{g}) \in \omega \times (\overline{A} \setminus \overline{m})$ with $\overline{f}^n \overline{g} = 0$. We have $(\overline{f}\overline{g})^{n+1} = 0$, and hence $\overline{f}\overline{g} = 0$ by Lemma 3.3. Take representatives $f \in A$ and $g \in A$ of $\overline{f}$ and $\overline{g}$ respectively. Since $\overline{f}\overline{g} = 0$, we have $fg \in I$. As $\overline{g} \notin \overline{m}$, we have $g \notin j_{A,I}(\overline{m}) = \widehat{\mu}_A(\Phi_A(j_{A,I}(\overline{m})))$, and hence $\|g\|_{A,\Phi_A(j_{A,I}(\overline{m}))} > 0$. Therefore there exists an $(\epsilon, U) \in (0, \infty) \times \Phi_A(j_{A,I}(\overline{m}))$ with $\inf_{x \in U} |g(x)| > \epsilon$ by Theorem 2.10. Since $\inf_{x \in X} |(1_{A,U} g + 1_{A,X \setminus U})(x)| \geq \min\{\epsilon, 1\} > 0$, $1_{A,U} g + 1_{A,X \setminus U}$ is invertible. We have $(1_{A,U} g + 1_{A,X \setminus U})^{-1} 1_{A,U} g = 1_{A,U}$, and hence $1_{A,U} f = (1_{A,U} g + 1_{A,X \setminus U})^{-1} 1_{A,U}(fg) \in I$. This implies that $f = 1_{A,U} f + 1_{A,X \setminus U} f \in 1_{A,X \setminus U} f + I$. As $1_{A,X \setminus U} + I \in \mathrm{Idem}_{\overline{A}}(\overline{m})$, we obtain $\overline{f} = 1_{A,X \setminus U} f + I \in \mathrm{Idem}_{\overline{A}}(\overline{m})$.

Secondly, suppose (ii). We show (iii). It suffices to show that $\Phi_A(j_{A,I}(\overline{m}))$ is a P-point of $I^{\perp}$ under the assumption that the topology of $\overline{A}/\overline{m}$ is not discrete. We have $I \subset j_{A,I}(\overline{m}) = \widehat{\mu}_A(\Phi_A(j_{A,I}(\overline{m})))$, and hence $\Phi_A(j_{A,I}(\overline{m})) \in I^{\perp}$. Let $\overline{V}$ be a family of neighbourhoods of $\Phi_A(j_{A,I}(\overline{m}))$ in $I^{\perp}$ indexed by ω. We show that $\bigcap_{n \in \omega} \overline{V}(n)$ is a neighbourhood of $\Phi_A(j_{A,I}(\overline{m}))$ in $I^{\perp}$. Since ζX is zero-dimensional, we may assume that $\overline{V}$ is given as a map $\omega \to \mathrm{CO}(\zeta X)$. By Lemma 3.5, there is a map $\mathscr{U}_0 \colon \omega \to \mathrm{CO}(X)$ such that for any $n \in \omega$ we have $I^{\perp} \cap \mathrm{Cl}_X(\mathscr{U}_0(n)) = \overline{V}(n)$. In particular, $\{\mathscr{U}_0(n) \mid n \in \omega\} \subset \Phi_A(j_{A,I}(\overline{m}))$. Since the topology of $\overline{A}/\overline{m}$ is not discrete, there is a pair (ϵ, ϖ) of maps $\epsilon \colon \omega \to (0,1)$ and $\varpi \colon \omega \to A$ such that ϵ converges to 0 and for any $n \in \omega$ we have $\epsilon(n+1) < \|(\varpi(n)+I)+\overline{m}\|_{\overline{A},\overline{m}} < \epsilon(n)$. The inequality in the second condition is equivalent to $\epsilon(n+1) < \|\varpi(n) + j_{A,I}(\overline{m})\|_{A,j_{A,I}(\overline{m})} < \epsilon(n)$ by the definition of the quotient norm. Therefore there is a map $\mathscr{U}_1 \colon \omega \to \Phi_A(j_{A,l}(\overline{m}))$ such that for any $n \in \omega$ we have $\epsilon(n+1) < \inf_{x \in \mathscr{U}_1(n)} |\varpi(n)(x)|$ and $\sup_{x \in \mathscr{U}_1(n)} |\varpi(n)(x)| < \epsilon(n)$ by Theorem 2.10. Denote by $\mathscr{U}_2$ the map $\omega \to \Phi_A(j_{A,I}(\overline{m}))$ which assigns $\bigcap_{n'=0}^{n}(\mathscr{U}_0(n') \cap \mathscr{U}_1(n')) \in \Phi_A(j_{A,I}(\overline{m}))$ to each $n \in \omega$. Then $\mathscr{U}_2$ is a decreasing sequence with $I^{\perp} \cap \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}_2(n)) \subset \bigcap_{n \in \omega} \overline{V}(n)$. Therefore it suffices to verify that $I^{\perp} \cap \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}_2(n))$ is a neighbourhood of $\Phi_A(j_{A,I}(\overline{m}))$ in $I^{\perp}$, i.e. there is a $U \in \Phi_A(j_{A,I}(\overline{m}))$ with $I^{\perp} \cap \mathrm{Cl}_X(U) \subset \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}_2(n))$.

Denote by ϖ' the map $\omega \to A$ which assigns $1_{\mathscr{U}_2(n) \setminus \mathscr{U}_2(n+1)} \varpi(n)$ to each $n \in \omega$. For any $N \in \omega$, we have

$$\sup_{a \geq b > N} \left\| \sum_{n=b}^{a} \varpi'(n) \right\| \leq \sup_{a \geq n \geq b > N} \sup_{x \in \mathscr{U}_2(n)} |\varpi(n)(x)| \leq \epsilon(N+1).$$

Therefore $(\sum_{n=0}^{N-1} \varpi'(n))_{N \in \omega} \in A^{\omega}$ forms a Cauchy sequence, which converges to a unique $f \in A$ by the completeness of A. We have $\|f\|_{A,\Phi_A(j_{A,I}(\overline{m}))} = 0$ by the construction, and hence $f \in \widehat{\mu}_A(\Phi_A(j_{A,I}(\overline{m}))) = j_{A,I}(\overline{m})$. This implies that $f + I \in \overline{m} = \mathrm{Idem}_{\overline{A}}(\overline{m})$. The idempotent $\overline{e} \in \overline{m}$ given as the supremum of idempotents appearing in a fixed presentation of $\overline{f}$ as an $\overline{A}$-linear combination of idempotents in $\overline{m}$ with respect to the lattice structure on idempotents satisfies $f + I = \overline{e}(f+I)$. By Lemma 3.6, there is a $U \in \mathrm{CO}(X)$ with $1_{A,U} + I = 1 - \overline{e}$. As $1_{A,U} + I = 1 - \overline{e} \in 1 + \overline{m}$, we have $1_{A,U} \notin j_{A,I}(\overline{m})$ and hence $U \in \Phi_A(j_{A,I}(\overline{m}))$. We show $I^{\perp} \cap \mathrm{Cl}_X(U) \subset \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}_2(n))$. Let $F \in I^{\perp} \cap \mathrm{Cl}_X(U)$. We have $1_{A,U} f \in (1-\overline{e})(f+I) = I \subset \widehat{\mu}_A(F)$ since $F \in I^{\perp}$ and $1_{A,U} \notin \widehat{\mu}_A(F)$ because $F \in \mathrm{Cl}_X(U)$. This implies that $f \in \widehat{\mu}_A(F)$ since $\widehat{\mu}_A(F) \in \mathrm{Max}(A) \subset \mathrm{Spec}(A)$. We show $\mathscr{U}_2(n) \in F$ for any $n \in \omega$ with $n > 0$. As $f \in \widehat{\mu}_A(F)$, there is a $U'_n \in F$ with $\sup_{x \in U'_n} |f(x)| < \epsilon(n+1)$. If there were an $x \in U'_n \setminus \mathscr{U}_2(n)$, then we would have $|f(x)| < \epsilon(n) < \inf_{n'=0}^{n-1} |\varpi'(n')(x)| \leq |f(x)|$. We obtain $U'_n \subset \mathscr{U}_2(n)$, and hence $\mathscr{U}_2(n) \in F$. This implies that $I^{\perp} \cap \mathrm{Cl}_X(U) \subset \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}_2(n))$.

Thirdly, suppose (iii). We show (i). It suffices to show that for any $\overline{f} \in \overline{m}$ the multiplicative subset $S \subset \overline{A}$ generated by $\overline{f}$ and $\overline{A} \setminus \overline{m}$ satisfies $0 \in S$. Take a representative $f \in A$ of $\overline{f}$. Since $f + I = \overline{f} \in \overline{m}$, we have $f \in j_{A,I}(\overline{m}) = \widehat{\mu_A}(\Phi_A(j_{A,I}(\overline{m})))$. Therefore there is a map $\mathscr{U} : \omega \to \Phi_A(j_{A,I}(\overline{m}))$ such that for any $n \in \omega$ we have $\sup_{x \in \mathscr{U}(n)} |f(x)| < 2^{-n}$. We show that there is a $U \in \Phi_A(j_{A,I}(\overline{m}))$ with $1_{A,U} f \in \mathrm{Jac}_A(I)$. First, suppose that the topology of $\overline{A}/\overline{m}$ is discrete. Then there is an $n \in \omega$ with $\{\overline{f} \in \overline{A}/\overline{m} \mid \|\overline{f}\|_{\overline{A},\overline{m}} < 2^{-n}\} = \{0\}$. Put $U := \mathscr{U}(n) \in \Phi_A(j_{A,I}(\overline{m}))$. For any $m \in \mathrm{Max}(A)$ with $I \subset m$, we have $\|1_{A,U} f + m\|_{A,m} = \|1_{A,U} f\|_{A,\Phi_A(m)} \leq \sup_{x \in U} |f(x)| < 2^{-n}$ by Lemma 3.4, and hence $1_{A,U} f \in m$. This implies that $1_{A,U} f \in \mathrm{Jac}_A(I)$. Next, suppose that $\Phi_A(j_{A,I}(\overline{m}))$ is a P-point of $I^{\perp}$. Then there is a $U \in \Phi_A(j_{A,I}(\overline{m}))$ with $I^{\perp} \cap \mathrm{Cl}_X(U) \subset \bigcap_{n \in \omega} \mathrm{Cl}_X(\mathscr{U}(n))$. For any $m \in \mathrm{Max}(A)$ with $I \subset m$, we have $\|1_{A,U} f + m\|_{A,m} = \|1_{A,U} f\|_{A,\Phi_A(m)} \leq \inf_{n \in \omega} \sup_{x \in \mathscr{U}(n)} |f(x)| = 0$ by Lemma 3.4, and hence $1_{A,U} f \in m$. This implies that $1_{A,U} f \in \mathrm{Jac}_A(I)$. Therefore in both cases, there is a $U \in \Phi_A(j_{A,I}(\overline{m}))$ with $1_{A,U} f \in \mathrm{Jac}_A(I) = I \subset j_{A,I}(\overline{m})$. Since $U \in \Phi_A(j_{A,I}(\overline{m}))$, we have $1_{A,U} \notin j_{A,I}(\overline{m})$, and hence $1_{A,U} + I \notin \overline{m}$. We obtain $0 = (1_{A,U} + I)\overline{f} \in S$. ∎

Applying Lemma 3.7 to $\{0\}$, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We have $\mathrm{Jac}_A(\{0\}) \subset \ker(\varepsilon_A) = \{0\}$, and $\{0\}^{\perp} = \{F \in \zeta X \mid \{0\} \subset \widehat{\mu}_A(F)\} = \zeta X$. Therefore the assertion follows from Lemma 3.7. ∎

In order to apply Lemma 3.7 to $\mathrm{Int}(A)$, we compute $\mathrm{Int}(A)^{\perp}$ and $\mathrm{Jac}_A(\mathrm{Int}(A))$.

LEMMA 3.8. *If X is a zero-dimensional Hausdorff topological space, then $\mathrm{Int}(A)^{\perp} = \overline{\partial}_{\zeta} X$.*

Proof. We show $\overline{\partial}_{\zeta} X \subset \mathrm{Int}(A)^{\perp}$. Let $F \in \partial_{\zeta} X$. We show $\mathrm{Int}(A) \subset \widehat{\mu}_A(F)$. Let $f \in \mathrm{Int}(A)$. We show that for any $\epsilon \in (0, \infty)$ there is a $U \in \mathrm{CO}(X)$ with $\sup_{x \in U} |f(x)| < \epsilon$. Since $f \in \mathrm{Int}(A)$, there is a compact subset $K \subset X$ with $\sup_{x \in X \setminus K} |f(x)| < \epsilon$. Denote by $\overline{K} \subset \zeta X$ the image of K by $\iota_{\zeta,X}$. Then $\overline{K}$ is compact by the continuity of $\iota_{\zeta,X}$, and hence is closed because ζX is Hausdorff. We have $\overline{K} \cap \partial_{\zeta} X = \emptyset$, and hence $F \notin \overline{K}$. Since $\overline{K}$ is closed, there is a $U \in \mathrm{CO}(X)$ with $U \in F$ and $\mathrm{Cl}_X(U) \cap \overline{K} = \emptyset$ by the definition of the Stone topology. We have $U \cap K = \emptyset$ by $\mathrm{Cl}_X(U) \cap \overline{K} = \emptyset$. We obtain $\sup_{x \in U} |f(x)| \leq \sup_{x \in X \setminus K} |f(x)| < \epsilon$. This implies that $\|f\|_{A,F} = 0$ and hence $f \in \widehat{\mu}_A(F)$. We obtain $\mathrm{Int}(A) \subset \widehat{\mu}_A(F)$, and hence $F \in \mathrm{Int}(A)^{\perp}$. This implies that $\partial_{\zeta} X \subset \mathrm{Int}(A)^{\perp}$. Since $\mathrm{Int}(A)^{\perp}$ is closed in ζX, we obtain $\overline{\partial}_{\zeta} X \subset \mathrm{Int}(A)^{\perp}$.

Now we show $\mathrm{Int}(A)^{\perp} \subset \overline{\partial}_{\zeta} X$. Let $F \in \mathrm{Int}(A)^{\perp}$. Assume $F \notin \overline{\partial}_{\zeta} X$. Since $\overline{\partial}_{\zeta} X$ is closed, there is a $U \in \mathrm{CO}(X)$ with $U \in F$ and $\mathrm{Cl}_X(U) \cap \overline{\partial}_{\zeta} X = \emptyset$

by the definition of the Stone topology. Since X is zero-dimensional and Hausdorff, $\iota_{\zeta,X}$ is a homeomorphism onto the image. As $\mathrm{Cl}_X(U) \cap \partial_\zeta X \subset \mathrm{Cl}_X(U) \cap \overline{\partial}_\zeta X = \emptyset$, the restriction of $\iota_{\zeta,X}$ to U is a homeomorphism onto the image, which coincides with $\mathrm{Cl}_X(U)$ by Proposition 1.3. Since $\mathrm{Cl}_X(U)$ is a clopen subset of the compact topological space ζX, it is compact. Therefore U is compact. This implies that $1_{A,U} \in \mathrm{Int}(A)$. As $F \in \mathrm{Int}(A)^\perp$, we have $\mathrm{Int}(A) \subset \widehat{\mu}_A(F)$, and hence $1_{A,U} \in \widehat{\mu}_A(F)$. We obtain $\|1_{A,U}\|_{A,F} = 0$, which contradicts $U \in F$. Thus we obtain $\mathrm{Int}(A)^\perp \subset \overline{\partial}_\zeta X$. ∎

By Lemma 3.8, the composite $\mu_A^\zeta \circ j_{A,\mathrm{Int}(A)}$ yields a homeomorphism $\mathrm{Max}(\partial A) \to \overline{\partial}_\zeta X$ in the case where X is a zero-dimensional Hausdorff topological space. In addition, if X is locally compact, then $\partial_\zeta X$ is closed in ζX. Therefore the notion of the growth of a non-Archimedean uniform ring can be seen as an analogue of the growth of a zero-dimensional Hausdorff locally compact topological space.

LEMMA 3.9. *If X is a zero-dimensional non-compact Hausdorff topological space, then* $\mathrm{Int}(A)$ *is a proper ideal with* $\mathrm{Jac}_A(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Proof. Since X is not compact, we have $1 \notin \mathrm{Int}(A)$ and $\overline{\partial}_\zeta X \neq \emptyset$. We show $f \in \mathrm{Int}(A)$ for any $f \in \mathrm{Jac}_A(\mathrm{Int}(A))$. By Lemma 3.8, we have $\{\widehat{\mu}_A(F) \mid F \in \overline{\partial}_\zeta X\} = \{m \in \mathrm{Max}(A) \mid \mathrm{Int}(A) \subset m\}$, and hence $f \in \bigcap_{F \in \overline{\partial}_\zeta X} \widehat{\mu}_A(F)$. We show that for any $\epsilon \in (0,\infty)$ there is a $V \in \mathrm{CO}(X)$ such that V is a compact subset with $\sup_{x \in X \setminus V} |f(x)| < \epsilon$. Put $\mathscr{U} := \bigcup_{F \in \overline{\partial}_\zeta X} \{U \in F \mid \sup_{x \in U} |f(x)| < \epsilon\}$. For each $F \in \overline{\partial}_\zeta X$, there is a $U \in F$ with $\sup_{x \in U} |f(x)| < \epsilon$ since $f \in \widehat{\mu}_A(F)$. Therefore $\{\mathrm{Cl}_X(U) \mid U \in \mathscr{U}\}$ forms a clopen covering of $\overline{\partial}_\zeta X$ in ζX. Since $\overline{\partial}_\zeta X$ is a closed subset of the compact topological space ζX, there is a finite subset $\mathscr{U}_0 \subset \mathscr{U}$ with $\overline{\partial}_\zeta X \subset \bigcup_{U \in \mathscr{U}_0} \mathrm{Cl}_X(U)$.

Put $V := X \setminus \bigcup_{U \in \mathscr{U}_0} U \in \mathrm{CO}(X)$. Then we have $V \cap U = \emptyset$ for any $U \in \mathscr{U}_0$, and hence $\mathrm{Cl}_X(V) \cap \overline{\partial}_\zeta X \subset \mathrm{Cl}_X(V) \cap \bigcup_{U \in \mathscr{U}_0} \mathrm{Cl}_X(U) \subset \bigcup_{U \in \mathscr{U}_0} \mathrm{Cl}_X(V \cap U) = \emptyset$. Since X is zero-dimensional and Hausdorff, $\iota_{\zeta,X}$ is a homeomorphism onto the image. As $\mathrm{Cl}_X(V) \cap \partial_\zeta X \subset \mathrm{Cl}_X(V) \cap \overline{\partial}_\zeta X = \emptyset$, the restriction of $\iota_{\zeta,X}$ to V is a homeomorphism onto the image, which coincides with $\mathrm{Cl}_X(V)$ by Proposition 1.3. Since $\mathrm{Cl}_X(V)$ is a clopen subset of the compact topological space ζX, it is compact. Therefore V is compact. We have $\sup_{x \in X \setminus V} |f(x)| = \sup_{U \in \mathscr{U}_0} \sup_{x \in U} |f(x)| < \epsilon$. This implies that $f \in \mathrm{Int}(A)$. ∎

Now we apply Lemma 3.7 to $I = \mathrm{Int}(A)$.

Proof of Theorem 3.2. If X is compact, then we have $\mathrm{Int}(A) = A$, and hence $\mathrm{Max}(\partial A) = \emptyset$. Therefore we may assume that X is not compact. Then the assertion immediately follows from Lemmata 3.7–3.9. ∎

3.2. Applications. Our many theorems yield several interesting corollaries on the existence of a maximal ideal which is also (resp. is not) a minimal prime ideal. We say that A is *Tate* if there exists an $f \in A^{\times}$ with $\|f\| < 1$, *discrete* if $\inf_{f \in A \setminus \{0\}} \|f\| > 0$, and *tame* if there exists an $r \in (0,1]$ such that for any $(f, \epsilon) \in A \times (0, r)$ the closure of the subset $\{x \in X \mid \epsilon < |\varepsilon(f)(x)| < r\}$ in X is compact.

Corollary 3.10. *If A is discrete, then every maximal ideal of A is also a minimal prime ideal. If X is a zero-dimensional Hausdorff topological space and A is tame, then every maximal ideal of ∂A is also a minimal prime ideal.*

Proof. If A is discrete, then the topology of A is discrete, and hence every maximal ideal of A is also a minimal prime ideal by Theorem 3.1. Suppose that X is a zero-dimensional locally compact topological space and A is tame. Take an $r \in (0,1]$ such that for any $(f, \epsilon) \in A \times (0, r)$ the closure of the subset $\{x \in X \mid \epsilon < |c| < r\}$ in X is compact. We show $\overline{f} = 0$ for any $\overline{f} \in \partial A$ with $\|\overline{f}\| < r$. Take a representative $f \in A$ of $\overline{f}$ with $\|f\| < r$. For any $\epsilon \in (0, r)$, the subset $\{x \in X \mid 2^{-1}\epsilon < |\varepsilon(f)(x)| < r\}$ is contained in a compact subset $K \subset X$, and we have $\sup_{x \in X \setminus K} |f(x)| \leq 2^{-1}\epsilon$ since $\|f\| < r$. This implies that $f \in \mathrm{Int}(A)$, and hence $\overline{f} = 0$. Therefore the topology of ∂A is discrete. Thus every maximal ideal of ∂A is also a minimal prime ideal by Theorem 3.2. ∎

Corollary 3.11. *For any field k equipped with the trivial valuation, every maximal ideal of $\mathrm{C}_{\mathrm{bd}}(X, k)$ is also a minimal prime ideal. In addition, if X is zero-dimensional and Hausdorff, then every maximal ideal of $\mathrm{C}_{\partial}(X, k)$ is also a minimal prime ideal.*

Proof. The assertions follow from Theorem 2.15 and Corollary 3.10 applied to the discrete tame non-Archimedean uniform ring $A_{X,k}$. ∎

Corollary 3.12. *Let κ be a family of multiplicative Banach fields indexed by a set Λ. If $\inf_{\lambda \in \Lambda} \inf_{c \in \kappa(\lambda)^{\times}} |c| > 0$, then every maximal ideal of $\prod_{\lambda \in \Lambda} \kappa(\lambda)$ is also a minimal prime ideal. If for any $\epsilon \in (0,1)$ the subset $\{\lambda \in \Lambda \mid \exists c \in \kappa(\lambda), \epsilon < |c| < 1\}$ is finite, then every maximal ideal of $\partial\kappa$ is also a minimal prime ideal.*

Proof. The assertions follow from Theorem 2.19 and Corollary 3.10 applied to the non-Archimedean uniform ring $\prod_{\lambda \in \Lambda} A_{\{\lambda\}, \kappa(\lambda)}$ on Λ equipped with the discrete topology. ∎

Corollary 3.13. *The Banach ring A admits a maximal ideal which is also (resp. is not) a minimal prime ideal if and only if ζX admits a P-point (resp. a point which is not a P-point). If X is a zero-dimensional Hausdorff topological space, then ∂A admits a maximal ideal which is also (resp. is* *not) a minimal prime ideal if and only if* $\overline{\partial}_\zeta X$ *admits a P-point* (*resp. a point which is not a P-point*).

In order to prove Corollary 3.13, we prepare a lemma.

LEMMA 3.14. *Let* $I \subset A$ *be a closed proper ideal. If* A *is Tate, then the topology of* A/I *is not discrete.*

Proof. Take an $f \in A^\times$ with $\|f\| < 1$. Since I is a proper ideal, we have $I \cap \{f^n \mid n \in \omega\} = \emptyset$. Therefore $0 < \|f^n + I\|_I \leq \|f\|^n$ for any $n \in \omega$. Hence for any $\epsilon \in (0, \infty)$, there exists an $n \in \omega$ such that $0 < \|f^n + I\|_I < \epsilon$. This implies that the topology of A/I is not discrete. ∎

Proof of Corollary 3.13. For any $m \in \mathrm{Max}(A)$, the topology of A/m is not discrete by Proposition 1.2 and Lemma 3.14. Therefore the first assertion immediately follows from Theorem 3.1. Suppose that X is a zero-dimensional Hausdorff topological space. For any $\overline{m} \in \mathrm{Max}(\partial A)$, the composite of the canonical projections $A \twoheadrightarrow \partial A$ and $\partial A \twoheadrightarrow (\partial A)/\overline{m}$ induces an isometric ring isomorphism $(\partial A)/\overline{m} \cong A/j_{A,\mathrm{Int}(A)}(\overline{m})$, and hence the topology of $(\partial A)/\overline{m}$ is not discrete by Proposition 1.2 applied to ∂A and by Lemma 3.14. Therefore the second assertion immediately follows from Theorem 3.2. ∎

COROLLARY 3.15. *Let* k *be a smooth Banach field admitting a* $c \in k^\times$ *with* $|c| < 1$. *When* (X, k) *is non-Archimedean, then* $\mathrm{C_{bd}}(X, k)$ *admits a maximal ideal which is also* (*resp. is not*) *a minimal prime ideal if and only if* ζX *admits a P-point* (*resp. a point which is not a P-point*). *If* X *is a zero-dimensional Hausdorff topological space and* (X, k) *is non-Archimedean, then* $\mathrm{C}_\partial(X, k)$ *admits a maximal ideal which is also* (*resp. is not*) *a minimal prime ideal if and only if* $\overline{\partial}_\zeta X$ *admits a P-point* (*resp. a point which is not a P-point*).

Proof. The assertions immediately follow from Theorem 2.15 and Corollary 3.13 applied to the Tate non-Archimedean uniform ring $A_{X,k}$. ∎

COROLLARY 3.16. *Let* κ *be a smooth family of Banach fields indexed by a discrete topological space* Λ. *Suppose that* $\prod_{\lambda \in \Lambda} \kappa(\lambda)$ *admits an invertible element* f *with* $\|f\| < 1$. *Then* $\partial\kappa$ *admits a maximal ideal which is also a minimal prime ideal if and only if* $\partial_\beta \Lambda$ *admits a P-point.*

Proof. If $\Lambda = \emptyset$, then $\partial\kappa$ is a zero-ring and $\partial_\beta \Lambda$ is empty. Therefore we may assume $\Lambda \neq \emptyset$. Then the assertion immediately follows from Theorem 2.15 and Corollary 3.13 applied to the Tate non-Archimedean uniform ring $\prod_{\lambda \in \Lambda} A_{\{\lambda\},\kappa(\lambda)}$ on Λ, because $\iota^\beta_{\zeta,\Lambda}$ induces a homeomorphism from $\partial_\beta \Lambda$ onto $\partial_\zeta \Lambda = \overline{\partial}_\zeta \Lambda$. ∎

EXAMPLE 3.17. We denote by $\mathbb{P} \subset \omega$ the subset of prime numbers equipped with the discrete topology. For a $(p, \epsilon) \in \mathbb{P} \times (0, 1)$, we denote by $|-|_{p,\epsilon} \colon \mathbb{Q}_p \to [0, \infty)$, $c \mapsto |c|_{p,\epsilon}$, the valuation uniquely determined by the normalisation condition $|p|_{p,\epsilon} = \epsilon$, and by $\mathbb{Q}_{p,\epsilon}$ the complete valuation field $(\mathbb{Q}_p, |-|_{p,\epsilon})$. Then both $(\mathbb{Q}_{p,p^{-1}})_{p \in \mathbb{P}}$ and $(\mathbb{Q}_{p,2^{-1}})_{p \in \mathbb{P}}$ are smooth families of Banach fields, but have the following distinct properties:

(i) For any $\epsilon \in (0,1)$, the subset $\{p \in \mathbb{P} \mid \exists c \in \mathbb{Q}_{p,p^{-1}}, \epsilon < |c| < 1\}$ is finite.
(ii) The sequence $(p)_{p \in \mathbb{P}} \in \prod_{p \in \mathbb{P}} \mathbb{Q}_p$ forms an invertible element $\boldsymbol{P}$ of $\prod_{\lambda \in \Lambda} A_{\{p\},\mathbb{Q}_{p,2^{-1}}}$ with $\|\boldsymbol{P}\| = 2^{-1} < 1$.

In particular, every maximal ideal of $\partial(\mathbb{Q}_{p,p^{-1}})_{p \in \mathbb{P}}$ is also a minimal prime ideal by Corollary 3.12, while the existence of a maximal ideal of $\partial(\mathbb{Q}_{p,2^{-1}})_{p \in \mathbb{P}}$ which is also a minimal prime ideal is independent of ZFC by Corollary 3.15 and the P-point independence theorem (cf. [Wim82, 6.5. Theorem] and [She98, VI.4.8]). We note that $\partial(\mathbb{Q}_{p,p^{-1}})_{p \in \mathbb{P}}$ is a significant object called the *ring of asymptotic numbers*, which appears in the study of truncated multiple zeta values in [Ros15].

We denote by CH the continuum hypothesis.

COROLLARY 3.18. *Assume* CH. *If X is a totally disconnected non-compact paracompact locally compact Hausdorff topological space, then A admits a maximal ideal which is also a minimal prime ideal.*

Proof. Every totally disconnected non-compact paracompact locally compact Hausdorff topological space is the disjoint union of infinitely many compact clopen subsets by [Eng89, 5.1.11. Theorem and 6.2.11. Theorem], and hence is non-pseudo-compact, i.e. admits an unbounded continuous map to $\mathbb{R}$. Therefore X is non-pseudo-compact. By [Dal00, Theorem 4.2.23], CH implies that $\partial_\beta \omega$ admits a P-point. By [Wal74, 4.32. Lemma], the existence of a P-point in $\partial_\beta \omega$ implies that $\partial_\beta Y$ admits a P-point for any non-pseudo-compact locally compact Hausdorff topological space Y. Therefore $\partial_\beta X$ admits a P-point. By [Eng89, 6.2.11. Theorem], X is strongly zero-dimensional, and hence $\iota^{\beta}_{\zeta,X}$ induces a homeomorphism $\partial_\beta X \to \partial_\zeta X = \overline{\partial}_\zeta X$. Therefore $\overline{\partial}_\zeta X$ admits a P-point. Thus ∂A admits a maximal ideal which is also a minimal prime ideal by Theorem 3.2. ■

We denote by MPMI_0 the sentence "For any non-compact totally disconnected paracompact locally compact Hausdorff topological space X, and any non-Archimedean uniform ring A on X, ∂A admits a maximal ideal which is also a minimal prime ideal", by MPMI_1 the sentence "For any non-compact totally disconnected paracompact locally compact Hausdorff topological space X, and any smooth Banach field k, $\mathrm{C}_\partial(X,k)$ admits a maximal ideal which is also a minimal prime ideal", and by MPMI_2 the sentence "For any infinite smooth family κ of Banach fields, $\partial\kappa$ admits a maximal ideal which is also a minimal prime ideal".

Corollary 3.19. *The sentences* MPMI_0*,* MPMI_1*, and* MPMI_2 *are independent of* ZFC.

Proof. By Corollary 3.18, MPMI_0 is provable under CH, and hence $\neg\mathrm{MPMI}_0$, $\neg\mathrm{MPMI}_1$, $\neg\mathrm{MPMI}_2$ are unprovable under ZFC. On the other hand, by Corollary 3.15(ii) applied to ω and an arbitrary complete discrete valuation field, e.g. $\mathbb{Q}_{2,2^{-1}}$, and by the P-point independence theorem (cf. [Wim82, 6.5. Theorem] and [She98, VI.4.8]), MPMI_0, MPMI_1, and MPMI_2 are unprovable under ZFC. ∎

We give a criterion for the existence of a maximal ideal which is not a minimal prime ideal.

Theorem 3.20. *If X is a totally disconnected locally compact Hausdorff topological space and A is Tate, then the following are equivalent:*

- (i) *The underlying set of X is finite.*
- (ii) *The underlying set of* $\mathrm{Spec}(A)$ *is finite.*
- (iii) *Every prime ideal of A is a maximal ideal which is also a minimal prime ideal.*
- (iv) *Every minimal prime ideal of A is closed.*

We note that the assumption that A is Tate is necessary by Corollary 3.10.

Proof of Theorem 3.20. First, suppose (i). We show (ii). Since X is a finite set and A is classical, ε_A is bijective by the Chinese remainder theorem. Since the underlying ring of $\prod_{x \in X} \kappa_A(x)$ is the finite direct product of fields again by the finiteness of X, $\mathrm{Spec}(\prod_{x \in X} \kappa_A(x))$ is a finite set.

Secondly, suppose (ii). We show (iii). Since $\mathrm{Spec}(A)$ is a finite set, so is the image $P \subset \mathrm{Spec}(A)$ of μ_A. Take a section $i \colon P \hookrightarrow X$ of μ_A. Since P is a finite set and A is classical, the ring homomorphism $A \to \prod_{\wp \in P} \kappa_A(i(\wp))$, $f \mapsto (f(i(\wp)))_{\wp \in P}$, is bijective by the Chinese remainder theorem. Therefore A is an Artinian ring. In particular, every prime ideal of A is a maximal ideal which is also a minimal prime ideal.

Thirdly, suppose (iii). We show (iv). Every minimal prime ideal of A is a maximal ideal by the assumption, and hence is a closed ideal by Proposition 1.2.

Finally, suppose (iv). We show (i). Since every minimal prime ideal of A is a closed ideal, it is also a maximal ideal by Lemma 2.12. This implies that every maximal ideal of A is also a minimal prime ideal, because it contains a minimal prime ideal. Therefore every point of ζX is a P-point by Corollary 3.13. By [Dal00, Corollary 4.2.19(i)], the underlying set of ζX is a finite set. By [Eng89, 6.2.9. Theorem], X is zero-dimensional and Hausdorff, and hence $\iota_{\zeta,X}$ is injective. This implies that the underlying set of X is a finite set. ∎

We have an immediate consequence of Theorems 2.15 and 3.20.

Corollary 3.21. *Let k be a smooth Banach field admitting a $c \in k^{\times}$ with $|c| < 1$. Suppose that X is a totally disconnected locally compact Hausdorff topological space and (X, k) is non-Archimedean. Then the following are equivalent:*

(i) *The underlying set of X is finite.*
(ii) *The underlying set of* $\mathrm{Spec}(\mathrm{C}_{\mathrm{bd}}(X, k))$ *is finite.*
(iii) *Every prime ideal of* $\mathrm{C}_{\mathrm{bd}}(X, k)$ *is a maximal ideal which is also a minimal prime ideal.*
(iv) *Every minimal prime ideal of* $\mathrm{C}_{\mathrm{bd}}(X, k)$ *is closed.*

3.3. Relation to the Berkovich spectrum. Suppose that $\kappa_A(x)$ is multiplicative for any $x \in X$. We give a comparison between the Berkovich spectrum and the Banaschewski compactification.

Proposition 3.22. *For any $F \in \zeta X$, $\|-\|_{A,F}$ is a bounded multiplicative seminorm on A.*

Proof. The assertion follows essentially by the same arguments as in the proofs of [Mih14, Theorem 2.1 and Proposition 2.10]. ∎

We denote by $\check{\mu}_A$ the family indexed by ζX which assigns $\|-\|_{A,F}$ to each $F \in \zeta X$. By Proposition 3.22, $\check{\mu}_A$ can be regarded as a map $\zeta X \to \mathscr{M}(A)$. We prove an analogue of Theorem 2.10.

Theorem 3.23. *The map $\check{\mu}_A \colon \zeta X \to \mathscr{M}(A)$ is a homeomorphism.*

Proof. We show the continuity of $\check{\mu}_A$ at any $F \in \zeta X$. Let $V \subset \mathscr{M}(A)$ be a neighbourhood of $\check{\mu}_A(F)$. By the definition of the topology of $\mathscr{M}(A)$, there exists an $(f, \epsilon) \in \mathrm{C}_{\mathrm{bd}}(X, k) \times (0, \infty)$ with $\{x \in \mathscr{M}(A) \mid |x(f) - \|f\|_{A,F}| < \epsilon\} \subset V$. Take a $U_0 \in F$ with $\sup_{x \in U_0} |f(x)| < \|f\|_{A,F} + \epsilon$. Since A is non-Archimedean, there is a $U_1 \in F$ with $\inf_{x \in U_1} |f(x)| > \|f\|_{A,F} - \epsilon$. Since $U_0 \in F$ and $U_1 \in F$, we have $F \in \mathrm{Cl}_X(U_0 \cap U_1)$. For any $G \in \mathrm{Cl}_X(U_0 \cap U_1)$, we have $\check{\mu}_A(G) \in V$ as $\|f\|_{A,F} - \epsilon < \|f\|_{A,G} < \|f\|_{A,F} + \epsilon$. Therefore $\check{\mu}_A$ is continuous at F.

We show the injectivity of $\check{\mu}_A$. Let $(F_0, F_1) \in (\zeta X)^2$ with $\check{\mu}_A(F_0) = \check{\mu}_A(F_1)$. We have $\widehat{\mu}_A(F_0) = \widehat{\mu}_A(F_1)$ by the definition of $\widehat{\mu}_A$, and hence $F_0 = \Phi_A(\widehat{\mu}_A(F_0)) = \Phi_A(\widehat{\mu}_A(F_1)) = F_1$ by Lemma 2.14. Therefore $\check{\mu}_A$ is injective.

We show the surjectivity of $\check{\mu}_A$. Let $x \in \mathscr{M}(A)$. Then $m := \{f \in A \mid x(f) = 0\}$ is a closed prime ideal of A by [Ber90, 1.2.2. Remark (i)], and hence is a maximal ideal by Lemma 2.12. Moreover, the composite of the canonical projection $A \twoheadrightarrow A/m$ and the norm $\|-\|_{A,m}$ of A/m coincides with $\check{\mu}_A(\Phi_A(m))$ by Lemma 3.4, and hence A/m is multiplicative. Since $x \colon A \to [0, \infty)$ decomposes into the composite of the canonical projection $A \twoheadrightarrow A/m$ and a bounded multiplicative seminorm on A/m, it coincides with

$\check{\mu}_A(\Phi_A(m))$ by [Ber90, 1.3.4. Corollary (i)]. Therefore $\check{\mu}_A$ is surjective. Since $\check{\mu}_A$ is a continuous bijective map between compact Hausdorff topological spaces, it is a homeomorphism. ■

We note that Theorem 3.23 is a common generalisation of [Mih14, Theorem 2.1], which states that there is a natural homeomorphism $\zeta X \cong \mathscr{M}(\mathrm{C}_{\mathrm{bd}}(X, k))$ for any complete valuation field k, and [Ber90, 1.2.3. Proposition] in the following sense:

Remark 3.24. (i) Let k be a multiplicative Banach field. If (X, k) is non-Archimedean, then Theorems 2.15 and 3.23 applied to $A_{X,k}$ ensure that there is a natural homeomorphism $\zeta X \cong \mathscr{M}(\mathrm{C}_{\mathrm{bd}}(X, k))$.

(ii) Let κ be a family of multiplicative Banach fields indexed by a discrete set Λ. Then Theorems 2.19 and 3.23 ensure that there is a natural homeomorphism $\beta\Lambda \cong \mathscr{M}(\prod_{\lambda \in \Lambda} \kappa(\lambda))$, because $\iota^{\beta}_{\zeta,\Lambda}$ is a homeomorphism. This result is precisely [Ber90, 1.2.3. Proposition]. We emphasise that the proof of Theorem 3.23 does not give an alternative proof of [Ber90, 1.2.3. Proposition], because we have used results heavily depending on the latter.

We give a complete description of bounded multiplicative seminorms on A. We denote by $\mathscr{P}(X)$ the Boolean algebra of subsets of X. For an ultrafilter $\mathscr{F}$ of $\mathscr{P}(X)$ and a bounded continuous function $g \colon X \to \mathbb{R}$, we denote by $\lim_{\mathscr{F}} g \in \mathbb{R}$ the limit of g along $\mathscr{F}$, i.e. the unique $c \in \mathbb{R}$ such that for any neighbourhood V of c, $\{x \in X \mid g(x) \in V\}$ lies in $\mathscr{F}$, which exists by the universality of the Stone–Čech compactification of the underlying set of X.

Corollary 3.25.

(i) *For any ultrafilter* $\mathscr{F}$ *of* $\mathscr{P}(X)$, $\mathscr{F} \cap \mathrm{CO}(X)$ *is an ultrafilter of* $\mathrm{CO}(X)$ *and any* $f \in A$ *satisfies* $\lim_{\mathscr{F}} |f|_{\varepsilon_A} = \|f\|_{A, \mathscr{F} \cap \mathrm{CO}(X)}$.

(ii) *For any* $x \in \mathscr{M}(A)$, *there exists an ultrafilter* $\mathscr{F}$ *of* $\mathscr{P}(X)$ *such that any* $f \in A$ *satisfies* $\lim_{\mathscr{F}} |f|_{\varepsilon_A} = x(f)$.

We note that Corollary 3.25 is a generalisation of [Mih14, Corollary 2.13], which is itself a generalisation of [EM10, Corollary 16.3].

Proof of Corollary 3.25. (i) The first assertion follows from the fact that the inclusion $\mathrm{CO}(X) \hookrightarrow \mathscr{P}(X)$ is a Boolean algebra homomorphism. Put $F := \mathscr{F} \cap \mathrm{CO}(X) \in \zeta X$. We show $\lim_{\mathscr{F}} |f|_{\varepsilon_A} = \|f\|_{A,F}$ for any $f \in A$. Put $\epsilon := \big| \|f\|_{A,F} - \lim_{\mathscr{F}} |f|_{\varepsilon_A} \big|$, and assume $\epsilon > 0$. By the definition of $\|f\|_{A,F}$, there is a $U_0 \in F$ such that $|f(x)| < \|f\|_{A,F} + 2^{-1}\epsilon$ for any $x \in U_0$. By the definition of $\lim_{\mathscr{F}} |f|_{\varepsilon_A}$, there is a $U_1 \in \mathscr{F}$ such that for any $x \in U_1$ we have $\big| |f(x)| - \lim_{\mathscr{F}} |f|_{\varepsilon_A} \big| < 2^{-1}\epsilon$. Since $U_0 \in F \subset F$ and $U_1 \in F$, we have $U_0 \cap U_1 \neq \emptyset$. Take an $x \in U_0 \cap U_1$. We obtain $\left| \|f\|_{A,F} - \lim_{\mathscr{F}} |f|_{\varepsilon_A} \right| \leq \left| \|f\|_{A,F} - f(x) \right| + |f(x) - \lim_{\mathscr{F}} |f|_{\varepsilon_A}| < \epsilon$, contrary to the definition of ϵ. This implies that $\epsilon = 0$, and hence $\lim_{\mathscr{F}} |f|_{\varepsilon_A} = \|f\|_{A,F}$.

(ii) By Theorem 3.23, there is an $F \in \zeta X$ with $x = \check{\mu}_A(F)$. Since F is a set of non-empty subsets of X closed under finite intersection, there is an ultrafilter $\mathscr{F}$ of $\mathscr{P}(X)$ with $\mathscr{F} \cap \mathrm{CO}(X) = F$. By the argument above, we have $\lim_{\mathscr{F}} |f|_{\varepsilon_A} = \|f\|_{A,F} = x(f)$. ∎

Corollary 3.26. *Suppose that any pair of disjoint closed subsets of X is separated by a clopen subset. Let $(\mathscr{F}_0, \mathscr{F}_1)$ be a pair of ultrafilters of $\mathscr{P}(X)$. Then the following are equivalent:*

(i) *Any $f \in A$ satisfies $\lim_{\mathscr{F}_0} |f|_{\varepsilon_A} = \lim_{\mathscr{F}_1} |f|_{\varepsilon_A}$.*
(ii) *$\mathscr{F}_n \cap \mathrm{CO}(X) = \mathscr{F}_1 \cap \mathrm{CO}(X)$.*
(iii) *For any $(K_0, K_1) \in \mathscr{F}_0 \times \mathscr{F}_1$, if K_0 and K_1 are closed, then $K_0 \cap K_1$ is non-empty.*
(iv) *For any $(K_0, K_1) \in \mathscr{F}_0 \times \mathscr{F}_1$, if K_0 and K_1 are clopen, then $K_0 \cap K_1$ is non-empty.*

We note that Corollary 3.26 is a generalisation of [Mih14, Corollary 2.15], which is itself a generalisation of [EM10, Theorems 1 and 4].

Proof of Corollary 3.26. By Theorem 3.23, (i) is equivalent to (ii). The implication (iii)⇒(iv) is obvious. Suppose (ii). We show (iii). If there were a $(K_0, K_1) \in \mathscr{F}_0 \times \mathscr{F}_1$ such that K_0 and K_1 are disjoint closed subsets, then there would be a $U \in \mathrm{CO}(X)$ with $K_0 \subset U$ and $K_1 \subset X \setminus U$. This contradicts $\mathscr{F}_0 \cap \mathrm{CO}(X) = \mathscr{F}_1 \cap \mathrm{CO}(X)$.

Suppose (iv). We show (i). Let $U \in \mathscr{F}_0 \cap \mathrm{CO}(X)$. Then U and $X \setminus U$ are disjoint closed subsets, and hence $X \setminus U \notin \mathscr{F}_1$ by the assumption. This implies that $U \in \mathscr{F}_1 \cap \mathrm{CO}(X)$. Therefore $\mathscr{F}_0 \cap \mathrm{CO}(X)$ is contained in $\mathscr{F}_1 \cap \mathrm{CO}(X)$. By a symmetric argument, $\mathscr{F}_1 \cap \mathrm{CO}(X)$ is contained in $\mathscr{F}_0 \cap \mathrm{CO}(X)$. We obtain $\mathscr{F}_0 \cap \mathrm{CO}(X) = \mathscr{F}_1 \cap \mathrm{CO}(X)$. This implies that for any $f \in A$, $\lim_{\mathscr{F}_0} |f|_{\varepsilon_A} = \|f\|_{A,\mathscr{F}_0 \cap \mathrm{CO}(X)} = \|f\|_{A,\mathscr{F}_1 \cap \mathrm{CO}(X)} = \lim_{\mathscr{F}_1} |f|_{\varepsilon_A}$ by Corollary 3.25(i). ∎

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References

[Ber90] V. G. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Math. Surveys Monogr. 33, Amer. Math. Soc., 1990.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry*, Grundlehren Math. Wiss. 261, Springer, 1984.

[Dal00] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.) 24, Oxford Univ. Press, 2000.

[Dou72] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Pure Appl. Math. 49, Academic Press, 1972.

[Eng89] R. Engelking, *General Topology*, Sigma Ser. Pure Math. 6, Heldermann, 1989.

[EM10] A. Escassut and N. Mainetti, *Multiplicative spectrum of ultrametric Banach algebras of continuous functions*, Topology Appl. 157 (2010), 2505–2515.

[Far11] I. Farah, *All automorphisms of the Calkin algebra are inner*, Ann. of Math. 173 (2011), 619–661.

[KZ] M. Kaneko and D. Zagier, *Finite multiple zeta values*, in preparation.

[Ked18] K. S. Kedlaya, *On commutative nonarchimedean Banach fields*, Doc. Math. 23 (2018), 171–188.

[Mih14] T. Mihara, *Characterisation of the Berkovich spectrum of the Banach algebra of bounded continuous functions*, Doc. Math. 19 (2014), 769–799.

[Neu99] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. 322, Springer, 1999.

[PW07] N. C. Phillips and N. Weaver, *The Calkin algebra has outer automorphisms*, Duke Math. J. 139 (2007), 185–202.

[Ros15] J. Rosen, *Asymptotic relations for truncated multiple zeta values*, J. London Math. Soc. 91 (2015), 554–572.

[She82] S. Shelah, *Proper Forcing*, Lecture Notes in Math. 940, Springer, 1982.

[She98] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic, Volume 5, Springer, 1998.

[Wal74] R. C. Walker, *The Stone–Čech Compactification*, Ergeb. Math. Grenzgeb. 83, Springer, 1974.

[Wim82] E. L. Wimmers, *The Shelah P-point independence theorem*, Israel J. Math. 43 (1982), 28–48.

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